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Rational Degree of Belief Ceases to Be Additive When the Dutch Book Argument Is Transported to a New Environment

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Abstract: The strength of a person's beliefs can be measured by the buying and selling prices they offer on contingent promissory notes. Consider a promissory note contingent on a proposition; it pays off one unit of money if the proposition is true and nothing otherwise. The more strongly a person believes the proposition, the higher the minimum price would be at which they would sell it. The same would apply to the maximum purchase price. The well-known Dutch Book Argument claims that, if the person's beliefs are rational, their buying/selling prices should combine additively, meaning that the price of a promissory note contingent on the disjunction of two incompatible propositions should be the sum of the prices of the promissory notes contingent on the individual incompatible propositions. This paper shows that the essence of the Dutch Book Argument is that rational belief is additive because money is additive. It is proved that, if the structure of the Dutch Book Argument is kept, but a nonadditive resource is substituted for money, then rational belief will follow a nonadditive combining rule. It is also shown how rational buying/selling prices behave when the pay-off amount of a contingent note changes.

Keywords: Dutch Book Argument; partial belief; rationality; consistency; coherence; additivity; Aczél Associativity Theorem; Cauchy Functional Equation

MSC: 91B08; 91C05; 39B22

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1. Consistency of Belief

Are our beliefs consistent?

How could I test whether my beliefs are consistent? One approach is the following.

1.1. Logical-Conjunction Criterion for Evaluating Consistency of Beliefs

We might test the consistency of my beliefs as follows. I might write down a finite subset of the propositions that I believe. Then, we could investigate whether those propositions could all be true. Thus we could use propositional logic and check whether or not the conjunction of the propositions that I wrote down was a logical contradiction. If it is a contradiction, then my written-down beliefs are inconsistent and, thus, irrational. (Note that the issue here is not whether my beliefs are correct; my beliefs might be consistent even though incorrect).

However, the logical-conjunction approach only makes sense for beliefs that are firm and unequivocal. Whereas, in real life, many of my beliefs are not firm; they are partial. I only partially believe them.

Among the propositions that I partially believe, I believe some more strongly others. Two propositions that I might partially believe to a greater or lesser degree are:

- The capital of Tennessee is Nashville.
- The capital of Tennessee is Memphis.

Because the capital of Tennessee cannot be both Nashville and Memphis, the conjunction of the two propositions listed above is a contradiction. So, by the logical-conjunction test, my beliefs are irrational. However, the logical-conjunction test does not seem appropriate for application to partially believed propositions.

I am aware that Nashville and Memphis are important cities. However, because I do not follow the progress of bills through the Tennessee legislature, I am unsure where that legislature is located. As a result, I think that Nashville *might* be the capital of Tennessee and I also think that Memphis *might* be the capital. Given that my beliefs are so unsure, it seems inappropriate to label my beliefs as inconsistent.

A different criterion for evaluating consistency of beliefs, one that is designed to be applied to partial beliefs, is the Dutch Book criterion. We turn to that next.

1.2. Dutch Book Criterion for Evaluating Consistency of Beliefs

Dutch Book Arguments are concerned with *rationality*. There are two main types:

- Arguments concerned with rationality of *current* beliefs.
- Arguments concerned with rationality of belief *change*.

This paper will be concerned only with the former and not with the latter.

Dutch Book Arguments were originally formulated independently by Frank Plumpton Ramsey in a 1926 paper that was published posthumously in 1931 [1] and by Bruno de Finetti in papers beginning in 1930 (see Gillies, ([2], p. 51)), notably [3,4]. By now, there is a large literature on Dutch Book Arguments and there are numerous versions of those arguments. Some reviews of that literature are [5–7].

A few words about the phrase "Dutch book". The origin of the term "Dutch book" is not entirely clear [8]. The "book" part of "Dutch book" seems to refer either to the collection of bets made with/by a bookie or to odds posted by the bookie. De Finetti [9] remarked that, in the English language, a collection of bets that takes advantage of an inconsistency in the bookie's odds is called a "Dutch book". However, why it was called that, de Finetti said he did not know. Peter Douglas [10] believes he knows why: As a result of the Anglo-Dutch wars [11], the English felt enmity toward the Dutch and came to use the adjective "Dutch" prejudicially in a derisive or pejorative sense with a general meaning of "inferior", "bad" or "awful". Douglas lists dozens of English phrases, among them "Dutch book", in which the adjective "Dutch" is so used. Backing up Douglas, the *Oxford English Dictionary* [12] indicates that historically the adjective "Dutch" has sometimes been used in an opprobrious or derisive sense.

The Dutch Book Argument for Additivity of Partial Belief (or the "Dutch Book Argument" for short) is an argument that claims to show that, for a person's currently held partial beliefs to be consistent, they must satisfy a certain additivity criterion (that will be explained below). Beliefs that do not satisfy the criterion are inconsistent and, thus, irrational. Beliefs that satisfy the criterion are consistent and rational.

Backstories. To motivate the abstract mathematics employed in the Dutch Book Argument, some backstory is often invented concerning the person whose beliefs are being evaluated. (In this paper, that person is called the *Protagonist*.) In these backstories, the Protagonist may be a bookie who takes bets from the public [6]; or the Protagonist may be a bookie who is also an epistemologist ([13], Chapter 2); or the Protagonist may be someone whose rationality is being studied by a psychologist ([2], Chapter 4).

In the next section, I will be the Protagonist and the rationality of my beliefs about the capital of Tennessee will be subjected to a Dutch Book Argument.

Terminology. In the Dutch Book literature, rational beliefs are referred to as being *coherent*; irrational beliefs as being *incoherent*.

1.3. A Dutch Book Argument Applied to My Beliefs

Suppose that I am indifferent between being given either

- (Nash) A contingent promissory note that pays off \$1 if Nashville is the capital of Tennessee, or
- (C70) A bearer check for \$0.70.

My being *indifferent* between (Nash) and (C70) means that I find them equally desirable and I would willingly trade either one for the other. The fact that I am indifferent between (Nash) and a check for $0.70 is a measure of the strength of my belief that Nashville is the capital of Tennessee. For example, if the strength of my belief that Nashville is the capital of Tennessee were slightly less, then I might be indifferent between (Nash) and a check for $0.60. Because I am indifferent between (a) and a check for $0.70, it may be said that 0.70 is the *strength* or *degree* of my belief that Nashville is the capital of Tennessee.

Next, suppose that 0.10 is the strength of my belief that Memphis is the capital of Tennessee. In other words, I am indifferent between being given either

- (Mem) A contingent promissory note that pays off $1 if Memphis is the capital of Tennessee, or
- (C10) A bearer check for $0.10.

Then, according to the Dutch Book Argument, 0.80 *should* be the strength of my belief that the capital of Tennessee is either Nashville or Memphis. In other words, I *should* be indifferent between being given either

- (Nash or Mem) A contingent promissory note that pays off $1 if either Nashville or Memphis is the capital of Tennessee, or
- (C80) A bearer check for $0.80.

However, what if 0.85, not 0.80, is the degree of my belief that either Nashville or Memphis is the capital of Tennessee? Suppose I am indifferent between (Nash or Mem) and

- (C85) A bearer check for $0.85.

If so, then the Dutch Book Argument says that my beliefs are incoherent because they can cause me to lose money. The argument goes like this:

- Suppose that I have an account with a brokerage and that I may keep (contingent) promissory notes and bearer checks in my account.
- Suppose that I start out out with the check (C85) in my account.
- The brokerage asks me whether I would be willing to trade the check (C85) for the promissory note (Nash or Mem). Because I am indifferent between (C85) and (Nash or Mem), I agree to the trade and the trade is carried out. I now have (Nash or Mem) in my account.
- The brokerage informs me that my owning the promissory note (Nash or Mem) is equivalent to my owning *both* of the promissory notes (Nash) and (Mem); whether I own the one note or the bundle of two, I will be paid $1 if either Nashville or Memphis is the capital of Tennessee. So, the brokerage takes the note (Nash or Mem) from my account and replaces it with the two notes (Nash) and (Mem).
- Next the brokerage asks me whether I would be willing to trade the promissory note (Nash) for the check (C70). Because I am indifferent between (Nash) and (C70), I agree to the trade and it is carried out. Now my account contains (C70) and (Mem).
- Then the brokerage asks me whether I would be willing to trade the promissory note (Mem) for the check (C10). Because I am indifferent between (Mem) and (C10), I agree to the trade and it is carried out. My account now contains (C70) and (C10).
- Next the brokerage informs me that, because money *combines additively*, it will take the two checks (C70) and (C10) from my account and replace them with the single check (C80). This is done and my account now contains the check (C80).
- However, my account started out containing the check (C85) for $0.85, but ended up with the check (C80) for $0.80. So I have lost money.

To summarize: In the sequence of takings and leavings listed aboce, the successive contents of my account are listed below. Interleaved between the account states is shown the reason for the transition from each account state to the next.

$$
\begin{aligned}
&\text{(C85)} && (1)\\
&\quad\Downarrow \text{ [substantive change to which I am indifferent]}\\
&\text{(Nash or Mem)} && (2)\\
&\quad\Downarrow \text{ [change of format only]}\\
&\text{(Nash) and (Mem)} && (3)\\
&\quad\Downarrow \text{ [substantive change to which I am indifferent]}\\
&\text{(C70) and (Mem)} && (4)\\
&\quad\Downarrow \text{ [substantive change to which I am indifferent]}\\
&\text{(C70) and (C10)} && (5)\\
&\quad\Downarrow \text{ [change of format only]}\\
&\text{(C80)} && (6)
\end{aligned}
$$

The transitions from State (1) to (2), from (3) to (4), and from (4) to (5) occurred as a result of trades to which I agreed because I was indifferent to the change.

The transitions from State (2) to State (3) and from State (5) to State (6) are of a distinct sort; they can be described as matters of formatting or of bookkeeping. Consider the transition from State (2) to State (3). If I own the promissory note (Nash or Mem), I will be paid \$1 if either Nashville or Memphis is the capital of Tennessee. Likewise, if I own both the promissory note (Nash) and the promissory note (Mem), then I will be paid \$1 if either Nashville or Memphis is the capital of Tennessee. Thus, there is no difference between owning the promissory note (Nash or Mem) and owning both of the notes (Nash) and (Mem). In other words, the promissory note (Nash or Mem) and the bundle of the two notes (Nash) and (Mem), are the same thing, just in different formats. (i.e., bookkeeping says they are the same thing.)

Next consider the transition from State (5) to State (6). If I own both the check (C70) and the check (C10), then I will receive one payment of \$0.70 and one payment of \$0.10. According to conventional bookkeeping, these two monetary amounts combine *additively*. Thus, I will be paid a *total* of \$0.80, which is the same amount that I would be paid if I owned the check (C80). Owning both of (C70) and (C10) (i.e., State (5)) is equivalent to owning (C80) (i.e., State (6)). Thus, according to conventional bookkeeping, State (5) and State (6) are the same thing, just in different formats.

Why did I lose money? The reason is that the strengths of my beliefs do not combine the same way that money combines (i.e., additively). The strength of my belief that Nashville is the capital of Tennessee is 0.70 and the strength of my belief that Memphis is the capital of Tennessee is 0.10. So, if my belief strengths combined additively, then the strength of my belief that either Nashville or Memphis is the capital of Tennessee would be 0.80. However, the strength of that belief is actually 0.85. That's why I lost money.

If the strength of my belief that either Nashville or Memphis is the capital of Tennessee were 0.80, then, if I owned the check (C80), I would trade it for the promissory note (Nash or Mem) and my account would be in State (2). Then, if I went through the sequence of trades from State (2) to State (6), I would once again own the check (C80) and I would not have lost any money.

In summary, the Dutch Book Argument claims that the reason I lost money is that my beliefs were incoherent or, more precisely, the strengths of my beliefs were incoherent. What made my belief strengths incoherent, according to the Dutch Book Argument, is that they did not combine the way money combines (i.e., additively).

1.4. What If Money Did Not Combine Additively?

Suppose that money did not combine additively. Suppose that \$0.70 and \$0.10 combined to yield \$0.85. In that case, my beliefs would add in the same way as money. Suppose I started with the check (C85) in my account (i.e., State (1)) and then made the sequence of trades that brought my account to State (5). Then, as a matter of bookkeeping, the checks (C70) and (C10) would be replaced with the check (C85). As a result, because my beliefs combined in the same way as money, I would not lose any money.

1.5. This Paper's Argument

The Dutch Book Argument's conclusion that, to be coherent, belief strengths must combine additively is not true in all environments. It is only true in environments where promissory notes pay off in a resource that combines additively, like money. In an environment where promissory notes pay off in a resource that combines nonadditively, coherent degree of belief will combine nonadditively.

This paper proceeds with a formal mathematical development, through a series of definitions, propositions and theorems, culminating in the paper's main theorem (Theorem 4). This theorem shows that, in an environment with a nonadditive resource, coherent belief is characterized by quasi-probability (Definition 12) rather than probability.

Furthermore, Section 7 shows beliefs that are coherent in one environment may be incoherent in another.

2. The Protagonist and the MacGuffin

We will be concerned with the beliefs and desires of a *Protagonist*. The Protagonist may be a person or an association of people such as a family or a corporation. Or, it may be a robot with artificial intelligence, or an alien being from another planet.

In his 1974 book, de Finetti imagines the reader of the book as having the role of the Protagonist ([14], p. 27) and refers to the Protagonist as "You", spelled with a capital-Y even in mid-sentence.

We (meaning the author of this paper and its readers), in our role as epistemic analysts, do not need to be concerned with what the Protagonist is. All that matters is that it has beliefs and desires and we wish to analyze whether its beliefs are coherent.

Resources

In the previous section, the strength of my belief that Memphis is the capital of Tennessee was said to be 0.10 because I was indifferent between (a) being given a promissory note that paid \$1 if Memphis is the capital of Tennessee vs. (b) being given a check that paid \$0.10. However, what if the promissory note and the check paid off, not with money, but with chocolate cake? Would it make a difference? Would I be indifferent between (c) being given a promissory note that paid one chocolate cake if Memphis is the capital of Tennessee vs. (d) being given a check that paid 0.10 part of a chocolate cake? Perhaps it would make a difference. Perhaps I would not trade the potential of receiving an entire cake for the certainty of receiving 0.10 part of a cake.

So, money is not the only resource that might be used to measure the strength of my beliefs. The use of money to measure belief is conventional and convenient but, in the end, arbitrary.

There are various kinds of resources—resources that come in greater or lesser quantities—that the Protagonist may desire to acquire. Such resources might be of many different kinds. Some examples are:

- Money.
- Chocolate cake.
- Time spent listening to an opera.
- Hints that help the Protagonist better play a computer game.
- Visitation hours, if the Protagonist is a prisoner.
- Offers of full-time jobs.

As epistemic analysts, we do not care what the desired resource is. The protagonist wants it and the more, the better.

When referring to the desired resource, we will borrow a term from the motion picture director Alfred Hitchcock, and refer to the desired resource as "the MacGuffin". When discussing his films, Hitchcock would sometimes flippantly refer to the object that the characters in the film were striving to obtain as "the MacGuffin". The exact nature of the MacGuffin was not important. Moreover, the nature of the MacGuffin varied from film to film. When explaining the concept of the MacGuffin in an interview with François Truffaut ([15], p. 138), Hitchcock said that it did not matter to him, the director of the film, what the MacGuffin was. However, what was important was that the MacGuffin be greatly desired by the characters in the film.

Analogously, for our purposes as epistemic analysts, we do not care what the desired resource is. We will call it "the MacGuffin". The nature of the MacGuffin can vary from one setting to another. We will assume that the MacGuffin comes in quantifiable amounts. Thus, if x is any nonnegative real number, then it is meaningful to speak of x amount of MacGuffin. We assume that the Protagonist's attitude is: The more MacGuffin I can get, the better.

3. Preference Relations Over Promissory Notes

This paper will frequently mention *statements* or, equivalently, *propositions*. It will be convenient to formally represent propositions as elements of a finite Boolean algebra.

3.1. Boolean Algebras

We will work with finite Boolean algebras, which are 6-tuples of the form:

$$\mathfrak{BA} = (\mathbf{St}, \wedge, \vee, \neg, \bot, \top), \tag{7}$$

where

- **St** is a finite set of elements called *statements* or, equivalently, *propositions*;
- $\wedge$ is a binary operator, called *conjunction*, that corresponds to the English-language connective *and*;
- $\vee$ is a binary operator, called *disjunction*, that corresponds to the English-language connective *or*;
- $\neg$ is a unary operator, called *negation*, that corresponds to the English-language connective *not*;
- $\bot$ is a proposition that cannot be true (i.e., a generic contradiction);
- $\top$ is a proposition that cannot be false (i.e., a generic tautology).

The propositions $\bot$ and $\top$ are members of **St**. If A and B are any members of **St**, then $A \wedge B$, $A \vee B$, and $\neg A$ are all members of **St**. The elements of **St** and the operators are required to obey certain axioms that may be found in standard texts, such as ([16], p. 52).

Given any two propositions A and B in **St**, we say that A *implies* B if $A \wedge B = A$. For any $A \in \mathbf{St}$, $\bot$ implies A and A implies $\top$. A proposition $A \in \mathbf{St}$ is an *atom* if the only members of **St** that imply A are A and $\bot$ and if $A \neq \bot$.

We say that A and B are *incompatible* if $A \wedge B = \bot$.

3.2. Promissory Notes

Recall that earlier, we mentioned promissory notes. An example of a promissory note is:

- The promise to pay \$1 if a flipped thumbtack lands point up (and nothing otherwise).

In this paper, we will consider a broader range of promissory notes. The general form of a promissory note is the promise to pay x amount of MacGuffin if some proposition A is true and to pay zero amount of MacGuffin if A is false.

Definition 1. *For any $A \in \mathbf{St}$ and any $x \geq 0$, let $\mathbf{PN}[A, x]$ denote a contingent promissory note that promises to pay off x amount of MacGuffin if A is true and zero amount of MacGuffin* *otherwise. In the promissory note* $\mathbf{PN}[A,x]$*, A is called the* prerequisite *and x is called the* payoff*. Let the set of all promissory notes be denoted:*

$$\mathcal{PN} = \{\mathbf{PN}[A,x] : A \in \mathbf{St},\ x \geq 0\}. \tag{8}$$

Notation 1. *Sometimes it is convenient to denote an arbitrary promissory note in* $\mathcal{PN}$ *by either* $\mathbf{PN}$ *or* $\mathbf{PN}'$ *or* $\mathbf{PN}''$*.*

Notation 2. *Let* $\mathbb{R}_+$ *denote the set of non-negative real numbers.*

Remark 1. *Promissory notes may be represented as functions on Boolean atoms. Let*

$$\mathbf{Atoms} = \{\omega \in \mathbf{St} : \omega \text{ is an atom}\}. \tag{9}$$

Then the promissory note $\mathbf{PN}[A,x]$ *may be represented by the function* $F_{A,x} : \mathbf{Atoms} \to \mathbb{R}_+$ *given by: For all* $\omega \in \mathbf{Atoms}$*,*

$$F_{A,x}(\omega) = \begin{cases} x & \text{if } \omega \text{ implies } A; \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Now, exactly one atom in **Atoms** *will be true. So, if* ω_{true} *is the atom that is true, then the promissory note* $\mathbf{PN}[A,x]$ *will pay off* $F_{A,x}(\omega_{\text{true}})$ *amount of MacGuffin.*

A special case. The promissory note $\mathbf{PN}[\top,x]$ pays off x amount of MacGuffin if $\top$ is true. However, it is *certain* that $\top$ is true. Therefore, being given the promissory note $\mathbf{PN}[\top,x]$ is equivalent to being given x amount of MacGuffin.

Let us reconsider the discussion concerning beliefs about the capital of Tennessee. If the Protagonist sort of believes that Nashville is the capital of Tennessee but is not sure, the Protagonist might be indifferent between being given *either*

- A promise to pay $1 if Nashville is the capital of Tennessee, *or*
- $0.70.

We can express the Protagonist's indifference using the notation of Definition 1. Let N denote the proposition that Nashville is the capital of Tennessee. Suppose that the MacGuffin resource is money and that the unit of MacGuffin is $1. Then the Protagonist is indifferent between given either the promissory note $\mathbf{PN}[N,1]$ or the promissory note $\mathbf{PN}[\top,0.70]$. Furthermore, because the Protagonist regards more MacGuffin as better than less MacGuffin, the Protagonist would strictly prefer $\mathbf{PN}[N,1]$ over $\mathbf{PN}[\top,0.65]$, but would strictly prefer $\mathbf{PN}[\top,0.75]$ over $\mathbf{PN}[N,1]$.

In order to analyze whether the Protagonist's beliefs are coherent, we will need to analyze the Protagonist's preferences over members of the set $\mathcal{PN}$ of Definition 1. It will be assumed that those preferences constitute a *weak order* over $\mathcal{PN}$. So, let us consider what weak orders are.

3.3. Weak Orders

There are various equivalent ways of defining a weak order. We use the following way taken from ([17], Section 1.3.1).

Definition 2. *A binary relation* $\precsim$ *on* $\mathcal{PN}$ *is a weak order if it has the following two properties:*

- *Connectivity. For all* $\mathbf{PN}$, $\mathbf{PN}' \in \mathcal{PN}$*, either* $\mathbf{PN} \precsim \mathbf{PN}'$ *or* $\mathbf{PN}' \precsim \mathbf{PN}$*.*
- *Transitivity. For all* $\mathbf{PN}$, $\mathbf{PN}'$, $\mathbf{PN}'' \in \mathcal{PN}$*, if* $\mathbf{PN} \precsim \mathbf{PN}'$ *and* $\mathbf{PN}' \precsim \mathbf{PN}''$*, then* $\mathbf{PN} \precsim \mathbf{PN}''$*.*

In addition, if $\precsim$ *is a weak order on* $\mathcal{PN}$*, two further binary relations* $\prec$ *and* $\sim$ *on* $\mathcal{PN}$ *may be defined. For all* $\mathbf{PN}$, $\mathbf{PN}' \in \mathcal{PN}$*,*

1. $\mathbf{PN} \prec \mathbf{PN}'$ *iff not* $\mathbf{PN}' \precsim \mathbf{PN}$*.*

2. $\mathbf{PN} \sim \mathbf{PN}'$ *iff both* $\mathbf{PN} \precsim \mathbf{PN}'$ *and* $\mathbf{PN}' \precsim \mathbf{PN}$.

3.4. Belief-Payoff Preference Relations

The following definition defines belief-payoff preference relations. These are weak orders, having certain properties, over the set of promissory notes $\mathcal{PN}$. A belief-payoff preference relation $\precsim$ may be interpreted as a desirability ordering over promissory notes. Thus, $\mathbf{PN}[A, x] \precsim \mathbf{PN}[B, y]$ means that the promissory note $\mathbf{PN}[B, y]$ is at least as desirable as the promissory note $\mathbf{PN}[A, x]$. In other words, it is at least as desirable to be *given* $\mathbf{PN}[B, y]$ as to be given $\mathbf{PN}[A, x]$. The desirability of any promissory note will be affected both by the strength of belief in the prerequisite and the size of the payoff.

Definition 3. *Suppose that* $\precsim$ *is a weak order on* $\mathcal{PN}$. *If the following properties hold, then* $\precsim$ *is a belief-payoff preference relation.*

PE: Extreme propositions. *Concerning* $\perp$ *and* $\top$:

PEZ: Zeros. $\mathbf{PN}[\perp, 0] \sim \mathbf{PN}[\top, 0]$.

PEM: Monotonicity. *If* $0 \leq x < y$:

- $\mathbf{PN}[\perp, x] \sim \mathbf{PN}[\perp, y]$.
- $\mathbf{PN}[\top, x] \prec \mathbf{PN}[\top, y]$.

PG: General propositions. *For any proposition* $A \in \mathbf{St}$ *and any* $x \in \mathbb{R}_+$,

$$\mathbf{PN}[\perp, x] \precsim \mathbf{PN}[A, x] \precsim \mathbf{PN}[\top, x]. \tag{11}$$

CE: Certainty equivalent. *If* $\mathbf{PN}$ *is any promissory note in* $\mathcal{PN}$, *then* $\mathbf{PN} \sim \mathbf{PN}[\top, z]$ *for some* $z \geq 0$. *The promissory note* $\mathbf{PN}[\top, z]$ *is called the certainty equivalent of* $\mathbf{PN}$.

Remark 2. *From [PEM], it follows that a promissory note's certainty equivalent is unique. If* $\mathbf{PN}[\top, x] \sim \mathbf{PN}[\top, y]$, *then* $x = y$.

Comments on Definition 3. Every belief-payoff preference relation has various properties that we would want it to have:

- For any proposition A in **St**, the promissory note $\mathbf{PN}[A, 0]$ will pay off zero no matter whether A is true or not. Thus, we would expect that the desirability of $\mathbf{PN}[A, 0]$ would be unaffected by what the proposition A is. Indeed, from [PG] and [PEZ],

$$\mathbf{PN}[\perp, 0] \sim \mathbf{PN}[A, 0] \sim \mathbf{PN}[\top, 0]. \tag{12}$$

- Because $\perp$ must be false, the size of the payoff in the promissory note $\mathbf{PN}[\perp, x]$ should be irrelevant. This is what the first part of [PEM] says.
- Because $\perp$ must be false and because $\top$ must be true, if $y > 0$, we would expect $\mathbf{PN}[\top, y]$ to be strictly preferred to $\mathbf{PN}[\perp, y]$ and, indeed, it is implied by the two parts of [PEM] and [PEZ] that

$$\mathbf{PN}[\perp, y] \sim \mathbf{PN}[\perp, 0] \sim \mathbf{PN}[\top, 0] \prec \mathbf{PN}[\top, y]. \tag{13}$$

On the other hand, *some* belief-payoff preference relations $\precsim$ have *some* absurd properties. For example: For some $A, B \in \mathbf{St}$,

$$\mathbf{PN}[A, 2] \prec \mathbf{PN}[A, 1]; \qquad \text{[Absurd!]} \tag{14}$$

$$\mathbf{PN}[B, 1] \prec \mathbf{PN}[A, 1], \text{ where } A \text{ implies } B. \qquad \text{[Absurd!]} \tag{15}$$

It will turn out that belief-payoff preference relations having these kind of absurd properties are *incoherent* as defined later in this paper (Definition 9). Without going into detail, suffice it to say that Proposition 3 will imply that the belief-payoff preference relation in Equation (14) is incoherent and Proposition 2 will imply that the belief-payoff preference relation in Equation (15) is incoherent.

Belief Strength

Recall that the strength of my belief that Nashville is the capital of Tennessee was said to be 0.70 because I was indifferent between being given a promissory note that paid off \$1 if Nashville is the capital of Tennessee vs. being given a check for \$0.70.

The following definition formalizes and generalizes the above concept of belief strength. In that definition, note that the promissory note $\mathbf{PN}[\top, z]$ pays off z units of MacGuffin if $\top$ is true and zero units otherwise. However, $\top$ is always true. Thus, $\mathbf{PN}[\top, z]$ is essentially the same thing as a check for z MacGuffin units.

Definition 4. *Given a belief-payoff preference relation $\precsim$, define its belief function $Blf_{\precsim} : \mathbf{St} \to [0,1]$ as follows. For each $A \in \mathbf{St}$, $Blf_{\precsim}(A)$ is the unique $z \in [0,1]$ such that $\mathbf{PN}[A,1] \sim \mathbf{PN}[\top, z]$. Then, $Blf_{\precsim}(A)$ is called the degree of belief in A under $\precsim$ or, more briefly, belief in A.*

The rationale for the above definition is that, the larger the z such that someone is indifferent between $\mathbf{PN}[A,1]$ and $\mathbf{PN}[\top, z]$, the greater must be that someone's confidence that A is true.

Thus, belief strength is defined in terms of a belief-payoff preference relation $\precsim$. So, if $\precsim$ has absurd properties, then the belief function $Blf_{\precsim}$ may have absurd properties. For example, if the absurd Equation (15) holds, then $Blf_{\precsim}$ will have the absurd property:

$$Blf_{\precsim}(B) < Blf_{\precsim}(A), \text{ where } A \text{ implies } B. \qquad \text{[Absurd!]} \tag{16}$$

However, $Blf_{\precsim}$ will not have absurd properties if $\precsim$ is coherent as defined later in this paper.

3.5. Worth Functions

Worth functions (to be defined below) provide a representation of belief-payoff preference relations.

Notation. Let $\mathbb{R}_+$ denote the set of all nonnegative real numbers.

Definition 5. *A worth function is function $\mathcal{W}$ that maps $\mathcal{PN}$ into $\mathbb{R}_+$ and that has the following properties:*

WE: Extreme propositions. *Concerning $\perp$ and $\top$. For all $x \geq 0$:*

WE.1. $\mathcal{W}(\mathbf{PN}[\perp, x]) = 0$.
WE.2. $\mathcal{W}(\mathbf{PN}[\top, x]) = x$.

WG: General propositions. *For any proposition $A \in \mathbf{St}$ and any $x \geq 0$,*

$$0 \leq \mathcal{W}(\mathbf{PN}[A, x]) \leq x. \tag{17}$$

Remark 3. *Considered as functions of $x \geq 0$:*

- *$\mathcal{W}(\mathbf{PN}[\perp, x])$ is the zero function.*
- *$\mathcal{W}(\mathbf{PN}[\top, x])$ is the identity function.*
- *$\mathcal{W}(\mathbf{PN}[A, x])$ is bounded below by the zero function and above by the identity function.*

Definition 6. *(Representation Definition.) Suppose that $\precsim$ is a belief-payoff preference relation and that $\mathcal{W} : \mathbf{PN} \to \mathbb{R}_+$ is a worth function. Suppose further that, for all promissory notes $\mathbf{PN}$ and $\mathbf{PN}'$ in $\mathcal{PN}$,*

$$\mathbf{PN} \precsim \mathbf{PN}' \textit{ iff } \mathcal{W}(\mathbf{PN}) \leq \mathcal{W}(\mathbf{PN}'). \tag{18}$$

Then $\mathcal{W}$ is a representation *of $\precsim$. Equivalently: $\mathcal{W}$* represents *$\precsim$.*

It is shown in Theorem A1 of the Appendix A that: (a) Every worth function represents some belief-payoff preference relation. (b) Every belief-payoff preference relation is represented by a unique worth function.

Why study worth functions? Worth functions provide a way of specifying belief-payoff preference relations. Thus, if you want to specify a particular belief-payoff preference relation, you can do so by specifying the worth function that represents it. Furthermore, you can study the properties of a belief-payoff preference relation by studying the properties of the worth function that represents it. More importantly, worth functions play a key role in the proof of Theorem 2.

Here is an important connection between a belief-payoff preference relation and the worth function that represents it.

Proposition 1. *If $\precsim$ is a belief-payoff preference relation and if $\mathcal{W}$ is the worth function that represents it, then, for all $A \in \mathbf{St}$,*

$$Blf_{\precsim}(A) = \mathcal{W}(\,\mathbf{PN}[A,1]\,). \tag{19}$$

Proof. Recall from Definition 4 that $Blf_{\precsim}(A)$ is the unique z such that $\mathbf{PN}[A,1] \sim \mathbf{PN}[\top,z]$. Then, because $\mathcal{W}$ represents $\precsim$ and because worth functions have Property [WE.2],

$$\mathcal{W}(\,\mathbf{PN}[A,1]\,) = \mathcal{W}(\,\mathbf{PN}[\top,z]\,) = z = Blf_{\precsim}(A). \tag{20}$$

□

4. Defining Coherence

We have defined belief-payoff preference relations $\precsim$, which are weak orders over the set of promissory notes $\mathcal{PN}$. We have not inquired how the Protagonist came to adopt its belief-payoff preference relation. Somehow—we do not know how—it was adopted by the Protagonist. We now turn to the issue of whether the Protagonist's belief-payoff preference relation is, in some sense, *coherent*.

Obviously, to investigate coherence, we need a definition of what it is. The goal of this section, which is several pages long, is to define what it means for a belief-payoff preference relation to be coherent. This is achieved with Definition 9 at the end of the section.

Recall that, in our earlier discussion of the standard Dutch Book Argument as applied to the rationality of my (partial) beliefs about the capital of Tennessee, a key role was played by the additivity property of money. Subsequently, it was argued that some other resources might not combine additively. Such a resource was called a MacGuffin resource (Section 2).

4.1. How Money Quantities Combine

Money combines additively. Thus, suppose one is given x units of money—the unit of money may be any convenient amount, say \$1. Next, one is given y units of money. Then one has $x + y$ units of money.

When we combine combine two or more quantities of money, the combination operation (i.e., the arithmetic operation of addition) has some desirable properties:

- Commutativity. Suppose I have some money in my left hand and some money in my right hand. My pocket is empty. I transfer the money in one of my hands into my pocket; then I transfer the contents of my other hand into the pocket. After so doing, the amount of money now in my pocket does not depend on whether I put into the pocket first the left-hand money and second the right-hand money, or vice versa.
- Associativity (plus commutativity). Suppose that I have money in my left pocket, money in my right pocket, and money in my hand. I take the money in my hand and put it into one of my two pockets. Then, I take all the money in my left pocket and put it into my right pocket. The amount of money now in my right pocket does not depend on whether the money that started out in my hand was put into my left pocket or into my right pocket.

- Zero incrementation. I have money in my pocket; I have no money in my hand. When I put the contents of my empty hand into my pocket, the amount of money in my pocket does not change.
- Monotonicity. My left and right pockets each contain the same amount of money. I have money in both my left hand and in my right hand, but there is more in my left hand than in my right. I transfer the contents of my left hand into my left pocket and transfer the contents of my right hand into my right pocket. After so doing, there is more money in my left pocket than in my right.

4.2. How MacGuffin Quantities Combine

Notation 3. *If one has x amount of MacGuffin and is given a further y amount of MacGuffin, then the amount of MacGuffin one possesses is denoted $x \oplus y$.*

MacGuffin quantities need not combine additively. However, it is assumed that, when MacGuffin quantities combine, they do so in a reasonable manner. Specifically, it is assumed that, when MacGuffin quantities combine, they do so with the properties of money combination listed above. This motivates the assumption that the MacGuffin combination operator $\oplus$ is a quasi-summative operator as defined below.

Definition 7. *A quasi-summative operator $\oplus$ is a continuous mapping of $\mathbb{R}_+ \times \mathbb{R}_+$ into $\mathbb{R}_+$ that has the following properties. For all $x, y, z \in \mathbb{R}_+$:*

- *Commutativity.* $x \oplus y = y \oplus x$.
- *Associativity.* $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.
- *Zero incrementation.* $x \oplus 0 = x$.
- *Monotonicity. If $y < z$, then $x \oplus y < x \oplus z$.*

Notation 4. *The above definition has been designed so that the ordinary addition operator is a quasi-summative operator; it will be denoted $\oplus^{\text{add}}$.*

Why are we concerned with quasi-summative operators? Because they will play a key role in the definition of *coherence* (Definition 9).

We Know What All the Quasi-Summative Operators Are

Based on a fundamental theorem proved by Aczél [18,19], p. 256, we know that there are many quasi-summative operators and we know what they all are.

Definition 8. *Let $\mathcal{MO}$ denote the set of all monotonic increasing functions f that map $\mathbb{R}_+$* onto *$\mathbb{R}_+$ and, for which $f(0) = 0$ and $f(1) = 1$.*

The functions in $\mathcal{MO}$ are necessarily continuous. If the function f is in $\mathcal{MO}$, then its inverse (denoted f^{-1}) exists and is also in $\mathcal{MO}$. If f and g are in $\mathcal{MO}$, then the function composition $f \circ g$ is also in $\mathcal{MO}$.

In the proof of the following theorem, the "heavy lifting" is provided by Aczél's Associativity Theorem and the Cauchy Functional Equation Theorem.

Theorem 1. *A binary operator $\oplus$ on $\mathbb{R}_+$ is a quasi-summative operator if and only if, for some $f \in \mathcal{MO}$,*

$$x \oplus y = f^{-1}[f(x) + f(y)], \text{ for all } x, y \in \mathbb{R}_+. \tag{21}$$

Moreover, the f in Equation (21) is unique.

Proof. (If.) It is straightforward to verify that, if Equation (21) holds for some $f \in \mathcal{MO}$, then $\oplus$ is a quasi-summative operator.

(Only if.) We will use the formulation of Aczél's Associativity Theorem as given by Alsina, Frank, and Schweizer ([20], Theorem 2.7.1). The theorem implies that, if $\oplus$ is a quasi-summative operator, then there exists a continuous strictly monotonic (increasing or decreasing) function a that maps $\mathbb{R}_+$ into $[-\infty, \infty]$ such that

$$x \oplus y = a^{-1}[\,a(x) + a(y)\,], \text{ for all } x, y \in \mathbb{R}_+. \quad (22)$$

Then, $a(0) = 0$ because, if it did not, we would not have $x \oplus 0 = x$. Because a is strictly monotonic, $a(1) \neq a(0) = 0$. Let $f(x) = a(x)/a(1)$ for all $x \in \mathbb{R}_+$. Then, regardless of whether the function a is monotonic increasing or decreasing, f is monotonic increasing, and satisfies Equation (21), and $f(0) = 0$ and $f(1) = 1$. Furthermore, as $x \to \infty$, $f(x) \to \infty$, because otherwise we could find $x, y \in \mathbb{R}_+$ such that $f(x) + f(y) > f(z)$ for all $z \in \mathbb{R}_+$. Therefore, by the Intermediate Value Theorem ([21], Theorem 4.5.6), f maps $\mathbb{R}_+$ *onto* $\mathbb{R}_+$ and, so, $f \in \mathcal{MO}$.

(Uniqueness.) Suppose that for some $f, g \in \mathcal{MO}$,

$$f^{-1}[f(x) + f(y)] = g^{-1}[g(x) + g(y)], \text{ for all } x, y \geq 0. \quad (23)$$

It will be shown that f and g are identical. Let $h = g \circ f^{-1}$. Then, $h \in \mathcal{MO}$ and $g = h \circ f$ and $g^{-1} = f^{-1} \circ h^{-1}$. Thus, the above equation may be rewritten:

$$f^{-1}[f(x) + f(y)] = f^{-1} \circ h^{-1}[h \circ f(x) + h \circ f(y)], \text{ for all } x, y \geq 0. \quad (24)$$

Apply the function $h \circ f$ to both sides of the above equation:

$$h[f(x) + f(y)] = h[f(x)] + h[f(y)], \text{ for all } x, y \geq 0. \quad (25)$$

Set $x' = f(x)$ and $y' = f(y)$. Then, because f maps $\mathbb{R}_+$ *onto* $\mathbb{R}_+$,

$$h[x' + y'] = h[x'] + h[y'], \text{ for all } x', y' \geq 0. \quad (26)$$

Because Equation (26) holds and, because $h \in \mathcal{MO}$ is continuous, we may apply the Cauchy Functional Equation Theorem ([19], p. 34). It follows from the theorem that, for some real number c, $h(z) = cz$ for all $z \geq 0$. Because $h \in \mathcal{MO}$, $h(1) = 1$ and, therefore, $c = 1$ and, thus, h is the identity function. Therefore $g = h \circ f = f$. $\square$

Notation 5. *A quasi-summative operator $\oplus$ that satisfies Equation (21) for some $f \in \mathcal{MO}$ may be denoted $\oplus_f$.*

It might seem, at first glance, that the ordinary addition operator is an exceptional member of the class of quasi-summative operators because it can be used, as shown by Equation (21), to express all the other members of the class. However, it is not the case that ordinary addition is exceptional in that sense. Consider any two quasi-summative operators $\oplus_f$ and $\oplus_g$, where $f, g \in \mathcal{MO}$. Then, $\oplus_f$ may be used to express $\oplus_g$. Let $h = f^{-1} \circ g$. Then, $h \in \mathcal{MO}$ and

$$x \oplus_g y = h^{-1}[h(x) \oplus_f h(y)], \text{ for all } x, y \in \mathbb{R}_+. \quad (27)$$

Some examples of quasi-summative operators $\oplus_f$ where $f \in \mathcal{MO}$:

- If f is the identity function, then $x \oplus_f y = x + y$. That is: $\oplus_f$ is $\oplus^{\text{add}}$.
- If $f(x) = x^2$, then $x \oplus_f y = (x^2 + y^2)^{1/2}$, Pythagorean combination.
- If $f(x) = x^{1/2}$, then $x \oplus_f y = x + y + 2(xy)^{1/2}$.
- If $f(x) = \log(1 + x)/\log 2$, then $x \oplus_f y = x + y + xy$.
- If $f(x) = (e^x - 1)/(e - 1)$, then $x \oplus_f y = \log(e^x + e^y - 1)$.

4.3. Examples of Quasi-Summation Applied to MacGuffin Resources

Let us consider some examples of MacGuffin resources that combine quasi-summatively.

4.3.1. Fuel in Spherical Tanks

A farmer stores fuel in spherical tanks. If a tank x meters in diameter and a tank y meters in diameter are both pumped into a tank $(x^3 + y^3)^{1/3}$ meters in diameter, the latter tank will be exactly filled. So, the farmer's quasi-summative operator is $\oplus_{cube}$, where the *cube*-function is: $cube(z) = z^3$ for all $z \geq 0$.

4.3.2. Wine Quality

There is a wine merchant who rates the quality of wines by assigning each wine a non-negative number. The merchant determines that the amount of enjoyment obtained from a bottle of wine having a quality rating of x is $w(x)$, where w is a function in $\mathcal{MO}$. Note that, because $w \in \mathcal{MO}$, $w(0) = 0$. Thus, being given a bottle of wine having quality rating zero is the equivalent of being given nothing.

For the merchant, the amount of enjoyment obtained from two bottles of wine, one with rating x and one with rating y, is the same amount of enjoyment as would be obtained from a single bottle of wine having rating $w^{-1}[w(x) + w(y)]$. So, the merchant's quasi-summative operator is $\oplus_w$.

4.3.3. Recorded Opera

An opera lover is a fan of the opera *NEO: The Never-Ending Opera*. The opera lover determines that the amount of enjoyment obtained from listening to a recording of the *first* x hours of *NEO* is $o(x)$, where o is a function in $\mathcal{MO}$. Suppose the opera lover is given two recordings of *NEO*, one of the first x hours and one of the first y hours; each recording can be listened to only once. The amount of enjoyment obtainable from the two recordings is equal to that obtainable from a single recording of the first $o^{-1}[o(x) + o(y)]$ hours of *NEO*. So, the opera lover's quasi-summative operator is $\oplus_o$.

4.4. This Paper's Backstory: The Brokerage

As mentioned earlier (Section 1.2), different versions of the Dutch Book Argument may have different backstories. These backstories are used to motivate the abstract mathematics involved in the argument. This paper's backstory will now be presented.

The Protagonist has an account with a brokerage that deals in promissory notes. The Protagonist's account may consist of either a single promissory note $\mathbf{PN}$, where $\mathbf{PN}$ is any member of $\mathcal{PN}$. Or it may consist of two promissory notes, denoted

$$\langle\, \mathbf{PN},\, \mathbf{PN}' \,\rangle, \tag{28}$$

where $\mathbf{PN}$ and $\mathbf{PN}'$ may be any promissory notes in $\mathcal{PN}$. The order in which the promissory notes are listed in Equation (28) is immaterial. If there are two promissory notes in the account, they could possibly be identical. (Having two identical promissory notes in the account is better than having only one, in the same way that having two identical coins in one's pocket is better than having only one).

Formal representation of two-note accounts. How should we formally define the entity $\langle\, \mathbf{PN},\, \mathbf{PN}' \,\rangle$? Because the order of promissory notes in in a two-note account immaterial, we may formally represent such an account as a set of of two ordered pairs:

$$\langle\, \mathbf{PN},\, \mathbf{PN}' \,\rangle = \{\, (\mathbf{PN}, \mathbf{PN}'),\, (\mathbf{PN}', \mathbf{PN}) \,\}. \tag{29}$$

Alternatively, if the reader prefers, the reader may think of the entity in Equation 28 as being a *multiset* [22,23]. In multisets, the number of occurrences of each member of the multiset is specified. Thus, if $\mathbf{PN} \neq \mathbf{PN}'$, then

$$\langle\, \mathbf{PN},\, \mathbf{PN}' \,\rangle \;=\; \{\, (\mathbf{PN}, 1),\, (\mathbf{PN}', 1) \,\}; \tag{30}$$

$$\langle\, \mathbf{PN},\, \mathbf{PN} \,\rangle \;=\; \{\, (\mathbf{PN}, 2) \,\}. \tag{31}$$

4.5. Trading in the Brokerage

The brokerage may make trades within the Protagonist's account. However, only certain types of trade are permitted.

Assume in the following that the Protagonist has a belief-payoff preference relation $\precsim$ and that this is represented (Definition 6) by a worth function

$$\mathcal{W} : \mathcal{PN} \to \mathbb{R}_+ \,. \tag{32}$$

Notation 6. *Two types of permitted trades, called "agreed trades" and "bookkeeping trades", will be described below. If AS and AS' are two possible states of the Protagonist's account, then $AS \overset{agree}{\Longrightarrow} AS'$ and $AS \overset{book}{\Longrightarrow} AS'$ denote agreed and bookkeeping trades, respectively, in which the brokerage changes the state of the Protagonist's account from AS to AS'. $AS \overset{agree}{\Longleftrightarrow} AS'$ and $AS \overset{book}{\Longleftrightarrow} AS'$ indicate that the trade may take place in either direction.*

The types of permitted trades will now be described in detail.

Agreed trades. These are trades that the brokerage has proposed to the Protagonist and to which the Protagonist has agreed. The brokerage need not have any information about the Protagonist's belief-payoff preference relation. Thus the brokerage may propose trades to the Protagonist without knowing whether or not the Protagonist will agree to those trades. Agreed trades are of two varieties.

- One-for-one trades.

$$\mathbf{PN} \overset{\text{agree}}{\Longrightarrow} \mathbf{PN}' \text{ only if } \mathbf{PN} \precsim \mathbf{PN}'. \tag{33}$$

  Such a trade is permitted only if the Protagonist agrees and the Protagonist will agree only if $\mathbf{PN} \precsim \mathbf{PN}'$. I.e., only if $\mathcal{W}(\,\mathbf{PN}\,) \leq \mathcal{W}(\,\mathbf{PN}'\,)$. Note that, if the Protagonist is indifferent between **PN** and **PN**$'$, the Protagonist will agree to make the trade. The Protagonist will not say, "Oh, I can't make up my mind whether to agree to that trade or not." If the Protagonist is indifferent between **PN** and **PN**$'$, the Protagonist will agree to trade either one for the other if asked to do so.
- One-out-of-two trades.

$$\langle\, \mathbf{PN}, \mathbf{PN}' \,\rangle \overset{\text{agree}}{\Longrightarrow} \langle\, \mathbf{PN}, \mathbf{PN}'' \,\rangle \text{ only if } \mathbf{PN}' \precsim \mathbf{PN}''. \tag{34}$$

  Notice that, in this type of trade, the Protagonist's account contains two promissory notes; one of the promissory notes remains in the account and one is replaced by another promissory note. Such a trade is permitted only if the Protagonist agrees and the Protagonist will agree only if $\mathbf{PN}' \precsim \mathbf{PN}''$. I.e., only if $\mathcal{W}(\,\mathbf{PN}'\,) \leq \mathcal{W}(\,\mathbf{PN}''\,)$. To be more specific, in such a trade, the brokerage asks the Protagonist whether it is willing to trade **PN**$'$ for **PN**$''$. If the Protagonist is willing, then the brokerage deems that the Protagonist is willing to trade $\langle\, \mathbf{PN}, \mathbf{PN}' \,\rangle$ for $\langle\, \mathbf{PN}, \mathbf{PN}'' \,\rangle$. Notice that the Protagonist is only required to express a preference between one promissory note and another; the Protagonist is never required to express a preference between a two-note account and another.

Bookkeeping trades. These are trades that, in the opinion of the brokerage, are guaranteed-fair to the Protagonist because all that such trades do is to change the format, but not the substance, of the contents of the Protagonist's account. Thus, the brokerage can ethically carry out such trades without getting consent from the Protagonist. These trades come in two varieties.

- Two-for-one Boolean trades.
  If A and B are incompatible propositions in **St** (i.e., $A \wedge B = \bot$) and if $x \geq 0$:

$$\mathbf{PN}[A \vee B, x] \overset{\text{book}}{\Longleftrightarrow} \langle\, \mathbf{PN}[A, x],\ \mathbf{PN}[B, x]\,\rangle. \tag{35}$$

  Such trades in either direction are fair to the Protagonist because the two sides of the trade are essentially the same thing. Both the left side and the right side of Equation (35) pay off the same. That is: the Protagonist receives x MacGuffin units if either A or B is true.
- Two-for-one quasi-summative trades.
  If $A \in \mathbf{St}$ and $x, y \geq 0$:

$$\mathbf{PN}[A, x \oplus y] \overset{\text{book}}{\Longleftrightarrow} \langle\, \mathbf{PN}[A, x],\ \mathbf{PN}[A, y]\,\rangle. \tag{36}$$

  Again the two sides of such trades are essentially the same thing and, thus, the trades are fair in either direction. On the left side of Equation (36), if A is true, the Protagonist will receive a single payoff of $x \oplus y$ MacGuffin units. On the right side of Equation (36), if A is true, the Protagonist will receive two payoffs: one of x MacGuffin units and another of y units. Thus, on the either side of Equation (36), the Protagonist gets what amounts to the same thing.

Terminology. Given a quasi-summative operator $\oplus$, if the Protagonist has an account with a brokerage that follows the above trading rules, we will say that the Protagonist is in a *quasi-summative environment with operator* $\oplus$ or, more briefly, a $\oplus$*-environment*. (Because different quasi-summative operators exist, there exist different quasi-summative environments. One of those environments is the environment where $\oplus$ is ordinary addition.) Furthermore, given a belief-payoff preference relation $\precsim$ in a $\oplus$-environment, the types of trades listed above (Equations (33)–(36)) are said to be $(\precsim, \oplus)$*-permitted*.

4.6. Coherence in a $\oplus$-Environment

We will now define incoherence and coherence of the Protagonist's belief-payoff preference relation $\precsim$. To be informal for a moment: We say that $\precsim$ is incoherent in a $\oplus$-environment if it is possible for the Protagonist to lose MacGuffin assets in a sequence of permitted trades. Here is the formal definition.

Definition 9. *Suppose, first, that the Protagonist has belief-payoff preference relation $\precsim$ and that this is represented by the worth function $\mathcal{W}$. Suppose, second, that the Protagonist has a brokerage account in a $\oplus$-environment.*

- *Incoherence of $\precsim$.*
  Suppose there exists a sequence of $(\precsim, \oplus)$-permitted trades that carry the Protagonist's account from a starting state of $\mathbf{PN}[\top, Start]$ to an ending state of $\mathbf{PN}[\top, End]$, where $Start > End$. If so, then, the Protagonist's belief-payoff preference relation $\precsim$ is said to be incoherent in the $\oplus$-environment.
- *Coherence of $\precsim$.*
  If $\precsim$ is not incoherent in the $\oplus$-environment, then it is coherent.
- *Coherence/incoherence of $\mathcal{W}$.*
  The worth function $\mathcal{W}$ that represents $\precsim$ is said to be coherent in the $\oplus$-environment if and only if $\precsim$ is coherent.

Note that, in the above definition, for a belief-payoff preference relation to be designated incoherent, it is not necessary for anyone to carry out the sequence of permittable trades that carries the account from $\mathbf{PN}[\top, Start]$ down to $\mathbf{PN}[\top, End]$. Neither is it necessary that either the brokerage or the Protagonist be able to discover the asset-losing sequence of trades. It suffices that the asset-losing sequence *exists*, regardless of whether the Protagonist or the brokerage can determine what it is.

5. Necessary and Sufficient Conditions for Coherence

The definition immediately below (Definition 10) defines two properties (obedience to the disjunctive constraint and obedience to the Cauchy constraint) that a belief-payoff preference relation may or may not possess. Then it is demonstrated that:

- Obeying the disjunctive constraint is a necessary condition for coherence (Proposition 2).
- Obeying the Cauchy constraint is also a necessary condition for coherence (Proposition 3).
- Obeying both constraints is a necessary and sufficient condition for coherence (Theorem 2).

Definition 10. *Constraints. Suppose that Protagonist's belief-payoff preference relation is* $\precsim$ *and that this is represented (Definition 6) by a worth function* $\mathcal{W}$*.*

- *Disjunctive constraint.*
  $\precsim$ *obeys the disjunctive constraint in a* $\oplus$*-environment if, for all incompatible* $A, B \in \mathbf{St}$ *and all* $x \geq 0$*,*
$$\mathcal{W}(\,\mathbf{PN}[A \vee B, x]\,) = \mathcal{W}(\,\mathbf{PN}[A, x]\,) \oplus \mathcal{W}(\,\mathbf{PN}[B, x]\,). \tag{37}$$
- *Cauchy constraint.*
  $\precsim$ *obeys the Cauchy constraint in a* $\oplus$*-environment if, for all* $A \in \mathbf{St}$ *and all* $x, y \geq 0$*,*
$$\mathcal{W}(\,\mathbf{PN}[A, x \oplus y]\,) = \mathcal{W}(\,\mathbf{PN}[A, x]\,) \oplus \mathcal{W}(\,\mathbf{PN}[A, y]\,). \tag{38}$$

The reason that Equation (38) is called the *Cauchy constraint* is that satisfaction of the constraint is a prerequisite to applying the Cauchy Functional Equation Theorem ([19], p. 34), as is done in the proof of Theorem 3.

Notation 7. *Consider any* $A \in \mathbf{St}$ *and any* $x \geq 0$*. To make equations easier to read,* $\mathcal{W}(\,\mathbf{PN}[A, x]\,)$ *will* sometimes *be abbreviated* $W(A, x)$*.*

Lemma 1. *Assume that the Protagonist's belief-payoff preference relation* $\precsim$ *is represented by the worth function* $\mathcal{W}$ *and that that the abbreviated notation of Notation 7 is being employed. Then, for any* $A \in \mathbf{St}$ *and any* $x \geq 0$ *and for any* $\mathbf{PN} \in \mathcal{PN}$*, the following are agreed trades:*

$$\mathbf{PN}[A, x] \overset{agree}{\Longleftrightarrow} \mathbf{PN}[\top, W(A, x)]. \qquad [\textit{1 for 1 trade}] \tag{39}$$
$$\langle\,\mathbf{PN},\, \mathbf{PN}[A, x]\,\rangle \overset{agree}{\Longleftrightarrow} \langle\,\mathbf{PN},\, \mathbf{PN}[\top, W(A, x)]\,\rangle. \qquad [\textit{1 out of 2 trade}] \tag{40}$$

Proof. By Property [WE.2] of worth functions, for any $y \geq 0$,
$$\mathcal{W}(\,\mathbf{PN}[\top, y]\,) = y. \tag{41}$$

In the above equation, set $y = \mathcal{W}(\,\mathbf{PN}[A, x]\,)$:
$$\mathcal{W}(\,\mathbf{PN}[\top, \mathcal{W}(\,\mathbf{PN}[A, x]\,)]\,) = \mathcal{W}(\,\mathbf{PN}[A, x]\,). \tag{42}$$

Then, because the worth function $\mathcal{W}$ represents $\precsim$,
$$\mathbf{PN}[\top, \mathcal{W}(\,\mathbf{PN}[A, x]\,)] \sim \mathbf{PN}[A, x]. \tag{43}$$

Using Notation 7, the above equation becomes:
$$\mathbf{PN}[\top, W(A, x)] \sim \mathbf{PN}[A, x], \tag{44}$$
which, via Equations (33) and (34), implies Equations (39) and (40). $\square$

Proposition 2. *If* $\precsim$ *violates the disjunctive constraint in a* $\oplus$*-environment, then* $\precsim$ *is incoherent in that environment.*

Proof. If $\precsim$ violates the disjunctive constraint then there exists some incompatible $A, B \in \mathbf{St}$ and some $x \geq 0$ such that

$$W(A \vee B, x) \neq W(A, x) \oplus W(B, x). \tag{45}$$

Consider the following sequence of trades, all of which are $(\precsim, \oplus)$-permitted in both directions. (Note: Lemma 1 is used to show that the $\overset{\text{agree}}{\Longleftrightarrow}$ trades are permitted.)

$$\begin{aligned}
&\mathbf{PN}[\top, W(A \vee B, x)] && (46)\\
&\overset{\text{agree}}{\Longleftrightarrow} \quad \mathbf{PN}[A \vee B, x] \qquad [\text{1 for 1 trade}]\\
&\overset{\text{book}}{\Longleftrightarrow} \quad \langle\, \mathbf{PN}[A, x],\, \mathbf{PN}[B, x]\,\rangle \qquad [\text{Boolean trade}]\\
&\overset{\text{agree}}{\Longleftrightarrow} \quad \langle\, \mathbf{PN}[A, x],\, \mathbf{PN}[\top, W(B, x)]\,\rangle \qquad [\text{1 out of 2 trade}]\\
&\overset{\text{agree}}{\Longleftrightarrow} \quad \langle\, \mathbf{PN}[\top, W(A, x)],\, \mathbf{PN}[\top, W(B, x)]\,\rangle \qquad [\text{1 out of 2 trade}]\\
&\overset{\text{book}}{\Longleftrightarrow} \quad \mathbf{PN}[\top, W(A, x) \oplus W(B, x)]. \qquad [\text{quasi-sum. trade}] && (47)
\end{aligned}$$

If the left side of Equation (45) is larger than the right, then the above sequence of trades from (46) to (47) is a net-loss sequence of trades. On the other hand, if the left side of Equation (45) is smaller than the right, then the above sequence of trades run backwards from (47) to (46) is a net-loss sequence of trades. □

Proposition 3. *If $\precsim$ violates the Cauchy constraint in a $\oplus$-environment, then $\precsim$ is incoherent in that environment.*

Proof. If $\precsim$ violates the Cauchy constraint then there exists some $A \in \mathbf{St}$ and some $x, y \geq 0$ such that

$$W(A, x \oplus y) \neq W(A, x) \oplus W(A, y). \tag{48}$$

Consider the following sequence of trades, all of which are $(\precsim, \oplus)$-permitted in both directions. (Note: Lemma 1 is used to show that the $\overset{\text{agree}}{\Longleftrightarrow}$ trades are permitted).

$$\begin{aligned}
&\mathbf{PN}[\top, W(A, x \oplus y)] && (49)\\
&\overset{\text{agree}}{\Longleftrightarrow} \quad \mathbf{PN}[A, x \oplus y] \qquad [\text{1 for 1 trade}]\\
&\overset{\text{book}}{\Longleftrightarrow} \quad \langle\, \mathbf{PN}[A, x],\, \mathbf{PN}[A, y]\,\rangle \qquad [\text{quasi-sum. trade}]\\
&\overset{\text{agree}}{\Longleftrightarrow} \quad \langle\, \mathbf{PN}[A, x],\, \mathbf{PN}[\top, W(A, y)]\,\rangle \qquad [\text{1 out of 2 trade}]\\
&\overset{\text{agree}}{\Longleftrightarrow} \quad \langle\, \mathbf{PN}[\top, W(A, x)],\, \mathbf{PN}[\top, W(A, y)]\,\rangle \qquad [\text{1 out of 2 trade}]\\
&\overset{\text{book}}{\Longleftrightarrow} \quad \mathbf{PN}[\top, W(A, x) \oplus W(A, y)]. \qquad [\text{quasi-sum. trade}] && (50)
\end{aligned}$$

If the left side of Equation (48) is greater than the right, then the above sequence of trades from (49) to (50) is a net-loss sequence of trades. On the other hand, If the left side of Equation (48) is less than the right, then the above sequence of trades run backwards from from (50) to (49) is a net-loss sequence of trades. □

Theorem 2. *A belief-payoff preference relation $\precsim$ is coherent in a $\oplus$-environment if and only if it obeys both the disjunctive constraint and the Cauchy constraint in that environment.*

Proof. Propositions 2 and 3 show that, if either constraint is violated, then $\precsim$ is incoherent. Suppose then that both constraints are obeyed.

Define a brokerage account's *appraisal* as follows. If the account contains a single promissory note $\mathbf{PN}[A, x]$, then the account's appraisal is $W(A, x)$. On the other hand, if the account contains two promissory notes, $\mathbf{PN}[A, x]$ and $\mathbf{PN}[B, y]$, then the account's appraisal is $W(A, x) \oplus W(B, y)$.

Examining each of the kinds of permitted trades (Equations (33)–(36)) shows that, if the disjunctive and Cauchy constraints are obeyed, then a permitted trade can never decrease an account's appraisal. So, a *sequence* of permitted trades can never decrease an account's appraisal. Thus, if an account starts with $\mathbf{PN}[\top, Start]$ having appraisal $W(\top, Start) = Start$ (i.e., Property [WE.2] of worth functions) and, after a sequence of permitted trades, becomes $\mathbf{PN}[\top, End]$ having appraisal $W(\top, End) = End$, it can never be the case that $End < Start$.

Thus, if $\precsim$ obeys both types of constraint in a $\oplus$-environment, then $\precsim$ is coherent in that environment. $\square$

What Has Not Been Assumed?

In the proofs of Propositions 2 and 3 and in the proof of Theorem 2, it has not been assumed that the Protagonist and the brokerage understand each other's operations. Consider what each is not assumed to know:

- Belief-payoff preference relation $\precsim$.
  There is no assumption that the brokerage knows the Protagonist's belief-payoff preference relation. (Equivalently, there is no assumption that the brokerage knows the Protagonist's worth function.) The brokerage is required to ask whether the Protagonist agrees to specific trades proposed by the brokerage. The brokerage may not how the Protagonist will answer and may have to guess which trades the Protagonist will agree to.
- Net-loss sequence of trades.
  If the Protagonist's belief-payoff preference relation is incoherent, then there exists a sequence of trades that, if carried out, would result in the Protagonist suffering a net loss of MacGuffin assets. When such a net-loss sequence of trades exists, there is no assumption that the brokerage knows how to find it.
- Quasi-summative operator $\oplus$.
  There is no assumption that the Protagonist knows the brokerage's quasi-summative operator $\oplus$.
- Appraisal.
  Who is it that determines an account's appraisal? It is not the Protagonist and not the brokerage. An account's appraisal is a quantity invented by the author of this paper; it is used by the author to prove Theorem 2. Thus, the author is the appraiser. So, the author knows an account's appraisal. However, there is no necessity for either the brokerage or the Protagonist to know the account's appraisal and, indeed, they may not know. Because the brokerage may not know the Protagonist's worth function, the brokerage may be unable to determine an account's appraisal. Because the Protagonist may not know the brokerage's quasi-summative operator, the Protagonist may be unable to determine an account's appraisal.

The key point made here is that the proofs of Propositions 2 and 3 and Theorem 2 do not require that either the Protagonist or the brokerage have knowledge of how each other operates.

6. Coherent Worth Functions: How Can They Be Expressed?

Theorem 2 has provided us with necessary and sufficient conditions for a belief-payoff preference relation to be coherent. However, it has not provided us with a mathematical expression for the worth function of a coherent belief-payoff preference relation.

Toward the goal of finding a mathematical expression for coherent worth functions, we will now introduce the concept of quasi-multiplication and prove some useful results concerning it.

6.1. Quasi-Multiplication

In addition to defining quasi-summative operators, it will also be useful define quasi-multiplicative operators. These will be used to express a theorem (Theorem 4) about coherent beliefs.

Definition 11. *Suppose that we are given any $f \in \mathcal{MO}$. Define a* quasi-multiplicative *operator $\boxdot_f$ as follows. For all $x, y \in \mathbb{R}_+$, let*

$$x \boxdot_f y = f^{-1}[f(x)f(y)]. \tag{51}$$

The next proposition describes some properties of quasi-multiplicative operators.

Proposition 4. *For any $f \in \mathcal{MO}$, the quasi-multiplicative operator $\boxdot_f$ has the followng properties: For all $x, y, z \in \mathbb{R}_+$:*

- *Commutativity. $x \boxdot_f y = y \boxdot_f x$.*
- *Associativity. $(x \boxdot_f y) \boxdot_f z = x \boxdot_f (y \boxdot_f z)$.*
- *Distributivity with $\oplus_f$. $x \boxdot_f (y \oplus_f z) = (x \boxdot_f y) \oplus_f (x \boxdot_f z)$.*
- *Zero multiple. $x \boxdot_f 0 = 0$.*
- *Unit multiple. $x \boxdot_f 1 = x$.*
- *Special f. When f is the identity function, then $\boxdot_f$ is ordinary multiplication. More generally, if $f(v) = v^\beta$ for any $v \in \mathbb{R}_+$ and where $\beta > 0$, then $\boxdot_f$ is the ordinary multiplication operator.*

Proof. It is straightforward to verify by calculation that $\boxdot_f$ has the above properties. $\square$

We will now use quasi-multiplicative operators to prove an analogue of the Cauchy Functional Equation Theorem ([19], p. 34); an analogue that will be used to prove Proposition 5, which will then be used to prove this paper's main theorem (Theorem 4).

Theorem 3. *Take any $f \in \mathcal{MO}$. Suppose that a function $\mathcal{L}$ maps $\mathbb{R}_+$ into $\mathbb{R}_+$ and is continuous at zero. Then $\mathcal{L}$ has the property*

$$\mathcal{L}(x \oplus_f y) = \mathcal{L}(x) \oplus_f \mathcal{L}(y) \text{ for all } x, y \in \mathbb{R}_+, \tag{52}$$

if and only if

$$\mathcal{L}(z) = \mathcal{L}(1) \boxdot_f z, \text{ for all } z \geq 0. \tag{53}$$

Proof. (If.) Suppose that Equation (53) holds. Take any $x, y \geq 0$ and set $z = x \oplus_f y$ in Equation (53). Then

$$\begin{aligned}
&\mathcal{L}(x \oplus_f y) && (54)\\
&= \mathcal{L}(1) \boxdot_f (x \oplus_f y) && (55)\\
&= [\mathcal{L}(1) \boxdot_f x] \oplus_f [\mathcal{L}(1) \boxdot_f y] \qquad \text{[by distributivity]} && (56)\\
&= \mathcal{L}(x) \oplus_f \mathcal{L}(y). && (57)
\end{aligned}$$

Thus, Equation (52) is satisfied.

(Only if.) Now suppose that $\mathcal{L}$ is continuous at zero and that Equation (52) is satisfied. Recall that $u \oplus_f v = f^{-1}[\,f(u) + f(v)\,]$ for all $u, v \geq 0$. Hence,

$$\mathcal{L}(\,f^{-1}[\,f(u) + f(v)\,]\,) = f^{-1}(\,f[\,\mathcal{L}(u)\,] + f[\,\mathcal{L}(v)\,]\,). \tag{58}$$

Define the function $\mathcal{L}^f : \mathbb{R}_+ \to \mathbb{R}_+$ by:

$$\mathcal{L}^f(z) = f \circ \mathcal{L} \circ f^{-1}(z) \text{ for all } z \in \mathbb{R}_+. \tag{59}$$

Then, for all $u \geq 0$, $f[\mathcal{L}(u)] = \mathcal{L}^f[f(u)]$. Thus Equation (58) may be rewritten:

$$\mathcal{L}(\, f^{-1}[\, f(u) + f(v)\,]\,) = f^{-1}[\, \mathcal{L}^f(f(u)) + \mathcal{L}^f(f(v))\,] \text{ for all } u, v \geq 0. \tag{60}$$

Applying the function f to both sides of Equation (60) yields:

$$\mathcal{L}^f(f(u) + f(v)) = \mathcal{L}^f(f(u)) + \mathcal{L}^f(f(v)) \text{ for all } u, v \geq 0. \tag{61}$$

Let $u' = f(u)$ and $v' = f(v)$. Because $f \in \mathcal{MO}$ and, thus, maps $\mathbb{R}_+$ *onto* $\mathbb{R}_+$, it follows from the above equation that

$$\mathcal{L}^f(u' + v') = \mathcal{L}^f(u') + \mathcal{L}^f(v') \text{ for all } u', v' \geq 0. \tag{62}$$

Because f and f^{-1} are continuous and $f^{-1}(0) = 0$ and $\mathcal{L}$ is continuous at $z = 0$, it follows that $\mathcal{L}^f$ is continuous at $z = 0$. Because Equation (62) holds and because $\mathcal{L}^f$ is continuous at zero, we may apply the Cauchy Functional Equation Theorem ([19], p. 34). That theorem implies that there exists some constant c' such that,

$$\mathcal{L}^f(z') = c'z' \text{ for all } z' \in \mathbb{R}_+. \tag{63}$$

Thus, because f maps $\mathbb{R}_+$ into $\mathbb{R}_+$, we may set $z' = f(z)$ in the above equation, yielding:

$$\mathcal{L}^f(f(z)) = c' f(z) \text{ for all } z \in \mathbb{R}_+. \tag{64}$$

Moreover, c' must be nonnegative because $\mathcal{L}^f(z')$ is nonnegative and because $f(z)$ is positive for $z > 0$. Thus, $c' \in \mathbb{R}_+$. Choose c such that $c' = f(c)$. Then, applying the function f^{-1} to both sides of Equation (63) after replacing c' with $f(c)$, we obtain:

$$f^{-1}[\, \mathcal{L}^f[\, f(z)\,]\,] = f^{-1}[\, f(c) f(z)\,] \text{ for all } z \in \mathbb{R}_+. \tag{65}$$

In other words,

$$\mathcal{L}(z) = c \boxdot_f z \text{ for all } z \in \mathbb{R}_+. \tag{66}$$

In particular,

$$\mathcal{L}(1) = c \boxdot_f 1 = c. \tag{67}$$

Then Equations (66) and (67) imply Equation (53), completing the proof. □

6.2. Cauchy-Constrained Worth Functions

Proposition 3 has shown that, for $\precsim$ to be coherent, its worth function must obey the Cauchy constraint. We will now use quasi-multiplication to prove a result concerning the form of Cauchy-constrained worth functions in a $\oplus_f$-environment.

Proposition 5. *Suppose that $\mathcal{W} : \mathcal{PN} \to \mathbb{R}_+$ is a worth function and that $f \in \mathcal{MO}$. Then $\mathcal{W}$ obeys the Cauchy constraint (Equation (38)) in the $\oplus_f$-environment if and only if, for all $A \in$ **St**,*

$$\mathcal{W}(\, \mathbf{PN}[A, z]\,) = \mathcal{W}(\, \mathbf{PN}[A, 1]\,) \boxdot_f z, \textit{ for all } z \geq 0. \tag{68}$$

Proof. Take any $A \in \mathbf{St}$ and hold it fixed. Then, by Equation (17) of the definition of *worth function* (Definition 5),

$$0 \leq \mathcal{W}(\, \mathbf{PN}[A, z]\,) \leq z \text{ for all } z \geq 0. \tag{69}$$

Therefore, $\mathcal{W}(\, \mathbf{PN}[A, z]\,)$, considered a function of z, is continuous at $z = 0$. Setting $\mathcal{L}(z) = \mathcal{W}(\, \mathbf{PN}[A, z]\,)$ for all $z \geq 0$ and applying Theorem 3 reveals that Equation (38) is satisfied if and only if Equation (68) is satisfied. □

Next we turn to finding an expression for $\mathcal{W}(\, \mathbf{PN}[A, 1]\,)$ in Equation (68) when the disjunctive constraint is obeyed. For that purpose, the concept of quasi-probability will be useful.

6.3. Quasi-Probability

Definition 12. *Suppose that $\mathfrak{BA} = (\mathbf{St}, \wedge, \vee, \neg, \perp, \top)$ is a finite Boolean algebra and that $\oplus$ is a quasi-summative operator. A function $\mathcal{QP} : \mathbf{St} \to [0,1]$ is called a* quasi-probability $\oplus$-measure *on $\mathfrak{BA}$ if it has the following properties.*

- $\mathcal{QP}(\perp) = 0$.
- $\mathcal{QP}(\top) = 1$.
- *For all $A, B \in \mathbf{St}$, if A and B are incompatible, then*

$$\mathcal{QP}(A \vee B) = \mathcal{QP}(A) \oplus \mathcal{QP}(B). \tag{70}$$

In the special case where the quasi-summative operator $\oplus$ is ordinary addition (i.e., $\oplus$ is $\oplus^{\text{add}}$), then $\mathcal{QP}$ is an ordinary Boolean probability measure, as studied in [24].

6.4. Theorem on Coherent Worth Functions

This section presents the main theorem of this paper (Theorem 4).

Notation 8. *Suppose that $\oplus$ is a quasi-summative operator and that f is the unique function in $\mathcal{MO}$ such that $\oplus = \oplus_f$. Sometimes it is desirable to suppress reference to f in our notation. When that is the case, $\boxdot_f$ may be denoted $\boxdot^{\oplus}$.*

Recall the following: Given a belief-payoff preference relation $\precsim$ and any $A \in \mathbf{St}$, Definition 4 defined $Blf_{\precsim}(A)$, the *degree of belief* in A. Specifically, $Blf_{\precsim}(A)$ is the unique value of z that $\mathbf{PN}[\top, z] \sim \mathbf{PN}[A, 1]$. If the worth function $\mathcal{W}$ represents $\precsim$ then by Proposition 1, $Blf_{\precsim}(A) = \mathcal{W}(\,\mathbf{PN}[A,1]\,)$.

Theorem 4. *Suppose that $\precsim$ is a belief-payoff preference relation on $\mathcal{PN}$ and that it is represented by the worth function $\mathcal{W}$. Suppose further that $\oplus$ is a quasi-summative operator. Then $\precsim$ is coherent in the $\oplus$-environment if and only if, first,*

$$\mathcal{W}(\,\mathbf{PN}[A,x]\,) = Blf_{\precsim}(A) \boxdot^{\oplus} x, \textit{ for all } A \in \mathbf{St} \textit{ and all } x \geq 0 \tag{71}$$

and, second, the belief function $Blf_{\precsim}$ is a quasi-probability $\oplus$-measure.

Proof. By Theorem 2, $\precsim$ is coherent in the $\oplus$-environment if and only if it obeys both the Cauchy constraint and the disjunctive constraint (Definition 10) in the $\oplus$-environment.

Now, $\precsim$ obeys the Cauchy constraint if and only if Equation (68) of Proposition 5 holds. However, that equation may be rewritten by replacing $\oplus_f$ and $\boxdot_f$ with $\oplus$ and $\boxdot^{\oplus}$ respectively and by replacing $\mathcal{W}(\,\mathbf{PN}[A,1]\,)$ with $Blf_{\precsim}(A)$ (as justified by Proposition 1). The result of rewriting Equation (68) is Equation (71).

Throughout the remainder of this proof, assume that the Cauchy constraint is obeyed and, thus, Equation (71) holds. It must now be shown that $\precsim$ obeys the disjunctive constraint in the $\oplus$-environment if and only if $Blf_{\precsim}$ is a quasi-probability $\oplus$-measure.

Suppose that $Blf_{\precsim}$ is a quasi-probability $\oplus$-measure. It will be shown that the disjunctive constraint is satisfied. Consider any incompatible $A, B \in \mathbf{St}$ and any $x \geq 0$. Then,

$$\begin{aligned}
&\mathcal{W}(\,\mathbf{PN}[A \vee B, x]\,) && (72)\\
&= Blf_{\precsim}(A \vee B) \boxdot^{\oplus} x \qquad [\text{by } (71)] && (73)\\
&= [\,Blf_{\precsim}(A) \oplus Blf_{\precsim}(B)\,] \boxdot^{\oplus} x \qquad [\text{by (70) of Definition 12}] && (74)\\
&= [\,Blf_{\precsim}(A) \boxdot^{\oplus} x\,] \oplus [\,Blf_{\precsim}(B) \boxdot^{\oplus} x\,] \qquad [\text{by distributivity}] && (75)\\
&= \mathcal{W}(\,\mathbf{PN}[A,x]\,) \oplus \mathcal{W}(\,\mathbf{PN}[B,x]\,) \qquad [\text{by } (71)]. && (76)
\end{aligned}$$

Thus the disjunctive constraint (Equation (37) of Definition 10) is satisfied.

Conversely, suppose that the disjunctive constraint is satisfied. Recall from Proposition 1 that

$$Blf_{\precsim}(A) = \mathcal{W}(\,\mathbf{PN}[A,1]\,), \text{ for all } A \in \mathbf{St}. \tag{77}$$

It will be shown that $Blf_{\precsim}$ is a quasi-probability $\oplus$-measure. Because the disjunctive constraint is obeyed, it follows that, for any incompatible $A, B \in \mathbf{St}$,

$$Blf_{\precsim}(A \vee B) = \mathcal{W}(\,\mathbf{PN}[A \vee B, 1]\,) \tag{78}$$
$$= \mathcal{W}(\,\mathbf{PN}[A,1]\,) \oplus \mathcal{W}(\,\mathbf{PN}[B,1]\,) \quad \text{[disj. constraint]} \tag{79}$$
$$= Blf_{\precsim}(A) \oplus Blf_{\precsim}(B). \tag{80}$$

Moreover, from the definition of *worth function* (Definition 5),

$$Blf_{\precsim}(\bot) = \mathcal{W}(\,\mathbf{PN}[\bot,1]\,) = 0 \text{ and } Blf_{\precsim}(\top) = \mathcal{W}(\,\mathbf{PN}[\top,1]\,) = 1. \tag{81}$$

Thus, by Definition 12, $Blf_{\precsim}$ is a quasi-probability $\oplus$-measure. $\square$

6.5. Coherent Belief

In Definition 9, we defined what it means for a belief-payoff preference relation and the worth function that represents it to be coherent. However, we have not yet defined what it means for a *belief function* to be coherent. Motivated by Theorem 4, we will now define coherence of belief functions.

Definition 13. *Suppose that $\precsim$ is a belief-payoff preference relation and that $\oplus$ is a quasi-summative operator. Then the belief function $Blf_{\precsim}$ is said to be* coherent *in a $\oplus$-environment if $Blf_{\precsim}$ is a quasi-probability $\oplus$-measure.*

Thus, by Theorem 4, if $\precsim$ is coherent in a $\oplus$-environment, then $Blf_{\precsim}$ is coherent in that environment. However, $Blf_{\precsim}$ could be coherent in that environment even though $\precsim$ was incoherent. This state of affairs could come about because Equation (71) of Theorem 4 failed to hold even though $Blf_{\precsim}$ was a quasi-probability $\oplus$-measure.

Theorem 4 showed that, in an environment where MacGuffin quantities combine via the quasi-summative operator $\oplus$, coherent belief in incompatible propositions must also combine (Equation (70)) via the quasi-summative operator $\oplus$. In the special case where the MacGuffin resource is money, which combines additively, coherent belief must combine additively.

Thus, in a $\oplus$-environment, where $\oplus$ is not ordinary addition, coherent belief conforms to a quasi-probability $\oplus$-measure rather than to a probability measure.

6.6. Quasi-Expectation

Suppose that $\precsim$ is coherent in the $\oplus$-environment and is represented by the worth function $\mathcal{W}$. Then Theorem 4 implies that there exists a quasi-probability $\oplus$-measure $\mathcal{QP}$, namely $Blf_{\precsim}$, such that, given any $A \in \mathbf{St}$ and any $x \geq 0$,

$$\mathcal{W}(\,\mathbf{PN}[A,x]\,) = \mathcal{QP}(A) \boxdot^{\oplus} x. \tag{82}$$

Recall that the promissory note $\mathbf{PN}[A,x]$ pays off x MacGuffin units if the proposition A is true and zero units otherwise. The right side of the above equation equals the payoff x quasi-multiplied by the quasi-probability of A. Thus, it may be interpreted as the quasi-expectation of $\mathbf{PN}[A,x]$. So the above equation says that worth of the promissory note $\mathbf{PN}[A,x]$ is its quasi-expectation.

6.7. An Aside: Proving vs. Assuming

Consider the special case where quasi-summative operator $\oplus$ is ordinary addition; that is, $x \oplus y = x + y$ for all $x, y \in \mathbb{R}_+$. In that case, $\boxdot^{\oplus}$ is ordinary multiplication. Suppose that the belief-payoff preference relation $\precsim$ is coherent in the $\oplus$-environment. Suppose that $\mathcal{W}$ is the unique worth function that represents $\precsim$. Then, by Theorem 4, Equation (71) must hold. When $\boxdot^{\oplus}$ is ordinary multiplication, that equation becomes: For all $A \in \mathbf{St}$ and all $x \geq 0$,

$$\mathcal{W}(\,\mathbf{PN}[A, x]\,) = Blf_{\precsim}(A)\,x. \tag{83}$$

Hence, applying Property [WE.2] of worth functions,

$$\mathcal{W}(\,\mathbf{PN}[\top, Blf_{\precsim}(A)\,x]\,) = Blf_{\precsim}(A)\,x = \mathcal{W}(\,\mathbf{PN}[A, x]\,). \tag{84}$$

Then, because $\mathcal{W}$ represents $\precsim$

$$\mathbf{PN}[A, x] \sim \mathbf{PN}[\top, Blf_{\precsim}(A)\,x]. \tag{85}$$

Consider the case where the MacGuffin resource is money (which, of course, combines additively) and where, the unit is \$1. Then, Equation (85) implies that, for each $A \in \mathbf{St}$, there exists a number $v_A \geq 0$ such that, for all $x \geq 0$, the Protagonist is indifferent between being given either of the following:

The promise to pay \$$x$ on the condition that A is true. (86)

The promise to pay \$$v_A x$ unconditionally. (87)

The indifference between (86) and (87) across all $x \geq 0$ is often *assumed* in the Dutch Book literature, for example, ([2], p. 55). Here, however, that indifference is not assumed. Rather, the indifference between (86) and (87) across all $x \geq 0$ has been *proved* to be a necessary condition for coherence in the $\oplus$-environment when $\oplus$ is ordinary addition.

7. Belief-Payoff Preference Relations That Are Coherent in One Environment May Be Incoherent in Another

Recall the situation in Section 4.3.2. There is a wine merchant who gives wines quality ratings and those quality ratings may be any non-negative real number.

Suppose that the MacGuffin resource is wine or, more accurately, *wine quality*. Let $\mathbf{PN}[A, x]$ denote the promissory note that pays off with a bottle of wine having quality x if the proposition A is true and pays off with a bottle of wine having quality zero if A is not true. (Recall that being given a bottle of wine having quality rating zero is equivalent to being given nothing at all.) Let $\mathcal{PN}_{\text{wine}}$ denote the set of all promissory notes $\mathbf{PN}[A, x]$, where $A \in \mathbf{St}$ and $x \geq 0$. As with earlier notation, let the Protagonist's belief-payoff preference relation over $\mathcal{PN}_{\text{wine}}$ be denoted $\precsim$ and let $Blf_{\precsim}$ denote the Protagonist's belief function (Definition 4) under $\precsim$.

Is the Protagonist's belief-payoff preference relation $\precsim$ coherent? To answer that question, we must specify the $\oplus$-environment in which the coherence of $\precsim$ is to be evaluated. In other words, we must specify a quasi-summative operator $\oplus$. Recall from Section 4.3.2 that, for the wine merchant, the amount of enjoyment obtained from a bottle of wine having a quality rating of x is $w(x)$, where w is a function in $\mathcal{MO}$. Thus, for the merchant, the amount of enjoyment obtained from two bottles of wine, one with rating x and one with rating y, is the same amount of enjoyment as would be obtained from a single bottle of wine having rating $w^{-1}[w(x) + w(y)]$. So, the merchant's wine-quality quasi-summative operator is $\oplus_w$.

Suppose, now, that the wine merchant sets up a brokerage that deals in the promissory notes of $\mathcal{PN}_{\text{wine}}$. Suppose, further, that the brokerage employs $\oplus_w$ as its wine-quality quasi-summative operator. In other words, the wine merchant's brokerage is a $\oplus_w$-environment. Then, by Theorem 4, the Protagonist's belief-payoff preference relation will be coherent in the $\oplus_w$-environment only if the Protagonist's belief function $Blf_{\precsim}$ is a quasi-probability $\oplus_w$-measure.

Now, suppose that a second wine merchant sets up shop. The second merchant gives wines exactly the same quality ratings as the first merchant. However, for the second merchant, the amount of enjoyment obtained from a bottle of wine having quality rating x is $v(x)$, where v is a function in $\mathcal{MO}$ that is different from w. So, the second wine merchant's wine-quality quasi-summative operator is $\oplus_v$.

Now imagine that the second wine merchant also sets up a brokerage that deals with the promissory notes of $\mathcal{PN}_{\text{wine}}$. The new brokerage adopts $\oplus_v$ as its wine-quality quasi-summative operator and, thus the brokerage is a $\oplus_v$-environment. By Theorem 4, the Protagonist's belief-payoff preference relation will be coherent in the $\oplus_v$-environment only if the Protagonist's belief function $Blf_{\precsim}$ is a quasi-probability $\oplus_v$-measure.

Now, it is possible for a belief function to be both a quasi-probability $\oplus_w$-measure and a quasi-probability $\oplus_v$-measure. However, as a general rule, if the belief function is a quasi-probability $\oplus_w$-measure, then it is not a quasi-probability $\oplus_v$-measure and vice versa.

This shows that a belief function may be coherent in one environment, but not in another.

Implication. One cannot say that a belief function is coherent, unless one has specified a $\oplus$-environment in which it is coherent.

7.1. Dutch Book Coherence Is an Exogenous Standard

If one wants to evaluate the coherence of the Protagonist's belief-payoff preference relation $\precsim$ one must bring in an external organization, the brokerage, to participate in the evaluation. What the brokerage brings to the evaluation is the quasi-summative operator $\oplus$. Once one has $\oplus$ in hand, one can evaluate whether $\precsim$ is coherent in the $\oplus$-environment.

Thus, the Protagonist supplies $\precsim$; the brokerage supplies $\oplus$. In other words, the brokerage supplies the grading standard by which the Protagonist's belief-payoff preference relation is evaluated for coherence. However, as has just been seen, more than one grading standard exists.

7.2. Can the Protagonist Supply Its Own Grading Standard?

Does the Protagonist have its own quasi-summative operator? Possibly, yes; possibly, no. The Protagonist could very well have a personal belief-payoff preference relation without having a personal quasi-summative operator.

Suppose, however, that the Protagonist does have both a personal belief-payoff preference relation $\precsim^{\text{Prot}}$ and a personal quasi-summative operator $\oplus^{\text{Prot}}$. If so, then we might ask the brokerage to employ $\oplus^{\text{Prot}}$ as its quasi-summative operator. Then we could evaluate the coherence of $\precsim^{\text{Prot}}$ within the $\oplus^{\text{Prot}}$-environment.

However, by so doing, we would not be evaluating the coherence of $\precsim^{\text{Prot}}$, but the coherence of the pair $(\precsim^{\text{Prot}}, \oplus^{\text{Prot}})$.

8. The Usual Dutch Book Argument Claims Too Much

The Dutch Book Argument addresses the problem of determining whether someone's partial beliefs are coherent/rational. This is a problem that has generated much interest as shown by the extensive literature (reviewed in [5–7]) on the Dutch Book Argument.

The problem with the usual Dutch Book Argument is that it implicitly assumes that, when studying coherence of beliefs, the only relevant resource is money and, because, money combines additively, the only relevant environment is the $\oplus^{\text{add}}$-environment. In other words, the usual Dutch Book Argument treats the $\oplus^{\text{add}}$-environment as though it were a *universal* environment and implicitly assumes that, if beliefs are coherent in that environment, they are coherent everywhere.

Nevertheless, although the money-based $\oplus^{\text{add}}$-environment is easy to describe and is mathematically convenient, it is not the only possible resource environment. There are other environments. Thus, Dutch Book Argument's consideration of only the $\oplus^{\text{add}}$-environment is arbitrary.

The standard Dutch Book Argument finds necessary and sufficient conditions for partial beliefs to be coherent in a $\oplus^{\text{add}}$-environment. However, it fails to consider that other environments exist. As shown in Section 7, beliefs that are coherent in one environment may be incoherent in another. Consequently, the Dutch Book Argument treats the necessary and sufficient conditions for coherence in a $\oplus^{\text{add}}$-environment as though they were universal conditions for coherence and not merely conditions for coherence in a specific environment.

Thus the standard Dutch Book Argument claims too much. It claims that its criterion for evaluating coherence of partial belief is applicable everywhere, when in fact that criterion is only applicable in environments where resources combine additively.

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Appendix A. Representation of Belief-Payoff Preference Relations by Worth Functions

How to construct a belief-payoff preference relation from a worth function

Definition A1. *Suppose that $\mathcal{W} : \mathcal{PN} \to \mathbb{R}_+$ is a worth function. Define binary relations $\precsim_{\mathcal{W}}$, $\prec_{\mathcal{W}}$, and $\sim_{\mathcal{W}}$ on $\mathcal{PN}$ as follows. For all propositions* **PN** *and* **PN**$'$ *in $\mathcal{PN}$, let*

$$\mathbf{PN} \precsim_{\mathcal{W}} \mathbf{PN}' \quad \textit{iff} \quad \mathcal{W}(\mathbf{PN}) \leq \mathcal{W}(\mathbf{PN}'); \tag{A1}$$

$$\mathbf{PN} \prec_{\mathcal{W}} \mathbf{PN}' \quad \textit{iff} \quad \mathcal{W}(\mathbf{PN}) < \mathcal{W}(\mathbf{PN}'); \tag{A2}$$

$$\mathbf{PN} \sim_{\mathcal{W}} \mathbf{PN}' \quad \textit{iff} \quad \mathcal{W}(\mathbf{PN}) = \mathcal{W}(\mathbf{PN}'). \tag{A3}$$

Remark A1. *If the binary relation $\precsim_{\mathcal{W}}$ defined in Definition A1 turns out to be a belief-payoff preference relation, then it is, of course, represented by $\mathcal{W}$ because Equation (A1) holds.*

Proposition A1. *Given a worth function $\mathcal{W}$, the binary relation $\precsim_{\mathcal{W}}$ constructed in Definition A1 is a belief-payoff preference relation on $\mathcal{PN}$ that is represented by $\mathcal{W}$.*

Proof. It is evident that $\precsim_{\mathcal{W}}$ is a weak order.

Because $\mathcal{W}$ is a worth function, it has properties [WE.1], [WE.2] and [WG] of Definition 5. Those properties of $\mathcal{W}$ will be used to demonstrate that $\precsim_{\mathcal{W}}$ has properties [PEZ], [PEM], [PG], and [CE] of Definition 3 and, thus, that $\precsim_{\mathcal{W}}$ is a belief-payoff preference relation.

- Demonstration of PEZ.
  From [WE.1] and [WE.2],

$$\mathcal{W}(\mathbf{PN}[\bot, 0]) = 0 = \mathcal{W}(\mathbf{PN}[\top, 0]) \tag{A4}$$

  and, so,

$$\mathbf{PN}[\bot, 0] \sim_{\mathcal{W}} \mathbf{PN}[\top, 0]. \tag{A5}$$

- Demonstration of PEM.
  Suppose $0 \leq x < y$. Then, by [WE.1] and [WE.2],

$$\mathcal{W}(\mathbf{PN}[\bot, x]) = \mathcal{W}(\mathbf{PN}[\bot, y]) \text{ and } \mathcal{W}(\mathbf{PN}[\top, x]) < \mathcal{W}(\mathbf{PN}[\top, y]). \tag{A6}$$

  Therefore,

$$\mathbf{PN}[\bot, x] \sim_{\mathcal{W}} \mathbf{PN}[\bot, y] \text{ and } \mathbf{PN}[\top, x] \prec_{\mathcal{W}} \mathbf{PN}[\top, y]. \tag{A7}$$

- Demonstration of PG.
  Take any $A \in \mathbf{St}$ and any $x \geq 0$. Then, by [WE.1], [WG], and [WE.2],

$$\mathcal{W}(\mathbf{PN}[\perp, x]) = 0 \leq \mathcal{W}(\mathbf{PN}[A, x]) \leq x = \mathcal{W}(\mathbf{PN}[\top, x]). \quad \text{(A8)}$$

  Therefore,

$$\mathbf{PN}[\perp, x] \precsim_{\mathcal{W}} \mathbf{PN}[A, x] \precsim_{\mathcal{W}} \mathbf{PN}[\top, x]. \quad \text{(A9)}$$

- Demonstration of CE.
  Consider any promissory note $\mathbf{PN} \in \mathcal{PN}$. We will show that it has a certainty equivalent. This is done as follows. By [WE.2], for all $x \geq 0$, $\mathcal{W}(\mathbf{PN}[\top, x]) = x$. Setting $x = \mathcal{W}(\mathbf{PN})$ yields:

$$\mathcal{W}(\mathbf{PN}[\top, \mathcal{W}(\mathbf{PN})\,]) = \mathcal{W}(\mathbf{PN}). \quad \text{(A10)}$$

  Hence, it follows from Equation (A3) of Definition A1 that

$$\mathbf{PN}[\top, \mathcal{W}(\mathbf{PN})\,] \sim_{\mathcal{W}} \mathbf{PN}. \quad \text{(A11)}$$

  Thus, the left side of Equation (A11) is the certainty equivalent of the right side.

We have now shown that $\precsim_{\mathcal{W}}$ is a belief-payoff preference relation. Therefore, by Remark A1, it is represented by $\mathcal{W}$. □

How to construct a worth function from a belief-payoff preference relation

Definition A2. *Suppose that $\precsim$ is a belief-payoff preference relation. Define a function*

$$\mathcal{U}^{\precsim} : \mathcal{PN} \to \mathbb{R}_+ \quad \text{(A12)}$$

as follows. From the [CE] property of belief-payoff preference relations and from Remark 2, we know that, for each $\mathbf{PN} \in \mathcal{PN}$, there exists a unique $z \in \mathbb{R}_+$ such that $\mathbf{PN}[\top, z] \sim \mathbf{PN}$. Set $\mathcal{U}^{\precsim}(\mathbf{PN}) = z$.

So, $\mathcal{U}^{\precsim}(\mathbf{PN}) = z$ where $\mathbf{PN} \sim \mathbf{PN}[\top, z]$. Hence, for all $\mathbf{PN} \in \mathcal{PN}$,

$$\mathbf{PN}[\top, \mathcal{U}^{\precsim}(\mathbf{PN})] \sim \mathbf{PN}. \quad \text{(A13)}$$

From the above equation and the monotonicity property [PEM] of Definition 3, it follows that, for all $\mathbf{PN}, \mathbf{PN}' \in \mathcal{PN}$,

$$\mathbf{PN} \precsim \mathbf{PN}' \text{ iff } \mathcal{U}^{\precsim}(\mathbf{PN}) \leq \mathcal{U}^{\precsim}(\mathbf{PN}') \quad \text{(A14)}$$

and

$$\mathbf{PN} \sim \mathbf{PN}' \text{ iff } \mathcal{U}^{\precsim}(\mathbf{PN}) = \mathcal{U}^{\precsim}(\mathbf{PN}'). \quad \text{(A15)}$$

Remark A2. *Given a belief-payoff preference relation $\precsim$ if the function $\mathcal{U}^{\precsim}$ constructed in Definition A2 is a worth function, then that worth function represents $\precsim$ because Equation (A14) holds.*

Proposition A2. *If $\precsim$ is a belief-payoff preference relation, then $\mathcal{U}^{\precsim}$ is a worth function that represents $\precsim$.*

Proof. By Remark A2, if $\mathcal{U}^{\precsim}$ is a worth function, then it represents $\precsim$.

Because $\precsim$ is a belief-payoff preference relation, it has properties [PEZ], [PEM], [PG], and [CE] of Definition 3. Those properties of $\precsim$ will be used to demonstrate that $\mathcal{U}^{\precsim}$ has properties [WE.1], [WE.2]and [WG] of Definition 5 and, thus, that $\mathcal{U}^{\precsim}$ is a worth function.

- Demonstration of WE.1.
  Take any $y \geq 0$. In Equation (A13), replace **PN** with $\mathbf{PN}[\bot, y]$ yielding:
  $$\mathbf{PN}[\top, \mathcal{U}^{\precsim}(\mathbf{PN}[\bot, y])] \sim \mathbf{PN}[\bot, y]. \tag{A16}$$
  However, by [PEM] and [PEZ],
  $$\mathbf{PN}[\bot, y] \sim \mathbf{PN}[\bot, 0] \sim \mathbf{PN}[\top, 0]. \tag{A17}$$
  Thus, from the above two equations,
  $$\mathbf{PN}[\top, \mathcal{U}^{\precsim}(\mathbf{PN}[\bot, y])] \sim \mathbf{PN}[\top, 0]. \tag{A18}$$
  So, by Remark 2,
  $$\mathcal{U}^{\precsim}(\,\mathbf{PN}[\bot, y]\,) = 0. \tag{A19}$$
- Demonstration of WE.2.
  Take any $x \geq 0$. In Equation (A13), replace **PN** with $\mathbf{PN}[\top, x]$ yielding:
  $$\mathbf{PN}[\top, \mathcal{U}^{\precsim}(\mathbf{PN}[\top, x])] \sim \mathbf{PN}[\top, x]. \tag{A20}$$
  So, by Remark 2,
  $$\mathcal{U}^{\precsim}(\mathbf{PN}[\top, x]) = x. \tag{A21}$$
- Demonstration of WG.
  Take any $A \in \mathbf{St}$ and any $x \geq 0$. From the definition of $\mathcal{U}^{\precsim}$, $\mathcal{U}^{\precsim}(\mathbf{PN}[A, x])$ must be non-negative. Next, by [PG], $\mathbf{PN}[A, x] \precsim \mathbf{PN}[\top, x]$. Applying Equations (A14) and A21 yields: $\mathcal{U}^{\precsim}(\mathbf{PN}[A, x]) \leq \mathcal{U}^{\precsim}(\mathbf{PN}[\top, x]) = x$.

□

Representation

Theorem A1. *Representation Theorem.*

R1. *Every worth function represents some belief-payoff preference relation.*

R2. *Every belief-payoff preference relation is represented by some worth function, and*

R3. *that worth function is unique.*

Proof.

- Demonstration of R1.
  This is an immediate consequence of Proposition A1.
- Demonstration of R2.
  This is an immediate consequence of Proposition A2.
- Demonstration of R3.
  Suppose that $\precsim$ is a belief-payoff preference relation on $\mathcal{PN}$ and suppose that $\mathcal{U}$ and $\mathcal{W}$ are both worth functions that represent $\precsim$. It will be shown that $\mathcal{U} = \mathcal{W}$ because, for every $\mathbf{PN} \in \mathcal{PN}$,
  $$\mathcal{U}(\mathbf{PN}) = \mathcal{W}(\mathbf{PN}). \tag{A22}$$
  To show Equation (A22), begin by taking any $\mathbf{PN} \in \mathcal{PN}$. By Property [CE] of belief-payoff preference relations, there exists a $z \geq 0$ such that $\mathbf{PN} \sim \mathbf{PN}[\top, z]$ and, by [PEM], that z is unique. So, because $\mathcal{U}$ and $\mathcal{W}$ represent $\precsim$,
  $$\mathcal{U}(\mathbf{PN}) = \mathcal{U}(\mathbf{PN}[\top, z]) \text{ and } \mathcal{W}(\mathbf{PN}) = \mathcal{W}(\mathbf{PN}[\top, z]). \tag{A23}$$
  However, by Property [WE.2] of worth functions, $\mathcal{U}(\mathbf{PN}[\top, z]) = z = \mathcal{W}(\mathbf{PN}[\top, z])$. Thus, $\mathcal{U}(\mathbf{PN}) = \mathcal{W}(\mathbf{PN})$ and, so, $\mathcal{U} = \mathcal{W}$.

□

References

1. Ramsey, F.P. Truth and probability (1926). In *The Foundations of Mathematics and Other Logical Essays*; Braithwaite, R.B., Ed.; Kegan Paul, Trench, Trubner & Co., Ltd.: London, UK, 1931; pp. 156–198. Available online: https://archive.org/details/in.ernet.dli.2015.218361/page/n177/mode/2up (accessed on 17 February 2022).
2. Gillies, D. *Philosophical Theories of Probability*; Routledge: New York, NY, USA, 2000.
3. de Finetti, B. On the subjective meaning of probability. In *Induction and Probability*; Monari, P., Cocchi, D., Eds.; CLUEB: Bologna, Italy, 1993; pp. 291–321.
4. de Finetti, B. Foresight: Its logical laws, its subjective sources. In *Studies in Subjective Probability*, 2nd ed.; Kyburg, H.E., Jr., Smokler, H.E., Eds.; Robert E. Krieger: Huntington, NY, USA, 1980; pp. 53–118.
5. Hájek, A. Dutch Book Arguments. In *The Handbook of Rational and Social Choice*; Anand, P., Pattanaik, P., Puppe, C., Eds.; Oxford University Press: Oxford, UK, 2009; pp. 173–195.
6. Vineberg, S. Dutch Book Arguments. In *Stanford Encyclopedia of Philosophy*, Spring 2016 ed.; Zalta, E.N., Ed.; Metaphysics Research Lab, Philosophy Department, Stanford University: Stanford, CA, USA, 2016. Available online: https://plato.stanford.edu/archives/spr2016/entries/dutch-book/ (accessed on 6 September 2021).
7. Pettigrew, R. *Dutch Book Arguments*; Cambridge University Press: Cambridge, UK, 2020.
8. Wakker, P.P. History of the Term Dutch Book. 2011. Available online: https://personal.eur.nl/wakker/miscella/dutchbk.htm (accessed on 12 February 2022).
9. de Finetti, B. Probability: Beware of falsifications! In *Studies in Subjective Probability*, 2nd ed.; Kyburg, H.E., Jr., Smokler, H.E., Eds.; Robert E. Krieger: Huntington, NY, USA, 1980; pp. 193–224.
10. Douglas, P. Dissing the Dutch: All's Fair in Love and War. Available online: https://www.newnetherlandinstitute.org/files/2613/6700/9122/DISSING_THE_DUTCH.pdf (accessed on 23 January 2022).
11. Anglo-Dutch Wars. Available online: https://en.wikipedia.org/wiki/Anglo-Dutch_Wars (accessed on 9 February 2022).
12. *The Compact Edition of the Oxford English Dictionary*; Oxford University Press: Oxford, UK, 1971.
13. Diaconis, P.; Skyrms, B. *Ten Great Ideas About Chance*; Princeton University Press: Princeton, NJ, USA, 2018.
14. de Finetti, B. *Theory of Probability: A Critical Introductory Treatment*; Wiley: London, UK, 1974; Volume 1.
15. Truffaut, F. *Hitchcock*, revised ed.; Simon and Schuster: New York, NY, USA, 1984.
16. Mendelson, E. *Schaum's Outline of Theory and Problems of Boolean Algebra and Switching Circuits*; McGraw-Hill: New York, NY, USA, 1970.
17. Krantz, D. H.; Luce, R. D.; Suppes, P.; Tversky, A. *Foundations of Measurement—Volume I: Additive and Polynomial Representations*; Academic Press: New York, NY, USA, 1971.
18. Aczél, J. Sur les opérations définies pour nombres réels. *Bull. de la Soc. Math. de France* **1948**, *76*, 59–64. [CrossRef]
19. Aczél, J. *Lectures on Functional Equations and Their Applications*; Academic Press: New York, NY, USA, 1966.
20. Alsina, C.; Frank, M. J.; Schweizer, B. *Associative Functions: Triangular Norms and Copulas*; World Scientific Publishing: Singapore, 2006.
21. Sohrab, H. H. *Basic Real Analysis*, 2nd ed.; Birkhäuser: New York, NY, USA, 2014.
22. Blizard, W. D. Multiset theory. *Notre Dame J. Form. Log.* **1989**, *30*, 36–66. [CrossRef]
23. Blizard, W. D. The development of multiset theory. *Mod. Log.* **1991**, *1*, 319–352.
24. Onicescu, O.; Cuculescu, I. *Probability Theory on Boolean Algebras of Events*; Editura Academiei Republicii Socialiste România: Bucharest, Romania, 1976.