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Variation Inequalities for the Hardy-Littlewood Maximal Function on Finite Directed Graphs

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Abstract: In this paper, the authors establish the bounds for the Hardy–Littlewood maximal operator defined on a finite directed graph $\vec{G}$ in the space $\mathrm{BV}_p(\vec{G})$ of bounded p-variation functions. More precisely, the authors obtain the BV_p norms of $M_{\vec{G}}$ for some directed graphs $\vec{G}$.

Keywords: finite directed graph; Hardy–Littlewood maximal operator; bounded variation

MSC: Primary 42B25; 05C12; Secondary 46E35; 05C20

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1. Introduction

The regularity theory of maximal operator has been an active research topic in harmonic analysis. A basic question related to this theory is whether the following inequality

$$\|\nabla \mathcal{M} f\|_{L^p(\mathbb{R}^n)} \leq C\|\nabla f\|_{L^p(\mathbb{R}^n)} \tag{1}$$

holds or not for some $p \in [1, \infty]$ under the condition that $f \in W^{1,p}(\mathbb{R}^n)$. Here, M denotes the centered Hardy–Littlewood maximal operator. This question was first addressed by Kinnunen [1] in 1997 when he established inequality (1) for all $1 < p \leq \infty$. Since then, Kinnunen's result was extended to various versions (see [2–5]). Since the above result did not include the endpoint case $p = 1$, whether inequality (1) holds for $p = 1$ is a certainly complex question. This question was well addressed in dimension $n = 1$. Particularly, Aldaz and Pérez Lázaro [6] stated that the one-dimension uncentered Hardy–Littlewood maximal function $\widetilde{\mathcal{M}} f$ is absolutely continuous and

$$\mathrm{Var}(\widetilde{\mathcal{M}} f) \leq \mathrm{Var}(f), \tag{2}$$

where f is of bounded variation on $\mathbb{R}$ and $\mathrm{Var}(f)$ denotes the total variation of f on $\mathbb{R}$. This yields (1) holds for $\widetilde{\mathcal{M}}$ with $p = 1$, $n = 1$ and $C = 1$. Recently, inequality (2) was extended to the fractional version in [7]. In [8], Carneiro et al. established the continuity of the map $f \mapsto (\widetilde{\mathcal{M}} f)'$ from $W^{1,1}(\mathbb{R})$ to $L^1(\mathbb{R})$. For the centered case, Kurka [9] proved inequality (1) with $p = 1$, $n = 1$ and C = 240,004 and inequality (2) for $\mathcal{M}$ with constant C = 240,004. When $n \geq 2$, Luiro [10] showed that inequality (1) holds for $\widetilde{\mathcal{M}}$ and all radial functions in $W^{1,1}(\mathbb{R}^n)$, which was later extended to the fractional case in [11]. We can consult [12–19] and the references therein for other interesting works.

Motivated by the works in [6–8], Liu and Xue [20] firstly investigated the bounds for the Hardy–Littlewood maximal operator in the space of bounded p-variation functions on finite connected graphs (see also [21]). We now introduce some definitions. We denote by $G = (V, E)$ an undirected simple and finite combinatorial graph, where V is the set of vertices and E is the set of edges. Let $A \subset V$, the notation $|A|$ is the cardinality of A. Let

$x, y \in V$. If there exists an edge $x \sim y \in E$, then x and y are called neighbors. For any $v \in V$, we use the notation $N_G(v)$ to denote the set of neighbors of v. If $|V| < \infty$, then G is said to be finite. If for any distinct $x, y \in V$, we can find a finite sequence of vertices $\{x_i\}_{i=0}^k$, $k \in \mathbb{N}$, such that $x = x_0 \sim x_1 \sim \cdots \sim x_k = y$, then the graph is said to be connected. For two given vertices $u, v \in V$, we denote by $d_G(u,v)$ the distance of u and v, which is the number of edges in a shortest path connecting u and v. Let $B_G(v,r)$ be the ball centered at v, with radius r on the graph G, i.e.,

$$B_G(v,r) = \{u \in V : d_G(u,v) \le r\}.$$

Particularly, $B_G(v,r) = \{v\}$ if $0 \le r < 1$ and $B_G(v,r) = \{v\} \cup N_G(v)$ if $1 \le r < 2$. Let f be a function defined on V, we define the Hardy–Littlewood maximal operator on G by

$$M_G f(v) = \sup_{r \ge 0} \frac{1}{|B_G(v,r)|} \sum_{w \in B_G(v,r)} |f(w)|.$$

In the past few years, many scholars have been devoted to studying the Hardy–Littlewood maximal operators on graphs (see [20–25]). In particularly, Liu and Xue [20] firstly studied the regularity results of maximal operators on graphs. In [20], the authors introduced the space of bounded p-variation functions on graph.

Definition 1 ($\mathrm{BV}_p(G)$ space)**.** *Let $G = (V, E)$ be a graph, where V is the set of vertices and E is the set of edges E. For $0 < p \le \infty$, the space of bounded p-variation functions is defined by*

$$\mathrm{BV}_p(G) := \{f : V \to \mathbb{R};\ \|f\|_{\mathrm{BV}_p(G)} = \mathrm{Var}_p(f) < \infty\},$$

where $\mathrm{Var}_p(f)$ is the p-variation of f defined by

$$\mathrm{Var}_p(f) = \begin{cases} \left(\sum_{u \sim v \in E} |f(u) - f(v)|^p \right)^{1/p}, & \text{if } 0 < p < \infty; \\ \sup_{u \sim v \in E} |f(u) - f(v)|, & \text{if } p = \infty. \end{cases}$$

Particularly, when $p = 1$, we denote $\mathrm{BV}_p(G) = \mathrm{BV}(G)$ and $\mathrm{Var}_p(f) = \mathrm{Var}(f)$.

It is not difficult to see that

$$\mathrm{Var}_q(f) \le \mathrm{Var}_p(f) \le \left(\frac{n(n-1)}{2} \right)^{1/p - 1/q} \mathrm{Var}_q(f), \text{ for } 0 < p \le q \le \infty. \tag{3}$$

Recently, Liu and Xue [20] investigated the boundedness for $M_G : \mathrm{BV}_p(G) \to \mathrm{BV}_p(G)$ and the BV_p-norm of M_G, which is given by

$$\|M_G\|_{\mathrm{BV}_p} := \sup_{\substack{f : V \to \mathbb{R} \\ \mathrm{Var}_p(f) \ne 0}} \frac{\mathrm{Var}_p(M_G f)}{\mathrm{Var}_p(f)}.$$

The main results of [20] can be shown as follows:

Theorem 1 ([20])**.** *Let G be a simple, finite and connected graph with $n \ge 2$ vertices and $0 < p \le \infty$.*

(*i*) *If $n = 2$, then $\|M_G\|_{\mathrm{BV}_p} = \frac{1}{2}$;*
(*ii*) *If $n = 3$ and $0 < p \le 1$, then $\|M_G\|_{\mathrm{BV}_p} = \frac{2}{3}$;*
(*iii*) *Let $K_n = (V, E)$ be the complete graph with n vertices, i.e., $|N_{K_n}(v)| = n - 1$ for any $v \in V$. If $n \ge 3$, then $1 - \frac{1}{n} \le \|M_{K_n}\|_{\mathrm{BV}_p} < 1$. Particularly, $\|M_{K_3}\|_{\mathrm{BV}_p} = \frac{2}{3}$;*

(*iv*) *Let* $S_n = (V, E)$ *be the star graph of n vertices, i.e., there exists an unique* $v \in V$ *such that* $|N_{S_n}(v)| = n - 1$ *and* $|N_{S_n}(w)| = 1$ *for every* $w \in V \setminus \{v\}$. *If* $n \geq 3$, *then* $1 - \frac{1}{n} \leq \|M_{S_n}\|_{\mathrm{BV}_p} < 1$;

(*v*) *The operator* M_G *is bounded from* $\mathrm{BV}_p(G)$ *to* $\mathrm{BV}_p(G)$ *for all* $0 < p \leq \infty$. *Moreover,*

$$\|M_G\|_{\mathrm{BV}_p} \leq \left(\frac{n}{2}\right)^{1/p} (n-1)^{\max\{1,1/p\}}.$$

Very recently, partial results in Theorem A were improved by González-Riquelme and Madrid [21] who obtained.

Theorem 2 ([21])**.** *Let* $n \geq 3$ *and* $0 < p \leq \infty$.

(*i*) *Then* $\|M_{K_n}\|_{\mathrm{BV}_p} = 1 - \frac{1}{n}$ *provided that one of the following conditions holds:*

- (*a*) $p \geq 1$;
- (*b*) $0 < p < 1$ *and* $n = 4$;
- (*c*) $n \geq 3$ *and* $\frac{\ln 4}{\ln 6} \leq p < 1$.

(*ii*) *If* $1 < p \leq \infty$, *then* $\|M_{S_n}\|_{\mathrm{BV}_p} = \frac{(1+2^{p/(p-1)})^{(p-1)/p}}{3}$. *Moreover, the equality* $\|M_{S_n}\|_{\mathrm{BV}_p} = 1 - \frac{1}{n}$ *holds provided that one of the following conditions holds:*

- (a') $p = 1$;
- (b') $0 < p < 1$ *and* $n = 4$;
- (c') $1/2 \leq p \leq 1$ *and* $n \geq 5$.

It should be pointed out that the graphs considered in the above references are finite undirected graphs. A natural question is that what happens when we consider the finite directed graphs? In this paper, we establish some new variation inequalities for the Hardy–Littlewood maximal operators on finite directed graphs.

Let us recall some notation and definitions. Let $\vec{G} = (V, E)$ be a finite graph, where V is the set of vertices V and E is the set of edges E. For an edge $u \sim v \in E$ satisfying $u \to v$, we then write $u \sim v = u \to v$. Moreover, v (resp., u) is called a right (resp., left) neighbor of u (resp., v). For $v \in V$, let $N_{\vec{G},+}(v)$ (resp., $N_{\vec{G},-}(v)$) be the set of right (resp., left) neighbors of v. If every edge in E has only a unique direction and $N_{\vec{G},+}(v) \cup N_{\vec{G},-}(v) \neq \emptyset$ for all $v \in V$, then $\vec{G}$ is said to be a directed graph. In addition, if for any distinct $x, y \in V$, we can find a finite sequence of vertices $\{x_i\}_{i=0}^{k}$, $k \in \mathbb{N}$, such that $x = x_0 \to x_1 \to \cdots \to x_k = y$, then $\vec{G}$ is said to be connected.

Throughout this paper, let $\vec{G} = (V, E)$ be the directed graph, where V is the set of vertices and E is the set of edges. We denote by $B_{\vec{G}}(v, r)$ the ball centered at v, with radius r on the graph $\vec{G}$, i.e.,

$$B_{\vec{G}}(v, r) = \{u \in V : d_{\vec{G}}(v, u) \leq r\}.$$

Here for any $u, v \in V$, the notation $d_{\vec{G}}(u, v)$ denotes the number of edges in a shortest path connecting from u to v. Clearly, $B_{\vec{G}}(v, r) = \{v\}$ if $0 \leq r < 1$ and $B_{\vec{G}}(v, r) = \{v\} \cup N_{\vec{G},+}(v)$ if $1 \leq r < 2$. Let f be a real function defined on V, the Hardy–Littlewood maximal operator on $\vec{G}$ is given by

$$M_{\vec{G}} f(v) = \sup_{r \geq 0} \frac{1}{|B_{\vec{G}}(v, r)|} \sum_{w \in B_{\vec{G}}(v, r)} |f(w)|.$$

Particulary, in the case $|V| = n$, we can rewrite $M_{\vec{G}}$ by

$$M_{\vec{G}} f(v) = \max_{k=0,\ldots,n-1} \frac{1}{|B_{\vec{G}}(v, k)|} \sum_{w \in B_{\vec{G}}(v, k)} |f(w)|.$$

It should be pointed out that the Hardy–Littlewood maximal operator on finite directed graphs was first introduced by Zhang et al. [26] who investigated the operator norm of the above operator on the Lebesgue spaces. By the definitions of M_G and $M_{\vec{G}}$, it is not difficult to find that there is no clear connections between M_G and $M_{\vec{G}}$. Based on Theorems A and B, a natural question is to study the BV_p-norm for $M_{\vec{G}}$, which is the main motivation of this paper.

This paper will be organized as follows. In Section 2, we establish the boundedness and continuity for $M_{\vec{G}}$ on $\mathrm{BV}_p(\vec{G})$. In Section 3, we discuss the BV_p-norm for $M_{\vec{G}}$ by introducing some directed graphs. Finally, some further comments will be given in Section 4.

2. Boundedness and Continuity for $M_{\vec{G}}$ on $\mathrm{BV}_p(\vec{G})$

This section concerns the boundedness and continuity for $M_{\vec{G}}$ on $\mathrm{BV}_p(\vec{G})$. Our main result can be formulated as follows.

Theorem 3. *Let $\vec{G}$ be a directed graph with n $(n \geq 2)$ vertices and $0 < p, q \leq \infty$. Then*

(i) *The operator $M_{\vec{G}}$ is bounded from $\mathrm{BV}_p(\vec{G})$ to $\mathrm{BV}_q(\vec{G})$. Specially,*

$$\mathrm{Var}_q(M_{\vec{G}}f) \leq (2n-4)^{\max\{1-1/p,0\}}\Big(\frac{n(n-1)}{2}\Big)^{1/q}\mathrm{Var}_p(f), \quad \forall f \in \mathrm{BV}_p(\vec{G}).$$

(ii) *If G is connected and $n = 2$, then $\|M_{\vec{G}}\|_{\mathrm{BV}_p} = 1$.*

(iii) *Let $\{f_j\} \subset \mathrm{BV}_p(\vec{G})$ be a sequence of functions such that $\|f_j - f\|_{\mathrm{BV}_p(\vec{G})} \to 0$ as $j \to \infty$. If*

$$\lim_{j\to\infty}\min_{u\in V}|f_j(u) - f(u)| = 0. \tag{4}$$

Then

$$\|M_{\vec{G}}f_j - M_{\vec{G}}f\|_{\mathrm{BV}_q(\vec{G})} \to 0 \text{ as } j \to \infty. \tag{5}$$

(iv) *The conclusion (5) could not holds without the assumption (4).*

Proof. We first prove part (i). Fix an edge $u \to v \in E$. By the definition of $M_{\vec{G}}$, there exist $r_u, r_v \in \{0, 1, \ldots, n-1\}$ such that

$$M_{\vec{G}}f(u) = \frac{1}{|B(u,r_u)|}\sum_{w\in B(u,r_u)}|f(w)|, \quad M_{\vec{G}}f(v) = \frac{1}{|B(v,r_v)|}\sum_{w\in B(v,r_v)}|f(w)|.$$

Without loss of generality, we may assume that $M_{\vec{G}}f(u) \geq M_{\vec{G}}f(v)$. Then there exist $\tau_u \in B(u,r_u)$ and $\tau_v \in B(v,r_v)$ such that

$$|f(\tau_u)| = \max\{|f(w)|; w \in B(u,r_u)\}, \quad |f(\tau_v)| = \min\{|f(w)|; w \in B(v,r_v)\}.$$

Then we have

$$|M_{\vec{G}}f(u) - M_{\vec{G}}f(v)| = M_{\vec{G}}f(u) - M_{\vec{G}}f(v) \leq |f(\tau_u)| - |f(\tau_v)| \leq |f(\tau_u) - f(\tau_v)|.$$

Note that $d_{\vec{G}}(u,\tau_u) \leq r_u$, $d_{\vec{G}}(v,\tau_v) \leq r_v$, $d_{\vec{G}}(u,\tau_u) + d_{\vec{G}}(v,\tau_v) \leq 2n-3$. Moreover $d_{\vec{G}}(u,\tau_v) = d_G(v,\tau_v) + 1$. There exist two finite sequences of vertices $\{u_i\}_{i=1}^{d_{\vec{G}}(u,\tau_u)}$ and $\{v_i\}_{i=1}^{d_{\vec{G}}(v,\tau_v)+1}$ such that $u = u_1 \to u_2 \to \cdots \to u_{d_{\vec{G}}(u,\tau_u)} = \tau_u$ and $u = v_0 \to v_1 \to v_2 \to \cdots \to v_{d_{\vec{G}}(v,\tau_v)} = \tau_v$. Then we have

$$
\begin{aligned}
&|f(\tau_u)-f(\tau_v)| \\
&\le |f(\tau_u)-f(u)|+|f(u)-f(\tau_v)| \\
&\le \sum_{i=1}^{d_{\vec{G}}(u,\tau_u)-1}|f(u_{i+1})-f(u_i)|+\sum_{i=0}^{d_{\vec{G}}(v,\tau_v)-1}|f(v_{i+1})-f(v_i)| \\
&\le (d_{\vec{G}}(u,\tau_u)-1+d_{\vec{G}}(v,\tau_v))^{\max\{1-1/p,0\}} \\
&\quad\times\Big(\sum_{i=1}^{d_{\vec{G}}(u,\tau_u)-1}|f(u_{i+1})-f(u_i)|^p+\sum_{i=0}^{d_{\vec{G}}(v,\tau_v)-1}|f(v_{i+1})-f(v_i)|^p\Big)^{1/p} \\
&\le (2n-4)^{\max\{1-1/p,0\}}\mathrm{Var}_p(f).
\end{aligned}
$$

Therefore

$$|M_{\vec{G}}f(u)-M_{\vec{G}}f(v)| \le (2n-4)^{\max\{1-1/p,0\}}\mathrm{Var}_p(f).$$

It follows that

$$\mathrm{Var}_p(M_{\vec{G}}f) \le (2n-4)^{\max\{1-1/p,0\}}\Big(\frac{n(n-1)}{2}\Big)^{1/q}\mathrm{Var}_p(f).$$

This proves part (i).

Next we prove part (ii). Let $n=2$ and $\vec{G}=(V,E)$ with $V=\{1,2\}$ and $E=\{1\to 2\}$. Given a function $f: V\to\mathbb{R}$, we have

$$M_{\vec{G}}f(1)=\max\{|f(1)|,\frac{1}{2}(|f(1)|+|f(2)|)\},\quad M_{\vec{G}}f(2)=|f(2)|.$$

When $|f(1)|\ge|f(2)|$, then

$$|M_{\vec{G}}f(1)-M_{\vec{G}}f(2)|=||f(1)|-|f(2)||\le|f(1)-f(2)|.$$

When $|f(1)|<|f(2)|$, then

$$|M_{\vec{G}}f(1)-M_{\vec{G}}f(2)|=\Big|\frac{1}{2}(|f(1)|+|f(2)|)-|f(2)|\Big|\le\frac{1}{2}|f(1)-f(2)|.$$

These facts yield that $\|M_{\vec{G}}\|_{\mathrm{BV}_p}\le 1$. On the other hand,

$$\|M_{\vec{G}}\|_{\mathrm{BV}_p}\ge\frac{\mathrm{Var}_p(M_{\vec{G}}\delta_1)}{\mathrm{Var}_p(\delta_1)}=1.$$

This proves $\|M_{\vec{G}}\|_{\mathrm{BV}_p}=1$.

We now prove part (iii). Fix $j\ge 1$, there exists $u\in V$ such that $\min_{x\in V}|f_j(x)-f(x)|=|f_j(u)-f(u)|$. Given $v\in V$, it holds that

$$
\begin{aligned}
|f_j(v)-f(v)| &\le |(f_j(v)-f(v))-(f_j(u)-f(u))|+|f_j(u)-f(u)| \\
&\le \mathrm{Var}(f_j-f)+|f_j(u)-f(u)| \\
&\le n^{\max\{1-1/q,0\}}\mathrm{Var}_q(f_j-f)+|f_j(u)-f(u)|,
\end{aligned}
$$

which gives

$$\|f_j-f\|_{\ell^\infty(\vec{G})}\le n^{\max\{1-1/q,0\}}\mathrm{Var}_q(f_j-f)+\min_{u\in V}|f_j(u)-f(u)|.$$

This, together with our assumptions, implies

$$\|M_{\vec{G}}f_j-M_{\vec{G}}f\|_{\ell^\infty(\vec{G})}\le\|M_{\vec{G}}(f_j-f)\|_{\ell^\infty(\vec{G})}\le\|f_j-f\|_{\ell^\infty(\vec{G})}\to 0 \text{ as } j\to\infty,$$

which leads to

$$\mathrm{Var}_p(M_{\vec{G}}f_j - M_{\vec{G}}f) \leq \Big(\frac{n(n-1)}{2}\Big)^{1/p} \|M_{\vec{G}}f_j - M_{\vec{G}}f\|_{\ell^\infty(\vec{G})} \to 0 \ \text{as} \ j \to \infty.$$

This gives (5).

Finally, we prove part (iv) by showing an example. Let $\vec{G} = \overrightarrow{S_{I,n}} = (V, E)$, where $V = \{1, 2, \ldots, n\}$ and $E = \{2 \to 1, 3 \to 1, \ldots, n \to 1\}$. Let $f(1) = 3$ and $f(i) = 1$ for all $i = 2, 3, \ldots, n$. Then $M_{\overrightarrow{S_{I,n}}}f(1) = 3$ and $M_{\overrightarrow{S_{I,n}}}f(i) = 2$ for all $i = 2, 3, \ldots, n$. Let $f_j(i) = f(i) - 2$ for all $i \in V$ and $j \geq 1$. Then we have $\mathrm{Var}_q(f_j - f) = 0$ and $M_{\overrightarrow{S_{I,n}}}f_j(i) = 1$ for all $i \in V$ and $j \geq 1$. It follows that $(M_{\overrightarrow{S_{I,n}}}f_j - M_{\overrightarrow{S_{I,n}}}f)(1) = 2$ and $(M_{\overrightarrow{S_{I,n}}}f_j - M_{\overrightarrow{S_{I,n}}}f)(i) = 1$ for all $i = 2, 3, \ldots, n$. Therefore, $\mathrm{Var}_p(M_{\overrightarrow{S_{I,n}}}f_j - M_{\overrightarrow{S_{I,n}}}f) = (n-1)^{1/p}$ for all $j \geq 1$, which gives $\|M_{\overrightarrow{S_{I,n}}}f_j - M_{\overrightarrow{S_{I,n}}}f\|_{\mathrm{BV}_p(G)} \nrightarrow 0$ as $j \to \infty$. □

Remark 1. *Let $\vec{G} = (V, E)$ be a directed connected graph with set of vertices $V = \{u, v\}$ and $E = \{u \to v\}$ and $0 < p \leq \infty$. Then $\|M_{\vec{G}}f\|_{\mathrm{BV}_p} = \|f\|_{\mathrm{BV}_p}$ if and only if $|f(u)| > |f(v)|$ and $f(u)f(v) \geq 0$.*

3. $\mathrm{BV}_p(\vec{G})$ Norms for $M_{\vec{G}}$

In this section, we study the $\mathrm{BV}_p(\vec{G})$ norm for $M_{\vec{G}}$. Before presenting our main results, let us introduce some directed graphs.

Definition 2.

(i) **(The inward star graph)**.*We say that $\vec{G} = (V, E)$ is an inward star graph with n vertices if there exists an unique $u \in V$ such that $N_{\vec{G},+}(u) = \emptyset$ and $N_{\vec{G},+}(v) = \{u\}$ for all $v \in V \setminus \{u\}$. We denote the inward star graph with n vertices by $\overrightarrow{S_{I,n}}$;*

(ii) **(The outward star graph)**. *We say that $\vec{G} = (V, E)$ is an outward star graph with n vertices if there exists an unique $u \in V$ such that $N_{\vec{G},+}(u) = V \setminus \{u\}$ and $N_{\vec{G},+}(v) = \emptyset$ for all $v \in V \setminus \{u\}$. We denote the outward star graph with n vertices by $\overrightarrow{S_{O,n}}$;*

(iii) **(The directed cyclic graph)**. *We say that $\vec{G} = (V, E)$ is a directed cyclic graph with n vertices if for any $u \in V$ we have $|N_{\vec{G},+}(u)| = |N_{\vec{G},-}(u)| = 1$. We denote the directed cyclic graph with n vertices by $\overrightarrow{C_n}$.*

(iv) **(The directed path graph)**. *We say that $\vec{G} = (V, E)$ is a directed line graph with n vertices if there exist unique two vertices u, $v \in V$ such that $N_{\vec{G},+}(u) = \emptyset$ and $|N_{\vec{G},+}(v)| = 1$ and $|N_{\vec{G},+}(w)| = |N_{\vec{G},-}(w)| = 1$ for all $w \in V \setminus \{u, v\}$. We denote the directed line graph with n vertices by $\overrightarrow{P_n}$.*

(v) **(The directed degraded graph)**. *We say that $\vec{G} = (V, E)$ is a directed degraded graph with n vertices if there exist n vertices $\{u_i\}_{i=1}^n = V$ such that $N_{\vec{G},+}(u_1) = \emptyset$, $N_{\vec{G},+}(u_2) = \{u_1\}$, $\cdots$, $N_{\vec{G},+}(u_n) = \{u_1, u_2, \ldots, u_{n-1}\}$.*
We denote the directed degraded graph with n vertices by $\overrightarrow{D_n}$.

The first one of main results can be listed as follows, which focuses on the graph $\overrightarrow{S_{I,n}}$.

Theorem 4. *Let $n \geq 3$ and $0 < p \leq \infty$. Then*

$$\|M_{\overrightarrow{S_{I,n}}}\|_{\mathrm{BV}_p} = 1.$$

Proof. Without loss of generality, we may assume that $\overrightarrow{S_{I,n}} = (V, E)$, where $V = \{1, 2, \ldots, n\}$ and $E = \{2 \to 1, 3 \to 1, \ldots, n \to 1\}$. We consider the Kronecker delta function

$$\delta_j(i) = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases}$$

It is not difficult to see that $\mathrm{Var}_p(\delta_2) = 1$ and

$$M_{\overrightarrow{S_{I,n}}}\delta_2(i) = \begin{cases} 1, & i = 2; \\ 0, & i = 1, 3, 4, \ldots, n. \end{cases}$$

Clearly, $\mathrm{Var}_p(\delta_2) = \mathrm{Var}_p(M_{\overrightarrow{S_{I,n}}}\delta_2) = 1$. It follows that

$$\|M_{\overrightarrow{S_{I,n}}}\|_{\mathrm{BV}_p} \geq 1.$$

Next we shall prove

$$\|M_{\overrightarrow{S_{I,n}}}\|_{\mathrm{BV}_p} \leq 1.$$

or, equivalently

$$\mathrm{Var}_p(M_{\overrightarrow{S_{I,n}}}f) \leq \mathrm{Var}_p(f) \tag{6}$$

for all $f : V \to \mathbb{R}$ with $\mathrm{Var}_p(f) > 0$. Fix $f = \sum_{i=1}^{n} a_i\delta_i$. We want to prove (6). Without loss of generality, we may assume that $f \geq 0$. Write

$$M_{\overrightarrow{S_{I,n}}}f(i) = \begin{cases} a_1, & i = 1; \\ \max\left\{a_i, \frac{1}{2}(a_i + a_1)\right\}, & i = 2, 3, \ldots, n. \end{cases}$$

Then we have

$$\mathrm{Var}_p(M_{\overrightarrow{S_{I,n}}}f) = \Big(\sum_{i=2}^{n}\Big|a_1 - \max\left\{a_i, \frac{1}{2}(a_i + a_1)\right\}\Big|^p\Big)^{1/p}.$$

Set

$$N_1 := \{j \in \{2, 3, \ldots, n\} : a_j \geq a_1\}, \quad N_2 := \{j \in \{2, 3, \ldots, n\} : a_j < a_1\}.$$

We further write

$$\begin{aligned}(\mathrm{Var}_p(M_{\overrightarrow{S_{I,n}}}f))^p &= \sum_{i \in N_1}|a_1 - a_i|^p + \sum_{i \in N_2}\Big|a_1 - \frac{1}{2}(a_i + a_1)\Big|^p \\ &\leq \sum_{i \in N_1}|a_1 - a_i|^p + \frac{1}{2^p}\sum_{i \in N_2}|a_1 - a_i|^p \\ &\leq \sum_{i=1}^{n}|a_1 - a_i|^p = (\mathrm{Var}_p(f))^p,\end{aligned}$$

which gives (6). This completes the proof of Theorem 4. □

Remark 2. *Let* $\overrightarrow{S_{I,n}} = (V, E)$*, where* $V = \{1, 2, \ldots, n\}$ *and* $E = \{2 \to 1, 3 \to 1, \ldots, n \to 1\}$*. Then* $\mathrm{Var}_p(M_{\overrightarrow{S_{I,n}}}f) = \mathrm{Var}_p(f)$ *if and only if* $|f(i)| \geq |f(1)|$ *and* $f(i)f(1) \geq 0$ *for all* $i = 2, 3, \ldots, n$*.*

Next we shall establish the BV_p-norm for $M_{\overrightarrow{S_{O,n}}}$.

Theorem 5. *Let* $n \geq 3$ *and* $0 < p \leq \infty$*. Then*

(i) $1 \leq \|M_{\overrightarrow{S_{O,n}}}\|_{\mathrm{BV}_p} \leq \frac{(n-1)^{\max\{1+1/p, 2\}}}{n}$;

(ii) If $n = 3$, then

$$\|M_{\overrightarrow{S_{O,n}}}\|_{\mathrm{BV}_p} = \begin{cases} 1, & \text{if } 1 \le p \le \infty; \\ \frac{(1+2^p)^{1/p}}{3}, & \text{if } 0 < p < 1. \end{cases}$$

Proof. The proof will be divided into two steps:

Step 1: Proof of part (i). We may assume without loss of generality that $\overrightarrow{S_{O,n}} = (V, E)$, where $V = \{1, 2, \ldots, n\}$ and $E = \{1 \to 2, 1 \to 3, \ldots, 1 \to n\}$. It is not difficult to see that $M_{\overrightarrow{S_{O,n}}}\delta_1 = \delta_1$. Then we have

$$\|M_{\overrightarrow{S_{O,n}}}\|_{\mathrm{BV}_p} \ge \frac{\mathrm{Var}_p(M_{\overrightarrow{S_{O,n}}}\delta_1)}{\mathrm{Var}_p(\delta_1)} = 1.$$

Therefore, to prove part (i), it suffices to show that

$$\mathrm{Var}_p(M_{\overrightarrow{S_{O,n}}} f) \le \frac{(n-1)^{\max\{1+1/p, 2\}}}{n} \mathrm{Var}_p(f) \tag{7}$$

for all f with $\mathrm{Var}_p(f) > 0$.

Fix $f = \sum_{i=1}^n a_i \delta_i$ with $\sum_{i=2}^n |a_i - a_1|^p > 0$. Without loss of generality, we may assume that $f \ge 0$, i.e., $a_i \ge 0$ for $i = 1, 2, \ldots, n$. We can write

$$M_{\overrightarrow{S_{O,n}}} f(i) = \begin{cases} \max\left\{a_1, \frac{1}{n}\sum_{j=1}^n a_j\right\}, & i = 1; \\ a_i, & i = 2, 3, \ldots, n. \end{cases}$$

Then we have

$$\mathrm{Var}_p(M_{\overrightarrow{S_{O,n}}} f) = \left(\sum_{i=2}^n \left|a_i - \max\left\{a_1, \frac{1}{n}\sum_{j=1}^n a_j\right\}\right|^p\right)^{1/p}.$$

We consider two cases:
(a) If $a_1 \ge \frac{1}{n}\sum_{j=1}^n a_j$, then

$$\mathrm{Var}_p(M_{\overrightarrow{S_{O,n}}} f) = \left(\sum_{i=2}^n |a_i - a_1|^p\right)^{1/p} = \mathrm{Var}_p(f).$$

This proves (7) in this case.
(b) If $a_1 < \frac{1}{n}\sum_{j=1}^n a_j$, we note that

$$\left|a_i - \frac{1}{n}\sum_{j=1}^n a_j\right| \le \frac{n-1}{n}\sum_{j=2}^n |a_j - a_1|, \quad \text{for } i = 2, 3, \ldots, n.$$

It follows that

$$\begin{aligned} \mathrm{Var}_p(M_{\overrightarrow{S_{O,n}}} f) &= \left(\sum_{i=2}^n \left|a_i - \frac{1}{n}\sum_{j=1}^n a_j\right|^p\right)^{1/p} \\ &\le \frac{(n-1)^{1+1/p}}{n}\sum_{j=2}^n |a_j - a_1| \\ &\le \frac{(n-1)^{\max\{1+1/p, 2\}}}{n}\mathrm{Var}_p(f). \end{aligned}$$

This proves (7) in this case.

Step 2: Proof of part (ii). At first, we shall prove that

$$\mathrm{Var}_p(M_{\overrightarrow{S_{O,3}}}f) \le \mathrm{Var}_p(f), \quad \forall p \ge 1; \tag{8}$$

and

$$\mathrm{Var}_p(M_{\overrightarrow{S_{O,3}}}f) \le \frac{(1+2^p)^{1/p}}{3}\mathrm{Var}_p(f), \ \ \forall p \in (0,1) \tag{9}$$

hold for all $f : V \to \mathbb{R}$ with $\mathrm{Var}_p(f) > 0$.

Let $\overrightarrow{S_{O,3}} = (V, E)$ with $V = \{1, 2, 3\}$ and $E = \{1 \to 2, 1 \to 3\}$. Fix a function $f = \sum_{i=1}^{3} a_i\delta_i$ with $|a_2 - a_1|^p + |a_3 - a_1|^p > 0$. Without loss of generality, we may assume that $a_i \ge 0\,(i = 1, 2, 3)$. By the proof of part (i), one gets

$$\mathrm{Var}_p(M_{\overrightarrow{S_{O,3}}}f) = \Big(\sum_{i=2}^{3}\Big|a_i - \max\Big\{a_1, \frac{1}{3}\sum_{j=1}^{3}a_j\Big\}\Big|^p\Big)^{1/p}.$$

We consider two cases:

(1) ($a_1 \ge \frac{1}{2}(a_2 + a_3)$). Then we have

$$\mathrm{Var}_p(M_{\overrightarrow{S_{O,3}}}f) = (|a_2 - a_1|^p + |a_3 - a_1|^p)^{1/p} = \mathrm{Var}_p(f).$$

This proves (8) and (9) in this case.

(2) If $a_1 < \frac{1}{2}(a_2 + a_3)$. For convenience, we set $\beta = |a_2 - a_1|$ and $\gamma = |a_3 - a_1|$. The following cases will be discussed:

(a) ($\max\{a_2, a_3\} \ge a_1 \ge \min\{a_2, a_3\}$). Without loss of generality, we may assume that $a_3 > a_2$. Then we have $\beta < \gamma$ because of $a_1 < \frac{1}{2}(a_2 + a_3)$. We write

$$\begin{aligned}\Big(\frac{\mathrm{Var}_p(M_{\overrightarrow{S_{O,3}}}f)}{\mathrm{Var}_p(f)}\Big)^p &= \frac{\Big|a_2 - \frac{1}{3}(a_1+a_2+a_3)\Big|^p + \Big|a_3 - \frac{1}{3}(a_1+a_2+a_3)\Big|^p}{|a_2 - a_1|^p + |a_3 - a_1|^p} \\ &= \frac{1}{3^p}\frac{(2\beta+\gamma)^p + (2\gamma+\beta)^p}{\beta^p + \gamma^p} \\ &= \frac{1}{3^p}\frac{\Big(2\frac{\beta}{\gamma}+1\Big)^p + \Big(2+\frac{\beta}{\gamma}\Big)^p}{1+\Big(\frac{\beta}{\gamma}\Big)^p}.\end{aligned}$$

Note that $\frac{\beta}{\gamma} \in [0, 1)$. Define the function $g(x) = \frac{(2x+1)^p+(2+x)^p}{1+x^p}$, $x \in [0, 1)$.

When $p \ge 1$, observing that $3^p - (2 + x)^p \ge (2x + 1)^p - (3x)^p$ for all $x \in [0, 1)$. It follows that $g(x) \le 3^p$ for all $x \in [0, 1)$. Hence,

$$\Big(\frac{\mathrm{Var}_p(M_{\overrightarrow{S_{O,3}}}f)}{\mathrm{Var}_p(f)}\Big)^p \le 1.$$

This proves (8) in this case.

When $p \in (0, 1)$, let $h(x) = (1 + 2^p)(1 + x^p) - (2x + 1)^p - (2 + x)^p$. It is clear that $h'(x) \ge 0$ for all $x \in [0, 1)$. Then $h(x) \ge h(0) = 0$. Hence, $g(x) \le 1 + 2^p$ and then

$$\frac{\mathrm{Var}_p(M_{\overrightarrow{S_{O,3}}}f)}{\mathrm{Var}_p(f)} \le \frac{(1+2^p)^{1/p}}{3}.$$

This proves (9) in this case.

(b) ($a_1 < \min\{a_2, a_3\}$). Without loss of generality, we may assume that $a_3 > a_2$. It is clear that $a_1 < a_2 < a_3$.

Assume that $a_2 \geq \frac{1}{2}(a_1 + a_3)$. Then we have

$$\left|a_2 - \frac{1}{3}(a_1 + a_2 + a_3)\right| = a_2 - \frac{1}{3}(a_1 + a_2 + a_3) \leq a_2 - a_1 = |a_2 - a_1|,$$

$$\left|a_3 - \frac{1}{3}(a_1 + a_2 + a_3)\right| = a_3 - \frac{1}{3}(a_1 + a_2 + a_3) \leq a_3 - a_1 = |a_3 - a_1|.$$

The above facts will give (8) and (9) in this case, since $\frac{1+2^p}{3^p} \geq 1$ when $p \in (0,1)$.
Assume that $a_2 < \frac{1}{2}(a_1 + a_3)$. Then $\gamma > 2\beta \geq 0$ and $\frac{\beta}{\gamma} \in [0, 1/2)$. We can write

$$\begin{aligned}
\left(\frac{\mathrm{Var}_p(M_{\overrightarrow{S_{O,3}}} f)}{\mathrm{Var}_p(f)}\right)^p &= \frac{\left|a_2 - \frac{1}{3}(a_1 + a_2 + a_3)\right|^p + \left|a_3 - \frac{1}{3}(a_1 + a_2 + a_3)\right|^p}{|a_2 - a_1|^p + |a_3 - a_1|^p} \\
&= \frac{1}{3^p} \frac{(\gamma - 2\beta)^p + (2\gamma - \beta)^p}{\beta^p + \gamma^p} \\
&= \frac{1}{3^p} \frac{\left(1 - 2\frac{\beta}{\gamma}\right)^p + \left(2 - \frac{\beta}{\gamma}\right)^p}{1 + \left(\frac{\beta}{\gamma}\right)^p} \leq \frac{1 + 2^p}{3^p}.
\end{aligned}$$

This proves (8) and (9) in this case since $\frac{1+2^p}{3^p} \leq 1$ when $p \geq 1$.

(8) together with part (i) leads to $\|M_{\overrightarrow{S_{O,n}}}\|_{\mathrm{BV}_p} = 1$ when $p \geq 1$. When $0 < p < 1$, let us consider the function $f : V \to \mathbb{R}$ defined by

$$f(1) = 1, \;\; f(2) = 1, \;\; f(3) = 4.$$

It is clear that $\mathrm{Var}_p(f) = 3$ and $M_{\overrightarrow{S_{O,3}}} f(1) = 1$, $M_{\overrightarrow{S_{O,3}}} f(2) = 2$ and $M_{\overrightarrow{S_{O,3}}} f(3) = 4$. Then we have

$$\|M_{\overrightarrow{S_{O,n}}}\|_{\mathrm{BV}_p} \geq \frac{\mathrm{Var}_p(M_{\overrightarrow{S_{O,3}}} f)}{\mathrm{Var}_p(f)} = \frac{1 + 2^p}{3^p}.$$

This, together with (9), leads to $\|M_{\overrightarrow{S_{O,n}}}\|_{\mathrm{BV}_p} = \frac{1+2^p}{3^p}$ when $p \in (0,1)$. This finishes the proof of Theorem 5. □

Remark 3. *It should be pointed out that part (i) of Theorem 5 implies* $\|M_{\overrightarrow{S_{O,n}}}\|_{\mathrm{BV}} = 1$. *Let* $\overrightarrow{S_{O,3}} = (V, E)$ *with* $V = \{1, 2, 3\}$ *and* $E = \{1 \to 2, 1 \to 3\}$ *and* f *be a function defined on* V. *Then*

(*i*) $\mathrm{Var}(M_{\overrightarrow{S_{O,3}}} f) = \mathrm{Var}(f)$ *if and only if* f *satisfies one of the following conditions:*
- (*a*) $f(i) \equiv C$ *for some* $C \in \mathbb{R}$ *and all* $i \in \{1, 2, 3\}$;
- (*b*) $|f(1)| = |f(2)| < |f(3)|$, $f(1)f(3) \geq 0$;
- (*c*) $|f(1)| = |f(3)| < |f(2)|$, $f(1)f(2) \geq 0$;
- (*d*) $|f(1)| > \frac{1}{2}(|f(2)| + |f(3)|)$, $a_1 a_i \geq 0$, $i = 2, 3$.

(*ii*) *If* $p > 1$, *then* $\mathrm{Var}_p(M_{\overrightarrow{S_{O,3}}} f) = \mathrm{Var}_p(f)$ *if and only if* f *satisfies the condition (a) or (d).*

Theorem 6. *Let* $n \geq 3$ *and* $0 < p \leq \infty$. *Then*

(*i*) $1 \leq \|M_{\overrightarrow{P_n}}\|_{\mathrm{BV}_p} \leq (n-1)^{\max\{1, 1/p\}}$;

(*ii*) *If* $n = 3$, *then*

$$\|M_{\overrightarrow{P_n}}\|_{\mathrm{BV}_p} = \begin{cases} 1, & \text{if } \; p_0 \leq p \leq \infty; \\ \frac{(1+3^p)^{1/p}}{6}, & \text{if } \; 0 < p < p_0. \end{cases}$$

Here p_0 *is the unique solution of equation* $1 + 3^x = 6^x$.

Proof. The proof will be divided into two steps.

Step 1: Proof of part (i). We may assume without loss of generality that $\overrightarrow{P_n} = (V, E)$, where $V = \{1, 2, \dots, n\}$ and $E = \{n \to n-1, n-1 \to n-2, \dots, 2 \to 1\}$. Let us consider the function $f : V \to \mathbb{R}$ defined by $f(i) = 1$, $i = 2, 3, \dots, n$ and $f(1) = 0$. Clearly, $M_{\overrightarrow{P_n}} f(i) = 1$, $i = 2, 3, \dots, n$ and $M_{\overrightarrow{P_n}} f(1) = 0$. Then we have

$$\|M_{\overrightarrow{P_n}}\|_{\mathrm{BV}_p} \geq \frac{\mathrm{Var}_p(M_{\overrightarrow{P_n}} f)}{\mathrm{Var}_p(f)} = 1.$$

We now prove

$$\|M_{\overrightarrow{P_n}}\|_{\mathrm{BV}_p} \leq (n-1)^{\max\{1/p, 1\}}. \tag{10}$$

Let $f = \sum_{i=1}^{n} a_i \delta_i$ and write

$$M_{\overrightarrow{P_n}} f(1) = |a_1|,$$

$$M_{\overrightarrow{P_n}} f(2) = \max\left\{|a_2|, \frac{1}{2}(|a_2| + |a_1|)\right\},$$

$$\cdots\cdots\cdots\cdots,$$

$$M_{\overrightarrow{P_n}} f(n-1) = \max\left\{|a_{n-1}|, \frac{1}{2}(|a_{n-1}| + |a_{n-2}|), \dots, \frac{1}{n-1}\sum_{i=1}^{n-1} |a_i|\right\},$$

$$M_{\overrightarrow{P_n}} f(n) = \max\left\{|a_n|, \frac{1}{2}(|a_n| + |a_{n-1}|), \dots, \frac{1}{n}\sum_{i=1}^{n} |a_i|\right\}.$$

One can easily check that

$$|M_{\overrightarrow{P_n}} f(i) - M_{\overrightarrow{P_n}} f(i+1)| \leq \sum_{i=1}^{n-1} |a_i - a_{i+1}| = \mathrm{Var}(f), \quad \text{for } i = 1, 2, \dots, n-1.$$

It follows that

$$\begin{aligned}
\mathrm{Var}_p(M_{\overrightarrow{P_n}} f) &\leq \Big(\sum_{i=1}^{n-1} |M_{\overrightarrow{P_n}} f(i) - M_{\overrightarrow{P_n}} f(i+1)|^p\Big)^{1/p} \\
&\leq (n-1)^{1/p}\mathrm{Var}(f) \\
&\leq (n-1)^{\max\{1/p, 1\}}\mathrm{Var}_p(f),
\end{aligned}$$

which gives (10) and proves part (i).

Step 2: Proof of part (ii). Let p_0 be given as in Theorem 6. We want to show that

$$\mathrm{Var}_p(M_{\overrightarrow{P_3}} f) \leq \mathrm{Var}_p(f), \quad \text{if } p_0 \leq p \leq \infty, \tag{11}$$

$$\mathrm{Var}_p(M_{\overrightarrow{P_3}} f) \leq \frac{(1+3^p)^{1/p}}{6}\mathrm{Var}_p(f), \quad \text{if } 0 < p < p_0, \tag{12}$$

for all $f : V \to \mathbb{R}$ with $\mathrm{Var}_p(f) > 0$.

Now we prove (11) and (12). Without loss of generality, we may assume that $\overrightarrow{P_3} = (V, E)$, where $V = \{1, 2, 3\}$ and $E = \{1 \to 2, 2 \to 3\}$. Let $f = \sum_{i=1}^{3} a_i \delta_i$. Without loss of generality, we may assume that $a_i \geq 0$ $(i = 1, 2, 3)$ and $|a_1 - a_2| + |a_2 - a_3| > 0$. One can easily check that

$$M_{\overrightarrow{P_3}} f(1) = \max\left\{a_1, \frac{1}{2}(a_1 + a_2), \frac{1}{3}(a_1 + a_2 + a_3)\right\},$$

$$M_{\overrightarrow{P_3}} f(2) = \max\left\{a_2, \frac{1}{2}(a_2 + a_3)\right\}, \quad M_{L_3} f(3) = a_3.$$

Then we have

$$\mathrm{Var}_p(M_{\overrightarrow{P_3}}f) = (|M_{\overrightarrow{P_3}}f(2) - M_{\overrightarrow{P_3}}f(1)|^p + |M_{\overrightarrow{P_3}}f(3) - M_{\overrightarrow{P_3}}f(2)|^p)^{1/p}.$$

For convenience, we set

$$\beta = |a_1 - a_2|, \quad \gamma = |a_2 - a_3|.$$

We consider the following cases:
(1) ($a_1 \geq a_2 \geq a_3$ and $a_1 > a_3$). In this case we have $M_{\overrightarrow{P_3}}f(1) = a_1$, $M_{\overrightarrow{P_3}}f(2) = a_2$ and

$$\frac{\mathrm{Var}_p(M_{\overrightarrow{P_3}}f)}{\mathrm{Var}_p(f)} = 1.$$

This proves (11) in this case. Noting that $1 < \frac{(1+3^p)^{1/p}}{6}$ when $0 < p < p_0$. This proves (12) in this case.

(2) ($a_1 \geq a_3 \geq a_2$ and $a_1 > a_2$). In this case we have $\beta \geq \gamma$, $\beta > 0$ and $\frac{\gamma}{\beta} \in [0,1]$. Moreover, $M_{\overrightarrow{P_3}}f(1) = a_1$ and $\mathrm{M}_{L_3}f(2) = \frac{1}{2}(a_2 + a_3)$. It holds that

$$\begin{aligned}
\frac{(\mathrm{Var}_p(M_{\overrightarrow{P_3}}f))^p}{(\mathrm{Var}_p(f))^p} &= \frac{\left|\frac{1}{2}(a_2+a_3) - a_1\right|^p + \left|\frac{1}{2}(a_3 - a_2)\right|^p}{(a_1 - a_2)^p + (a_3 - a_2)^p} \\
&= \frac{1}{2^p}\frac{(2\beta - \gamma)^p + \gamma^p}{\beta^p + \gamma^p} \\
&= \frac{1}{2^p}\frac{(2 - \frac{\gamma}{\beta})^p + (\frac{\gamma}{\beta})^p}{1 + (\frac{\gamma}{\beta})^p}.
\end{aligned}$$

Let $h(x) = \frac{(2-x)^p + x^p}{1+x^p}$, $x \in [0,1]$. Note that $h'(x) \leq 0$ for all $x \in [0,1]$ and $h(x) \leq 2^p$ for all $x \in [0,1]$. Hence,

$$\frac{(\mathrm{Var}_p(M_{\overrightarrow{P_3}}f))^p}{(\mathrm{Var}_p(f))^p} < 1.$$

This proves (11) and (12) in this case.

(3) ($a_2 \geq a_1 \geq a_3$ and $a_2 > a_3$). In this case we have $\mathrm{M}_{P_3}f(1) = \frac{1}{2}(a_1 + a_2)$ and $\mathrm{M}_{P_3}f(2) = a_2$. Hence,

$$\frac{(\mathrm{Var}_p(M_{\overrightarrow{P_3}}f))^p}{(\mathrm{Var}_p(f))^p} = \frac{\left|\frac{1}{2}(a_2 - a_1)\right|^p + |a_2 - a_3|^p}{|a_2 - a_1|^p + |a_3 - a_2|^p} \leq 1,$$

which proves (11) and (12) in this case.

(4) ($a_2 \geq a_3 \geq a_1$ and $a_2 > a_1$). If $a_3 \leq \frac{1}{2}(a_1 + a_2)$. In this case we have $M_{\overrightarrow{P_3}}f(1) = \frac{1}{2}(a_1 + a_2)$ and $M_{\overrightarrow{P_3}}f(2) = a_2$. Then we have

$$\frac{(\mathrm{Var}_p(M_{\overrightarrow{P_3}}f))^p}{(\mathrm{Var}_p(f))^p} = \frac{\left|\frac{1}{2}(a_2 - a_1)\right|^p + |a_2 - a_3|^p}{|a_2 - a_1|^p + |a_3 - a_2|^p} < 1.$$

If $a_3 > \frac{1}{2}(a_1 + a_2)$. In this case we have $M_{\overrightarrow{P_3}}f(1) = \frac{1}{3}(a_1 + a_2 + a_3)$ and $M_{\overrightarrow{P_3}}f(2) = a_2$. Note that $|2a_2 - (a_1 + a_3)| \leq 2|a_2 - a_1|$. Then

$$\frac{(\mathrm{Var}_p(M_{\overrightarrow{P_3}}f))^p}{(\mathrm{Var}_p(f))^p} = \frac{\left|\frac{1}{3}(2a_2 - a_1 - a_3)\right|^p + |a_2 - a_3|^p}{(a_2 - a_1)^p + (a_2 - a_3)^p} < 1.$$

Then (11) and (12) are proved in this case.

(5) ($a_3 \geq a_1 \geq a_2$ and $a_3 > a_2$). If $a_1 \geq \frac{1}{2}(a_2+a_3)$. Then we have $M_{\overrightarrow{P_3}} f(1) = a_1$ and $M_{\overrightarrow{P_3}} f(2) = \frac{1}{2}(a_2+a_3)$. Note that $|2a_1 - (a_2+a_3)| \leq a_1 - a_2$. Then

$$\frac{(\mathrm{Var}_p(M_{\overrightarrow{P_3}} f))^p}{(\mathrm{Var}_p(f))^p} = \frac{\left|\frac{1}{2}(2a_1 - a_2 - a_3)\right|^p + \left|\frac{1}{2}(a_3 - a_2)\right|^p}{(a_1-a_2)^p + (a_3-a_2)^p} \leq \frac{1}{2^p} < 1.$$

This gives (11) and (12) in this case.

If $a_1 < \frac{1}{2}(a_2+a_3)$. Then $\gamma > 2\beta$, $\gamma > 0$ and $\frac{\beta}{\gamma} \in [0, 1/2)$. Moreover, $M_{\overrightarrow{P_3}} f(1) = \frac{1}{3}(a_1+a_2+a_3)$ and $M_{\overrightarrow{P_3}} f(2) = \frac{1}{2}(a_2+a_3)$. It holds that

$$\begin{aligned}\frac{(\mathrm{Var}_p(M_{\overrightarrow{P_3}} f))^p}{(\mathrm{Var}_p(f))^p} &= \frac{\left|\frac{1}{6}(a_2+a_3-2a_1)\right|^p + \left|\frac{1}{2}(a_3-a_2)\right|^p}{(a_1-a_2)^p + (a_3-a_2)^p} \\ &= \frac{1}{6^p}\frac{(\gamma-2\beta)^p + (3\gamma)^p}{\gamma^p+\beta^p} \\ &= \frac{1}{6^p}\frac{(1-2\frac{\beta}{\gamma})^p + 3^p}{1+(\frac{\beta}{\gamma})^p} \leq \frac{1+3^p}{6^p}.\end{aligned}$$

Noting that $\frac{1+3^p}{6^p} \leq 1$ when $p \geq p_0$. Here, p_0 is the unique solution of equation $1+3^x = 6^x$. This proves (11) and (12) in this case.

(6) ($a_3 \geq a_2 \geq a_1$ and $a_3 > a_1$). In this case we have $\gamma \geq \beta$ and $\gamma > 0$. Moreover, $M_{\overrightarrow{P_3}} f(1) = \frac{1}{3}(a_1+a_2+a_3)$ and $M_{\overrightarrow{P_3}} f(2) = \frac{1}{2}(a_2+a_3)$. It follows that

$$\frac{(\mathrm{Var}_p(M_{\overrightarrow{P_3}} f))^p}{(\mathrm{Var}_p(f))^p} = \frac{\left|\frac{1}{6}(a_2+a_3-2a_1)\right|^p + \left|\frac{1}{2}(a_3-a_2)\right|^p}{(a_2-a_1)^p + (a_3-a_2)^p} = \frac{1}{6^p}\frac{(\gamma+2\beta)^p + (3\gamma)^p}{\beta^p+\gamma^p}.$$

When $p \in (0,1]$. It holds that

$$\frac{1}{6^p}\frac{(\gamma+2\beta)^p + (3\gamma)^p}{\beta^p+\gamma^p} \leq \frac{1}{6^p}\frac{\gamma^p + 2^p\beta^p + (3\gamma)^p}{\beta^p+\gamma^p} < \frac{1+3^p}{6^p}.$$

When $p > 1$. Then

$$\begin{aligned}\frac{1}{6^p}\frac{(\gamma+2\beta)^p + (3\gamma)^p}{\beta^p+\gamma^p} &\leq \frac{1}{6^p}\frac{2^{p-1}\gamma^p + 2^{2p-1}\beta^p + (3\gamma)^p}{\beta^p+\gamma^p} \\ &\leq \frac{\max\{2^{p-1}+3^p, 2^{2p-1}\}}{6^p} < 1.\end{aligned}$$

These prove (11) and (12) in this case.

Finally, let us consider the function $f: V \to \mathbb{R}$ defined by

$$f(1) = 1, \ \ f(2) = 1, \ \ f(3) = 7.$$

It is clear that $\mathrm{Var}_p(f) = 6$ and $M_{\overrightarrow{P_3}} f(1) = 3$, $\mathrm{M}_{P_3} f(2) = 4$ and $M_{\overrightarrow{P_3}} f(3) = 7$. Then we have $\mathrm{Var}_p(M_{\overrightarrow{P_3}} f) = (1+3^p)^{1/p}$. This gives $\frac{\mathrm{Var}_p(M_{\overrightarrow{P_3}} f)}{\mathrm{Var}_p(f)} \leq \frac{(1+3^p)^{1/p}}{6}$, which together with part (i), (11) and (12) implies the conclusion in part (ii). This finishes the proof of Theorem 6. □

Remark 4. *Let $\overrightarrow{P_3} = (V, E)$ with $V = \{1,2,3\}$ and $E = \{1 \to 2, 2 \to 3\}$ and f be a function defined on V. Let $0 < p \leq \infty$. Then $\mathrm{Var}(M_{\overrightarrow{P_3}} f) = \mathrm{Var}(f)$ if and only if f satisfies $|f(1)| \geq |f(2)| \geq |f(3)|$ and $f(i)f(j) \geq 0$ for all $i, j \in \{1,2,3\}$.*

Theorem 7. *Let $n \geq 3$ and $0 < p \leq \infty$. Then*

(i) $1 \leq \|M_{\overrightarrow{D_n}}\|_{\mathrm{BV}_p} \leq \left(\frac{n(n-1)}{2}\right)^{1/p}(2n-4)^{\max\{1-1/p,0\}}$;

(ii) *If $p \geq 1$ and $n = 3$, then* $\|M_{\overrightarrow{D_3}}\|_{\mathrm{BV}_p} = 1$.

Proof. We shall divide the proof into two steps:

Step 1: Proof of part (i). Without loss of generality, we may assume that $\overrightarrow{D_n} = (V, E)$, where $V = \{1, 2, \ldots, n\}$ and $E = \{i \to j : 1 \leq j < i \leq n\}$. Taking $f : V \to \mathbb{R}$ by $f(i) = 1$ for $i = 2, \ldots, n$ and $f(1) = 0$. Clearly, $M_{\overrightarrow{D_n}} f(1) = 0$, $M_{\overrightarrow{D_n}} f(i) = 1$ for $i = 2, 3, \ldots, n$. Moreover, $\mathrm{Var}_p(f) = \mathrm{Var}_p(M_{\overrightarrow{D_n}} f) = (n-1)^{1/p}$. It follows that

$$\|M_{\overrightarrow{D_n}}\|_{\mathrm{BV}_p} \geq \frac{\mathrm{Var}_p(\mathrm{M}_{D_n} f)}{\mathrm{Var}_p(f)} = 1.$$

This together with Theorem 3 yields part (i) of Theorem 7.

Step 2: Proof of part (ii). By part (i), it suffices to show that

$$\|M_{\overrightarrow{D_3}}\|_{\mathrm{BV}_p} \leq 1. \tag{13}$$

Let $D_3 = (V, E)$ with $V = \{1, 2, 3\}$ and $E = \{1 \to 2, 2 \to 3, 1 \to 3\}$. Let $f = \sum_{i=1}^{3} a_i \delta_i$ with $|a_2 - a_1|^p + |a_3 - a_2|^p + |a_1 - a_3|^p > 0$. One can easily check that

$$M_{\overrightarrow{D_3}} f(1) = \max\left\{a_1, \frac{1}{3}\sum_{i=1}^{3}(a_1 + a_2 + a_3)\right\},$$

$$M_{\overrightarrow{D_3}} f(2) = \max\left\{a_2, \frac{1}{2}(a_2 + a_3)\right\}, \quad \mathrm{M}_{D_3} f(3) = a_3.$$

To prove (13), it is enough to show that

$$\mathrm{Var}_p(M_{\overrightarrow{D_3}} f) \leq \mathrm{Var}_p(f). \tag{14}$$

Without loss of generality, we may assume that $a_i \geq 0\,(i = 1, 2, 3)$. For convenience, we set

$$\beta = |a_1 - a_2|, \quad \gamma = |a_1 - a_3|.$$

We consider the follows cases:

(1) ($a_1 \geq a_2 \geq a_3$ and $a_1 > a_3$). In this case we have $M_{\overrightarrow{D_3}} f(1) = a_1$ and $M_{\overrightarrow{D_3}} f(2) = a_2$. It holds that

$$\frac{(\mathrm{Var}_p(M_{\overrightarrow{D_3}} f))^p}{(\mathrm{Var}_p(f))^p} = \frac{|a_2 - a_1|^p + |a_3 - a_2|^p + |a_3 - a_1|^p}{|a_2 - a_1|^p + |a_3 - a_2|^p + |a_3 - a_1|^p} = 1,$$

which gives (14) in this case.

(2) ($a_1 \geq a_3 \geq a_2$ and $a_1 > a_2$). In this case we have $M_{\overrightarrow{D_3}} f(1) = a_1$, $M_{\overrightarrow{D_3}} f(2) = \frac{1}{2}(a_2 + a_3)$ and $|M_{\overrightarrow{D_3}} f(2) - M_{\overrightarrow{D_3}} f(1)| = \frac{1}{2}(2a_1 - (a_2 + a_3)) \leq |a_1 - a_2|$. It follows that

$$\begin{aligned}\frac{(\mathrm{Var}_p(M_{\overrightarrow{D_3}} f))^p}{(\mathrm{Var}_p(f))^p} &= \frac{\left|\frac{1}{2}(2a_1 - (a_2 + a_3))\right|^p + \left|\frac{1}{2}(a_3 - a_2)\right|^p + |a_3 - a_1|^p}{|a_2 - a_1|^p + |a_3 - a_2|^p + |a_3 - a_1|^p} \\ &\leq \frac{|a_2 - a_1|^p + |a_3 - a_2|^p + |a_3 - a_1|^p}{|a_2 - a_1|^p + |a_3 - a_2|^p + |a_3 - a_1|^p} = 1.\end{aligned}$$

This proves (14) in this case.

(3) ($a_2 \geq a_1 \geq a_3$ and $a_2 > a_3$). If $a_1 \geq \frac{1}{2}(a_2 + a_3)$. In this case we have $M_{\overrightarrow{D_3}} f(1) = a_1$ and $M_{\overrightarrow{D_3}} f(2) = a_2$. It holds that

$$\frac{(\mathrm{Var}_p(M_{\overrightarrow{D_3}} f))^p}{(\mathrm{Var}_p(f))^p} = \frac{|a_2 - a_1|^p + |a_3 - a_2|^p + |a_3 - a_1|^p}{|a_2 - a_1|^p + |a_3 - a_2|^p + |a_3 - a_1|^p} = 1.$$

If $a_1 < \frac{1}{2}(a_2 + a_3)$. Then we have $M_{\overrightarrow{D_3}} f(1) = \frac{1}{3}(a_1 + a_2 + a_3)$ and $M_{\overrightarrow{D_3}} f(2) = a_2$. Note that $\beta > \gamma \geq 0$. We have

$$\begin{aligned}\frac{(\mathrm{Var}_p(M_{\overrightarrow{D_3}} f))^p}{(\mathrm{Var}_p(f))^p} &= \frac{\left|\frac{1}{3}(2a_2 - (a_1 + a_3))\right|^p + |a_3 - a_2|^p + \left|\frac{1}{3}(a_1 + a_2 - 2a_3)\right|^p}{|a_2 - a_1|^p + |a_3 - a_2|^p + |a_3 - a_1|^p} \\ &\leq \frac{1}{3^p} \frac{(2\beta + \gamma)^p + 3^p(\beta + \gamma)^p + (\beta + 2\gamma)^p}{\beta^p + (\beta + \gamma)^p + \gamma^p} \\ &\leq \frac{1}{3^p} \frac{(2 + \frac{\gamma}{\beta})^p + 3^p(1 + \frac{\gamma}{\beta})^p + (1 + 2\frac{\gamma}{\beta})^p}{1 + (1 + \frac{\gamma}{\beta})^p + (\frac{\gamma}{\beta})^p}.\end{aligned}$$

Noting that $h_1(x) = \frac{(2+x)^p + 3^p(1+x)^p + (1+2x)^p}{1 + x^p + (1+x)^p} \leq 3^p$ for all $x \in [0, 1)$ since $p \geq 1$. This implies

$$\frac{(\mathrm{Var}_p(M_{\overrightarrow{D_3}} f))^p}{(\mathrm{Var}_p(f))^p} \leq 1.$$

This proves (14) in this case.

(4) ($a_2 \geq a_3 \geq a_1$ and $a_2 > a_1$). In this case we have $M_{\overrightarrow{D_3}} f(1) = \frac{1}{3}(a_1 + a_2 + a_3)$ and $M_{\overrightarrow{D_3}} f(2) = a_2$. If $a_3 \geq \frac{1}{2}(a_1 + a_2)$. Then we have

$$|M_{\overrightarrow{D_3}} f(1) - M_{\overrightarrow{D_3}} f(2)| = \frac{1}{3}(2a_2 - (a_1 + a_3)) < a_2 - a_1$$

and

$$|M_{\overrightarrow{D_3}} f(3) - M_{\overrightarrow{D_3}} f(1)| = \frac{1}{3}(2a_3 - (a_1 + a_2)) < a_3 - a_1.$$

It holds that

$$\begin{aligned}\frac{(\mathrm{Var}_p(M_{\overrightarrow{D_3}} f))^p}{(\mathrm{Var}_p(f))^p} &= \frac{\left|\frac{1}{3}(2a_2 - (a_1 + a_3))\right|^p + |a_3 - a_2|^p + \left|\frac{1}{3}(2a_3 - (a_1 + a_2))\right|^p}{|a_2 - a_1|^p + |a_3 - a_2|^p + |a_3 - a_1|^p} \\ &\leq \frac{|a_2 - a_1|^p + |a_3 - a_2|^p + |a_3 - a_1|^p}{|a_2 - a_1|^p + |a_3 - a_2|^p + |a_3 - a_1|^p} = 1.\end{aligned}$$

If $a_3 < \frac{1}{2}(a_1 + a_2)$. Then $\beta > 2\gamma$ and $\frac{\gamma}{\beta} \in [0, 1/2)$. Note that

$$h_2(x) = \frac{(2 - x)^p + 3^p(1 - x)^p + (1 - 2x)^p}{1 + x^p + (1 - x)^p} \leq 3^p, \quad \text{for all } x \in [0, \frac{1}{2}].$$

Then we have

$$\begin{aligned}\frac{(\mathrm{Var}_p(M_{\overrightarrow{D_3}} f))^p}{(\mathrm{Var}_p(f))^p} &= \frac{\left|\frac{1}{3}(2a_2 - (a_1 + a_3))\right|^p + |a_3 - a_2|^p + \left|\frac{1}{3}(a_1 + a_2 - 2a_3)\right|^p}{|a_2 - a_1|^p + |a_3 - a_2|^p + |a_3 - a_1|^p} \\ &\leq \frac{1}{3^p} \frac{(2\beta - \gamma)^p + 3^p(\beta - \gamma)^p + (\beta - 2\gamma)^p}{\beta^p + (\beta - \gamma)^p + \gamma^p} \\ &\leq \frac{1}{3^p} \frac{(2 - \frac{\gamma}{\beta})^p + 3^p(1 - \frac{\gamma}{\beta})^p + (1 - 2\frac{\gamma}{\beta})^p}{1 + (1 - \frac{\gamma}{\beta})^p + (\frac{\gamma}{\beta})^p} \leq 1.\end{aligned}$$

This proves (14) in this case.

(5) ($a_3 \geq a_1 \geq a_2$ and $a_3 > a_2$). If $a_1 \geq \frac{1}{2}(a_2 + a_3)$. Then we have $M_{\overrightarrow{D_3}} f(1) = a_1$ and $M_{\overrightarrow{D_3}} f(2) = \frac{1}{2}(a_2 + a_3)$. It follows that

$$|M_{\overrightarrow{D_3}} f(1) - M_{\overrightarrow{D_3}} f(2)| = \frac{1}{2}(2a_1 - (a_2 + a_3)) \leq (a_1 - a_2).$$

Therefore,

$$\begin{aligned}\frac{(\mathrm{Var}_p(M_{\overrightarrow{D_3}} f))^p}{(\mathrm{Var}_p(f))^p} &= \frac{\left|\frac{1}{2}(2a_1 - (a_2 + a_3))\right|^p + \left|\frac{1}{2}(a_2 - a_3)\right|^p + |a_3 - a_1|^p}{|a_2 - a_1|^p + |a_3 - a_2|^p + |a_3 - a_1|^p} \\ &< \frac{|a_1 - a_2|^p + |a_2 - a_3|^p + |a_3 - a_1|^p}{|a_2 - a_1|^p + |a_3 - a_2|^p + |a_3 - a_1|^p} = 1.\end{aligned}$$

This proves (14) in this case.

If $a_1 < \frac{1}{2}(a_2 + a_3)$. Then we have $\beta < \gamma$. Moreover, $M_{\overrightarrow{D_3}} f(1) = \frac{1}{3}(a_1 + a_2 + a_3)$ and $M_{\overrightarrow{D_3}} f(2) = \frac{1}{2}(a_2 + a_3)$. We have

$$\begin{aligned}\frac{(\mathrm{Var}_p(M_{\overrightarrow{D_3}} f))^p}{(\mathrm{Var}_p(f))^p} &= \frac{\left|\frac{1}{6}(a_2 + a_3 - 2a_1)\right|^p + \left|\frac{1}{2}(a_3 - a_2)\right|^p + \left|\frac{1}{3}(2a_3 - (a_1 + a_2))\right|^p}{|a_2 - a_1|^p + |a_3 - a_2|^p + |a_3 - a_1|^p} \\ &\leq \frac{1}{6^p} \frac{(\gamma - \beta)^p + 3^p(\beta + \gamma)^p + 2^p(\beta + 2\gamma)^p}{\beta^p + (\beta + \gamma)^p + \gamma^p} \\ &\leq \frac{1}{6^p} \frac{(1 - \frac{\beta}{\gamma})^p + 3^p(1 + \frac{\beta}{\gamma})^p + 2^p(2 + \frac{\beta}{\gamma})^p}{1 + (1 + \frac{\beta}{\gamma})^p + (\frac{\beta}{\gamma})^p}.\end{aligned}$$

Note that $\frac{\beta}{\gamma} \in [0, 1)$ and let $h_3(x) = \frac{(1-x)^p + 3^p(1+x)^p + 2^p(2+x)^p}{1 + x^p + (1+x)^p}$ for all $x \in [0, 1)$. Since $(1 - x)^p + 3^p(1 + x)^p \leq (4 + 2x)^p$ for all $x \in [0, 1)$ because $p \geq 1$, then

$$h_3(x) \leq \frac{2(4 + 2x)^p}{1 + x^p + (1 + x)^p} \leq 6^p, \quad \text{for all } x \in [0, 1).$$

Hence, we have

$$\frac{(\mathrm{Var}_p(M_{\overrightarrow{D_3}} f))^p}{(\mathrm{Var}_p(f))^p} \leq 1.$$

This proves (14) in this case.

(6) ($a_3 \geq a_2 \geq a_1$ and $a_3 > a_1$). Then $\gamma > \beta$, $\gamma > 0$ and $\frac{\beta}{\gamma} \in [0, 1)$. Moreover, $M_{\overrightarrow{D_3}} f(1) = \frac{1}{3}(a_1 + a_2 + a_3)$ and $M_{\overrightarrow{D_3}} f(2) = \frac{1}{2}(a_2 + a_3)$. Then we have

$$\begin{aligned}\frac{(\mathrm{Var}_p(M_{\overrightarrow{D_3}} f))^p}{(\mathrm{Var}_p(f))^p} &= \frac{\left|\frac{1}{6}(a_2 + a_3 - 2a_1)\right|^p + \left|\frac{1}{2}(a_3 - a_2)\right|^p + \left|\frac{1}{3}(2a_3 - (a_1 + a_2))\right|^p}{|a_2 - a_1|^p + |a_3 - a_2|^p + |a_3 - a_1|^p} \\ &\leq \frac{1}{6^p} \frac{(\gamma + \beta)^p + 3^p(\gamma - \beta)^p + 2^p(2\gamma - \beta)^p}{\beta^p + (\gamma - \beta)^p + \gamma^p} \\ &\leq \frac{1}{6^p} \frac{(1 + \frac{\beta}{\gamma})^p + 3^p(1 - \frac{\beta}{\gamma})^p + 2^p(2 - \frac{\beta}{\gamma})^p}{1 + (1 - \frac{\beta}{\gamma})^p + (\frac{\beta}{\gamma})^p}.\end{aligned}$$

Let $h_4(x) = \frac{(1+x)^p + 3^p(1-x)^p + 2^p(2-x)^p}{1 + x^p + (1-x)^p}$, $x \in [0, 1)$. Notice that $(1 + x)^p + 3^p(1 - x)^p \leq (4 - 2x)^p$ for all $x \in [0, 1)$ since $p \geq 1$. Then

$$h(x) \leq \frac{2(4 - 2x)^p}{1 + x^p + (1 - x)^p} \leq 6^p, \quad \text{for all } x \in [0, 1).$$

Therefore,

$$\frac{(\mathrm{Var}_p(M_{\overrightarrow{D_3}}f))^p}{(\mathrm{Var}_p(f))^p} \leq 1.$$

This gives (14) in this case. The proof of Theorem 7 is now complete. □

Remark 5. *Let $\overrightarrow{D_3} = (V, E)$ with $V = \{1, 2, 3\}$ and $E = \{1 \to 2, 2 \to 3, 1 \to 3\}$ and f be a function defined on V. Then $\mathrm{Var}(M_{\overrightarrow{D_3}}f) = \mathrm{Var}(f)$ if and only if f satisfies one of the following conditions:*

(*a*) $f(i) \equiv C$ *for some* $C \in \mathbb{R}$ *and all* $i \in \{1, 2, 3\}$;
(*b*) $|f(1)| \geq |f(2)| \geq |f(3)|$, $|f(1)| > |f(3)|$, $f(i)f(j) \geq 0$ *for all* $i, j \in \{1, 2, 3\}$;
(*c*) $|f(2)| \geq |f(1)| \geq |f(3)|$, $|f(2)| > |f(3)$, $|f(1)| \geq \frac{1}{2}(|f(2)| + |f(3)|)$, $f(i)f(j) \geq 0$ *for all* $i, j \in \{1, 2, 3\}$.

Theorem 8. *Let $n \geq 3$ and $0 < p \leq \infty$. Then*

(*i*) $\left(1 - \frac{1}{n}\right)\left(\frac{n}{2}\right)^{\min\{1/p-1,0\}} \leq \|M_{\overrightarrow{C_n}}\|_{\mathrm{BV}_p} \leq n^{\max\{1,1/p\}}(\sum_{j=1}^{n-1} \frac{1}{j})$;
(*ii*) *If $p \geq 1$, then* $\|M_{\overrightarrow{C_3}}\|_{\mathrm{BV}_p} \leq \frac{2}{3}$;
(*iii*) $\|M_{\overrightarrow{C_3}}\|_{\mathrm{BV}} = \frac{2}{3}$.

Proof. The proof will be divided into two steps.

Step 1: Proof of part (i). Without loss of generality, we may assume that $\overrightarrow{C_n} = (V, E)$, where $V = \{1, 2, \ldots, n\}$ and $E = \{1 \to 2, 2 \to 3, \ldots, n-1 \to n, n \to 1\}$. Let $f = \sum_{i=1}^n a_i \delta_i$. It is clear that

$$M_{\overrightarrow{C_n}}f(1) = \max\left\{|a_1|, \frac{1}{2}(|a_1| + |a_2|), \cdots, \frac{1}{n}\sum_{i=1}^n |a_i|\right\},$$

$$M_{\overrightarrow{C_n}}f(2) = \max\left\{|a_2|, \frac{1}{2}(|a_2| + |a_3|), \cdots, \frac{1}{n}\sum_{i=1}^n |a_i|\right\},$$

$$\cdots\cdots\cdots\cdots,$$

$$M_{\overrightarrow{C_n}}f(n-1) = \max\left\{|a_{n-1}|, \frac{1}{2}(|a_{n-1}| + |a_n|), \cdots, \frac{1}{n}\sum_{i=1}^n |a_i|\right\},$$

$$M_{\overrightarrow{C_n}}f(n) = \max\left\{|a_n|, \frac{1}{2}(|a_n| + |a_1|), \cdots, \frac{1}{n}\sum_{i=1}^n |a_i|\right\}.$$

Invoking Lemma 2.1 in [20], we get

$$\begin{aligned}
&|M_{\overrightarrow{C_n}}f(1) - M_{\overrightarrow{C_n}}f(2)| \\
&\leq |a_1 - a_2| + \frac{1}{2}|a_1 - a_3| + \cdots + \frac{1}{n-2}|a_1 - a_{n-1}| + \frac{1}{n-1}|a_1 - a_n| \\
&\leq \Big(\sum_{j=1}^{n-1} \frac{1}{j}\Big)|a_1 - a_2| + \Big(\sum_{j=2}^{n-1} \frac{1}{j}\Big)|a_2 - a_3| + \cdots + \frac{1}{n-1}|a_{n-1} - a_n| \\
&\leq \Big(\sum_{j=1}^{n-1} \frac{1}{j}\Big)\mathrm{Var}(f).
\end{aligned}$$

Similarly one has

$$|M_{\overrightarrow{C_n}}f(i) - M_{\overrightarrow{C_n}}f(i+1)| \leq \Big(\sum_{j=1}^{n-1} \frac{1}{j}\Big)\mathrm{Var}(f), \ \text{ for } \ i = 2, \ldots, n-1;$$

$$|M_{\overrightarrow{C_n}} f(n) - M_{\overrightarrow{C_n}} f(1)| \leq \Big(\sum_{j=1}^{n-1} \frac{1}{j}\Big)\mathrm{Var}(f).$$

Thus we have

$$\begin{aligned}\mathrm{Var}_p(M_{\overrightarrow{C_n}} f) &= \Big(\sum_{i=2}^{n} |M_{\overrightarrow{C_n}} f(i) - M_{\overrightarrow{C_n}} f(i-1)|^p + |M_{\overrightarrow{C_n}} f(n) - M_{\overrightarrow{C_n}} f(1)|^p\Big)^{1/p} \\ &\leq n^{1/p}\Big(\sum_{j=1}^{n-1} \frac{1}{j}\Big)\mathrm{Var}(f) \leq n^{\max\{1,1/p\}}\Big(\sum_{j=1}^{n-1} \frac{1}{j}\Big)\mathrm{Var}_p(f).\end{aligned}$$

This gives

$$\|M_{\overrightarrow{C_n}}\|_{\mathrm{BV}_p} \leq n^{\max\{1,1/p\}}\Big(\sum_{j=1}^{n-1} \frac{1}{j}\Big).$$

On the other hand, we get

$$\begin{aligned}\|M_{\overrightarrow{C_n}}\|_{\mathrm{BV}_p} &\geq \frac{\mathrm{Var}_p(M_{\overrightarrow{C_n}}\delta_1)}{\mathrm{Var}_p(\delta_1)} = \frac{\Big((1-\frac{1}{n})^p + \sum_{j=1}^{n-1}(\frac{1}{j} - \frac{1}{j+1})^p\Big)^{1/p}}{2^{1/p}} \\ &\geq \Big(1-\frac{1}{n}\Big)\Big(\frac{n}{2}\Big)^{\min\{1/p-1,0\}}.\end{aligned}$$

This proves part (i) of Theorem 8.

Step 2: Proofs of parts (ii) and (iii). At first, we prove

$$\|M_{\overrightarrow{C_3}}\|_{\mathrm{BV}_p} \leq \frac{2}{3}. \tag{15}$$

Without loss of generality, we assume that $\overrightarrow{C_3} = (V, E)$ with $V = \{1, 2, 3\}$ and $E = \{1 \to 2, 2 \to 3, 3 \to 1\}$. Let $f = \sum_{i=1}^{3} a_i\delta_i$ with $|a_2 - a_1|^p + |a_3 - a_2|^p + |a_1 - a_3|^p > 0$. We write

$$\begin{aligned}M_{\overrightarrow{C_3}} f(1) &= \max\Big\{|a_1|, \frac{1}{2}(|a_1| + |a_2|), \frac{1}{3}(|a_1| + |a_2| + |a_3|)\Big\}, \\ M_{\overrightarrow{C_3}} f(2) &= \max\Big\{|a_2|, \frac{1}{2}(|a_2| + |a_3|), \frac{1}{3}(|a_1| + |a_2| + |a_3|)\Big\}, \\ M_{\overrightarrow{C_3}} f(3) &= \max\Big\{|a_3|, \frac{1}{2}(|a_3| + |a_1|), \frac{1}{3}(|a_1| + |a_2| + |a_3|)\Big\}.\end{aligned}$$

Then we have

$$\begin{aligned}\mathrm{Var}_p(M_{\overrightarrow{C_3}} f) = (&|M_{\overrightarrow{C_3}} f(1) - M_{\overrightarrow{C_3}} f(2)|^p + |M_{\overrightarrow{C_3}} f(2) - M_{\overrightarrow{C_3}} f(3)|^p \\ &+ |M_{\overrightarrow{C_3}} f(3) - M_{\overrightarrow{C_3}} f(1)|^p)^{1/p}.\end{aligned}$$

To prove (15), it suffices to show that

$$\mathrm{Var}_p(M_{\overrightarrow{C_3}} f) \leq \frac{2}{3}\mathrm{Var}_p(f). \tag{16}$$

Without loss of generality we may assume that all $a_i \geq 0\,(i = 1, 2, 3)$. Let

$$\beta = |a_1 - a_2|, \quad \gamma = |a_2 - a_3|.$$

We only consider the following two cases since the other cases are analogous.

(1) ($a_1 \geq a_2 \geq a_3$ and $a_1 > a_3$). When $a_2 \geq \frac{1}{2}(a_1+a_3)$, then $\gamma \geq \beta$ and $\gamma > 0$. Moreover, $M_{\overrightarrow{C_3}}f(1) = a_1$, $M_{\overrightarrow{C_3}}f(2) = a_2$ and $M_{\overrightarrow{C_3}}f(3) = \frac{1}{3}(a_1+a_2+a_3)$. It follows that

$$\begin{aligned}\frac{(\mathrm{Var}_p(M_{\overrightarrow{C_3}}f))^p}{\mathrm{Var}_p(f)} &= \frac{(a_1-a_2)^p + \left(\frac{2a_2-(a_1+a_3)}{3}\right)^p + \left(\frac{2a_1-(a_2+a_3)}{3}\right)^p}{(a_1-a_2)^p+(a_2-a_3)^p+(a_1-a_3)^p} \\ &= \frac{\beta^p + \left(\frac{\gamma-\beta}{3}\right)^p + \left(\frac{2\beta+\gamma}{3}\right)^p}{\beta^p+\gamma^p+(\beta+\gamma)^p} \\ &= \frac{1}{3^p}\frac{(3\frac{\beta}{\gamma})^p + (1-\frac{\beta}{\gamma})^p + (2\frac{\beta}{\gamma}+1)^p}{(\frac{\beta}{\gamma})^p + (1+\frac{\beta}{\gamma})^p + 1}.\end{aligned}$$

Note that $\frac{\beta}{\gamma} \in [0,1]$. Let $g(x) = \frac{(3x)^p+(1-x)^p+(2x+1)^p}{x^p+(1+x)^p+1}$, $x \in [0,1]$. Noting that $g(x) < 2^p$ for all $x \in [0,1]$ when $p \geq 1$. Then

$$\frac{(\mathrm{Var}_p(M_{\overrightarrow{C_3}}f))^p}{(\mathrm{Var}_p(f))^p} \leq \left(\frac{2}{3}\right)^p.$$

This proves (16) in this case.

When $a_2 < \frac{1}{2}(a_1+a_3)$. Then $\gamma < \beta$ and $M_{\overrightarrow{C_3}}f(1) = a_1$, $M_{\overrightarrow{C_3}}f(2) = \frac{1}{3}(a_1+a_2+a_3)$ and $M_{\overrightarrow{C_3}}f(3) = \frac{1}{2}(a_1+a_3)$. It follows that

$$\begin{aligned}\frac{(\mathrm{Var}_p(M_{\overrightarrow{C_3}}f))^p}{\mathrm{Var}_p(f)} &= \frac{\left(\frac{2a_1-(a_2+a_3)}{3}\right)^p + \left(\frac{(a_1+a_3)-2a_2}{6}\right)^p + \left(\frac{a_1-a_3}{2}\right)^p}{(a_1-a_2)^p+(a_2-a_3)^p+(a_1-a_3)^p} \\ &= \frac{\left(\frac{2\beta+\gamma}{3}\right)^p + \left(\frac{\beta-\gamma}{6}\right)^p + \left(\frac{\beta+\gamma}{2}\right)^p}{\beta^p+(\beta+\gamma)^p+\gamma^p} \\ &= \frac{1}{6^p}\frac{(2(2+\frac{\gamma}{\beta}))^p + (1-\frac{\gamma}{\beta})^p + (3(1+\frac{\gamma}{\beta}))^p}{1+(1+\frac{\gamma}{\beta})^p + (\frac{\gamma}{\beta})^p}.\end{aligned}$$

Note that $\frac{\gamma}{\beta} \in [0,1)$. Let $g(x) = \frac{(2(2+x))^p+(1-x)^p+(3(1+x))^p}{1+(1+x)^p+x^p}$, $x \in [0,1)$. Since $p \geq 1$, we have

$$g(x) \leq 4^p \frac{2(1+\frac{x}{2})^p}{1+(1+x)^p+x^p} \leq 4^p, \quad \text{for } x \in [0,1).$$

This yields

$$\frac{(\mathrm{Var}_p(M_{\overrightarrow{C_3}}f))^p}{(\mathrm{Var}_p(f))^p} \leq \left(\frac{2}{3}\right)^p,$$

which proves (16) in this case.

(2) ($a_1 \geq a_3 \geq a_2$ and $a_1 > a_2$). In this case we have $\beta \geq \gamma$ and $\beta > 0$. Moreover, $M_{\overrightarrow{C_3}}f(1) = a_1$, $M_{\overrightarrow{C_3}}f(2) = \frac{1}{3}(a_1+a_2+a_3)$ and $M_{\overrightarrow{C_3}}f(3) = \frac{1}{2}(a_1+a_3)$. It follows that

$$\begin{aligned}\frac{(\mathrm{Var}_p(M_{\overrightarrow{C_3}}f))^p}{(\mathrm{Var}_p(f))^p} &= \frac{\left(\frac{2a_1-(a_2+a_3)}{3}\right)^p + \left(\frac{(a_1+a_3)-2a_2}{6}\right)^p + \left(\frac{a_1-a_3}{2}\right)^p}{(a_1-a_2)^p+(a_3-a_2)^p+(a_1-a_3)^p} \\ &= \frac{\left(\frac{2\beta-\gamma}{3}\right)^p + \left(\frac{\beta+\gamma}{6}\right)^p + \left(\frac{\beta-\gamma}{2}\right)^p}{\beta^p+\gamma^p+(\beta-\gamma)^p} \\ &= \frac{1}{6^p}\frac{(2(2-\frac{\gamma}{\beta}))^p + (1+\frac{\gamma}{\beta})^p + (3(1-\frac{\gamma}{\beta}))^p}{1+(\frac{\gamma}{\beta})^p + (1-\frac{\gamma}{\beta})^p}.\end{aligned}$$

Let $h(x)=\frac{(2(2-x))^p+(1+x)^p+(3(1-x))^p}{1+x^p+(1-x)^p}$, $x\in[0,1]$. Notice that

$$h(x)\le\frac{2(2(2-x))^p}{1+x^p+(1-x)^p}\le 4^p\frac{2(1-\frac{x}{2})^p}{1+x^p+(1-x)^p}\le 4^p,\ \ \text{for}\ \ x\in[0,1]$$

since $p\ge 1$. This proves (16) in this case.

On the other hand, let $f:\{1,2,3\}\to\mathbb{R}$ be defined by $f(1)=2$, $f(2)=f(3)=1$. Then we have $M_{\vec{C_3}}f(1)=2$, $M_{\vec{C_3}}f(2)=\frac{4}{3}$ and $M_{\vec{C_3}}f(3)=\frac{3}{2}$. It follows that

$$\frac{\mathrm{Var}(M_{\vec{C_3}}f)}{\mathrm{Var}(f)}=\frac{(2-\frac{4}{3})+(\frac{3}{2}-\frac{4}{3})+(2-\frac{3}{2})}{2}=\frac{2}{3},$$

which together with part (ii) leads to

$$\|M_{\vec{C_3}}\|_{\mathrm{BV}}=\frac{2}{3}.$$

□

4. Conclusions and Further Comments

It should be pointed out that our main results represent some significant extensions of the main results in [20,21]. In the works [20,21], the authors established some BV_p norms of the Hardy–Littlewood maximal operator on undirected graphs. Here, we focus on the BV_p norms of the Hardy–Littlewood maximal operator on some directed graphs. Combining with the undirected graph case, the directed graph case is often more complex. Our main results not only enrich the variation inequalities for Hardy–Littlewood maximal operators defined on finite graphs, but also explore some new graphs to serve our aim. This is the novelty of this paper.

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