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Article

On a New Generalization of Bernstein-Type Rational Functions and Its Approximation

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Abstract: In this study, we introduce a new generalization of a Bernstein-type rational function possessing better estimates than the classical Bernstein-type rational function. We investigate its error of approximation globally and locally in terms of the first and second modulus of continuity and a class of Lipschitz-type functions. We present graphical comparisons of its approximation with illustrative examples.

Keywords: linear positive operator; rate of convergence; Bernstein-type rational function

MSC: 41A25; 41A36

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1. Introduction

Bernstein polynomials [1] are defined to prove the well-known convergence theorem of Weierstreiss for each real-valued function f defined on $[0,1]$ by

$$B_n(f;x) = \sum_{k=0}^{n} f(k/n)\binom{n}{k}x^k(1-x)^{n-k}, n = 1,2,\ldots \tag{1}$$

In 1975, Balázs [2] defined an operator for each real-valued function f defined on $[0,\infty)$ and appropriately chosen real sequences (a_n) and (b_n) such that $a_n = \frac{b_n}{n}$ by

$$R_n(f;x) = \frac{1}{(1+a_nx)^n}\sum_{k=0}^{n} f\left(\frac{k}{b_n}\right)\binom{n}{k}(a_nx)^k, n = 1,2,\ldots \tag{2}$$

When $b_n = n$, this operator possesses the following relation with a Bernstein polynomial:

$$R_n(f;x) = B_n\left(f;\frac{a_nx}{1+a_nx}\right),$$

which is known as a Bernstein-type rational function. Balàzs estimated its rate of convergence for each continuous function f defined on $[0,\infty)$ and proved an asymptotic approximation theorem under the condition that $f(x) = O(e^{\tau x})$, $x \to \infty$ for some real number τ. In [3], Balàzs and Szabados improved the estimates given in [2] under more restrictive conditions by choosing $a_n = n^{\zeta-1}$ and $b_n = n^{\zeta}$ for $0 < \zeta \leq \frac{2}{3}$, $n = 1,2,\ldots$ by assumming that f is uniformly continuous on $[0,\infty)$. Additionally, in [4], Balázs presented approximation results for Balázs–Szabados operators on all real axes. Totik investigated in [5] saturation properties of Balázs–Szabados operators, and Abel and Veccia [6] obtained Voronovskaja type asiymtotic results for Balázs–Szabados operators. In [7], Holhoş have presented new approximation results for Balázs–Szabados operators by means of super-exponential functions.

In [8], İspir and Atakut gave a generalization of Bernstein-type rational functions as follows:

$$G_n(f;x) = \frac{1}{\phi(a_n x)} \sum_{k=0}^{\infty} \frac{\phi^{(k)}(0)}{k!} (a_n x)^k f\left(\frac{k}{b_n}\right),$$

where a_n and b_n are suitably chosen positive numbers, and ϕ_n is a sequence of functions satisfying certain conditions. Recently, Agratini [9] has studied a class of Bernstein-type rational functions by choosing a strictly decreasing positive real sequence (λ_n) such that $\lim_{n\to\infty} \lambda_n = 0$ as follows:

$$L_n(f;x) = \frac{1}{(1+\lambda_n x)^n} \sum_{k=0}^{n} f\left(\frac{k}{n\lambda_n}\right) \binom{n}{k} (\lambda_n x)^k, n = 1,2,\ldots, \tag{3}$$

where f is continuous on $[0,\infty)$ satisfying a certain growing condition. Agratini has investigated both a local and global estimation of rate of convergence and has presented a weighted approximation result by using weighted modulus of continuity. Researchers can also find approximation results of some other Bernstein-type rational functions in those references [10–20].

Denoted by $C_B([0,\infty))$ is the Banach space of all real-valued continuous and bounded functions on $[0,\infty)$ endowed with the sup-norm $\|f\|_\infty = \sup_{x\in[0,\infty)} |f(x)|$.

For a compact subinterval $[a,b] \subset [0,\infty)$, the same norm is valid and reduced to $\|f\|_{[a,b]} = \sup_{x\in[a,b]} |f(x)|$.

In this paper, we construct a new generalization of Bernstein-type rational function, which is reducible to (2) and (3), and it is a rational function associated with the Bernstein polynomial given in (1). In an effort to define a well-defined Bernstein-type rational function, we choose non-negative real sequences $(\alpha_n), (\beta_n)$ and (γ_n) such that $\gamma_n = n\alpha_n$ satisfying the property

$$\lim_{n\to\infty} \alpha_n = 0, \ \lim_{n\to\infty} \beta_n = 1 \text{ and } \lim_{n\to\infty} \gamma_n = \infty. \tag{4}$$

We consider a newly defined Bernstein-type rational function as follows:

$$R_n^G(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{\gamma_n}\right) \binom{n}{k} \frac{(\alpha_n x)^k (\beta_n)^{n-k}}{(\beta_n + \alpha_n x)^n}, x \geq 0, n \in \mathbb{N}, \tag{5}$$

where f is a real-valued continuous function on $[0,\infty)$, $(\alpha_n), (\beta_n)$ and (γ_n) are real sequences such that $\gamma_n = n\alpha_n$ satisfies the property (4). It is clear that R_n^G is a well-defined, linear and positive operator. When $\beta_n = 1, \alpha_n = a_n$ and $\gamma_n = b_n$, under the condition that $f(x) = O(e^{\tau x}), x \to \infty$ for some real number τ, it is reduced to the Bernstein-type rational functions given by (2). When $\alpha_n := \lambda_n$ is a strictly decreasing positive real sequence, and f is continuous on $[0,\infty)$ satisfying a certain growing condition, it is induced to Agratini's modification given by (3). Additionally, since it has the following connection with the Bernstein polynomial given by (1):

$$R_n^G(f;x) = B_n\left(f; \frac{\alpha_n x}{\beta_n + \alpha_n x}\right),$$

when $\gamma_n = n$ for $\beta_n = 1$, it can be called a generalized Bernstein-type rational function.

2. Approximation Results

Firstly, we present the following auxiliary result, which will be used throughout the paper:

Lemma 1. *We have the following values of the generalized Bernstein-type rational function at monomials:*

$$R_n^G(e_0;x) = 1, \tag{6}$$

$$R_n^G(e_1;x) = \frac{x}{\beta_n+\alpha_n x}, \tag{7}$$

$$R_n^G(e_2;x) = \frac{\left(1-\frac{1}{n}\right)x^2}{(\beta_n+\alpha_n x)^2} + \frac{x}{\gamma_n(\beta_n+\alpha_n x)}, \tag{8}$$

where $e_i(t) = t^i$ for $i = 0,1,2$, (α_n), (β_n) and (γ_n) are real sequences such that $\gamma_n = n\alpha_n$.

Proof. By considering

$$(\beta_n+\alpha_n x)^n = \sum_{k=0}^{n} \binom{n}{k} (\alpha_n x)^k (\beta_n)^{n-k}, \tag{9}$$

We calculate that

$$\begin{aligned}
R_n^G(e_0;x) &= R_n^G(1;x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(\alpha_n x)^k (\beta_n)^{n-k}}{(\beta_n+\alpha_n x)^n} = 1, \\
R_n^G(e_1;x) &= R_n^G(t;x) = \sum_{k=0}^{n} \frac{k}{\gamma_n} \binom{n}{k} \frac{(\alpha_n x)^k (\beta_n)^{n-k}}{(\beta_n+\alpha_n x)^n} \\
&= \frac{n\alpha_n x}{\gamma_n(\beta_n+\alpha_n x)} \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{\alpha_n x}{\beta_n+\alpha_n x}\right)^k \left(\frac{\beta_n}{\beta_n+\alpha_n x}\right)^{n-1-k} \\
&= \frac{n\alpha_n x}{\gamma_n(\beta_n+\alpha_n x)}.1 \\
&= \frac{x}{\beta_n+\alpha_n x}, \\
R_n^G(e_2;x) &= R_n^G(t^2;x) = \sum_{k=0}^{n} \frac{k^2}{\gamma_n^2} \binom{n}{k} \frac{(\alpha_n x)^k (\beta_n)^{n-k}}{(\beta_n+\alpha_n x)^n} \\
&= \frac{n(n-1)\alpha_n^2 x^2}{\gamma_n^2(\beta_n+\alpha_n x)^2} \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{(\alpha_n x)^k (\beta_n)^{n-2-k}}{(\beta_n+\alpha_n x)^{n-2}} \\
&\quad + \frac{n\alpha_n x}{\gamma_n^2(\beta_n+\alpha_n x)} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(\alpha_n x)^k (\beta_n)^{n-1-k}}{(\beta_n+\alpha_n x)^{n-1}} \\
&= \frac{n(n-1)\alpha_n^2 x^2}{\gamma_n^2(\beta_n+\alpha_n x)^2} + \frac{n\alpha_n x}{\gamma_n^2(\beta_n+\alpha_n x)} \\
&= \frac{\left(1-\frac{1}{n}\right)x^2}{(\beta_n+\alpha_n x)^2} + \frac{x}{\gamma_n(\beta_n+\alpha_n x)}.
\end{aligned}$$

□

Remark 1. *We have the following first- and second-order central moments by considering Lemma 1:*

$$R_n^G(e_1-x;x) = \frac{(1-\beta_n)x}{\beta_n+\alpha_n x} - \frac{\alpha_n x^2}{\beta_n+\alpha_n x}, \tag{10}$$

$$\begin{aligned}
R_n^G((e_1-x)^2;x) &= \frac{\beta_n x}{\gamma_n(\beta_n+\alpha_n x)^2} + \frac{\left(\alpha_n+(\beta_n-1)^2-\frac{1}{n}\right)x^2}{(\beta_n+\alpha_n x)^2} \\
&\quad + \frac{2\alpha_n(\beta_n-1)x^3}{(\beta_n+\alpha_n x)^2} + \frac{\alpha_n^2 x^4}{(\beta_n+\alpha_n x)^2}.
\end{aligned} \tag{11}$$

Theorem 1. *Let R_n^G, $n \in \mathbb{N}$, be the generalized Bernstein-type rational function defined by (5). If (α_n), (β_n) and (γ_n) are non-negative real sequences satisfying (4) for each $n \in \mathbb{N}$, then $R_n^G(f;x)$ converges to $f(x)$ uniformly with respect to x on $[0,r] \subset [0,\infty)$, $r > 0$, for each $f \in C([0,r])$.*

Proof. The proof can be fulfilled easily from the well-known Bohman–Korovkin theorem [21]. From (6) of Lemma 1, it is clear that

$$\lim_{n\to\infty}\left\|R_n^G(e_0;.) - e_0(.)\right\|_{[0,r]} = 0. \tag{12}$$

Since $\frac{1}{\beta_n+\alpha_n x} \leq \frac{1}{\beta_n}$ for each $x \in [0,r]$, by Remark 1, we can write

$$\begin{aligned}\left|R_n^G(e_1;x) - e_1(x)\right| &= \left|R_n^G(e_1 - x;x)\right| = \left|\frac{(1-\beta_n)x}{\beta_n+\alpha_n x} - \frac{\alpha_n x^2}{\beta_n+\alpha_n x}\right| \\ &\leq \frac{|1-\beta_n|r}{\beta_n} + \frac{\alpha_n r^2}{\beta_n} := \mu_n^1. \end{aligned} \tag{13}$$

By considering relation (4), in (13), since $\lim_{n\to\infty}\mu_n^1 = 0$, we obtain

$$\lim_{n\to\infty}\left\|R_n^G(e_1;.) - e_1(.)\right\|_{[0,r]} = 0. \tag{14}$$

By considering (8) of Lemma 1, we can calculate that

$$\begin{aligned}\left|R_n^G(e_2;x) - e_2(x)\right| &= \left|\frac{x}{\gamma_n(\beta_n+\alpha_n x)} + \frac{\left(1-\frac{1}{n}\right)x^2}{(\beta_n+\alpha_n x)^2} - x^2\right| \\ &\leq \frac{x}{\gamma_n(\beta_n+\alpha_n x)} + \frac{\left|1-\frac{1}{n}-\beta_n^2\right|x^2}{(\beta_n+\alpha_n x)^2} \\ &\quad + \frac{2\beta_n\alpha_n x^3}{(\beta_n+\alpha_n x)^2} + \frac{\alpha_n^2 x^4}{(\beta_n+\alpha_n x)^2} \\ &\leq \frac{r}{\gamma_n\beta_n} + \frac{\left|1-\frac{1}{n}-\beta_n^2\right|r^2}{\beta_n^2} + \frac{2\alpha_n r^3}{\beta_n} + \frac{\alpha_n^2 r^4}{\beta_n^2} := \mu_n^2. \end{aligned} \tag{15}$$

Under conditions of relation (4), from (15), since $\lim_{n\to\infty}\mu_n^2 = 0$, we get

$$\lim_{n\to\infty}\left\|R_n^G(e_2;.) - e_2(.)\right\|_{[0,r]} = 0. \tag{16}$$

From relations (12), (14) and (16), the criterion of the Bohman–Korovkin theorem is satisfied. Therefore, the proof of the theorem is completed. □

3. Local and Global Approximation

In this part, we present local and global results of approximation with the help of the first and second modulus of continuity and a Lipschitz class of functions.

For any $\mu > 0$, modulus of continuity of $f \in C_B([0,\infty))$ is defined as

$$\omega(f;\mu) = \sup_{0<h<\mu}\ \sup_{x\in[0,\infty)} |f(x+h) - f(x)|, \tag{17}$$

which possesses the following property:

$$\omega(f;\kappa\mu) \leq (\kappa+1)\omega(f;\mu), \tag{18}$$

for $\kappa, \mu > 0$, and $\lim_{\mu\to 0^+}\omega(f;\mu) = 0$, when f is uniformly continuous [22].

Theorem 2. *Let $(\alpha_n), (\beta_n)$ and (γ_n) be real sequences such that $\gamma_n = n\alpha_n$, satisfying the property (4). For any $f \in C_B([0,\infty))$, we have*

$$\left|R_n^G(f;x) - f(x)\right| \leq 2\omega\left(f; \sqrt{\mu_n^x}\right),$$

where

$$\begin{aligned}\mu_n^x \quad : \quad &= \frac{\beta_n x}{\gamma_n(\beta_n + \alpha_n x)^2} + \frac{\left(\alpha_n + (\beta_n - 1)^2 - \frac{1}{n}\right)x^2}{(\beta_n + \alpha_n x)^2} \\ &+ \frac{2\alpha_n(\beta_n - 1)x^3}{(\beta_n + \alpha_n x)^2} + \frac{\alpha_n^2 x^4}{(\beta_n + \alpha_n x)^2}.\end{aligned} \tag{19}$$

Proof. Let $f \in C_B([0,\infty))$. By (18), we have

$$|f(t) - f(x)| \leq \left(1 + \frac{|t - x|}{\mu}\right)\omega(f;\mu). \tag{20}$$

By applying the operator R_n^G to (20), by taking linearity and positivity of the operator R_n^G into account and by applying Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\left|R_n^G(f;x) - f(x)\right| &\leq R_n^G(|f(t) - f(x)|;x) \\ &\leq \omega(f;\mu)\left(1 + \frac{1}{\mu}R_n^G(|e_1 - x|;x)\right) \\ &\leq \omega(f;\mu)\left(1 + \frac{1}{\mu}\sqrt{R_n^G\left((e_1 - x)^2;x\right)}\right).\end{aligned} \tag{21}$$

From (11) of Remark 1, by choosing

$$\begin{aligned}\mu_n^x \quad : \quad &= R_n^G\left((e_1 - x)^2;x\right) = \frac{\beta_n x}{\gamma_n(\beta_n + \alpha_n x)^2} + \frac{\left(\alpha_n + (\beta_n - 1)^2 - 1\right)x^2}{(\beta_n + \alpha_n x)^2} \\ &+ \frac{2\alpha_n(\beta_n - 1)x^3}{(\beta_n + \alpha_n x)^2} + \frac{\alpha_n^2 x^4}{(\beta_n + \alpha_n x)^2},\end{aligned}$$

and by replacing $\mu := \sqrt{\mu_n^x}$, we complete the proof of the theorem. □

Remark 2. *In Theorem 2, μ_n^x is dependent on x and choosing of $(\alpha_n), (\beta_n)$ and (γ_n). $(\alpha_n), (\beta_n)$ and (γ_n) must be non-negative real sequences satisfying $\mu_n^x \geq 0$. Otherwise, Theorem 2 becomes invalid. For example, if $\beta_n \geq 1$ and $\alpha_n + (\beta_n - 1)^2 \geq \frac{1}{n}$ then $\mu_n^x \geq 0$. This is not the only possible condition as $\mu_n^x \geq 0$.*

Moreover, for $f \in C([0,r])$, Theorem 2 is reduced to the following inequality:

$$\left\|R_n^G(f;.) - f\right\|_{[0,r]} \leq 2\omega(f;\sqrt{\mu_n}),$$

where

$$\begin{aligned}\mu_n \quad : \quad &= \frac{r}{\gamma_n\beta_n} + \frac{\left(\alpha_n + (\beta_n - 1)^2 - \frac{1}{n}\right)r^2}{\beta_n^2} \\ &+ \frac{2\alpha_n(\beta_n - 1)r^3}{\beta_n^2} + \frac{\alpha_n^2 r^4}{\beta_n^2}.\end{aligned} \tag{22}$$

For $f \in C_B([0,\infty))$, Petree's K-functional is defined by

$$K_2(f;\mu) = \inf_{g\in C_B^{(2)}([0,\infty))} \{\|f-g\|_\infty + \mu\|g''\|_\infty\}, \tag{23}$$

where
$$C_B^{(2)}([0,\infty)) := \{g \in C_B([0,\infty)) : g', g'' \in C_B([0,\infty))\}.$$

We have the following connection (see p. 192 in [23]) between Petree's K-functional and the second modulus of continuity $\omega_2(f;.)$

$$K_2(f;\mu) \le C\omega_2(f;\sqrt{\mu}), \tag{24}$$

where
$$\omega_2(f;\sqrt{\mu}) = \sup_{0<h<\sqrt{\mu}} \sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|. \tag{25}$$

Theorem 3. *Let (α_n), (β_n) and (γ_n) be real sequences such that $\gamma_n = n\alpha_n$ satisfying property (4). For each $f \in C_B([0,\infty))$, then there exists a $C > 0$ such that*

$$\left|R_n^G(f;x) - f(x)\right| \le C\left\{\omega_2\left(f;\sqrt{\mu_n^x}\right) + \sqrt{\mu_n^x}\right\}, x \ge 0,$$

where μ_n^x is given as in (19).

Proof. We initially define an auxilary operator by

$$E_n^G(f;x) := R_n^G(f;x) + f(x) - f(\eta_n^x), \tag{26}$$

where
$$\eta_n^x := R_n^G(e_1 - x;x) = \frac{(1-\beta_n)x - \alpha_n x^2}{\beta_n + \alpha_n x}. \tag{27}$$

By (26), we obtain

$$E_n^G(e_1 - x;x) = R_n^G(e_1 - x;x) - x - \eta_n^x = 0. \tag{28}$$

For any $g \in C_B^{(2)}([0,\infty))$, from Taylor's formula, we can write

$$g(t) - g(x) = g'(x)(t-x) + \int_x^t (t-u)g''(u)du. \tag{29}$$

By applying the operator R_n^G to (29), by (28), we obtain

$$\begin{aligned} \left|R_n^G(g;x) - g(x)\right| &\le |g'(x)|\left|R_n^G(t-x;x)\right| + \left|R_n^G\left(\int_x^t (t-u)g''(u)du;x\right)\right| \\ &\le \|g'\|_\infty\left|R_n^G(t-x;x)\right| + \|g''\|_\infty R_n^G\left((t-x)^2;x\right). \end{aligned} \tag{30}$$

Since $g' \in C_B([0,\infty))$, there exists a $k_0 > 0$ such that $\|g'\|_\infty = k_0$. Therefore, by applying Cauchy–Schwarz inequality, we obtain

$$\left|R_n^G(g;x) - g(x)\right| \le k_0\sqrt{R_n^G\left((t-x)^2;x\right)} + \|g''\|_\infty R_n^G\left((t-x)^2;x\right). \tag{31}$$

Additionally, for $f \in C_B([0,\infty))$, we have

$$\begin{aligned} \left|R_n^G(f;x)\right| &\le \sum_{k=0}^n \left|f\left(\frac{k}{\gamma_n}\right)\right|\binom{n}{k}\frac{(\alpha_n x)^k\beta_n^{n-k}}{(\beta_n+\alpha_n x)^n} \\ &\le \|f\|_\infty R_n^G(e_0;x) = \|f\|_\infty. \end{aligned} \tag{32}$$

By considering (30) and (32), we can write

$$\begin{aligned}\left|R_n^G(f;x)-f(x)\right| &\leq \left|R_n^G((f-g);x)-(f-g)(x)\right|+\left|R_n^G(g;x)-g(x)\right| \\ &\leq R_n^G(|f-g|;x)+|(f-g)(x)|+\left|R_n^G(g;x)-g(x)\right| \\ &\leq 2\|f-g\|_\infty+\|g''\|_\infty R_n^G\left((e_1-x)^2;x\right) \\ &\quad +k_0\sqrt{R_n^G\left((e_1-x)^2;x\right)}. \end{aligned} \tag{33}$$

In (33), considering Remark 1 and choosing μ_n^x as in (19), by taking the infimum right-hand side of the last inequality, for $g \in C_B^{(2)}([0,\infty))$, we obtain

$$\left|R_n^G(f;x)-f(x)\right| \leq 2K(f;\mu_n^x)+k_0\sqrt{\mu_n^x}. \tag{34}$$

Lastly, by applying (24) to (34), we acquire

$$\left|R_n^G(f;x)-f(x)\right| \leq c_0\omega_2\left(f;\sqrt{\mu_n^x}\right)+k_0\sqrt{\mu_n^x},$$

where $c_0 > 0$. By choosing $C = \max\{k_0, c_0\}$, we obtain the desired result. □

Remark 3. *For $f \in C([0,r])$, Theorem 3 is induced to the following result:*

$$\left\|R_n^G(f;.)-f\right\|_{[0,r]} \leq C\{\omega_2(f;\sqrt{\mu_n})+\sqrt{\mu_n}\}, C>0,$$

where μ_n is as in (22).

Let E be any subset of $\mathbb{R}$ and $\theta \in (0,1]$. Let $Lip_{M_f}(E,\theta)$ denote a class of Lipschitz functions in $C_B([0,\infty))$ satisfying

$$|f(t)-f(x)| \leq M_f|t-x|^\theta, t \in \overline{E}, x \geq 0, \tag{35}$$

where M_f is a constant, and $\overline{E}$ is the closure of E in $[0,\infty)$.

Theorem 4. *Let $(\alpha_n), (\beta_n)$ and (γ_n) be real sequences such that $\gamma_n = n\alpha_n$, satisfying the property (4). For any $f \in Lip_{M_f}(E,\theta)$, we have*

$$\left|R_n^G(f;x)-f(x)\right| \leq M_f\left\{\left(\sqrt{\mu_n^x}\right)^\theta+2(d(x,E))^\theta\right\},$$

where μ_n^x is given as in (19), M_f is a constant depending on f, and E is any subset of $[0,\infty)$, $x \in [0,\infty)$ and $\theta \in (0,1]$.

Proof. Let $x \in [0,\infty)$ and $x \in \overline{E}$ such that $d(x,x_0) = |x-x_0|$. We can write

$$|f-f(x)| \leq |f-f(x_0)|+|f(x_0)-f(x)|. \tag{36}$$

By applying R_n^G to (36), and by considering the linearity and positivity of R_n^G and (35), we obtain

$$\begin{aligned}\left|R_n^G(f;x)-f(x)\right| &\leq R_n^G(|f-f(x_0)|;x)+R_n^G(|f(x)-f(x_0)|;x) \\ &\leq M_f\left\{R_n^G\left(|e_1-x_0|^\theta e_0;x\right)+R_n^G\left(|x-x_0|^\theta e_0;x\right)\right\} \\ &= M_f\left\{R_n^G\left(|e_1-x_0|^\theta e_0;x\right)+|x-x_0|^\theta R_n^G(e_0;x)\right\}. \end{aligned} \tag{37}$$

In (37), by using Hölder's inequality for $p = \frac{2}{\theta}$ and $q = \frac{2}{2-\theta}$ such that $\frac{1}{p} + \frac{1}{q} = 1$, by considering Lemma 1 and (11) of Remark 1, we obtain

$$\begin{aligned} \left|R_n^G(f;x) - f(x)\right| &\leq M_f\left\{\left(R_n^G\left(|e_1 - x|^{\theta p};x\right)\right)^{1/p}\left(R_n^G((e_0)^q;x)\right)^{1/q} + 2(d(x,E))^\theta\right\} \\ &= M_f\left\{\left(R_n^G\left((e_1 - x)^2;x\right)\right)^{\theta/2}(1)^{1/q} + 2(d(x,E))^\theta\right\} \\ &= M_f\left\{\left(\sqrt{\mu_n^x}\right)^\theta + 2(d(x,E))^\theta\right\}, \end{aligned}$$

which completes the proof of theorem. □

Remark 4. *When $x \in [0,r] := E \subset [0,\infty)$, it is clear that $d(x,E) = 0$. From Theorem 4, we have the following inequality:*

$$\left\|R_n^G(f;.) - f\right\|_{[0,r]} \leq M_f(\sqrt{\mu_n})^\theta,$$

where μ_n is as in (22).

4. Graphical Comparison

In this part, we present some graphical results produced in Maple software.

Example 1. *Let us choose $f(x) = x\left(x + \frac{1}{2}\right)\left(x + \frac{1}{3}\right)$, $\alpha_n = \frac{1}{\sqrt{n}}$, $\gamma_n = \sqrt{n}$ for $x \geq 0$ and $n \in \mathbb{N}$.*

In Figures 1 and 2, by choosing $\beta_n = 1 - \frac{1}{n}$, graphical comparison of approximation of $R_n^G(f;x)$ to f for $n = 50, 75$ and 100 is presented on $[0,\infty)$ and $[0,3]$. It is clear that approximation of $R_n^G(f;x)$ to f is better for increasing the value of n.

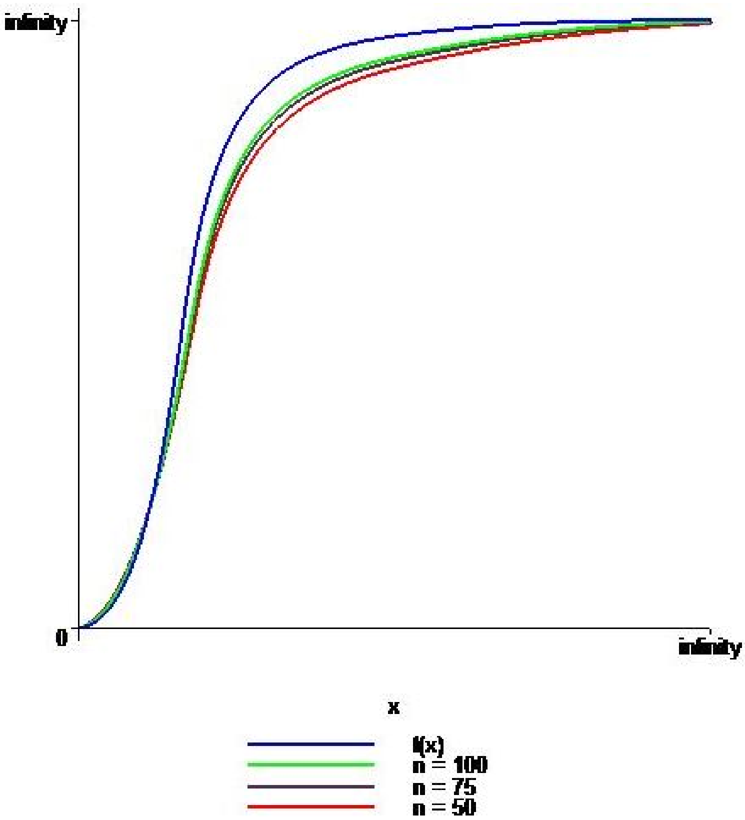

Figure 1. Approximation by R_n^G on $[0,\infty)$, for $\beta_n = 1 - \frac{1}{n}$, $n = 50, 75$ and 100.

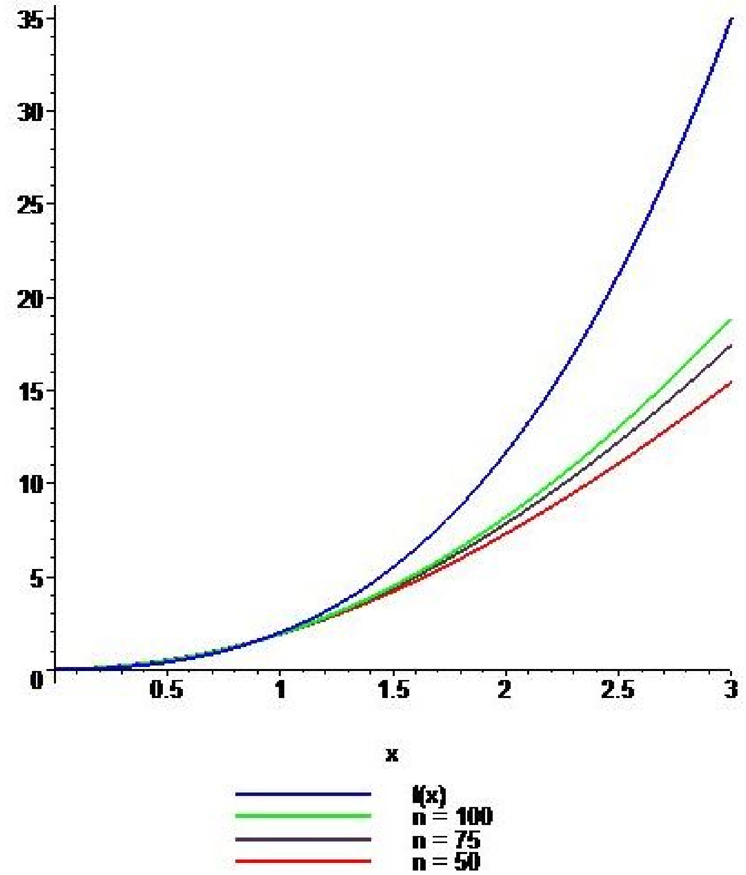

Figure 2. Approximation by R_n^G on $[0,3]$, for $\beta_n = 1 - \frac{1}{n}$, $n = 50, 75$ and 100.

In Figures 3 and 4, by choosing $\beta_n = 1 - \frac{1}{n}$, $R_n^G(f;x)$ is compared graphically with the classical Bernstein-type rational function $R_n(f;x)$ given by (2) to f for $n = 25$ on $[0,\infty)$ and $[0,3]$. It is obvious that approximation of $R_n^G(f;x)$ to f is better than approximation by $R_n(f;x)$ to f on $[0,\infty)$ and $[0,3]$.

In Figures 5 and 6, by denoting $R_n^G(f;x) := R_n^G(f;x,\beta_n)$ and choosing $\beta_n = 1 - \frac{2}{n}, 1$ and $1 + \frac{2}{n}$, $R_n^G(f;x,1-\frac{2}{n})$, $R_n^G(f;x,1)$ and $R_n^G(f;x,1+\frac{2}{n})$ are graphically compared. Here, it is clear that $R_n^G(f;x,1)$ is reduced to $R_n(f;x)$ for $\beta_n = 1$. If we choose β_n as a real sequence such that $\lim_{n\to\infty} \beta_n = 1$, and β_n is not constant, then we see that approximation by $R_n^G(f;x,1-\frac{2}{n})$ is better than $R_n^G(f;x,1) = R_n(f;x)$ and $R_n^G(f;x,1+\frac{2}{n})$ on $[0,\infty)$ and $[0,3]$.

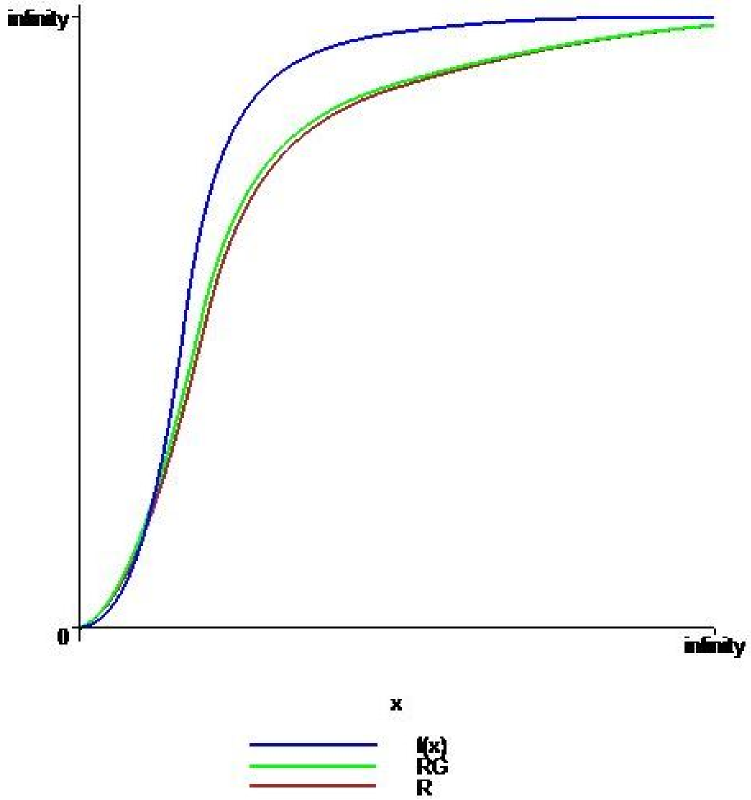

Figure 3. Comparison by R_n^G and R_n on $[0, \infty)$, for $\beta_n = 1 - \frac{1}{n}$, $n = 25$.

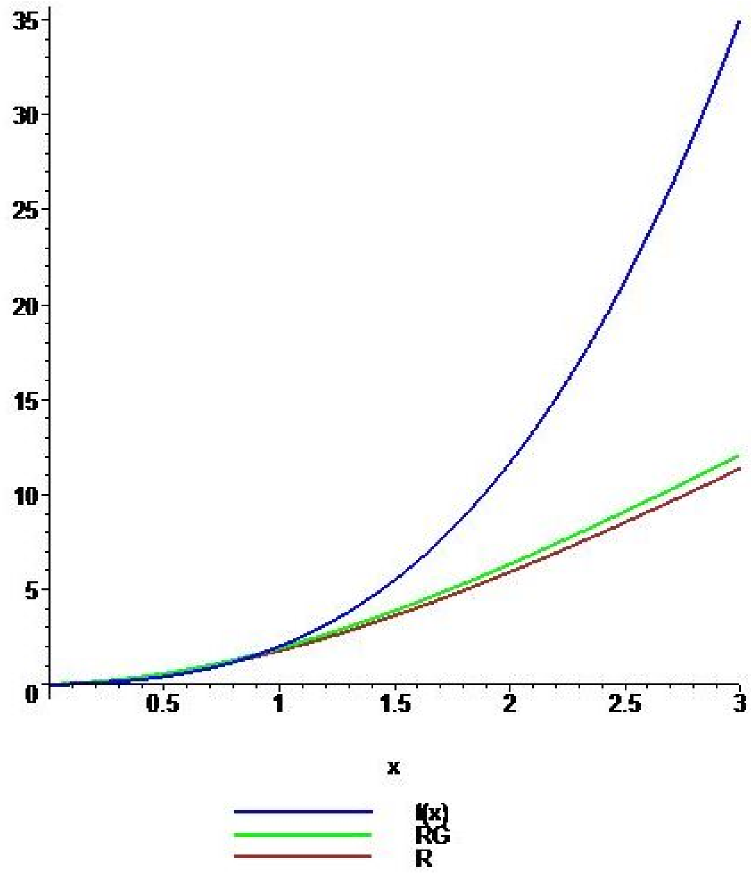

Figure 4. Comparison of R_n^G and R_n on $[0, 3]$, for $\beta_n = 1 - \frac{1}{n}$, $n = 25$.

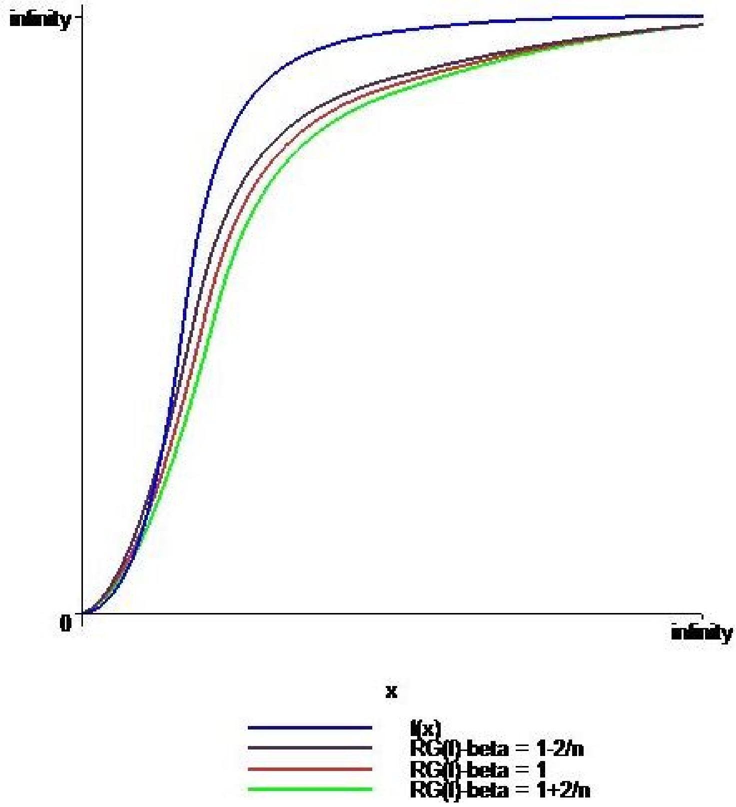

Figure 5. Approximation by R_n^G on $[0,\infty)$, for $\beta_n = 1 - \frac{2}{n}, 1$ and $1 + \frac{2}{n}$, $n = 25$.

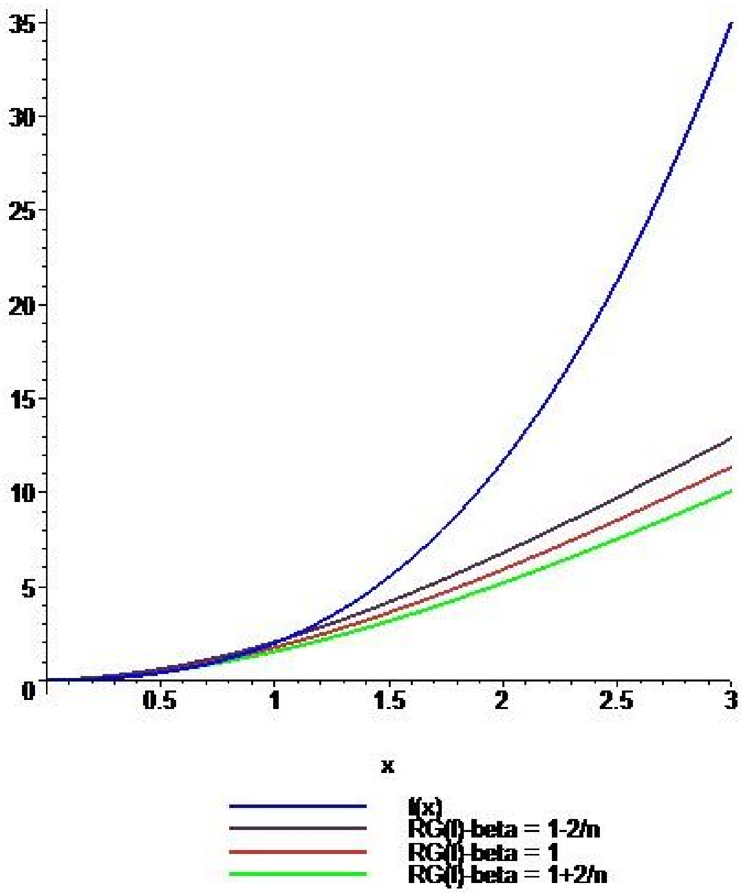

Figure 6. Approximation by R_n^G on $[0,3]$, for $\beta_n = 1 - \frac{2}{n}, 1$ and $1 + \frac{2}{n}$, $n = 25$.

5. Conclusions

In this study, we have introduced a newly defined Bernstein-type rational function R_n^G, which is a generalized Bernstein-type rational function in terms of including the classical Bernstein-type rational function defined by (2) and Agratini's modification, defined by (3). We have estimated the error of its approximation for conveniently chosen non-negative real sequences (α_n) and (β_n). Consequently, the newly defined generalized Bernstein-type rational function possesses better results than the classical Bernstein-type rational function defined by (2) for certain functions.

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