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An Application of Hayashi's Inequality for Differentiable Functions

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Abstract: In this work, we offer new applications of Hayashi's inequality for differentiable functions by proving new error estimates of the Ostrowski- and trapezoid-type quadrature rules.

Keywords: Hayashi's inequality; Ostrowski's inequality; trapezoidal rule; differentiable functions

MSC: 26D15

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1. Introduction

In ([1], pp. 311–312) Hayashi proved the following theorem.

Theorem 1. *Let $p : [a,b] \to \mathbb{R}$ be a nonincreasing mapping on $[a,b]$ and $h : [a,b] \to \mathbb{R}$ an integrable mapping on $[a,b]$ with $0 \le h(x) \le A$ for all $x \in [a,b]$. Then, the inequality*

$$A\int_{b-\lambda}^{b} p(x)dx \le \int_a^b p(x)h(x)dx \le A\int_a^{a+\lambda} p(x)dx \tag{1}$$

holds, where $\lambda = \frac{1}{A}\int_a^b h(x)dx$.

Inequality (1), called Hayashi's inequality, is a simple generalization of Steffensen's inequality which holds with same assumptions with $A = 1$. For recent results concerning Hayashi's inequality see [2].

In 1996, Agarwal and Dragomir [3] presented an application of this inequality as follows.

Theorem 2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° (the interior of I) and $[a,b] \subseteq I^\circ$ with $M = \sup_{x\in[a,b]} f'(x) < \infty$, $m = \inf_{x\in[a,b]} f'(x) < \infty$ and $M > m$. If f' is integrable on $[a,b]$, then the following inequality holds*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{f(a)+f(b)}{2}\right| \\ &\le \frac{[f(b)-f(a)-m(b-a)][M(b-a)-f(b)+f(a)]}{2(M-m)(m-a)} \\ &\le \frac{(M-m)(b-a)}{8}.\end{aligned} \tag{2}$$

This elegant inequality presents an error estimation for the trapezoidal rule.

In 2002, Gauchman [4] generalized (2) for n-times differentiable functions using the Taylor expansion so that (2) becomes a special case of Gauchman's result when $n = 0$.

In this paper, we present a generalization of (2). In the same argument, two other inequalities of the Ostrowski and trapezoidal type are also introduced.

2. The Results

Let us begin with a generalization of (2).

Theorem 3. *Let $g : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$ with $0 \le g'(t) \le (b-a)$, and suppose that g' is integrable on $[a,b]$. Then*

$$\left|\frac{1}{b-a}\int_a^b g(t)dt - \frac{(x-a)g(a)+(b-x)g(b)}{b-a} - \lambda\left(x - \frac{a+b}{2}\right)\right| \le \frac{\lambda}{2}(b-a-\lambda) \le \frac{(b-a)^2}{8} \quad (3)$$

for all $x \in [a,b]$, where $\lambda = \frac{g(b)-g(a)}{b-a}$. In particular, for $x = \frac{a+b}{2}$, the following inequality holds

$$\left|\frac{1}{b-a}\int_a^b g(t)dt - \frac{g(a)+g(b)}{2}\right| \le \frac{\lambda}{2}(b-a-\lambda) \le \frac{(b-a)^2}{8}.$$

Proof. Let $f(t) = x - t$, $t \in [a,b]$. Applying Hayashi's inequality (1) by setting $p(t) = f(t)$ and $h(t) = g'(t)$, we get

$$(b-a)\int_{b-\lambda}^b (x-t)dt \le \int_a^b (x-t)g'(t)dt \le (b-a)\int_a^{a+\lambda}(x-t)dt \quad (4)$$

where, $A = b - a$ or we write

$$\lambda = \frac{1}{b-a}\int_a^b g'(t)dt = \frac{g(b)-g(a)}{b-a}.$$

Also, we have

$$\int_{b-\lambda}^b (x-t)dt = \lambda(x-b) + \frac{1}{2}\lambda^2,$$

$$\int_a^b (x-t)g'(t)dt = -(x-a)g(a) - (b-x)g(b) + \int_a^b g(t)dt,$$

and

$$\int_a^{a+\lambda}(x-t)dt = \lambda(x-a) - \frac{1}{2}\lambda^2.$$

Substituting the above equalities in (4) and dividing by $(b-a)$, we get

$$\begin{aligned}
&\ell_1(x) := \lambda(x-b) + \frac{1}{2}\lambda^2 \\
&\le -\frac{(x-a)g(a)+(b-x)g(b)}{b-a} + \frac{1}{b-a}\int_a^b g(t)dt := I \\
&\le \lambda(x-a) - \frac{1}{2}\lambda^2 := \ell_2(x).
\end{aligned}$$

We also have

$$\left|I - \frac{\ell_1(x) + \ell_2(x)}{2}\right| = \left|I - \lambda\left(x - \frac{a+b}{2}\right)\right| \leq \frac{\ell_2(x) - \ell_1(x)}{2}$$
$$= \frac{\lambda}{2}(b - a - \lambda)$$

which proves the first inequality in (3). The second inequality follows by applying the same technique as in ([3], pp. 96–97). □

Remark 1. *For some results closely related to Theorem 3 we refer the reader to [5–13].*

A corrected generalized version of the Agarwal-Dragomir result (2) is incorporated in the following corollary.

Corollary 1. *Let $g : [a, b] \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$ with $\gamma \leq g'(t) \leq \Gamma$, and suppose that g' is integrable on $[a, b]$. Then*

$$\left|\frac{1}{b-a}\int_a^b g(t)dt - \frac{(x-a)g(a) + (b-x)g(b)}{b-a} - \frac{g(b) - g(a)}{b-a} \cdot \left(x - \frac{a+b}{2}\right)\right|$$
$$\leq \frac{\Gamma - \gamma}{2} \cdot \lambda \cdot \frac{(b - a - \lambda)}{b-a} \leq \frac{(\Gamma - \gamma)(b-a)}{8} \quad (5)$$

for all $x \in [a, b]$, where $\lambda = \frac{g(b)-g(a)-\gamma(b-a)}{\Gamma-\gamma}$. In particular, for $x = \frac{a+b}{2}$, the following inequality holds

$$\left|\frac{1}{b-a}\int_a^b g(t)dt - \frac{g(a) + g(b)}{2}\right| \leq \frac{\Gamma - \gamma}{2} \cdot \lambda \cdot \frac{(b - a - \lambda)}{b-a} \leq \frac{(\Gamma - \gamma)(b-a)}{8}.$$

Proof. Repeating the proof of Theorem 3, with $h(t) = g'(t) - \gamma$, $t \in [a, b]$, we get the first inequality. The second inequality in (5) follows by applying the same technique as in ([3], pp. 96–97). Analogous manipulation for $x = \frac{a+b}{2}$ gives the same result as in (2). □

Remark 2. *Let the assumptions of Corollary 1 be satisfied. Then Corollaries 3 and 4 and Remarks 1 and 2 in [3] (p. 97) also hold.*

In 1997, Dragomir and Wang [11] introduced an inequality of Ostrowski-Grüss type as follows: inequality

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b) - f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \leq \frac{1}{4}(b-a)(\Gamma - \gamma) \quad (6)$$

holds for all $x \in [a, b]$, where f is assumed to be differentiable on $[a, b]$ with $f' \in L^1[a, b]$ and $\gamma \leq f'(x) \leq \Gamma$, $\forall x \in [a, b]$.

In 1998, another result for twice differentiable was proved in [10]. In 2000, the constant $\frac{1}{4}$ in (6) was improved by $\frac{1}{\sqrt{3}}$ in [14].

A better improvement of (6) can be deduced by applying Hayashi's inequality as it is presented in the following theorem.

Theorem 4. *Let $g : [a, b] \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$ with $0 \leq g'(t) \leq (b - a)$ and suppose that g' is integrable on $[a, b]$. Then*

$$\left|\frac{1}{b-a}\int_a^b g(t)dt - g(x) + \lambda\left(x - \frac{a+b}{2}\right)\right| \leq \lambda\frac{b-a}{2} - \lambda^2 \leq \frac{(b-a)^2}{16} \quad (7)$$

for all $x \in [a, b]$, *where* $\lambda = \frac{g(b)-g(a)}{b-a}$. *In particular, for* $x = \frac{a+b}{2}$, *the following inequality holds*

$$\left| \frac{1}{b-a} \int_a^b g(t)dt - g\left(\frac{a+b}{2}\right) \right| \leq \lambda \frac{b-a}{2} - \lambda^2 \leq \frac{(b-a)^2}{16}.$$

Proof. Fix $x \in [a, b]$ and let $f(t) = a - t$, $t \in [a, x]$. Applying Hayashi's inequality (1) by setting $p(t) = f(t)$ and $h(t) = g'(t)$, we get

$$(b-a) \int_{x-\lambda}^{x} (a-t)dt \leq \int_a^x (a-t)g'(t)dt \leq (b-a) \int_a^{a+\lambda} (a-t)dt \tag{8}$$

where,

$$\lambda = \frac{1}{b-a} \int_a^b g'(t)dt = \frac{g(b)-g(a)}{b-a}.$$

Also, we have

$$\int_{x-\lambda}^{x} (a-t)dt = -\lambda(x-a) + \frac{1}{2}\lambda^2,$$

$$\int_a^x (a-t)g'(t)dt = -(x-a)g(x) + \int_a^x g(t)dt,$$

and

$$\int_a^{a+\lambda} (a-t)dt = -\frac{1}{2}\lambda^2.$$

Substituting in (8), we get

$$(b-a)\left(-\lambda(x-a) + \frac{1}{2}\lambda^2\right) \leq -(x-a)g(x) + \int_a^x g(t)dt \leq -\frac{1}{2}\lambda^2(b-a). \tag{9}$$

Now, let $f(t) = b - t$, $t \in [x, b]$. Applying Hayashi's inequality (1) again we get

$$(b-a) \int_{b-\lambda}^{b} (b-t)dt \leq \int_x^b (b-t)g'(t)dt \leq (b-a) \int_x^{x+\lambda} (b-t)dt \tag{10}$$

where,

$$\int_{b-\lambda}^{b} (b-t)dt = \frac{1}{2}\lambda^2,$$

$$\int_x^b (b-t)g'(t)dt = -(b-x)g(x) + \int_x^b g(t)dt,$$

and

$$\int_x^{x+\lambda} (b-t)dt = \lambda(b-x) - \frac{1}{2}\lambda^2.$$

Substituting in (10), we obtain

$$\frac{1}{2}\lambda^2(b-a) \leq -(b-x)g(x) + \int_x^b g(t)dt \leq (b-a)\left(\lambda(b-x) - \frac{1}{2}\lambda^2\right). \tag{11}$$

Adding (9) and (11) we get

$$
\begin{aligned}
(b-a)\left(-\lambda(x-a)+\lambda^2\right) &\le \int_a^b g(t)dt-(b-a)g(x) \\
&\le (b-a)\left(\lambda(b-x)-\lambda^2\right).
\end{aligned}
$$

Setting

$$
I := \frac{1}{b-a}\int_a^b g(t)dt - g(x)
$$

$$
\ell_1(x) = -\lambda(x-a)+\lambda^2,
$$

and

$$
\ell_2(x) = \lambda(b-x)-\lambda^2.
$$

Therefore,

$$
\begin{aligned}
\left|I-\frac{\ell_1(x)+\ell_2(x)}{2}\right| &= \left|\frac{1}{b-a}\int_a^b g(t)dt - g(x) + \lambda\left(x-\frac{a+b}{2}\right)\right| \\
&\le \frac{\ell_2(x)-\ell_1(x)}{2} \\
&= \lambda\frac{b-a}{2}-\lambda^2,
\end{aligned}
$$

which proves the first inequality in (7). To prove the second inequality, define the mapping $\phi(t) = -t^2 + \frac{b-a}{2}t$. Then $\max\phi(t) = \phi\left(\frac{b-a}{4}\right) = \left(\frac{b-a}{4}\right)^2$, so that $\phi(\lambda) = -\lambda^2 + \frac{b-a}{2}\lambda \le \left(\frac{b-a}{4}\right)^2$, which completes the proof of this theorem. □

Corollary 2. *Let $g : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$ with $\gamma \le g'(t) \le \Gamma$ and suppose that g' is integrable on $[a,b]$. Then*

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b g(t)dt - g(x) + \frac{g(b)-g(a)}{b-a}\cdot\left(x-\frac{a+b}{2}\right)\right| \\
&\le \left(\frac{\Gamma-\gamma}{b-a}\right)\left(\lambda\frac{b-a}{2}-\lambda^2\right) \le \frac{(b-a)(\Gamma-\gamma)}{16}
\end{aligned}
\tag{12}
$$

for all $x \in [a,b]$, where $\lambda = \frac{g(b)-g(a)-\gamma(b-a)}{\Gamma-\gamma}$. In particular, for $x = \frac{a+b}{2}$, the following inequality holds

$$
\begin{aligned}
\left|\frac{1}{b-a}\int_a^b g(t)dt - g\left(\frac{a+b}{2}\right)\right| &\le \left(\frac{\Gamma-\gamma}{b-a}\right)\left(\lambda\frac{b-a}{2}-\lambda^2\right) \\
&\le \frac{(b-a)(\Gamma-\gamma)}{16}.
\end{aligned}
$$

Proof. Repeating the proof of Theorem 4, with $h(t) = g'(t)-\gamma$, $t \in [a,b]$, we get the first inequality. The second inequality (12) follows by applying the same technique. □

Remark 3. *As we notice, (12) improves (6) by $\frac{1}{4}$, which is better than Matić et al. result from [14].*

In [6], under the assumptions of Theorem 4, Alomari proved the following version of a Guessab–Schmeisser-type inequality (see [12]):

$$
\left|\frac{g(x)+g(a+b-x)}{2} - \frac{1}{b-a}\int_a^b g(t)dt\right| \le \frac{(\Gamma-\gamma)(b-a)}{8},
\tag{13}
$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

Next we give an improvement of (13).

Theorem 5. *Let $g : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$ with $0 \le g'(t) \le (b-a)$ and suppose that g' is integrable on $[a,b]$. Then*

$$\left| \frac{1}{b-a} \int_a^b g(t)dt - \frac{g(x) + g(a+b-x)}{2} \right| \le \lambda \left[\frac{b-a}{2} - \frac{3}{2}\lambda \right] \le \frac{(b-a)^2}{24} \tag{14}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$, where $\lambda = \frac{g(b)-g(a)}{b-a}$.

Proof. Fix $x \in \left[a, \frac{a+b}{2}\right]$ and let $f(t) = a - t$, $t \in [a,x]$. Applying Hayashi's inequality (1) by setting $p(t) = f(t)$ and $h(t) = g'(t)$, we get that (8) holds, i.e.,

$$\begin{aligned} (b-a)\left(-\lambda(x-a) + \frac{1}{2}\lambda^2\right) &\le -(x-a)g(x) + \int_a^x g(t)dt \\ &\le -\frac{1}{2}\lambda^2(b-a). \end{aligned} \tag{15}$$

where

$$\lambda = \frac{1}{b-a} \int_a^b g'(t)dt = \frac{g(b) - g(a)}{b-a}.$$

Now, let $f(t) = \frac{a+b}{2} - t$, $t \in [x, a+b-x]$. Applying Hayashi's inequality (1) again we get

$$\begin{aligned} (b-a) \int_{a+b-x-\lambda}^{a+b-x} \left(\frac{a+b}{2} - t\right) dt &\le \int_x^{a+b-x} \left(\frac{a+b}{2} - t\right) g'(t)dt \\ &\le (b-a) \int_x^{x+\lambda} \left(\frac{a+b}{2} - t\right) dt \end{aligned} \tag{16}$$

where

$$\int_{a+b-x-\lambda}^{a+b-x} \left(\frac{a+b}{2} - t\right) dt = -\lambda \left(\frac{a+b}{2} - x\right) + \frac{1}{2}\lambda^2,$$

$$\begin{aligned} &\int_x^{a+b-x} \left(\frac{a+b}{2} - t\right) g'(t)dt \\ &\qquad = -\left(\frac{a+b}{2} - x\right)(g(x) + g(a+b-x)) + \int_x^{a+b-x} g(t)dt, \end{aligned}$$

and

$$\int_x^{x+\lambda} \left(\frac{a+b}{2} - t\right) dt = \lambda \left(\frac{a+b}{2} - x\right) - \frac{1}{2}\lambda^2.$$

Substituting in (16), we get

$$\begin{aligned}
&-(b-a)\left[\lambda\left(\frac{a+b}{2}-x\right)-\frac{1}{2}\lambda^2\right]\\
&\le -\left(\frac{a+b}{2}-x\right)(g(x)+g(a+b-x))+\int_x^{a+b-x}g(t)dt \\
&\le (b-a)\left[\lambda\left(\frac{a+b}{2}-x\right)-\frac{1}{2}\lambda^2\right].
\end{aligned} \tag{17}$$

Now, let $f(t) = b-t$, $t \in [a+b-x, b]$. Applying Hayashi's inequality (1) again we obtain

$$\begin{aligned}
(b-a)\int_{b-\lambda}^{b}(b-t)dt &\le \int_{a+b-x}^{b}(b-t)g'(t)dt \\
&\le (b-a)\int_{a+b-x}^{a+b-x+\lambda}(b-t)dt
\end{aligned} \tag{18}$$

where

$$\int_{b-\lambda}^{b}(b-t)dt = \frac{1}{2}\lambda^2,$$

$$\int_{a+b-x}^{b}(b-t)g'(t)dt = -(x-a)g(a+b-x)+\int_{a+b-x}^{b}g(t)dt,$$

and

$$\int_{a+b-x}^{a+b-x+\lambda}(b-t)dt = \lambda(x-a)-\frac{1}{2}\lambda^2.$$

Substituting in (18) we get

$$\begin{aligned}
\frac{1}{2}\lambda^2(b-a) &\le -(x-a)g(a+b-x)+\int_{a+b-x}^{b}g(t)dt \\
&\le (b-a)\left[\lambda(x-a)-\frac{1}{2}\lambda^2\right].
\end{aligned} \tag{19}$$

Adding (15), (17) and (19) we obtain

$$\begin{aligned}
-\lambda(b-a)\left[\frac{b-a}{2}-\frac{3}{2}\lambda\right] &\le \int_a^b g(t)dt-(b-a)\frac{g(x)+g(a+b-x)}{2} \\
&\le \lambda(b-a)\left[\frac{b-a}{2}-\frac{3}{2}\lambda\right],
\end{aligned}$$

which is equivalent to the first inequality in (14). To prove the second inequality in (14), define the mapping $\phi(t) = -\frac{3}{2}(b-a)t^2+\frac{(b-a)^2}{2}t$. Then $\max\phi(t) = \phi\left(\frac{b-a}{6}\right) = \frac{(b-a)^2}{24}$, so that $\phi(\lambda) = -\frac{3}{2}(b-a)\lambda^2+\frac{(b-a)^2}{2}\lambda \le \frac{(b-a)^2}{24}$, which completes the proof. □

A generalization of (13) and (14) is incorporated in the following result.

Corollary 3. *Let $g : [a, b] \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$ with $\gamma \le g'(t) \le \Gamma$ and suppose that g' is integrable on $[a, b]$. Then*

$$\left|\frac{1}{b-a}\int_a^b g(t)dt - \frac{g(x)+g(a+b-x)}{2}\right| \le \frac{1}{2}\lambda\left(\frac{\Gamma-\gamma}{b-a}\right)[(b-a)-3\lambda] \le \frac{(\Gamma-\gamma)(b-a)}{24} \quad (20)$$

for all $x \in \left[a, \frac{a+b}{2}\right]$, where $\lambda = \frac{g(b)-g(a)-\gamma(b-a)}{\Gamma-\gamma}$.

Proof. Repeating the proof of Theorem 5, with $h(t) = g'(t) - \gamma$, $t \in [a, b]$, we get the first inequality. The second inequality in (20) follows by applying the same technique. □

3. Applications

Let X be a random variable taking values in the finite interval $[a, b]$, with the probability density function $f : [a, b] \to [0, 1]$ with the cumulative distribution function $F(x) = Pr(X \le x) = \int_a^b f(t)dt$.

Theorem 6. *With the assumptions of Theorem 4, we have the inequality*

$$\left|\frac{1}{2}[F(x)+F(a+b-x)] - \frac{b-E(X)}{b-a}\right| \le \frac{1}{2}\lambda\left(\frac{\Gamma-\gamma}{b-a}\right)[(b-a)-3\lambda] \le \frac{(\Gamma-\gamma)(b-a)}{24}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$, where $\lambda = \frac{F(b)-F(a)-\gamma(b-a)}{\Gamma-\gamma}$, and $E(X)$ is the expectation of X.

Proof. In the proof of Corollary 3, we set $f = F$, and take into account that

$$E(X) = \int_a^b t dF(t) = b - \int_a^b F(t)dt.$$

We leave the details to the interested reader. □

4. Conclusions

This paper summarises several types of general quadrature rules, such as the general trapezoidal rule or the so-called Ostrowski trapezoidal, Ostrowski midpoint and Guessab–Schmeisser quadrature rules for symmetric points. Using the presented inequalities, several error estimates of the above quadrature rules could therefore be derived with corresponding numerical experiments. This work thus represents a very good application of Hayashi's inequality in quadrature approximation. One future research direction might be to use Fink's generalization of the Ostrowski inequality to obtain some Hayashi–Ostrowski-type results. Further applications of Hayashi's inequality we leave to the interested reader.

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