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Article

Uniform (C_k, P_{k+1})-Factorizations of $K_n - I$ When k Is Even

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Abstract: Let H be a connected subgraph of a graph G. An H-factor of G is a spanning subgraph of G whose components are isomorphic to H. Given a set $\mathcal{H}$ of mutually non-isomorphic graphs, a uniform $\mathcal{H}$-factorization of G is a partition of the edges of G into H-factors for some $H \in \mathcal{H}$. In this article, we give a complete solution to the existence problem for uniform (C_k, P_{k+1})-factorizations of $K_n - I$ in the case when k is even.

Keywords: graph factorization; complete graph; block design

MSC: 05B30

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1. Introduction

Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G, respectively. As per standard notations, K_n denotes the complete graph on n vertices, C_k is the k-cycle (i.e., the cycle of length k) and P_{k+1} is the path on $k+1$ vertices. For missing notions and terms that are not explicitly defined in this paper, we point the reader to [1] and its online updates. If $\mathcal{H}$ is a set of mutually non-isomorphic connected graphs, an *$\mathcal{H}$-decomposition* of a graph G is a partition of $E(G)$ into subgraphs (*blocks*) that are isomorphic to some element of $\mathcal{H}$. An *$\mathcal{H}$-factor* of G is a spanning subgraph of G, i.e., a subgraph of G with the same vertex set as G, whose connected components are isomorphic to some element of $\mathcal{H}$. An *$\mathcal{H}$-factorization* of G is an $\mathcal{H}$-decomposition of G whose set of blocks admits a partition into $\mathcal{H}$-factors. An $\mathcal{H}$-factorization of G is also known as a *resolvable* $\mathcal{H}$-decomposition of G and an $\mathcal{H}$-factor of G can be called a *parallel class* of G. When $\mathcal{H} = \{H\}$, then we simply write H-factor and H-factorization. A K_2-factorization of G is better known as a 1-*factorization* and its factors are said 1-*factors*; a 1-factor of K_n is a set of $\frac{n}{2}$ mutually vertex disjoint edges of K_n and a 1-factorization of K_n exists if and only if n is even [2]. A C_k-factorization of K_n exists if and only if $3 \leq k \leq n$, n and k are odd and $n \equiv 0 \pmod{k}$ [3]. An $\mathcal{H}$-factorization of a graph G is said to be *uniform* if each factor is an H-factor for some $H \in \mathcal{H}$ (sometimes it is referred to as a uniformly resolvable $\mathcal{H}$-decomposition of G).

In the context of graph factorizations, and in particular of cycle factorizations, the most famous problems are the *Oberwolfach Problem* and the *Hamilton–Waterloo Problem*. The first one was first posed in 1967 by G. Ringel and asks whether it is possible to seat n mathematicians at m round tables in $(n-1)/2$ dinners so that every two mathematicians sit next to each other exactly once. This puzzle can be formalized in terms of graph factorizations as follows. If integers $p_1, p_2, \ldots, p_m$ denote the sizes of the m round tables, then the solution of the Oberwolfach Problem is a factorization of K_n where each factor has m components which are isomorphic to cycles of length $p_1, p_2, \ldots, p_m$, $\sum_{i=1}^{m} p_i = n$. It is well known that such a factorization can exist only if n is odd. If the number n is even, then an analogous problem is reformulated in terms of decomposition of $K_n - I$, that is the graph obtained by removing a 1-factor from K_n. The version where all cycles of a factor have the same size is called the uniform Oberwolfach Problem, which has been completely solved by Alspach and Häggkvist [4] and Alspach, Schellenberg, Stinson and Wagner [3]. The Hamilton–Waterloo Problem is a variation of the Oberwolfach Problem and requires that the dining mathematicians have their dinners in two different venues. In this case, the factors of the sought decomposition of K_n (when n is odd) or $K_n - I$ (when n is even) can have either s components that are isomorphic to cycles of length $p_1, p_2, \ldots, p_s$ or t components that are isomorphic to cycles of length $q_1, q_2, \ldots, q_t$, $\sum_{i=1}^{s} p_i = \sum_{i=1}^{t} q_i = n$. If the tables in one venue sit p mathematicians and those in the other venue sit q each, then the problem is called the uniform Hamilton–Waterloo Problem, which asks for a decomposition of K_n or $K_n - I$ into C_p-factors and C_q-factors. For both problems, the Hamilton–Waterloo Problem and the non-uniform case of the Oberwolfach Problem, many partial results are known, but a complete solution is far to be achieved.

Existence problems for $\mathcal{H}$-factorizations are usually considered for the complete graph K_n or the graph $K_n - I$. For these graph families, many results have been obtained especially in the uniform case; just to give some examples, when $\mathcal{H}$ contains two complete graphs of order $k \leq 5$ [5–8], when $\mathcal{H}$ contains two or three paths of order $2 \leq k \leq 4$ [9,10], for $\mathcal{H} = \{K_2, K_{1,3}\}$ [11,12], for $\mathcal{H} = \{K_2, K_{1,4}\}$ [13], and for $\mathcal{H} = \{C_{2k}, K_{1,2k}\}$ [14].

A uniform $\{H_1, H_2\}$-factorization of G with r_i H_i-factors, $i = 1, 2$, is denoted by $\mathrm{URD}(G; H_1^{r_1}, H_2^{r_2})$. When $G = K_n$ we simply write $\mathrm{URD}(n; H_1^{r_1}, H_2^{r_2})$. In this paper, we deal with uniform $\mathcal{H}$-factorizations of K_n or $K_n - I$ in the case when $\mathcal{H} = \{C_k, P_{k+1}\}$. In [9], a solution to the existence problem of a $\mathrm{URD}(n; C_k^{r_1}, P_{k+1}^{r_2})$ is given for $k = 2$ (note that $C_2 = K_2$). Here, we are interested in the case when $k \equiv 0 \pmod 2$ and $k \geq 4$. As for the k even case, it is known that a $\mathrm{URD}(n; C_k^0, P_{k+1}^{r_2})$ exists if and only if $n \equiv 0 \pmod{k+1}$ and $(k+1)(n-1)n \equiv 0 \pmod{2k}$, see [15,16], while no $\mathrm{URD}(n; C_k^{r_1}, P_{k+1}^0)$ exists because n must be odd and divisible by k; a $\mathrm{URD}(K_n - I; C_k^{r_1}, P_{k+1}^0)$ exists if and only if $n \equiv 0 \pmod 2$ and k divides n, see [17]. When n is even, no $\mathrm{URD}(n; C_k^{r_1}, P_{k+1}^{r_2})$ exists with $r_1 > 0$ because, otherwise, the resolvability implies $2(k+1)r_1 + 2kr_2 = (k+1)(n-1)$ and, clearly, this is impossible. Therefore, it would be interesting to prove whether or not there exist uniformly resolvable decompositions of $K_n - I$ in terms of factors belonging to $\mathcal{H} = \{C_k, P_{k+1}\}$. For brevity, we introduce the notation $\mathrm{URD}^*(n; C_k^{r_1}, P_{k+1}^{r_2})$ for a decomposition of this kind. Moreover, since k and $k+1$ must divide n, we assume $n \equiv 0 \pmod{k(k+1)}$. Finally, we must have $r_1 > 0$ because $(n-2)(k+1)/(2k)$ is not an integer.

The goal of this paper is to characterize the existence of $\mathrm{URD}^*(n; C_k^{r_1}, P_{k+1}^{r_2})$ in the previously defined cases, namely, $k \equiv 0 \pmod 2$, $k \geq 4$, and $n \equiv 0 \pmod{k(k+1)}$. Our main result, shown in the last section, proves that such decompositions exist if and only if the (ordered) pair (r_1, r_2) belongs to the set $I(n)$ defined in Table 1.

Table 1. The set $I(n)$.

n	$I(n)$
$0 \pmod{2k(k+1)}$	$(\frac{n-2}{2} - kx, (k+1)x)$, $x = 0, 1, \ldots, \frac{n-2k}{2k}$
$k(k+1) \pmod{2k(k+1)}$	$(\frac{n-2}{2} - kx, (k+1)x)$, $x = 0, 1, \ldots, \frac{n-k}{2k}$

We remark that, since n is an even positive integer, $I(n)$ is a set of ordered pairs of integers.

To this goal, we firstly recall in Section 2 known decompositions of simple cases and two basic constructions, the so-called GDD Construction and Filling Construction, which allow to derive decompositions of more general cases from the knowledge of simpler cases. Our main theorem is crucially based on a clever use of these two constructions. In Section 3, we derive the necessary conditions for the existence of $\mathrm{URD}^*(n; C_k^{r_1}, P_{k+1}^{r_2})$, see Lemma 2. Moreover, we set up preliminary results for later use, consisting of particular decompositions of certain simple graphs. On the basis of these results, in Section 4, we prove that the necessary conditions derived in Lemma 2 are also sufficient. Section 5 essentially contains the statement of our main theorem, the proof of which boils down to a recall of the partial results of the previous sections.

2. Two Constructions

In what follows, $K_{u(g)}$ denotes the complete multipartite graph with u partite sets of size g. An $\mathcal{H}$-decomposition of $K_{u(g)}$ is known as a *group divisible decomposition* (briefly, $\mathcal{H}$-GDD) of type g^u; the partite sets are called *groups*. An $\mathcal{H}$-decomposition of K_n can be regarded as an $\mathcal{H}$-GDD of type 1^n. When $\mathcal{H} = \{H\}$, we simply write H-GDD. In what follows, a (uniformly) resolvable $\mathcal{H}$-GDD is denoted by $\mathcal{H}$-(U)RGDD. More precisely, a $\{H_1, H_2\}$-URGDD with r_i H_i-factors is denoted by URGDD$(H_1^{r_1}, H_2^{r_2})$. It is not hard to see that the number of H-factors of an H-RGDD is

$$\alpha = \frac{g(u-1)|V(H)|}{2|E(H)|}.$$

Let H be a given graph. For any positive integer t, $H_{(t)}$ denotes the graph with vertex set $V(H) \times \mathbb{Z}_t$ and edge set $\{\{x_i, y_j\} : \{x, y\} \in E(H), i, j \in \mathbb{Z}_t\}$, where the subscript notation a_i denotes the pair (a, i). We say that the graph $H_{(t)}$ is obtained from H by expanding each vertex t times. When $H = K_m$, the graph $H_{(t)}$ is the complete equipartite graph

$$K_{\underbrace{t, t, \ldots, t}_{m \text{ times}}}$$

with m partite sets of size t and is denoted by $K_{m(t)}$. Analogously, $C_{m(t)}$ denotes the graph $H_{(t)}$ where H is an m-cycle.

Remark 1. *The graph $H_{(t)}$ admits t 1-factors for each 1-factor of G. Therefore, since a $2k$-cycle has two 1-factors, then $C_{2k(t)}$ admits $2t$ 1-factors.*

Given two pairs (r_1, r_2) and (r_1', r_2') of non-negative integers, define $(r_1, r_2) + (r_1', r_2') = (r_1 + r_1', r_2 + r_2')$. Given two sets I and I' of pairs of non-negative integers and a positive integer α, then $I + I'$ denotes the set

$$\{(r_1, r_2) + (r_1', r_2') : (r_1, r_2) \in I, (r_1', r_2') \in I'\}.$$

Moreover, we denote $\alpha * I$ the set whose elements are all pairs of non-negative integers obtained by adding any α elements of I (repetitions of elements of I are allowed).

To obtain our main result we firstly construct RGDDs with appropriate parameters by means of the GDD-Construction defined here below, see Theorem 1. Subsequently, we fill their groups using the Filling Construction, stated in Theorem 2. The GDD-Construction can be derived from the more general construction described in [14]. The Filling Construction is a minor variation of the corresponding construction in [14].

Theorem 1 (GDD-Construction)**.** *Let t be a positive integer and suppose there exists an $\mathcal{H}$-RGDD of type g^u, whose blocks are graphs of order at least 2 and whose factors are F_i, $i = 1, 2, \ldots, \alpha$. If for any $i = 1, 2, \ldots, \alpha$ there exists a URD$(B_{(t)}; C_k^{\bar{r}_1}, P_{k+1}^{\bar{r}_2})$ for each $B \in F_i$ and for each $(\bar{r}_1, \bar{r}_2) \in I_i$, then so does URGDD$(C_k^{r_1}, P_{k+1}^{r_2})$ of type $(gt)^u$ for each $(r_1, r_2) \in I_1 + I_2 + \cdots + I_\alpha$.*

Theorem 2 (Filling Construction)**.** *Suppose there exists a URGDD$(C_k^{r_1}, P_{k+1}^{r_2})$ of type g^u for each $(r_1, r_2) \in I$. If there exists a URD$^*(g; C_k^{r_1'}, P_{k+1}^{r_2'})$, for each $(r_1', r_2') \in I'$, then so does a URD$^*(ug; C_k^{\bar{r}_1}, P_{k+1}^{\bar{r}_2})$, for each $(\bar{r}_1, \bar{r}_2) \in I' + I$.*

Proof. Fixed any pairs $(r_1, r_2) \in I$ and $(r_1', r_2') \in I'$, for every group G_i, $i = 1, 2, \ldots, u$, of a URGDD$(C_k^{r_1}, P_{k+1}^{r_2})$ of type g^u, place a copy of a URD$^*(g; C_k^{r_1'}, P_{k+1}^{r_2'})$ on G_i so to obtain a URD$^*(gu; C_k^{r_1'+r_1}, P_{k+1}^{r_2'+r_2})$. □

We conclude this section by quoting from [18] the following result for a later use.

Lemma 1. *Let $m \geq 3$ and $u \geq 2$. There exists a C_m-RGDD of type g^u if and only if $g(u-1) \equiv 0 \pmod 2$, $gu \equiv 0 \pmod m$, $m \equiv 0 \pmod 2$ if $u = 2$, and $(g, u, m) \neq (2, 3, 3)$, $(2, 6, 3)$, $(6, 2, 6)$, or $(6, 3, 3)$.*

3. Necessary Conditions and Basic Decompositions

Let $k \equiv 0 \pmod 2$, $k \geq 4$. In this section, we start by giving necessary conditions for the existence of a URD$^*(n; C_k^{r_1}, P_{k+1}^{r_2})$, and then, we construct the basic decompositions which will be used as ingredients in the GDD and Filling Constructions. From now on, throughout the paper, we set $p = k(k+1)$. Recall that the set $I(n)$ is defined in Table 1.

Lemma 2. *Let $n \equiv 0 \pmod p$. If there exists a URD$^*(n; C_k^{r_1}, P_{k+1}^{r_2})$ then $(r_1, r_2) \in I(n)$.*

Proof. The resolvability implies

$$\frac{r_1 kn}{k} + \frac{r_2 kn}{k+1} = \frac{n(n-2)}{2}$$

and so

$$2(k+1)r_1 + 2kr_2 = (k+1)(n-2). \tag{1}$$

Since $k+1$ cannot divide $2k$, Equation (1) implies $r_2 = (k+1)x$. Replacing $r_2 = (k+1)x$ in the above equation gives $r_1 = \frac{n-2}{2} - kx$, where $x < \frac{n-2}{2k}$ (because r_1 is a positive integer) and so $0 \leq x \leq \lfloor \frac{n-2}{2k} \rfloor$. □

From now on, we denote by $(a_1, a_2, \ldots, a_k)$ the k-cycle on $\{a_1, a_2, \ldots, a_k\}$ with edge set $\{\{a_1, a_2\}, \{a_2, a_3\}, \ldots, \{a_{k-1}, a_k\}, \{a_k, a_1\}\}$, and by $[a_1, a_2, \ldots, a_{k+1}]$ the path P_{k+1} on the vertex set $\{a_1, a_2, \ldots, a_{k+1}\}$ with edge set $\{\{a_1, a_2\}, \{a_2, a_3\}, \ldots, \{a_k, a_{k+1}\}\}$.

Lemma 3. *There exists a C_{2l-2}-decomposition of $P_{l(2)}$ for any integer $l \geq 3$.*

Proof. Let $P_{l(2)}$ be the graph obtained from the path $[1, 2, \ldots, l]$ by expanding each vertex twice. Consider the $(2l-2)$-cycles

$$C = (1_0, 2_0, \ldots, (l-1)_0, l_1, (l-1)_1, (l-2)_1, \ldots, 2_1),$$

and

$$\bar{C} = (1_1, 2_0, 3_1, 4_0, \ldots, (l-1)_1, l_0, (l-1)_0, (l-2)_1, (l-3)_0, (l-4)_1, \ldots, 3_0, 2_1)$$

if l is even, or

$$\bar{C} = (1_1, 2_0, 3_1, 4_0, \ldots, (l-2)_1, (l-1)_0, l_0, (l-1)_1, (l-2)_0, (l-3)_1, (l-4)_1, \ldots, 3_0, 2_1),$$

if l is odd. It is easy to see that C and $\bar{C}$ decompose the graph $P_{l(2)}$. □

Lemma 4. *Let $q = \frac{k}{2}(1+k)$. There exists a C_k-factorization of $C_{q(2)}$.*

Proof. Start from the cycle $C_q = (1, 2, \ldots, q)$ and decompose it into the following copies of P_l, with $l = 1 + \frac{k}{2}$,

$$P^{(i)} = \left[1 + \frac{k}{2}i, 2 + \frac{k}{2}i, \ldots, 1 + \frac{k}{2}(1+i)\right],\ i = 0, 1, \ldots, k.$$

Expand twice each vertex of C_q and for every $i = 0, 1, \ldots, k$, decompose the graph $P_{l(2)}$ on $V(P^{(i)}) \times \mathbb{Z}_2$ into the k-cycles C_i and $\bar{C}_i$ by using Lemma 3. The set of k-cycles $\{C_i, \bar{C}_i\}_{i=0,1,\ldots,k}$ is a decomposition of $C_{q(2)}$ whose cycles can be partitioned into the factors $\{C_i\}_{i=0,1,\ldots,k}$ and $\{\bar{C}_i\}_{i=0,1,\ldots,k}$. □

Lemma 5. *There exists a* 1*-factorization of* $C_{l(2)}$ *for any integer* $l \geq 3$.

Proof. If l is even, since C_l can be decomposed into two 1-factors, then $C_{l(2)}$ can be decomposed into four 1-factors (see Remark 1). If l is odd, $C_{l(2)}$ can be decomposed into the $2l$-cycles

$$C_1 = (1_0, 2_1, 3_0, 4_1, \ldots, l_0, 1_1, l_1, (l-1)_0, (l-2)_1 (l-3)_0, \ldots, 2_0)$$

and

$$C_2 = (1_0, l_0, (l-1)_0, (l-2)_0, (l-3)_0, \ldots, 2_0, 1_1, 2_1, 3_1, 4_1, \ldots, l_1),$$

each of which provides two 1-factors and so a 1-factorization of $C_{l(2)}$ is given. □

The following lemma follows by a result first proved by R. Laskar in [19]. For the ease of the reader, here, we propose an alternative proof which uses Graeco-Latin squares.

Lemma 6. *Let* $k \neq 4, 12$ *and* $m = k + 1$. *Then, there exists a Hamiltonian cycle decomposition of* $C_{m(\frac{k}{2})}$.

Proof. Consider the graph $C_{m(\frac{k}{2})}$ obtained from the cycle $(1, 2, \ldots, m)$ by expanding each vertex $\frac{k}{2}$ times. Let Q be a Graeco-Latin square of order $\frac{k}{2}$ on the sets $X_1 = \{1\} \times Z_{\frac{k}{2}}$ and $X_2 = \{2\} \times \mathbb{Z}_{k/2}$, which exists for any $\frac{k}{2} \neq 2, 6$, see [20]. The columns of Q give a 1-factorization F_j, $j \in \mathbb{Z}_{k/2}$, of the complete bipartite graph with partite sets X_1 and X_2. For $i = 1, 2, \ldots, m$ and $j \in \mathbb{Z}_{k/2}$, consider the $\frac{k}{2} \times 1$ matrices $A_i^j = [i_j\ \ i_j\ \ \cdots\ \ i_j]^t$ and

$$A_{(i,j)} = [i_j\ \ i_{j+1}\ \ \cdots\ \ i_{j+\frac{k}{2}-1}]^t.$$

Now, for each $j \in Z_{\frac{k}{2}}$, construct the $\frac{k}{2} \times m$ matrix

$$A_j = [F_j\ \ A_3^j\ \ A_{(4,j)}\ \ A_5^j\ \ A_{(6,j)}\ \ \cdots\ \ A_{(m-1,j)}\ \ A_m^j].$$

The rows of the $\frac{k}{2} \times \frac{km}{2}$ matrix $A = [A_0\ \ A_1\ \ \cdots\ \ A_{\frac{k}{2}-1}]$ give a Hamiltonian cycle decomposition of $C_{m(k/2))}$. □

Lemma 7. *A* URD$(C_{m(k)}; C_k^{r_1}, P_{k+1}^{r_2})$ *with* $m = k+1$ *exists for every* $(r_1, r_2) \in \{(k, 0), (0, k+1)\}$.

Proof. The proof is divided into two parts, which respectively cover the case $(r_1, r_2) = (k, 0)$ and $(r_1, r_2) = (0, k+1)$.

1. *Case* $(r_1, r_2) = (k, 0)$. If $k \neq 4, 12$, start from a Hamiltonian cycle decomposition of $C_{m(\frac{k}{2})}$ (which exists by Lemma 6 and has $\frac{k}{2}$ cycles) and, after expanding each vertex twice, for each cycle C on $V(C) \times \mathbb{Z}_2$, place a C_k-factorization of $C_{q(2)}$, $q = \frac{k}{2}(1+k)$ (which exists by Lemma 4 and has two factors) so to obtain a C_k-factorization of $C_{m(k)}$ with k factors, i.e., a URD$(C_{m(k)}; C_k^k, P_{k+1}^0)$. For $k = 4, 12$, start from a 1-factorization of

$C_{m(2)}$ (by Lemma 5, it exists and has four factors) and after expanding each vertex $\frac{k}{2}$ times, for each 1-factor F and each edge $e \in F$ on $e \times \mathbb{Z}_{\frac{k}{2}}$, place a C_k-RGDD of type $(\frac{k}{2})^2$, which is known to exist and have $\frac{k}{4}$ C_k-factors [21], so obtain a URD$(C_{m(k)}; C_k^k, P_{k+1}^0)$.

2. *Case* $(r_1, r_2) = (0, k+1)$. Starting from $C = (0, 1, \ldots, k)$ on $\mathbb{Z}_{k+1}$, expand each vertex k times and take the factors

$$F_j = \{[j_i, (1+j)_i, (2+j)_{i+1}, (3+j)_i, (4+j)_{i+2}, \ldots, (k-1+j)_i, (k+j)_{i+\frac{k}{2}}] : i \in \mathbb{Z}_k\},$$

for $j \in \mathbb{Z}_{k+1}$.

□

4. Sufficient Conditions

Lemma 8. *If $n \equiv 0 \pmod{2p}$, then a URD$^*(n; C_k^{r_1}, P_{k+1}^{r_2})$ exists for every $(r_1, r_2) \in I(n)$.*

Proof. Let $n = 2ph$, $h \geq 1$. Apply the GDD-Construction with $t = k$ to a C_{k+1}-RGDD of type $2^{(k+1)h}$ (which exists by Lemma 1 and has $\alpha = (k+1)h - 1$ factors) to obtain a URGDD$(C_k^{\bar{r}_1}, P_{k+1}^{\bar{r}_2})$ of type $(2k)^{(k+1)h}$ for each

$$(\bar{r}_1, \bar{r}_2) \in [(k+1)h - 1] * \{(k, 0), (0, k+1)\}$$

using as ingredients designs from Lemma 7. Finally, apply the Filling Construction by using copies of a URD$^*(2k; C_k^{k-1}, P_{k+1}^0)$ (see [17]) to get a URD$^*(2ph; C_k^{r_1}, P_{k+1}^{r_2})$ for every

$$\begin{aligned}(r_1, r_2) &\in \{(k-1, 0)\} + [(k+1)h - 1] * \{(k, 0), (0, k+1)\} \\ &= \{(ph - 1 - kx, (k+1)x) : x = 0, 1, \ldots, (k+1)h - 1\} \\ &= I(2ph) = I(n).\end{aligned}$$

□

Lemma 9. *If $n \equiv p \pmod{2p}$, then a URD$^*(n; C_k^{r_1}, P_{k+1}^{r_2})$ exists for every $(r_1, r_2) \in I(n)$.*

Proof. Let $n = p(1+2h)$, $h \geq 0$. Apply the GDD-Construction with $t = k$ to a C_{k+1}-RGDD of type $1^{(k+1)(1+2h)}$ (which exists by Lemma 1 and has $\alpha = \frac{(k+1)(1+2h)-1}{2}$ factors) to obtain a URGDD$(C_k^{\bar{r}_1}, P_{k+1}^{\bar{r}_2})$ of type $k^{(k+1)(1+2h)}$ for each

$$(\bar{r}_1, \bar{r}_2) \in \frac{(k+1)(1+2h) - 1}{2} * \{(k, 0), (0, k+1)\}$$

using as ingredients the designs from Lemma 7. Finally, apply the Filling Construction by using copies of a URD$^*(k; C_k^{\frac{k-2}{2}}, P_{k+1}^0)$ (see [17]) to get a URD$^*(p(1+2h); C_k^{r_1}, P_{k+1}^{r_2})$ for every

$$\begin{aligned}(r_1, r_2) &\in \left\{\left(\frac{k-2}{2}, 0\right)\right\} + \frac{(k+1)(1+2h) - 1}{2} * \{(k, 0), (0, k+1)\} \\ &= \left\{\left(\frac{p(1+2h) - 2}{2} - kx, (k+1)x\right) : x = 0, 1, \ldots, \frac{(k+1)(1+2h) - 1}{2}\right\} \\ &= I(p(1+2h)) = I(n).\end{aligned}$$

□

5. Conclusions

Combining together Lemmas 2, 8 and 9 gives our main result.

Theorem 3. *Let* $n \equiv 0 \pmod{k(k+1)}$. *There exists a* $URD^*(n; C_k^{r_1}, P_{k+1}^{r_2})$ *if and only if* $(r_1, r_2) \in I(n)$.

We emphasize that our main result fits in the context of a series of papers, where the authors investigated the existence of $\mathcal{H}$-factorizations of K_n or $K_n - I$ in the case that $\mathcal{H}$ contains at least one cycle. As a final note, we stress that determining necessary and sufficient conditions for the existence of analogous decompositions for odd values of k is still an open problem of definite interest for further research.

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