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On Statistical and Semi-Weyl Manifolds Admitting Torsion

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Abstract: We introduce the concept of quasi-semi-Weyl structure, we provide a couple of ways for constructing quasi-statistical and quasi-semi-Weyl structures by means of a pseudo-Riemannian metric, an affine connection and a tensor field on a smooth manifold, and we place these structures in relation with one another.

Keywords: statistical structure; quasi-statistical structure; semi-Weyl structure; quasi-semi-Weyl structure; dual; semi-dual connections

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1. Introduction

Statistical structures, introduced by Lauritzen in [1], are pairs (g, ∇) of a pseudo-Riemannian metric g and a torsion-free affine connection ∇ such that ∇g is totally symmetric. Kurose firstly considered statistical manifolds admitting torsion (or quasi-statistical manifolds) in order to describe geometric structures on quantum state spaces [2]. These naturally appear in the geometry of affine distributions and can be regarded as the quantum version of statistical manifolds. In [3], Norden used the notion of generalized dual connections to study Weyl geometry; hence, another generalization of a statistical manifold (and also of a Weyl manifold) will be the semi-Weyl manifold [4], appearing in affine differential geometry. A natural example is provided by a nondegenerate affine immersion [5]. Moreover, if the nondegenerate immersion is also equiaffine, then it gives rise to a statistical manifold [5]. It is worth mentioning that Weyl introduced the concept of Weyl manifold [6] in order to unify the laws of gravitation and of electromagnetism, which he did not succeed in, but, still, Weyl manifolds are studied in mathematics as an important class of manifolds in conformal geometry.

In this paper, we introduce the concept of quasi-semi-Weyl structure, provide a couple of ways to construct quasi-statistical and quasi-semi-Weyl structures by means of a pseudo-Riemannian metric, an affine connection and a tensor field on a smooth manifold and place these structures in relation with one another. This study complements the ideas of Tao and Zhang, who analyzed in [7] transformations preserving Codazzi coupling of conjugate connections and, hence, the statistical structures.

2. Statistical and Quasi-Statistical Structures—Dual Connections

Let M be a smooth manifold and let g be a pseudo-Riemannian metric on M. Throughout the paper, we shall denote the tangent bundle of M by TM, its cotangent bundle by T^*M and the smooth sections of TM (respectively, of T^*M) by $\Gamma^\infty(TM)$ (respectively, by $\Gamma^\infty(T^*M)$).

For an affine connection ∇ on M, its torsion tensor T^{∇} and its curvature tensor R^{∇} are defined, respectively, by the following:

$$T^{\nabla}(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y],$$

$$R^{\nabla}(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]},$$

for $X, Y \in \Gamma^{\infty}(TM)$, where $[\cdot,\cdot]$ is the Lie bracket. Thus, an affine connection is said to be *torsion-free* if its torsion tensor is zero and *flat* if its curvature tensor is zero.

We shall recall the notions of statistical, quasi-statistical structures and dual connections.

Definition 1 ([8]). *Let (M,g) be a pseudo-Riemannian manifold and let ∇ be a torsion-free affine connection on M. Then, (g,∇) is called a* statistical structure *on M (and (M,g,∇) a* statistical manifold*) if the following equation is satisfied:*

$$(\nabla_X g)(Y,Z) = (\nabla_Y g)(X,Z),$$

for any $X, Y, Z \in \Gamma^{\infty}(TM)$.

Remark 1. *Notice that (M,g,∇) is a statistical manifold if and only if ∇g is totally symmetric.*

A trivial example of statistical structure is provided by a pseudo-Riemannian metric g with its Levi–Civita connection ∇^g.

In all the rest of the paper, we shall denote by ∇^g the Levi–Civita connection of the pseudo-Riemannian metric g.

In 2007, Kurose introduced the notion of statistical manifold admitting torsion.

Definition 2 ([2]). *Let (M,g) be a pseudo-Riemannian manifold and let ∇ be an affine connection on M with torsion tensor T^{∇}. Then, (g,∇) is called a* quasi-statistical structure *on M (and (M,g,∇) a* quasi-statistical manifold *or* statistical manifold admitting torsion*) if $d^{\nabla}g = 0$, where*

$$(d^{\nabla}g)(X,Y,Z) := (\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z) + g(T^{\nabla}(X,Y),Z),$$

for any $X, Y, Z \in \Gamma^{\infty}(TM)$.

The notion of dual connection was firstly introduced by Amari, which he used in treating statistical inference problems.

Definition 3 ([1,8]). *Let (M,g) be a pseudo-Riemannian manifold. Two affine connections ∇ and ∇^* on M are said to be* dual connections *with respect to g if the following equation is satisfied:*

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z),$$

for any $X, Y, Z \in \Gamma^{\infty}(TM)$, and we call (g,∇,∇^*) a dualistic structure*.*

We remark that, from the symmetry of g, it follows that $(\nabla^*)^* = \nabla$.

Remark 2. *Notice that $\nabla = \nabla^*$ if and only if ∇ is a metric connection, that is, $\nabla g = 0$. Moreover, if ∇ is torsion-free, then, $\nabla = \nabla^*$ if and only if ∇ is the Levi–Civita connection of g.*

For any vector field X and any 1-form η, we will denote further by $X^{\flat}$ and $\eta^{\sharp}$ their images through the musical isomorphisms $\flat$ and $\sharp$, that is, $X^{\flat}(Y) := g(X,Y)$ and $g(\eta^{\sharp},Y) := \eta(Y)$, for any vector field Y.

Direct computations provide the following.

Lemma 1. *If ∇ is an affine connection on (M, g), then, the dual connection ∇^* of ∇ with respect to g is given by:*

$$\nabla^*_X Y = (\nabla_X Y^\flat)^\sharp,$$

for any $X, Y \in \Gamma^\infty(TM)$. Moreover, we have $\nabla^* g = -\nabla g$.

Lemma 2. *If ∇ and ∇^* are dual connections with respect to g, then, their curvature and torsion tensors are related by the following formulas:*

$$g(R^{\nabla^*}(Z, W)X, Y) + g(R^{\nabla}(Z, W)Y, X) = 0,$$

$$g(T^{\nabla^*}(X, Y), Z) = g(T^{\nabla}(X, Y), Z) + (\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z),$$

for any $X, Y, Z, W \in \Gamma^\infty(TM)$.

From the previous two lemmas, we can state the following.

Corollary 1. *If ∇ and ∇^* are dual connections with respect to g, then we have:*
(i) $R^{\nabla} = 0 \Leftrightarrow R^{\nabla^*} = 0$;
(ii) $T^{\nabla^*} = 0 \Leftrightarrow (M, g, \nabla)$ is a quasi-statistical manifold;
(iii) $T^{\nabla} = 0 \Leftrightarrow (M, g, \nabla^*)$ is a quasi-statistical manifold;
(iv) $T^{\nabla^*} = 0$, $T^{\nabla} = 0 \Leftrightarrow (M, g, \nabla)$ and (M, g, ∇^*) are both statistical manifolds.

Proof. It follows from Lemmas 1 and 2. □

Definition 4. *A dualistic structure (g, ∇, ∇^*) on M such that ∇ and ∇^* are flat and torsion-free is called a* dually flat structure *on M (and (M, g, ∇, ∇^*) a* dually flat manifold*).*

3. Quasi-Statistical Structures with the Same Metric

Let (M, g) be a pseudo-Riemannian manifold and let ∇ be an affine connection on M.

3.1. Quasi-Statistical Structures Defined by a Tensor Field

Starting from an arbitrary quasi-statistical structure, we will construct other quasi-statistical structures by means of a $(1, 1)$ or $(1, 2)$-tensor field.

Direct computations provide the following.

Lemma 3. *For any $(1, 2)$-tensor field S on (M, g), the affine connection $\bar{\nabla} := \nabla + S$ satisfies the following:*

$$T^{\bar{\nabla}}(X, Y) = T^{\nabla}(X, Y) + S(X, Y) - S(Y, X),$$

$$(\bar{\nabla}_X g)(Y, Z) = (\nabla_X g)(Y, Z) - g(S(X, Y), Z) - g(Y, S(X, Z)),$$

$$(d^{\bar{\nabla}} g)(X, Y, Z) = (d^{\nabla} g)(X, Y, Z) + g(X, S(Y, Z)) - g(Y, S(X, Z)),$$

for any $X, Y, Z \in \Gamma^\infty(TM)$.

Thus, we can state the following.

Proposition 1. *Let S be a $(1, 2)$-tensor field on (M, g) satisfying $g(X, S(Y, Z)) = g(S(X, Z), Y)$, for any $X, Y, Z \in \Gamma^\infty(TM)$. Then, (g, ∇) is a quasi-statistical structure if and only if $(g, \bar{\nabla} := \nabla + S)$ is a quasi-statistical structure.*

Proof. Just remark that $d^{\bar{\nabla}} g = d^{\nabla} g$. □

If we denote by ∇^* and $\bar{\nabla}^*$ the dual connection of ∇ and of $\bar{\nabla}$, with respect to g, by a direct computation, we obtain the following:

Lemma 4.
$$g(\bar{\nabla}^*_X Y, Z) = g(\nabla^*_X Y, Z) - g(Y, S(X, Z)),$$
for any $X, Y, Z \in \Gamma^\infty(TM)$.

Proposition 2. *Let* ϕ *be a g-self-adjoint Codazzi* $(1,1)$*-tensor field on* (M, g)*, that is,* $g(\phi X, Y) = g(X, \phi Y)$ *and* $(\nabla^g_X \phi)Y = (\nabla^g_Y \phi)X$*, for any* $X, Y \in \Gamma^\infty(TM)$*. Then,* (g, ∇) *is a quasi-statistical structure if and only if* $(g, \bar{\nabla} := \nabla + \nabla^g \phi)$ *is a quasi-statistical structure. In this case,* $\bar{\nabla}^* = \nabla^* - \nabla^g \phi$*.*

Proof. For any $X, Y, Z \in \Gamma^\infty(TM)$, we have the following:
$$g((\nabla^g_X \phi)Y, Z) = g((\nabla^g_X \phi)Z, Y) = g((\nabla^g_Z \phi)X, Y) = g((\nabla^g_Z \phi)Y, X),$$
and we obtain the conclusion from Proposition 1. Moreover, from Lemma 4, we find the expression of the dual connection from the next relation:
$$g(\bar{\nabla}^*_X Y, Z) = g(\nabla^*_X Y, Z) - g(Y, (\nabla^g_X \phi)Z) = g(\nabla^*_X Y, Z) - g(Z, (\nabla^g_X \phi)Y). \quad \square$$

Corollary 2. *If* ϕ *is a g-self-adjoint Codazzi* $(1,1)$*-tensor field on* (M, g) *and* (g, ∇) *is a statistical structure, then* $(g, \bar{\nabla} := \nabla + \nabla^g \phi)$ *is a statistical structure too.*

In particular, we have the following.

Corollary 3. *If* ϕ *is a g-self-adjoint Codazzi* $(1,1)$*-tensor field on* (M, g)*, then* $(g, \bar{\nabla} := \nabla^g + \nabla^g \phi)$ *is a statistical structure.*

Proposition 3. *Let* ϕ *be a* $(1,1)$*-tensor field on* (M, g) *and let* $\tilde{\nabla}$ *be an affine connection satisfying* $g((\tilde{\nabla}_X \phi)Y, Z) = g(X, (\tilde{\nabla}_Z \phi)Y)$*, for any* $X, Y, Z \in \Gamma^\infty(TM)$*. Then,* (g, ∇) *is a quasi-statistical structure if and only if* $(g, \bar{\nabla} := \nabla + \tilde{\nabla} \phi)$ *is a quasi-statistical structure.*

Proof. It follows from Proposition 1. $\square$

Corollary 4. *If* ϕ *is a* $(1,1)$*-tensor field on* (M, g)*,* $\tilde{\nabla}$ *is an affine connection satisfying* $g((\tilde{\nabla}_X \phi)Y, Z) = g(X, (\tilde{\nabla}_Z \phi)Y)$*, for any* $X, Y, Z \in \Gamma^\infty(TM)$*, and* (g, ∇) *is a statistical structure, then* $(g, \bar{\nabla} := \nabla + \tilde{\nabla} \phi)$ *is a quasi-statistical structure.*

In particular, we have the following.

Corollary 5. *If* ϕ *is a* $(1,1)$*-tensor field on* (M, g) *and* $\tilde{\nabla}$ *is an affine connection satisfying* $g((\tilde{\nabla}_X \phi)Y, Z) = g(X, (\tilde{\nabla}_Z \phi)Y)$*, for any* $X, Y, Z \in \Gamma^\infty(TM)$*, then* $(g, \bar{\nabla} := \nabla^g + \tilde{\nabla} \phi)$ *is a quasi-statistical structure.*

3.2. Quasi-Statistical Structures Defined by a $(1,1)$*-Tensor Field and a* 1*-Form*

Starting from an arbitrary quasi-statistical structure, we will construct other quasi-statistical structures by means of a $(1,1)$-tensor field and a 1-form.

Proposition 4. *Let* ϕ *be a g-self-adjoint* $(1,1)$*-tensor field on* (M, g) *and let* η *be a* 1*-form. Then,* (g, ∇) *is a quasi-statistical structure if and only if* $(g, \bar{\nabla} := \nabla + \phi \otimes \eta)$ *is a quasi-statistical structure. In this case,* $\bar{\nabla}^*_X Y = \nabla^*_X Y - g(\phi X, Y)\eta^\sharp$*, for any* $X, Y \in \Gamma^\infty(TM)$*.*

Proof. By a direct computation, for any $X, Y, Z \in \Gamma^\infty(TM)$ we obtain the following:
$$T^{\bar{\nabla}} = T^\nabla + \phi \otimes \eta - \eta \otimes \phi,$$
$$(\bar{\nabla}_X g)(Y, Z) = (\nabla_X g)(Y, Z) - \eta(Y) g(\phi X, Z) - \eta(Z) g(\phi X, Y).$$

Hence, $d^{\bar{\nabla}} g = d^{\nabla} g$ and the conclusion follows from Proposition 1. Moreover, from Lemma 4, we find the expression of the dual connection from the next relation:

$$g(\bar{\nabla}^*_X Y, Z) = g(\nabla^*_X Y, Z) - g(Y, \phi X)\eta(Z). \quad \square$$

Remark that if ∇ is torsion-free, then $\bar{\nabla}$ is a quarter-symmetric connection [9].

Corollary 6. *If ϕ is a g-self-adjoint $(1,1)$-tensor field on (M,g), η is a nonzero 1-form and (g,∇) is a statistical structure, then $(g, \bar{\nabla} := \nabla + \phi \otimes \eta)$ is a quasi-statistical structure.*

In particular, we have the following.

Corollary 7. *If ϕ is a g-self-adjoint $(1,1)$-tensor field on (M,g) and η is a nonzero 1-form, then $(g, \bar{\nabla} := \nabla^g + \phi \otimes \eta)$ is a quasi-statistical structure.*

Examples of quasi-statistical structures obtained by using the previous constructions can be found in the framework of Norden structures [10].

Definition 5. *Let (M,g) be an even dimensional pseudo-Riemannian manifold and let J be a g-self-adjoint almost complex structure on M, that is, $J : TM \to TM$, $J^2 = -I$ and $g(JX,Y) = g(X,JY)$, for any $X, Y \in \Gamma^{\infty}(TM)$. Then, (g,J) is called a* Norden structure *on M (and (M,g,J) a* Norden manifold*). Moreover, the metric $\tilde{g}$ defined by $\tilde{g}(X,Y) := g(X,JY)$ is called the* twin metric *associated with (g,J).*

Proposition 5. *Let (g,J) be a Norden structure on M. If $d^{\nabla^g} J = 0$, then $(g, \bar{\nabla} := \nabla^g + \nabla^g J)$ is a statistical structure, and the dual connection of $\bar{\nabla}$ with respect to g is given by $\bar{\nabla}^*_g = \nabla^g - \nabla^g J$. Moreover, if $\tilde{g}$ is the twin metric, then $(\tilde{g}, \nabla^g)$ is a statistical structure, and the dual connection of ∇^g with respect to $\tilde{g}$ is given by $(\nabla^g)^*_{\tilde{g}} = \nabla^g - J(\nabla^g J)$.*

Proof. We have that J is a g-self-adjoint $(1,1)$-tensor field and $\nabla^g J$ is a g-self-adjoint $(1,2)$-tensor field. Moreover, from $d^{\nabla^g} J = 0$, it follows that $\nabla^g J$ is a Codazzi tensor field and we apply Corollary 2. Then, the dual connection is given by Lemma 4. Moreover, remark that $d^{\nabla^g} J = 0$ is the condition for $(\tilde{g}, \nabla^g)$ to be a statistical structure, and a direct computation (see [11]) provides $(\nabla^g)^*_{\tilde{g}}$. $\square$

From Proposition 4 and Corollary 7, we obtain the following.

Proposition 6. *Let (g,J) be a Norden structure on M and let η be a nonzero 1-form. Then, $(g, \bar{\nabla} := \nabla^g + J \otimes \eta)$ is a quasi-statistical structure, and the dual connection of $\bar{\nabla}$ with respect to g is given by $\bar{\nabla}^*_X Y = \nabla^g_X Y - g(JX,Y)\eta^\sharp$, for any $X, Y \in \Gamma^{\infty}(TM)$.*

A direct computation provides the following.

Proposition 7. *Let (g,J) be a Norden structure on M, let η be a 1-form and let $\bar{\nabla} := \nabla^g + J \otimes \eta$. If $\bar{\nabla}^*$ is the dual connection of $\bar{\nabla}$ with respect to g, then the curvature operator of $\bar{\nabla}^*$ is provided by the following formula:*

$$R^{\bar{\nabla}^*}(X,Y) = R^{\nabla^g}(X,Y) - \eta^\sharp \otimes ((d^{\nabla^g} J)(X,Y))^\flat$$

$$+ \left(\eta(JX)\eta^\sharp - \nabla^g_X \eta^\sharp\right) \otimes (JY)^\flat - \left(\eta(JY)\eta^\sharp - \nabla^g_Y \eta^\sharp\right) \otimes (JX)^\flat,$$

for any $X, Y \in \Gamma^{\infty}(TM)$.

4. Quasi-Statistical Structures with Conformal Metrics

Starting from an arbitrary quasi-statistical structure, we will construct other quasi-statistical structures for which the pseudo-Riemannian metric is conformal to the initial one.

Let (M, g) be a pseudo-Riemannian manifold and let ∇ be an affine connection on M. Direct computations provide the following.

Lemma 5. *For any positive smooth function f on (M, g), the affine connection $\bar{\nabla} := \nabla + F_f(df \otimes I)$ and the pseudo-Riemannian metric $\bar{g} := G_f g$, with F_f and G_f two smooth functions, satisfy the following:*

$$T^{\bar{\nabla}} = T^{\nabla} + F_f(df \otimes I - I \otimes df),$$

$$(\bar{\nabla}_X \bar{g})(Y, Z) = G_f(\nabla_X g)(Y, Z) + \{X(G_f) - 2F_f G_f X(f)\} g(Y, Z),$$

$$(d^{\bar{\nabla}} \bar{g})(X, Y, Z) = G_f(d^{\nabla} g)(X, Y, Z)$$

$$+\{X(G_f) - F_f G_f X(f)\} g(Y, Z) - \{Y(G_f) - F_f G_f Y(f)\} g(X, Z),$$

for any $X, Y, Z \in \Gamma^{\infty}(TM)$.

Thus, we can state the following.

Lemma 6. *Let $f > 0$, F_f and G_f be smooth functions on (M, g) and let $\bar{\nabla} := \nabla + F_f(df \otimes I)$, $\bar{g} := G_f g$. Then, $d^{\bar{\nabla}} \bar{g} = G_f d^{\nabla} g$ if and only if $d(G_f) = F_f G_f df$.*

By means of the previous two lemmas, we construct the following quasi-statistical structures given in the next two propositions.

Proposition 8. *Let f be a positive smooth function on (M, g) and let $\bar{\nabla} := \nabla - \frac{1}{f} df \otimes I$, $\bar{g} := \frac{1}{f} g$. Then we obtain:*

$$T^{\bar{\nabla}} = T^{\nabla} - \frac{1}{f}(df \otimes I - I \otimes df),$$

$$\bar{\nabla} \bar{g} = \frac{1}{f}\left(\nabla g + \frac{1}{f} df \otimes g\right).$$

Thus, (g, ∇) is a quasi-statistical structure if and only if $(\bar{g}, \bar{\nabla})$ is a quasi-statistical structure.

Proof. We immediately obtain $d^{\bar{\nabla}} \bar{g} = \frac{1}{f} d^{\nabla} g$, and the conclusion follows from Lemmas 5 and 6. $\square$

In particular, we have the following.

Corollary 8. *If f is a positive smooth function on (M, g), then $\left(\bar{g} := \frac{1}{f} g, \bar{\nabla} := \nabla^g - \frac{1}{f} df \otimes I\right)$ is a quasi-statistical structure.*

Proposition 9. *Let f be a positive smooth function on (M, g) and let $\bar{\nabla} := \nabla + df \otimes I$, $\bar{g} := e^f g$. Then we obtain:*

$$T^{\bar{\nabla}} = T^{\nabla} + df \otimes I - I \otimes df,$$

$$\bar{\nabla} \bar{g} = e^f(\nabla g - df \otimes g).$$

Thus, (g, ∇) is a quasi-statistical structure if and only if $(\bar{g}, \bar{\nabla})$ is a quasi-statistical structure.

Proof. We immediately obtain $d^{\bar{\nabla}} \bar{g} = e^f d^{\nabla} g$, and the conclusion follows from Lemmas 5 and 6. $\square$

Remark that if ∇ is torsion-free, then $\bar{\nabla}$ is a semi-symmetric connection [9].
In particular, we have the following.

Corollary 9. *If f is a positive smooth function on (M, g), then $(\bar{g} := e^f g, \bar{\nabla} := \nabla^g + df \otimes I)$ is a quasi-statistical structure.*

We denote by ∇^*_g and $\bar{\nabla}^*_g$ the dual connection of ∇ and of $\bar{\nabla}$, with respect to g, and the dual connection of ∇ and of $\bar{\nabla}$, with respect to $\bar{g}$, by $\nabla^*_{\bar{g}}$ and $\bar{\nabla}^*_{\bar{g}}$. Then, we can state the following.

Proposition 10. *Let f be a positive smooth function on (M, g) and let (g, ∇) be a quasi-statistical structure.*

(i) If $\bar{g} := \frac{1}{f}g$ and $\bar{\nabla} := \nabla - \frac{1}{f}df \otimes I$, then we obtain:

$$\bar{\nabla}^*_{\bar{g}} = \nabla^*_g \ \textit{and} \ \bar{\nabla}^*_{\bar{g}} - \nabla^*_{\bar{g}} = \frac{1}{f}df \otimes I = \bar{\nabla}^*_g - \nabla^*_g.$$

(ii) If $\bar{g} := e^f g$ and $\bar{\nabla} := \nabla + df \otimes I$, then we obtain:

$$\bar{\nabla}^*_{\bar{g}} = \nabla^*_g \ \textit{and} \ \bar{\nabla}^*_{\bar{g}} - \nabla^*_{\bar{g}} = -df \otimes I = \bar{\nabla}^*_g - \nabla^*_g.$$

Proof. For any $X, Y, Z \in \Gamma^\infty(TM)$, we have the following:

$$(i) \ \ \bar{g}(\bar{\nabla}^*_X Y, Z) = X(\bar{g}(Y, Z)) - \bar{g}(Y, \bar{\nabla}_X Z) = X(\bar{g}(Y, Z)) - \bar{g}(Y, \nabla_X Z) + \frac{df(X)}{f}\bar{g}(Y, Z)$$

$$= \bar{g}(\nabla^*_X Y, Z) + \frac{df(X)}{f}\bar{g}(Y, Z),$$

hence, $\bar{\nabla}^*_{\bar{g}} = \nabla^*_{\bar{g}} + \frac{1}{f}df \otimes I$.

$$\bar{g}(\bar{\nabla}^*_X Y, Z) = X(\bar{g}(Y, Z)) - \bar{g}(Y, \bar{\nabla}_X Z)$$

$$= -\frac{X(f)}{f^2}g(Y, Z) + \frac{1}{f}\{X(g(Y, Z)) - g(Y, \nabla_X Z)\} + \frac{df(X)}{f^2}g(Y, Z)$$

$$= \frac{1}{f}g(\nabla^*_X Y, Z) = \bar{g}(\nabla^*_X Y, Z),$$

hence, $\bar{\nabla}^*_{\bar{g}} = \nabla^*_g$.

$$g(\bar{\nabla}^*_X Y, Z) = X(g(Y, Z)) - g(Y, \bar{\nabla}_X Z) = X(g(Y, Z)) - g(Y, \nabla_X Z) + \frac{df(X)}{f}g(Y, Z)$$

$$= g(\nabla^*_X Y, Z) + \frac{df(X)}{f}g(Y, Z),$$

hence, $\bar{\nabla}^*_g = \nabla^*_g + \frac{1}{f}df \otimes I$.

$$(ii) \ \ \bar{g}(\bar{\nabla}^*_X Y, Z) = X(\bar{g}(Y, Z)) - \bar{g}(Y, \bar{\nabla}_X Z) = X(\bar{g}(Y, Z)) - \bar{g}(Y, \nabla_X Z) - df(X)\bar{g}(Y, Z)$$

$$= \bar{g}(\nabla^*_X Y, Z) - df(X)\bar{g}(Y, Z),$$

hence, $\bar{\nabla}^*_{\bar{g}} = \nabla^*_{\bar{g}} - df \otimes I$.

$$\bar{g}(\bar{\nabla}^*_X Y, Z) = X(\bar{g}(Y, Z)) - \bar{g}(Y, \bar{\nabla}_X Z)$$

$$= e^f X(f)g(Y, Z) + e^f\{X(g(Y, Z)) - g(Y, \nabla_X Z)\} - e^f df(X)g(Y, Z)$$

$$= e^f g(\nabla^*_X Y, Z) = \bar{g}(\nabla^*_X Y, Z),$$

hence, $\bar{\nabla}^*_{\bar{g}} = \nabla^*_g$.

$$g(\bar{\nabla}^*_X Y, Z) = X(g(Y, Z)) - g(Y, \bar{\nabla}_X Z) = X(g(Y, Z)) - g(Y, \nabla_X Z) - df(X)g(Y, Z)$$

$$= g(\nabla^*_X Y, Z) - df(X)g(Y, Z),$$

hence, $\bar{\nabla}^*_{\bar{g}} = \nabla^*_g - df \otimes I$. $\square$

In particular, we have the following.

Corollary 10. *Let f be a positive smooth function on (M, g). If (i) $\bar{g} := \frac{1}{f} g$ and $\bar{\nabla} := \nabla^g - \frac{1}{f} df \otimes I$, or (ii) $\bar{g} := e^f g$ and $\bar{\nabla} := \nabla^g + df \otimes I$, then $\bar{\nabla}^*_{\bar{g}} = \nabla^g$, equivalently $\bar{\nabla} = (\nabla^g)^*_{\bar{g}}$.*

Proof. We remark that for the Levi–Civita connection ∇^g, we have $\nabla^g = (\nabla^g)^*_g$. Then, the following is obtained:

$$X(\bar{g}(Y, Z)) = \bar{g}(\bar{\nabla}_X Y, Z) + \bar{g}(Y, \nabla^g_X Z). \quad \square$$

5. Semi-Weyl and Quasi-Semi-Weyl Structures—Semi-Dual Connections

We introduce the notion of quasi-semi-Weyl structure and study its relation with the quasi-statistical structure. Then, we construct semi-Weyl and quasi-semi-Weyl structures starting from a statistical one.

Definition 6 ([5])**.** *Let (M, g) be a pseudo-Riemannian manifold, let ∇ be a torsion-free affine connection on M and let η be a 1-form. Then, (g, η, ∇) is called a* semi-Weyl structure *on M (and (M, g, η, ∇) a* semi-Weyl manifold*) if the following equation is satisfied:*

$$(\nabla_X g)(Y, Z) + \eta(X)g(Y, Z) = (\nabla_Y g)(X, Z) + \eta(Y)g(X, Z),$$

for any $X, Y, Z \in \Gamma^\infty(TM)$.

Remark 3. *Notice that (M, g, η, ∇) is a semi-Weyl manifold if and only if $\nabla g + \eta \otimes g$ is totally symmetric. In particular, if $\eta = 0$, then (M, g, ∇) is a statistical manifold. Moreover, if $\nabla g + \eta \otimes g = 0$, then (M, g, η, ∇) is a* Weyl manifold.

In the framework of Weyl geometry, the corresponding concept of duality is provided by the semi-dual connection introduced by Norden.

Definition 7 ([3,12])**.** *Let (M, g) be a pseudo-Riemannian manifold and let η be a nonzero 1-form on M. Two affine connections ∇ and ∇^* on M are said to be* semi-dual connections *(or* generalized dual connections*) with respect to (g, η) if the following equation is satisfied:*

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) - \eta(X)g(Y, Z),$$

for any $X, Y, Z \in \Gamma^\infty(TM)$, and we call $(g, \eta, \nabla, \nabla^*)$ a semi-dualistic structure.

We remark that, from the symmetry of g, it follows that $(\nabla^*)^* = \nabla$.

Remark 4. *Notice that if ∇ is torsion-free, then, $\nabla = \nabla^*$ if and only if (M, g, η, ∇) is a Weyl manifold.*

Direct computations provide the following expressions, which generalize from dual to semi-dual connections.

Lemma 7. *If η is a nonzero 1-form and ∇ is an affine connection on (M, g), then the semi-dual connection ∇^* of ∇ with respect to (g, η) is given by:*

$$\nabla^*_X Y = (\nabla_X Y^\flat)^\sharp + \eta(X)Y,$$

for any $X, Y \in \Gamma^\infty(TM)$. *Moreover, we have:*

$$(\nabla^*_X g)(Y,Z) + \eta(X)g(Y,Z) = -\{(\nabla_X g)(Y,Z) + \eta(X)g(Y,Z)\},$$

for any $X, Y, Z \in \Gamma^\infty(TM)$.

Lemma 8. *If* ∇ *and* ∇^* *are semi-dual connections with respect to* (g,η)*, then, their curvature and torsion tensors are related by the following formulas:*

$$g(R^{\nabla^*}(Z,W)X,Y) + g(R^{\nabla}(Z,W)Y,X) = 0,$$

$$\begin{aligned} g(T^{\nabla^*}(X,Y),Z) &= g(T^{\nabla}(X,Y),Z) + (\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z) \\ &\quad + \eta(X)g(Y,Z) - \eta(Y)g(X,Z), \end{aligned}$$

for any $X, Y, Z, W \in \Gamma^\infty(TM)$.

Now, we introduce the following definition that generalizes the notion of semi-Weyl manifold to semi-Weyl manifold admitting torsion.

Definition 8. *Let* (M,g) *be a pseudo-Riemannian manifold, let* ∇ *be an affine connection on* M *with torsion tensor* T^∇ *and let* η *be a nonzero* 1*-form. Then,* (g,η,∇) *is called a* quasi-semi-Weyl structure *on* M *(and* (M,g,η,∇) *a* quasi-semi-Weyl manifold *or* semi-Weyl manifold admitting torsion*) if the following equation is satisfied:*

$$(\nabla_X g)(Y,Z) + \eta(X)g(Y,Z) = (\nabla_Y g)(X,Z) + \eta(Y)g(X,Z) - g(T^\nabla(X,Y),Z),$$

for any $X, Y, Z \in \Gamma^\infty(TM)$.

Example 1. *Let* η *be a nonzero* 1*-form on* M *and let* (g,∇) *be a quasi-statistical structure on* M *with* $T^\nabla \neq I \otimes \eta - \eta \otimes I$*. Then* $(g,\eta,\bar{\nabla} := \nabla + \eta \otimes I)$ *is a quasi-semi-Weyl structure on* M*. Indeed, the torsion of* $\bar{\nabla}$ *is equal to:*

$$T^{\bar{\nabla}}(X,Y) = T^\nabla(X,Y) + \eta(X)Y - \eta(Y)X$$

and we obtain:

$$\begin{aligned} &\{(\bar{\nabla}_X g)(Y,Z) + \eta(X)g(Y,Z)\} - \{(\bar{\nabla}_Y g)(X,Z) + \eta(Y)g(X,Z)\} \\ &= (\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z) + g(T^\nabla(X,Y),Z) - g(T^\nabla(X,Y) + \eta(X)Y - \eta(Y)X, Z), \end{aligned}$$

for any $X, Y, Z \in \Gamma^\infty(TM)$.

Example 2. *Let* (g,∇) *be a statistical structure on* M *and let* η *be a nonzero* 1*-form. Then,* $(g,\eta,\bar{\nabla} := \nabla + \eta \otimes I)$ *is a quasi-semi-Weyl structure on* M*.*

Remark 5. *If we denote the dual connection of* ∇ *with respect to* g *by* ∇^*_g *and the semi-dual connection of* ∇ *with respect to* (g,η) *by* $\nabla^*_{(g,\eta)}$*, then* $\nabla^*_g = \nabla^*_{(g,\eta)} - \eta \otimes I$ *and* $\nabla = (\nabla^*_{(g,\eta)})^*_g + \eta \otimes I$*.*

From the previous two lemmas and Corollary 1, we can state the following.

Corollary 11. *If* ∇ *and* $\nabla^*_{(g,\eta)}$ *are semi-dual connections with respect to* (g,η) *and* ∇^*_g *is the dual connection of* ∇ *with respect to* g*, then we have:*

(i) $R^\nabla = 0 \Leftrightarrow R^{\nabla^*_{(g,\eta)}} = 0 \Leftrightarrow R^{\nabla^*_g} = 0$;

(ii) $T^{\nabla^*_{(g,\eta)}} = 0 \Leftrightarrow (M,g,\eta,\nabla)$ *is a quasi-semi-Weyl manifold;*

(iii) $T^\nabla = 0 \Leftrightarrow (M,g,\eta,\nabla^*_{(g,\eta)})$ *is a quasi-semi-Weyl manifold;*

(iv) $T^{\nabla^*_{(g,\eta)}} = 0$, $T^{\nabla} = 0 \Leftrightarrow (M, g, \eta, \nabla)$ *and* $(M, g, \eta, \nabla^*_{(g,\eta)})$ *are both semi-Weyl manifolds;*
(v) $(M, g, \eta, \nabla^*_{(g,\eta)})$ *is a quasi-semi-Weyl manifold* $\Leftrightarrow (M, g, \nabla^*_g)$ *is a quasi-statistical manifold.*

Proof. (i), (ii), (iii) and (iv) follow from Lemmas 7 and 8. (v) follows from (iii) and Corollary 1. $\square$

Moreover, from Lemma 3, we obtain the following.

Proposition 11. *Let S be a symmetric $(1,2)$-tensor field on (M, g) and let η be a nonzero 1-form satisfying $g(X, S(Y,Z) - \eta(Y)Z) = g(S(X,Z) - \eta(X)Z, Y)$, for any $X, Y, Z \in \Gamma^\infty(TM)$. Then, (g, ∇) is a statistical structure if and only if $(g, \eta, \bar{\nabla} := \nabla + S)$ is a semi-Weyl structure.*

Proof. We remark that:

$$\{(\bar{\nabla}_X g)(Y,Z) + \eta(X)g(Y,Z)\} - \{(\bar{\nabla}_Y g)(X,Z) + \eta(Y)g(X,Z)\}$$
$$= (\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z),$$

for any $X, Y, Z \in \Gamma^\infty(TM)$. $\square$

If we denote the dual connection of ∇ with respect to g by ∇^*, and the semi-dual connection of $\bar{\nabla}$ with respect to (g, η) by $\bar{\nabla}^*$, by a direct computation, we obtain the following:

Lemma 9.
$$g(\bar{\nabla}^*_X Y, Z) = g(\nabla^*_X Y, Z) - g(Y, S(X,Z)) + \eta(X)g(Y,Z),$$
for any $X, Y, Z \in \Gamma^\infty(TM)$.

We can construct semi-Weyl structures by means of statistical structures for which the connections are projectively equivalent.

Proposition 12. *Let η be a nonzero 1-form on (M, g). Then, (g, ∇) is a statistical structure if and only if $(g, \eta, \bar{\nabla} := \nabla + \eta \otimes I + I \otimes \eta)$ is a semi-Weyl structure. In this case, $\bar{\nabla}^*_{(g,\eta)} = \nabla^*_g - g \otimes \eta^\sharp$; hence, the dual connection of ∇ with respect to g and the semi-dual connection of $\bar{\nabla}$ with respect to (g, η) are dual projectively equivalent.*

Proof. For any $X, Y, Z \in \Gamma^\infty(TM)$, we have the following:

$$g(X, (\eta(Y)Z + \eta(Z)Y) - \eta(Y)Z) = g((\eta(X)Z + \eta(Z)X) - \eta(X)Z, Y),$$

and we obtain the conclusion from Proposition 11. Moreover, from Lemma 9, we find the expression of the dual connection from the next relation:

$$g(\bar{\nabla}^*_X Y, Z) = g(\nabla^*_X Y, Z) - g(Y, \eta(X)Z + \eta(Z)X) + \eta(X)g(Y,Z)$$
$$= g(\nabla^*_X Y, Z) - g(X,Y)\eta(Z). \quad \square$$

We denote the dual connections of ∇ with respect to g and $\bar{g}$ by ∇^*_g and $\nabla^*_{\bar{g}}$, and the semi-dual connections of $\bar{\nabla}$ with respect to (g, η) and $(\bar{g}, \eta)$ by $\bar{\nabla}^*_{(g,\eta)}$ and $\bar{\nabla}^*_{(\bar{g},\eta)}$. Then we can state.

Proposition 13. *Let f be a positive smooth function on (M, g) and let η be a nonzero 1-form. Then, (g, ∇) is a statistical structure if and only if $(\bar{g} := e^f g, \eta, \bar{\nabla} := \nabla + (\eta + df) \otimes I + I \otimes (\eta + df))$ is a semi-Weyl structure. In this case, $\bar{\nabla}^*_{(\bar{g},\eta)} = \nabla^*_g - g \otimes (\eta + df)^\sharp$; hence, the dual connection of ∇ with respect to g and the semi-dual connection of $\bar{\nabla}$ with respect to $(\bar{g}, \eta)$ are dual projectively equivalent. Moreover, we have $\bar{\nabla}^*_{(\bar{g},\eta)} - \bar{\nabla}^*_{(g,\eta)} = df \otimes I = \nabla^*_{\bar{g}} - \nabla^*_g$.*

Proof. For any $X, Y, Z \in \Gamma^{\infty}(TM)$, we have the following:

$$(\bar{\nabla}_X \bar{g})(Y,Z) + \eta(X)\bar{g}(Y,Z) = e^f(\nabla_X g)(Y,Z)$$

$$-e^f\{(\eta(X) + df(X))g(Y,Z) - (\eta(Y) + df(Y))g(Z,X) - (\eta(Z) + df(Z))g(X,Y)\}$$

and we obtain the conclusion. Moreover, we find the expression of the semi-dual connection from the next relation:

$$\bar{g}(\bar{\nabla}^*_X Y, Z) = e^f\{g(\nabla^*_X Y, Z) - g(X,Y)(\eta + df)(Z)\}.$$

Also, we obtain:

$$\bar{g}((\bar{\nabla}^*_{(\bar{g},\eta)})_X Y, Z) = X(\bar{g}(Y,Z)) - \bar{g}(Y, \bar{\nabla}_X Z) + \eta(X)\bar{g}(Y,Z)$$

$$= e^f\{df(X)g(Y,Z) + X(g(Y,Z)) - g(Y, \bar{\nabla}_X Z) + \eta(X)g(Y,Z)\}$$
$$= e^f\{df(X)g(Y,Z) + g((\bar{\nabla}^*_{(g,\eta)})_X Y, Z)\},$$

hence, $\bar{\nabla}^*_{(\bar{g},\eta)} = \bar{\nabla}^*_{(g,\eta)} + df \otimes I$, and:

$$\bar{g}((\nabla^*_{\bar{g}})_X Y, Z) = X(\bar{g}(Y,Z)) - \bar{g}(Y, \nabla_X Z)$$

$$= e^f\{df(X)g(Y,Z) + X(g(Y,Z)) - g(Y, \nabla_X Z)\}$$
$$= e^f\{df(X)g(Y,Z) + g((\nabla^*_g)_X Y, Z)\},$$

hence, $\nabla^*_{\bar{g}} = \nabla^*_g + df \otimes I$. □

We remark that, to a quasi-statistical structure with semi-symmetric connection, one can associate a semi-Weyl structure in the following manner.

Proposition 14. *Let η be a nonzero 1-form on (M, g). If (g, ∇) is a quasi-statistical structure with $T^{\nabla} = \eta \otimes I - I \otimes \eta$, then $(g, \eta, \bar{\nabla} := \nabla + \eta \otimes I + 2I \otimes \eta)$ is a semi-Weyl structure. In this case, $\bar{\nabla}^*_{(g,\eta)} = \nabla^*_g - 2g \otimes \eta^{\sharp}$; hence, the dual connection of ∇ with respect to g and the semi-dual connection of $\bar{\nabla}$ with respect to (g, η) are dual projectively equivalent.*

Proof. We immediately obtain $T^{\bar{\nabla}} = 0$. For any $X, Y, Z \in \Gamma^{\infty}(TM)$, we have the following:

$$(\bar{\nabla}_X g)(Y,Z) + \eta(X)g(Y,Z) = (\bar{\nabla}_Y g)(X,Z) + \eta(Y)g(X,Z)$$

and we obtain the conclusion. Moreover, we find the expression of the semi-dual connection from the next relation:

$$g(\bar{\nabla}^*_X Y, Z) = g(\nabla^*_X Y, Z) - 2g(X,Y)\eta(Z). \quad \square$$

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