


Article

# Family of Distributions Derived from Whittaker Function

Maha A. Omair, Yusra A. Tashkandy, Sameh Askar \*  and Abdulhamid A. Alzaid

Department of Statistics and Operations Research, College of Sciences, King Saud University, Riyadh 145111, Saudi Arabia; maomair@ksu.edu.sa (M.A.O.); ytashkandi@ksu.edu.sa (Y.A.T.); alzaid@ksu.edu.sa (A.A.A.)

\* Correspondence: saskar@ksu.edu.sa

**Abstract:** In this paper, we introduce a general family of distributions based on Whittaker function. The properties of obtained distributions, moments, ordering, percentiles, and unimodality are studied. The distributions' parameters are estimated using methods of moments and maximum likelihood. Furthermore, a generalization of Whittaker distribution that contains a wider class of distributions is developed. Validation of the obtained results is applied to real life data containing four data sets.

**Keywords:** Whittaker function; confluent hypergeometric series; Whittaker distribution; generalized Whittaker distribution

## 1. Introduction

There is no single definition of “special functions”, as the term has been used for a variety of functions. They are important to many mathematicians, such as mathematical analysis, functional analysis, applied mathematics, and distribution theory. Such functions can be written as a summation or an integration. Examples include the relatively exponential, gamma, Bessel, hypergeometric, and Whittaker functions. For more instances of special functions, see [1–3]. Most of these functions are non-negative and finite over a certain range, and thus can be used to define probability distributions. Typical examples are the Poisson, gamma, and Bessel distributions.

There was an early interest in special functions in the theory of distribution. This included studies by [4,5], who introduced Bessel function distributions of types I and II. The exact distribution of the product of two independent generalized gamma variables with the same shape parameters using the modified Bessel function of the second kind was given by [6]. A power series distribution generated by the first type of modified Bessel function has been studied [7]. The generalized beta of the first and second kinds of distribution was considered by [8]. The Kummer-beta distribution in the problem of common value action, which features confluent hypergeometric functions, was defined [9]. The type I distribution of the inverted hypergeometric function that includes a Gaussian hypergeometric function is obtained [10]. A generalized Laplacian distribution using the modified Bessel function of second kind has been introduced [11]. The product distribution of two independent random variables featuring the Kummer-beta distribution was obtained by [12] using a confluent hypergeometric series of a two-variable function. The distribution of the product of two independent random variables, beta type II (beta prime) and beta type III, featuring the hypergeometric function of two variables is derived [13]. The distribution of the difference between independent gamma random variables with different shapes and scale parameters, called the bilateral gamma, is introduced [14]. This distribution has used the Whittaker function. The density of the product of type I variables of the Kummer-beta and the inverted hypergeometric functions containing a confluent hypergeometric series of a two-variable function was developed by [15]. They also derived the distribution of the product of the beta type III and type I variables of the inverted hypergeometric function that uses the hypergeometric function of two variables. Several properties of the



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extended Gauss hypergeometric and extended confluent hypergeometric functions have been studied [16]. The distribution of the products of independent central normal variables involving Meijer G-functions has been developed [17]. A new distribution was obtained as a solution of the generalized Pearson differential equation by [18], and a generalization of this distribution was also defined using the Whittaker functions. This Distribution includes as special cases distributions obtained as the product of well-known distributions. A distribution based on the generalized Pearson differential equation using a modified Bessel function of the second kind was introduced by [19] and named as generalized inverse Gaussian distribution (GGIG). The GGIG distribution is also defined in terms of the Whittaker function.

The current paper belongs to the above studies. We use the Whittaker function to introduce a new life distribution called the Whittaker distribution. This distribution is a generalization of many important continuous distributions. The gamma, exponential, chi-square, generalized Lindley, Lindley, beta prime, and Lomax are special cases of the Whittaker distribution. In some applications, the Whittaker distribution has outperformed the generalized gamma and log-normal distributions. We also define the generalized Whittaker distribution as  $Y = X^{1/c}$ , where  $X$  is the Whittaker distribution and  $c \neq 0$ . The generalized gamma, half-normal, Weibull, half-t, half-Cauchy, Nakagami, Rayleigh, Dagum, Lévy, type-2 Gumbel, inverse gamma, inverse chi-square, inverse exponential, inverse Weibull, and inverse-Rayleigh distributions are special cases of the proposed generalized Whittaker distribution.

The current paper is organized as follows: In Section 2, we proposed the Whittaker distribution and examined some of its properties. Several important special distributions are presented in Section 3. The parameters are estimated by methods of moments and maximum likelihood in Section 4. To illustrate the usefulness of the proposed Whittaker distribution, four data sets are analyzed in Section 5. Finally, some concluding remarks are given in Section 6.

## 2. Whittaker Distribution

The Whittaker function is defined as in [2] (Ch. Confluent Hypergeometric Functions, page 1024),

$$W_{\alpha,\beta}(z) = \frac{z^\alpha e^{-\frac{z}{2}}}{\Gamma(\beta - \alpha + \frac{1}{2})} \int_0^\infty e^{-t} t^{\beta - \alpha - \frac{1}{2}} \left(1 + \frac{t}{z}\right)^{\alpha + \beta - \frac{1}{2}} dt, \quad \beta - \alpha > -\frac{1}{2} \quad (1)$$

This function can be used to introduce a probability distribution on  $(0, \infty)$  as follows:

**Theorem 1.** *The function:*

$$f(x) = C e^{-\lambda x} x^{\alpha-1} \left(\frac{\gamma}{\lambda} + x\right)^{\beta-1}; \quad x > 0, \quad (2)$$

where  $C = \frac{e^{-\frac{\gamma}{\lambda}}}{\Gamma(\alpha)} \left(\frac{\lambda}{\sqrt{\gamma}}\right)^{\alpha+\beta-1} \frac{\sqrt{\gamma}}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)}$ ,  $\alpha > 0$ ,  $\lambda > 0$ ,  $\gamma > 0$ , and  $-\infty < \beta < \infty$  is a probability density function.

**Proof of Theorem 1.** As  $b > a - \frac{1}{2}$  and  $z > 0$ , the Whittaker function  $W_{a,b}(z)$  is non-negative and finite, hence  $f(x) \geq 0$   $\alpha > 0$ ,  $\lambda > 0$ ,  $\gamma > 0$ , and  $-\infty < \beta < \infty$ .

To show that  $\int_0^\infty f(x) dx = 1$ , we will use the following relation [3] (Ch. Laplace Transforms-Arbitrary Powers p. 139 Equation (22),

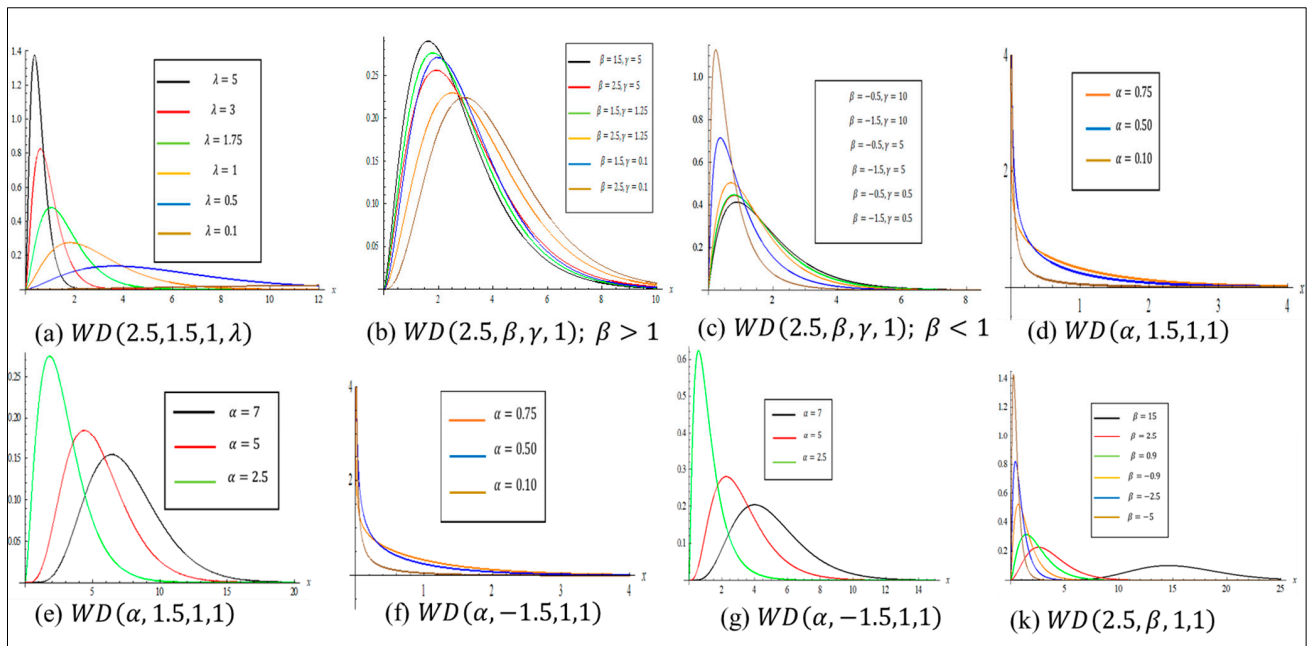
$$\int_0^\infty (t+a)^{2u-1} (t-b)^{2v-1} e^{-pt} dt = \frac{(a+b)^{u+v-1}}{p^{u+v}} e^{[\frac{(a-b)p}{2}] \Gamma(2v)} W_{u-v, u+v-\frac{1}{2}}(ap+bp), \quad (3)$$

for  $t > b$ ,  $\text{Re } v > 0$ ,  $|\arg(a+b)| < \pi$ ,  $\text{Re } p > 0$ .

Therefore, by setting  $b = 0$ ,  $u = \frac{\beta}{2}$ ,  $v = \frac{\alpha}{2}$ ,  $p = \lambda$ , and  $a = \frac{\gamma}{\lambda}$ . The proof is complete.  $\square$

**Definition 1.** Let  $X$  be a non-negative random variable with probability density function (pdf) (2). Then, we say that  $X$  has a Whittaker distribution and is denoted by  $WD(\alpha, \beta, \gamma, \lambda)$ .

Figure 1 gives sample plots of the Whittaker probability density function for different values of the relevant parameters. We first note that  $\lambda$  is a scale parameter as exhibited in Figure 1. In Figure 1b–k, we fix  $\lambda = 1$ .



**Figure 1.** Plots of density function  $WD(\alpha, \beta, \gamma, \lambda)$  for some parameters values.

From the plot in Figure 1b, when  $\alpha + \beta > 2$ , we note that the change in the value of  $\gamma$  has a negligible effect on the shape of the Whittaker probability density function. In Figure 1c, when  $\alpha + \beta \leq 2$ , note that the change in the value of  $\gamma$  is taken as a scale parameter, and has a greater effect when  $\gamma < 1$ . For  $\beta > 1$  ( $\beta < 1$ ), the distribution peaks further as  $\gamma$  increases (decreases). In Figure 1d–k, we fix  $\gamma = 1$ . For  $\alpha < 1$ , the Whittaker pdf attains  $\infty$  and decreases sharply to zero as shown in Figure 1d,f. In Figure 1e,g, as the value of  $\alpha$  decreases, the distribution peaks more.

From the plot in Figure 1k, we note that when the value of  $\beta$  increases, the mode increases. Finally, we note that the Whittaker distribution is unimodal.

**Theorem 2.** The cumulative distribution function of the  $WD(\alpha, \beta, \gamma, \lambda)$  is given by,

$$F(x) = \frac{e^{-\frac{\gamma}{2}}}{\Gamma(\alpha + 1)} \frac{\lambda^\alpha}{\gamma^{\frac{\alpha-\beta}{2}}} x^\alpha \frac{\Phi_1\left(\alpha, 1 - \beta, \alpha + 1, -\lambda x, -\frac{\lambda x}{\gamma}\right)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)}; \quad x > 0, \tag{4}$$

where  $\alpha > 0$ ,  $\lambda > 0$ ,  $\gamma > 0$ ,  $-\infty < \beta < \infty$ , and  $\Phi_1(\alpha, \beta, \gamma, x, y)$  is the confluent hypergeometric series of two variables defined in [2] (Ch. Exponentials and arbitrary powers, page 349),

$$\Phi_1(\alpha, \beta, \gamma, x, y) = \frac{1}{B(\alpha, \gamma - \alpha)} \int_0^1 t^{\alpha-1} (1-t)^{\gamma-\alpha-1} (1-yt)^{-\beta} e^{xt} dt, \quad \text{for } \gamma > \alpha > 0 \text{ arg}(1-\beta) < \pi. \tag{5}$$

**Proof of Theorem 2.**  $F(x) = A \int_0^x u^{\alpha-1} \left(\frac{\gamma}{\lambda} + u\right)^{\beta-1} e^{-\lambda u} du; y > 0$ , where  $A = \frac{1}{\Gamma(\alpha)} \frac{\lambda^{\alpha+\beta-1}}{\gamma^{\frac{\alpha+\beta}{2}-1}} e^{-\frac{\gamma}{2}}$

$$\left(W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)\right)^{-1}.$$

Using (5) one acquires,

$$\int_0^x u^{\alpha-1} \left(\frac{\gamma}{\lambda} + u\right)^{\beta-1} e^{-\lambda u} du = x^\alpha \left(\frac{\gamma}{\lambda}\right)^{\beta-1} \int_0^1 u^{\alpha-1} \left(1 + \frac{\lambda x}{\gamma} u\right)^{-(1-\beta)} e^{-\lambda x u} du = \frac{x^\alpha}{\alpha} \left(\frac{\gamma}{\lambda}\right)^{\beta-1} \Phi_1\left(\alpha, 1 - \beta, \alpha + 1, -\lambda x, -\frac{\lambda x}{\gamma}\right);$$

$\lambda > 0, \alpha > 0. \quad \square$

Using the properties of the cumulative distribution function, the following lemma provides new properties of the confluent hypergeometric series of two variables:

**Lemma 1.** Let  $\alpha > 0$ , the confluent hypergeometric series of two variables  $\Phi_1(\alpha, \beta, \gamma, x, y)$  satisfies the following conditions:

- 1-  $\lim_{x \rightarrow 0} \left[ x^\alpha \Phi_1\left(\alpha, 1 - \beta, \alpha + 1, -\frac{\lambda x}{\gamma}, -\lambda x\right) \right] = 0$
- 2-  $\lim_{x \rightarrow \infty} \left[ x^\alpha \Phi_1\left(\alpha, 1 - \beta, \alpha + 1, -\frac{\lambda x}{\gamma}, -\lambda x\right) \right] = 1$
- 3-  $x^\alpha \frac{\Phi_1\left(\alpha, 1 - \beta, \alpha + 1, -\lambda x, -\frac{\lambda}{\gamma} x\right)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)}$  is a non-decreasing and right continuous function.

The proof is straightforward using the properties of the cumulative distribution function (cdf).

### 3. Statistical Properties

This section presents the statistical properties and main characteristics of  $WD(\alpha, \beta, \gamma, \lambda)$ . We summarize these properties in the following theorem:

**Theorem 3.**

- I. The Laplace transformation is  $L(t) = \left(\frac{\lambda}{\lambda+t}\right)^{\frac{\alpha+\beta}{2}} e^{\frac{\gamma t}{2\lambda}} \frac{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma(1+\frac{t}{\lambda}))}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)}$ .
- II. The  $r$ th moment about the origin is  $\mu'_r = E(X^r) = \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \left(\frac{\gamma}{\lambda^2}\right)^{\frac{r}{2}} \frac{W_{\frac{\beta-\alpha-r}{2}, \frac{\alpha+\beta+r-1}{2}}(\gamma)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)}$ ;  $r = 1, 2, \dots$
- III. The mean is  $\mu = \alpha \frac{\sqrt{\gamma}}{\lambda} \frac{W_{\frac{\beta-\alpha-1}{2}, \frac{\alpha+\beta}{2}}(\gamma)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)}$ .
- IV. The variance is  $\sigma^2 = \alpha(\alpha+1) \frac{\gamma}{\lambda^2} \frac{W_{\frac{\beta-\alpha-1}{2}, \frac{\alpha+\beta+1}{2}}(\gamma)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)} - \alpha^2 \frac{\gamma}{\lambda^2} \left[ \frac{W_{\frac{\beta-\alpha-1}{2}, \frac{\alpha+\beta}{2}}(\gamma)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)} \right]^2$ .
- V. The mode  $x_{mo}$  of  $WD(\alpha, \beta, \gamma, \lambda)$  for  $\alpha \geq 1$  is  $x_{mo} = \frac{\alpha+\beta-\gamma-2+\sqrt{(\alpha+\beta-\gamma-2)^2+4\gamma(\alpha-1)}}{2\lambda}$ .
- VI. The measures of skewness ( $\gamma_1$ ), kurtosis ( $\gamma_2$ ), and the coefficient of variation  $C_v$  of  $WD(\alpha, \beta, \gamma, \lambda)$  distribution are, respectively,  $\gamma_1 = \frac{\left[ \frac{\Gamma(\alpha+3)}{\Gamma(\alpha)} \frac{W_{\frac{\beta-\alpha-3}{2}, \frac{\alpha+\beta+1}{2}}(\gamma)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)} - 3\alpha^2(\alpha+1) \frac{W_{\frac{\beta-\alpha-1}{2}, \frac{\alpha+\beta}{2}}(\gamma)W_{\frac{\beta-\alpha-1}{2}, \frac{\alpha+\beta+1}{2}}(\gamma)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)^2} + 2\alpha^3 \left[ \frac{W_{\frac{\beta-\alpha-1}{2}, \frac{\alpha+\beta}{2}}(\gamma)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)} \right]^3 \right]}{\left[ \alpha(\alpha+1) \frac{W_{\frac{\beta-\alpha-1}{2}, \frac{\alpha+\beta+1}{2}}(\gamma)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)} - \alpha^2 \left[ \frac{W_{\frac{\beta-\alpha-1}{2}, \frac{\alpha+\beta}{2}}(\gamma)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)} \right]^2 \right]^{3/2}}$ ,  $\gamma_2 = \frac{\left[ \frac{\Gamma(\alpha+4)}{\alpha\Gamma(\alpha+1)} W_{\frac{\beta-\alpha-2}{2}, \frac{\alpha+\beta+3}{2}}(\gamma)W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma) - 4 \frac{\Gamma(\alpha+3)}{\Gamma(\alpha+1)} W_{\frac{\beta-\alpha-1}{2}, \frac{\alpha+\beta}{2}}(\gamma)W_{\frac{\beta-\alpha-3}{2}, \frac{\alpha+\beta+1}{2}}(\gamma) + \right.}{\left. \left[ \alpha(\alpha+1) \frac{W_{\frac{\beta-\alpha-1}{2}, \frac{\alpha+\beta+1}{2}}(\gamma)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)} - \alpha^2 \left[ \frac{W_{\frac{\beta-\alpha-1}{2}, \frac{\alpha+\beta}{2}}(\gamma)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)} \right]^2 \right]^{3/2}} \right]$ .

$$6\alpha(\alpha + 1) \frac{W_{\frac{\beta-\alpha-1}{2}, \frac{\alpha+\beta}{2}}(\gamma)^2 W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta+1}{2}}(\gamma)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)} - 3\alpha^2 \frac{W_{\frac{\beta-\alpha-1}{2}, \frac{\alpha+\beta}{2}}(\gamma)^4}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)^2} \Bigg] /$$

$$\left[ \frac{(\alpha+1)}{\alpha} W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta+1}{2}}(\gamma) - \frac{W_{\frac{\beta-\alpha-1}{2}, \frac{\alpha+\beta}{2}}(\gamma)^2}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)} \right]^2,$$

and  $C_v = \sqrt{\frac{\alpha+1}{\alpha} \frac{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta+1}{2}}(\gamma) W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)}{W_{\frac{\beta-\alpha-1}{2}, \frac{\alpha+\beta}{2}}(\gamma)^2} - 1}.$

VII. The survival function is  $\bar{F}(t) = 1 - F(t) = 1 - \frac{e^{-\frac{\gamma}{2}}}{\Gamma(\alpha+1)} \frac{\lambda^\alpha}{\gamma^{\frac{\alpha-\beta}{2}}} t^\alpha \frac{\Phi_1(\alpha, 1-\beta, \alpha+1, -\lambda t, -\frac{\lambda}{\gamma} t)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)}$ .

VIII. The hazard function is  $h(t) = \frac{\alpha \gamma \left(\frac{\lambda^2}{\gamma}\right)^{\frac{\alpha+\beta}{2}} e^{-\lambda x - \frac{\gamma}{2} x^{\alpha-1}} \left(\frac{\gamma}{\lambda} + x\right)^{\beta-1}}{\Gamma(\alpha+1) W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma) - \gamma^{\frac{\beta-\alpha}{2}} \lambda^\alpha e^{-\frac{\gamma}{2} x} \Phi_1(\alpha, 1-\beta, \alpha+1, -\lambda x, -\frac{\lambda x}{\gamma})}$ .

**Proof of Theorem 3.**

I. The Laplace transformation:  $L(t) = \frac{\lambda^{\alpha+\beta-1}}{\gamma^{\frac{\alpha+\beta}{2}-1} \Gamma(\alpha) W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)} \int_0^\infty x^{\alpha-1} \left(\frac{\gamma}{\lambda} + x\right)^{\beta-1} e^{-(\lambda+t)x} dx = \frac{\gamma^{-\frac{\alpha+\beta}{2}+1} \lambda^{\alpha+\beta-1} e^{-\frac{\gamma}{2}}}{\Gamma(\alpha) W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)} \left(\frac{\gamma}{\lambda}\right)^{\frac{\alpha+\beta}{2}-1} \frac{\Gamma(\alpha)}{(\lambda+t)^{\frac{\alpha+\beta}{2}}} \exp\left[\frac{\gamma(\lambda+t)}{2\lambda}\right] W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}\left(\gamma\left(\frac{\lambda+t}{\lambda}\right)\right) = \left(\frac{\lambda}{\lambda+t}\right)^{\frac{\alpha+\beta}{2}} e^{\frac{\gamma t}{2\lambda}} \frac{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}\left(\gamma\left(\frac{\lambda+t}{\lambda}\right)\right)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)}$ , by using (3).

II. The  $r$ th moment function is,  $\mu'_r = E(X^r) = \frac{\lambda^{\alpha+\beta-1}}{\gamma^{\frac{\alpha+\beta}{2}-1} \Gamma(\alpha) W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)} \int_0^\infty x^{\alpha+r-1} \left(\frac{\gamma}{\lambda} + x\right)^{\beta-1} e^{-\lambda x} dx = \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \frac{\gamma^{\frac{r}{2}}}{\lambda^r} \frac{W_{\frac{\beta-\alpha-r}{2}, \frac{\alpha+\beta+r-1}{2}}(\gamma)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)}$ , By using (3).

III. We have from II that,  $\mu'_1 = \mu = \alpha \frac{\sqrt{\gamma}}{\lambda} \frac{W_{\frac{\beta-\alpha-1}{2}, \frac{\alpha+\beta}{2}}(\gamma)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)}$ , and  $\mu'_2 = \alpha(\alpha + 1) \frac{\gamma}{\lambda^2} \frac{W_{\frac{\beta-\alpha-1}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)}$ .

IV. The variance is given by  $\sigma^2 = \mu'_2 - \mu^2$ .

V. The mode  $x_{m0}$  can be obtained by differentiating the pdf of  $WD(\alpha, \beta, \gamma, \lambda)$  with respect to  $x$  as follows,  $f'(x) = A \left[ (\alpha - 1)x^{\alpha-2}(\gamma + \lambda x)^{\beta-1} e^{-\lambda x} + \lambda(\beta - 1)x^{\alpha-1} (\gamma + \lambda x)^{\beta-2} e^{-\lambda x} - \lambda x^{\alpha-1} (\gamma + \lambda x)^{\beta-1} e^{-\lambda x} \right] = \left[ \frac{(\alpha-1)}{x} + \frac{\lambda(\beta-1)}{(\gamma+\lambda x)} - \lambda \right] f(x)$ .

However, the mode  $x_{m0}$  is obtained as the solution of  $f'(x_{m0}) = 0$ . Therefore, we have  $\left[ \frac{(\alpha-1)}{x_{m0}} + \frac{\lambda(\beta-1)}{(\gamma+\lambda x_{m0})} - \lambda \right] f(x_{m0}) = 0$ , that leads to  $f(x_{m0}) = 0$ , or  $\left[ \frac{(\alpha-1)}{x_{m0}} + \frac{\lambda(\beta-1)}{(\gamma+\lambda x_{m0})} - \lambda \right] = 0$ .

By solving  $\left[ \frac{(\alpha-1)}{x_{m0}} + \frac{\lambda(\beta-1)}{(\gamma+\lambda x_{m0})} - \lambda \right] = 0$ , one acquires,  $x_{m0} = \frac{(\alpha+\beta-\gamma-2) \pm \sqrt{(\alpha+\beta-\gamma-2)^2 + 4\gamma(\alpha-1)}}{2\lambda}$ .

Since the distribution is defined on the set  $(0, \infty)$ , we acquire  $x_{m0}$  as in the form,  $x_{m0} = \frac{(\alpha+\beta-\gamma-2) + \sqrt{(\alpha+\beta-\gamma-2)^2 + 4\gamma(\alpha-1)}}{2\lambda}$ .  $\square$

Note: The skewness  $(\gamma_1)$ , kurtosis  $(\gamma_2)$ , and the coefficient of variation  $C_v$  of  $WD(\alpha, \beta, \gamma, \lambda)$  do not depend on  $\lambda$ .

Some recurrence relations of the Whittaker function are provided in [1]. From the above theorem, we can deduce other recurrence relations for the Whittaker function as follows.

**Corollary 1.** For  $z > 0$  and  $b > a - \frac{1}{2}$ , we have,

$$1- W_{a-\frac{1}{2}, b+\frac{1}{2}}(z) = \frac{2(a+b-z)+1}{[2(b-a)+1]\sqrt{z}} W_{a,b}(z) + \frac{2}{[2(b-a)+1]\sqrt{z}} W_{a+1,b}(z).$$

$$\begin{aligned}
 2- & W_{a+2,b}(z) = \left(b - a + \frac{1}{2}\right) \left(b - a + \frac{3}{2}\right) z W_{a-1,b}(z) - \left(a + b + \frac{1}{2} + \left(a + b + \frac{1}{2} - z\right)^2\right) \\
 & W_{a,b}(z) - 2\left(a + b - z + \frac{3}{2}\right) W_{a+1,b}(z). \\
 3- & \frac{2(b-a)+3}{2(b-a)+1} W_{a-1,b+1}(z) W_{a,b}(z) > \left[W_{a-\frac{1}{2},b+\frac{1}{2}}(z)\right]^2.
 \end{aligned}$$

**Proof of Corollary 1.** From Theorem 3, one can acquire the mean and the second order moment using the Laplace transformation as follows,  $\mu'_1 = \mu = -\frac{\partial}{\partial t} L(t)\Big|_{t=0} = \frac{\beta}{\lambda} -$

$$\frac{\gamma}{\lambda} + \frac{1}{\lambda} \frac{W_{\frac{\beta-\alpha}{2}+1, \frac{\alpha+\beta-1}{2}}(\gamma)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)}, \text{ and } \mu'_2 = E(X^2) = \frac{\partial^2}{\partial t^2} L(t)\Big|_{t=0} = \frac{\beta}{\lambda^2} + \frac{1}{\lambda^2}(\beta - \gamma)^2 + \frac{2}{\lambda^2}(\beta - \gamma + 1) \frac{W_{\frac{\beta-\alpha}{2}+1, \frac{\alpha+\beta-1}{2}}(\gamma)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)} + \frac{1}{\lambda^2} \frac{W_{\frac{\beta-\alpha}{2}+2, \frac{\alpha+\beta-1}{2}}(\gamma)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)}.$$

The mean and the second order moment can also be derived from the  $r$ th moment formula in Theorem 3,  $\mu'_1 = \mu = \alpha \frac{\sqrt{\gamma}}{\lambda} \frac{W_{\frac{\beta-\alpha-1}{2}, \frac{\alpha+\beta}{2}}(\gamma)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)}$ , and  $\mu'_2 = E(X^2) = \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} \frac{\gamma}{\lambda^2} \frac{W_{\frac{\beta-\alpha-1}{2}, \frac{\alpha+\beta+1}{2}}(\gamma)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)}$ .

Hence, by equating the two formulas of  $\mu$ , we acquire  $W_{\frac{\beta-\alpha-1}{2}, \frac{\alpha+\beta}{2}}(\gamma) = \frac{\beta-\gamma}{\alpha\sqrt{\gamma}} W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma) + \frac{1}{\alpha\sqrt{\gamma}} W_{\frac{\beta-\alpha}{2}+1, \frac{\alpha+\beta-1}{2}}(\gamma)$ .

By replacing  $\alpha, \beta$ , and  $\gamma$  with  $b - a + \frac{1}{2}, a + b + \frac{1}{2}$ , and  $z$ , respectively, we acquire,  $W_{a-\frac{1}{2},b+\frac{1}{2}}(z) = \frac{2(a+b-z)+1}{[2(b-a)+1]\sqrt{z}} W_{a,b}(z) + \frac{2}{[2(b-a)+1]\sqrt{z}} W_{a+1,b}(z)$ .

Similarly, by equating the two formulas of  $E(X^2)$ , we acquire,  $(\beta + (\beta - \gamma)^2) W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma) = \alpha(\alpha + 1)\gamma W_{\frac{\beta-\alpha}{2}-1, \frac{\alpha+\beta-1}{2}}(\gamma) - 2(\beta - \gamma + 1) W_{\frac{\beta-\alpha}{2}+1, \frac{\alpha+\beta-1}{2}}(\gamma) - W_{\frac{\beta-\alpha}{2}+2, \frac{\alpha+\beta-1}{2}}(\gamma)$ .

By replacing  $\alpha, \beta$ , and  $\gamma$  with  $b - a + \frac{1}{2}, a + b + \frac{1}{2}$ , and  $z$ , respectively, we acquire,  $W_{a+2,b}(z) = \left(b - a + \frac{1}{2}\right) \left(b - a + \frac{3}{2}\right) z W_{a-1,b}(z) - \left(a + b + \frac{1}{2} + \left(a + b + \frac{1}{2} - z\right)^2\right) W_{a,b}(z) - 2\left(a + b - z + \frac{3}{2}\right) W_{a+1,b}(z)$ .

$$\text{Since } \sigma^2 = \alpha(\alpha + 1) \frac{\gamma}{\lambda^2} \frac{W_{\frac{\beta-\alpha}{2}-1, \frac{\alpha+\beta+1}{2}}(\gamma)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)} - \alpha^2 \frac{\gamma}{\lambda^2} \left[ \frac{W_{\frac{\beta-\alpha-1}{2}, \frac{\alpha+\beta}{2}}(\gamma)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)} \right]^2 > 0.$$

By replacing  $\alpha, \beta$ , and  $\gamma$  with  $b - a + \frac{1}{2}, a + b + \frac{1}{2}$ , and  $z$ , respectively, we acquire,  $\frac{2(b-a)+3}{2(b-a)+1} W_{a-1,b+1}(z) W_{a,b}(z) > \left[W_{a-\frac{1}{2},b+\frac{1}{2}}(z)\right]^2$ .

The proof is complete.  $\square$

**Corollary 2.** For  $a > 0$  and  $z > 0$ , the Whittaker function satisfies the following conditions,

- I.  $W_{\frac{1}{2}-a,a}(z) = z^{\frac{1}{2}-a} e^{-\frac{z}{2}}$ .
- II.  $W_{1-a, a+\frac{1}{2}}(z) = z^{1-a} e^{-\frac{z}{2}} (2az^{-1} + 1); a, z > 0$ .

**Proof of Corollary 2.**

- I.  $W_{\frac{1}{2}-a, a}(z) = \frac{z^{\frac{a}{2}+\frac{1}{2}} e^{-\frac{z}{2}}}{\Gamma(2a)} \int_0^\infty e^{-zt} t^{2a-1} dt = z^{\frac{1}{2}-a} e^{-\frac{z}{2}}$ .
- II.  $W_{1-a, a+\frac{1}{2}}(z) = \frac{z^{a+1} e^{-\frac{z}{2}}}{\Gamma(2a)} \int_0^\infty e^{-zt} t^{2a-1} (1+t) dt = \frac{z^{a+1} e^{-\frac{z}{2}}}{\Gamma(2a)} \left( z^{-2a} \Gamma(2a) + z^{-(2a+1)} \Gamma(2a+1) \right) = z^{1-a} e^{-\frac{z}{2}} (2az^{-1} + 1). \quad \square$

**Corollary 3.** For  $\alpha + \beta < 1$  and  $\alpha > 0$ , the confluent hypergeometric function of the second kind satisfies the relation,

$$U(\alpha, \alpha + \beta, 0) = \frac{\Gamma(1 - \alpha - \beta)}{\Gamma(1 - \beta)}. \tag{6}$$

**Proof of Corollary 3.** The confluent hypergeometric function of the second kind is defined in [1] (Ch. Confluent Hypergeometric functions/Integral Representations page 505),

$$U(a, b, z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt} t^{a-1} (1+t)^{b-a-1} dt.$$

At  $z = 0$ , we acquire  $U(\alpha, \alpha + \beta, 0) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} (1+t)^{\beta-1} dt = \frac{\Gamma(1-\alpha-\beta)}{\Gamma(1-\beta)}$ .  $\square$

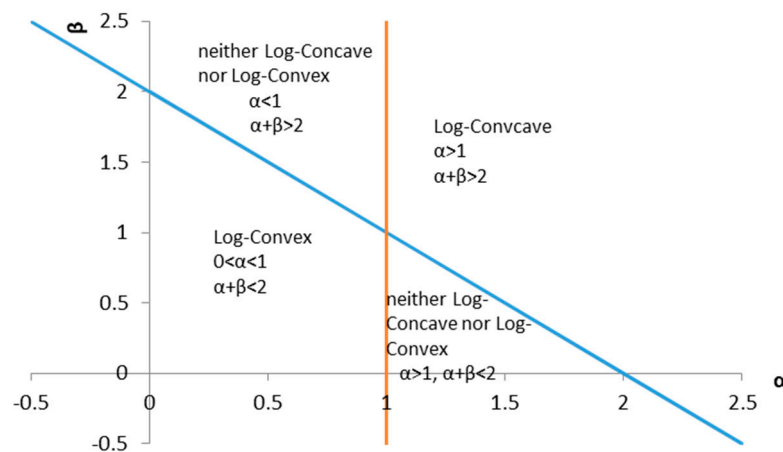
**Theorem 4.** Let  $X \sim WD(\alpha, \beta, \gamma, \lambda)$ . Then,

- (a) For  $\alpha > 1$  and  $\alpha + \beta > 2$ ,  $f(x)$  is log-concave.
- (b) For  $\alpha < 1$  and  $\alpha + \beta < 2$ ,  $f(x)$  is log-convex.

**Proof of Theorem 4.** It is simple to see that,  $\frac{\partial}{\partial x} \log f(x) = \frac{(\alpha+\beta-2)}{(\frac{\gamma}{\lambda}+x)} + \frac{(\alpha-1)\frac{\gamma}{\lambda}}{x(\frac{\gamma}{\lambda}+x)} - \lambda$ .

It is a decreasing (increasing) function for  $\alpha + \beta > 2$  ( $< 2$ ), and  $\alpha > 1$  ( $< 1$ ). Thus,  $f(x)$  is log-concave (convex).  $\square$

Theorem 4 does not cover all ranges of  $\alpha$  and  $\beta$  as shown in Figure 2. In the following example, we illustrate that there is no obvious conclusion outside the above ranges.



**Figure 2.** The relation between  $\alpha$  and  $\beta$ , and the concavity of  $WD(\alpha, \beta, \gamma, \lambda)$ .

Example: The second derivative of the log of density is,

$$g(x) = \frac{\partial^2}{\partial x^2} \log f(x) = -\frac{(\alpha - 1)}{x^2} - \frac{(\beta - 1)}{(\frac{\gamma}{\lambda} + x)^2}.$$

Figure 3 shows the function  $g(x)$  for the two cases where  $(\alpha, \beta) = (0.5, 4)$  and  $(\alpha, \beta) = (1.5, -4)$ . When  $(\alpha, \beta) = (0.5, 4)$ ,  $g(x)$  is positive for  $x < 0.69$ , and for  $x \geq 0.69$ ,  $g(x)$  is negative. Therefore  $f(x)$  is neither log-convex nor log-concave. Further, when  $(\alpha, \beta) = (1.5, -4)$ ,  $g(x)$  is negative for  $x < 0.46$ , and for  $x \geq 0.46$  is positive. Therefore,  $f(x)$  is neither log-convex nor log-concave.

The distributions with log-concave density are unimodal and have increasing failure rate property.

**Theorem 5.**

- I. The Whittaker distribution has a monotonic likelihood ratio in  $x$  with respect to  $\alpha$  when the other parameters are constant.
- II. The Whittaker distribution has a monotonic likelihood ratio in  $x$  with respect to  $\beta$  when the other parameters are constant.

III. The Whittaker distribution has a monotonic likelihood ratio in  $x$  with respect to  $\gamma$  when the other parameters are constant and  $\beta < 1$ .

**Proof of Theorem 5.**

I Let  $\alpha_2 > \alpha_1$ . Then,  $\frac{\partial}{\partial x} \left( \frac{f_{\alpha_2}(x)}{f_{\alpha_1}(x)} \right) = C(\alpha_2 - \alpha_1)x^{\alpha_2 - \alpha_1 - 1} > 0$ , where  $C = \left[ \frac{\Gamma(\alpha_1)}{\Gamma(\alpha_2)} \right]^n \frac{\lambda^{n(\alpha_2 - \alpha_1)}}{\gamma^{n \frac{\alpha_2 - \alpha_1}{2}}}$   
 $\left[ \frac{W_{\frac{\beta - \alpha_1}{2}, \frac{\alpha_1 + \beta - 1}{2}}(\gamma)}{W_{\frac{\beta - \alpha_2}{2}, \frac{\alpha_2 + \beta - 1}{2}}(\gamma)} \right]^n$ .

II. The proof is similar to that of I.

1. For  $\gamma_1 < \gamma_2$ , we acquire  $\frac{\partial}{\partial x} \left( \frac{f_{\gamma_2}(x)}{f_{\gamma_1}(x)} \right) = (\beta - 1)(\gamma_1 - \gamma_2) \frac{(\gamma_2 + \lambda x)^{\beta - 2}}{(\gamma_1 + \lambda x)^\beta}$ , where  $C = \lambda e^{-\frac{\gamma_2 - \gamma_1}{2}} \left( \frac{\gamma_1}{\gamma_2} \right)^{\frac{\alpha + \beta}{2}} \frac{W_{\frac{\beta - \alpha}{2}, \frac{\alpha + \beta - 1}{2}}(\gamma_1)}{W_{\frac{\beta - \alpha}{2}, \frac{\alpha + \beta - 1}{2}}(\gamma_2)}$

If  $\beta < 1$ , then  $\frac{\partial}{\partial x} \left( \frac{f_{\gamma_2}(x)}{f_{\gamma_1}(x)} \right) > 0$ . □

Note: The Whittaker distribution is not a member of the monotonic likelihood ratio family with respect to  $\lambda$  when other parameters are constant.

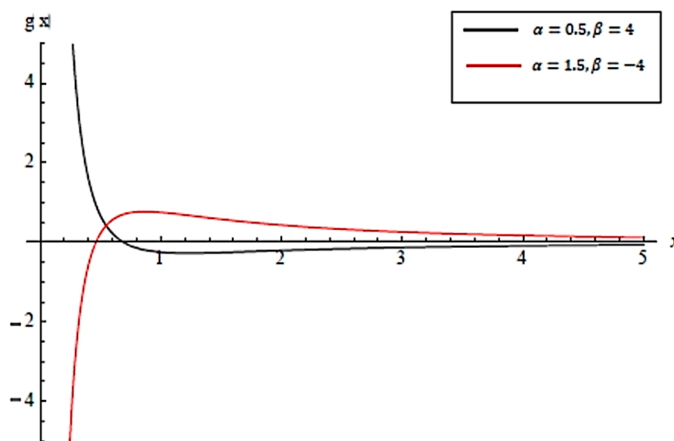


Figure 3.  $g(x)$  for  $WD(0.5, 4, 1, 1)$ , and  $WD(1.5, -4, 1, 1)$ .

*Percentiles*

Now, we compute the percentiles of Whittaker distribution. For any  $0 < q < 1$ , the  $100q$ -th percentile is a number  $x_q$  such that the area under  $f(x)$  to the left of  $x_q$  is  $q$ . That is,  $x_q$  is any root of the equation,

$$F(x_q) = \int_0^{x_q} f(x)dx = q$$

Using numerical simulation, the percentage points  $x_q$  associated with the cdf of  $X$  are computed for some selected values of the parameters using Mathematica software (The code is provided in Appendix A).

**Definition 2.** Let  $X \sim WD(\alpha, \beta, \gamma, \lambda)$  and  $Y = X^{1/c}$ , then,  $Y$  is defined as the generalized Whittaker distribution given by,

$$f(y) = \frac{e^{-\frac{\gamma}{c}} |c| \lambda^\alpha}{\Gamma(\alpha) \gamma^{\frac{\alpha + \beta}{2} - 1}} e^{-\lambda y^c} \frac{y^{\alpha c - 1} (\gamma + \lambda y^c)^{\beta - 1}}{W_{\frac{\beta - \alpha}{2}, \frac{\alpha + \beta - 1}{2}}(\gamma)}; y > 0, \tag{7}$$

where  $\alpha > 0, \lambda > 0, \gamma > 0, -\infty < \beta < \infty$ , and  $\forall c \neq 0$ . For simplicity, we use the notation  $GWD(\alpha, \beta, \gamma, \lambda, c)$ .



### 4. Special Cases

Many well-known distributions can be viewed as special cases of the WD and GWD. Figure 4 summarizes all these findings. Whereas the distribution defined in [18,19] are neither a generalization nor a special cases of the WD.

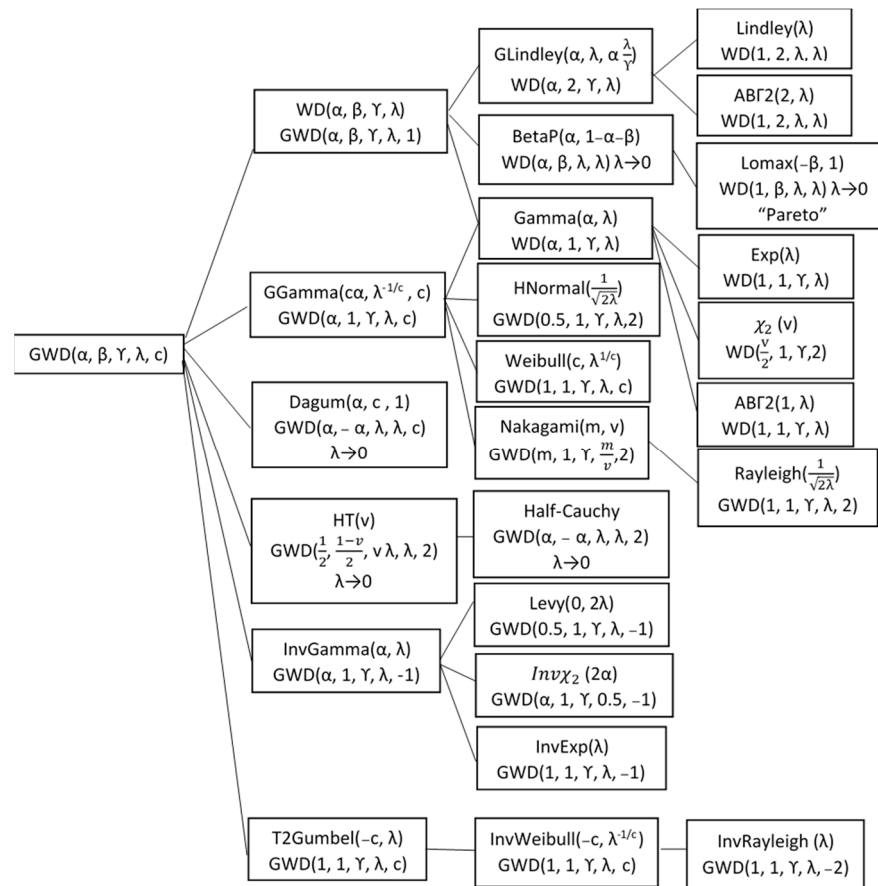


Figure 4. The special cases of  $GWD(\alpha, \beta, \gamma, \lambda, c)$  and  $WD(\alpha, \beta, \gamma, \lambda)$ .

#### 4.1. Generalized Lindley Distribution

If  $\beta = 2$ , then the Whittaker distribution is reduced to the generalized Lindley distribution proposed by [20], as follows:

$$\text{At } \beta = 2, \text{ the mgf } M_X(t) = \left(\frac{\lambda}{\lambda-t}\right)^{\frac{\alpha}{2}+1} e^{-\frac{\gamma t}{2\lambda}} \frac{W_{\frac{2-\alpha}{2}, \frac{\alpha+1}{2}}(\gamma(\frac{\lambda-t}{\lambda}))}{W_{\frac{2-\alpha}{2}, \frac{\alpha+1}{2}}(\gamma)}$$

By using Corollary 2(II) at  $a = \frac{\alpha}{2}$ , we acquire  $W_{\frac{2-\alpha}{2}, \frac{\alpha+1}{2}}(z) = z^{-\frac{\alpha}{2}+1} e^{-\frac{z}{2}} (\alpha z^{-1} + 1)$ . Then,

$$M_X(t) = \left(\frac{\lambda}{\lambda-t}\right)^{\frac{\alpha}{2}+1} e^{-\frac{\gamma t}{2\lambda}} \frac{\left(\gamma\left(\frac{\lambda-t}{\lambda}\right)\right)^{-\frac{\alpha}{2}+1} e^{-\frac{\gamma}{2}\left(\frac{\lambda-t}{\lambda}\right)} \left(\alpha\left(\gamma\left(\frac{\lambda-t}{\lambda}\right)\right)^{-1} + 1\right)}{\gamma^{-\frac{\alpha}{2}+1} e^{-\frac{\gamma}{2}} (\alpha\gamma^{-1} + 1)}, M_X(t) = \left(\frac{\lambda}{\lambda-t}\right)^{\alpha+1} \frac{\alpha\lambda + \gamma(\lambda-t)}{\alpha\lambda + \gamma\lambda}$$

Thus, if  $\alpha = 1, \beta = 2$ , and  $\lambda = \gamma$ , the Whittaker distribution is reduced to the Lindley distribution [21].

#### 4.2. Beta Prime Distribution

If  $\beta < \min(0, (1 - \alpha))$ ,  $\gamma = \lambda$ , and  $\lambda \rightarrow 0$ , the Whittaker distribution is reduced to the beta prime distribution [22].

For  $\gamma = \lambda$ , the pdf of  $WD(\alpha, \beta, \lambda, \lambda)$  is reduced to  $f(x) = \frac{1}{\Gamma(\alpha)} \lambda^{\frac{\alpha+\beta}{2}} e^{-\lambda(x+\frac{1}{2})} \frac{x^{\alpha-1}(1+x)^{\beta-1}}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\lambda)}$ .  
 By using [1] (Ch. Confluent Hypergeometric functions, page 505),

$$W_{\lambda, \mu}(z) = \exp(-z/2) z^{\mu+\frac{1}{2}} U\left(\mu - \lambda + \frac{1}{2}, 1 + 2\mu; z\right). \tag{8}$$

we acquire  $W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\lambda) = e^{-\frac{\lambda}{2}} \lambda^{\frac{\alpha+\beta}{2}} U(\alpha, \alpha + \beta; \lambda)$ .

Then,  $f(x) = \frac{1}{\Gamma(\alpha)} e^{-\lambda x} \frac{x^{\alpha-1}(1+x)^{\beta-1}}{U(\alpha, \alpha+\beta; \lambda)}$ . By taking the limit as  $\lambda$  tends to zero, one acquires  $\lim_{\lambda \rightarrow 0} \left( \frac{1}{\Gamma(\alpha)} e^{-\lambda x} \frac{x^{\alpha-1}(1+x)^{\beta-1}}{U(\alpha, \alpha+\beta; \lambda)} \right) = \frac{1}{\Gamma(\alpha)} \frac{1}{U(\alpha, \alpha+\beta, 0)} x^{\alpha-1}(1+x)^{\beta-1}$ .

Using Corollary 3, we acquire the pdf  $f(x) = \frac{\Gamma(1-\beta)}{\Gamma(\alpha)\Gamma(1-\alpha-\beta)} x^{\alpha-1}(1+x)^{\beta-1}; y > 0$ .

Thus, if  $\alpha = 1, \beta < 0, \gamma = \lambda$ , and  $\lambda \rightarrow 0$ , the Whittaker distribution is reduced to Lomax [23] and the Pareto distribution [24].

### 4.3. Generalized Gamma Distribution

If  $X \sim GWD(\alpha, 1, \gamma, \lambda, c)$ , then  $X$  follows a generalized gamma distribution [25]. Hence the corresponding pdf is,

$$f(x) = \frac{e^{-\frac{\gamma}{2}} c \lambda^\alpha}{\Gamma(\alpha) \gamma^{\frac{\alpha-1}{2}}} e^{-\lambda x^c} \frac{x^{\alpha c-1}}{W_{\frac{1-\alpha}{2}, \frac{\alpha}{2}}(\gamma)}$$

From Corollary 2 (I), we acquire  $W_{\frac{1-\alpha}{2}, \frac{\alpha}{2}}(\gamma) = \gamma^{\frac{1-\alpha}{2}} e^{-\frac{\gamma}{2}}$ .

Therefore,  $f(x) = \frac{c}{\Gamma(\alpha)} \lambda^\alpha x^{\alpha c-1} e^{-\lambda x^c}$ .

Notes:

- (a) If  $\beta = 1, c = 2$ , and  $\lambda = \frac{\alpha}{v}$ , the generalized Whittaker distribution is reduced to the Nakagami distribution [26].
- (b) If  $\alpha = 1, \beta = 1$ , and  $c = 2$ , the generalized Whittaker distribution is reduced to Rayleigh distribution [27].
- (c) If  $\alpha = 0.5, \beta = 1$ , and  $c = 2$ , the generalized Whittaker distribution is reduced to the half-normal distribution.
- (d) If  $\alpha = 1$ , and  $\beta = 1$ , the generalized Whittaker distribution is reduced to the Weibull distribution.
- (e) If  $\beta = 1$ , and  $c = 1$ , the generalized Whittaker distribution is reduced to the gamma distribution.
- (f) If  $\alpha = 1$  and  $\beta = 1$ , the Whittaker distribution is reduced to an exponential distribution.
- (g) If  $\beta = 1, 2\alpha = n$ , and  $\lambda = 2$ , the Whittaker distribution is reduced to a chi-square distribution with  $n$  degrees of freedom.

### 4.4. Dagum Distribution

If  $X \sim GWD(\alpha, -\alpha, \lambda, \lambda, c)$ ,  $X$  follows the Dagum distribution [28] with  $f(y) = \frac{c}{\Gamma(\alpha)} e^{-\lambda y^c} \frac{y^{\alpha c-1}(1+y^c)^{-(\alpha+1)}}{U(\alpha, 0; \lambda)}$ .

Recalling Corollary 3, we acquire the pdf as  $f(y) = c \lambda y^{\alpha c-1} (1+y^c)^{-(\alpha+1)}$ .

### 4.5. Half-t Distribution

If  $X$  follows a generalized Whittaker distribution with  $\frac{\gamma}{\lambda} = v, \alpha = \frac{1}{2}, \beta = \frac{1-v}{2}, c = 2$ , and at  $\lambda \rightarrow 0$ , the generalized Whittaker distribution is reduced to the half-t distribution.

The pdf of  $GWD\left(\frac{1}{2}, \frac{1-v}{2}, \lambda v, \lambda, 2\right)$  is given by  $f(x) = \frac{e^{-\frac{\lambda v}{2}} 2 \lambda^{\frac{2-v}{4}}}{\Gamma\left(\frac{1}{2}\right) v^{-\frac{v-2}{4}}} e^{-\lambda x^2} \frac{(v+x^2)^{-\frac{1-v}{2}}}{W_{\frac{-v}{4}, \frac{-v}{4}}(\lambda v)}$ .

From (8), we acquire  $W_{\frac{-v}{4}, \frac{-v}{4}}(\lambda v) = e^{-\frac{\lambda v}{2}} (\lambda v)^{\frac{2-v}{4}} U\left(\frac{1}{2}, 1 - \frac{v}{2}; \lambda v\right)$ . Then,

$$f(x) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{2}{v^{\frac{-v}{2}}} e^{-\lambda x^2} \frac{(v+x^2)^{\frac{-1-v}{2}}}{U\left(\frac{1}{2}, 1-\frac{v}{2}; \lambda v\right)} \lim_{\lambda \rightarrow 0} \left( \frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{2}{v^{\frac{-v}{2}}} e^{-\lambda x^2} \frac{(v+x^2)^{\frac{-1-v}{2}}}{U\left(\frac{1}{2}, 1-\frac{v}{2}; \lambda v\right)} \right) = \frac{v^{\frac{v}{2}}}{\Gamma\left(\frac{1}{2}\right)} \frac{2}{U\left(\frac{1}{2}, 1-\frac{v}{2}, 0\right)} (v+x^2)^{\frac{-1-v}{2}}$$

From Corollary 3, at  $\alpha = \frac{1}{2}$ ,  $\beta = \frac{1-v}{2}$ , we acquire  $U\left(\frac{1}{2}, 1-\frac{v}{2}, 0\right) = \frac{\Gamma\left(\frac{v}{2}\right)}{\Gamma\left(\frac{v+1}{2}\right)}$ .

Therefore,

$$f(y) = 2v^{-\frac{v+1}{2}} v^{\frac{v}{2}} \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{v}{2}\right)} \left(1 + \frac{y^2}{v}\right)^{-\frac{v+1}{2}} = 2 \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{v\pi}\Gamma\left(\frac{v}{2}\right)} \left(1 + \frac{y^2}{v}\right)^{-\frac{v+1}{2}}.$$

Thus, if  $\alpha = \frac{1}{2}$ ,  $\beta = 0$ ,  $c = 2$ ,  $\gamma = \lambda$ , and  $\lambda \rightarrow 0$ , the generalized Whittaker distribution is reduced to a half-Cauchy distribution.

#### 4.6. Inverse Gamma Distribution

If  $X \sim \text{GWD}(\alpha, 1, \gamma, \lambda, -1)$ ,  $X$  follows the inverse gamma distribution [23], as,

$$f(x) = \frac{e^{-\frac{\gamma}{x}}}{\Gamma(\alpha)} \frac{\lambda^\alpha}{\gamma^{\frac{\alpha-1}{2}}} e^{-\frac{\lambda}{x}} \frac{x^{-\alpha-1}}{W_{\frac{1-\alpha}{2}, \frac{\alpha}{2}}(\gamma)}$$

From Corollary 2 (I), we acquire  $W_{\frac{1-\alpha}{2}, \frac{\alpha}{2}}(\gamma) = \gamma^{\frac{\alpha-1}{2}} e^{-\frac{\gamma}{2}}$ .

Therefore,

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\frac{\lambda}{x}}.$$

Notes:

- (a) If  $\alpha = \frac{1}{2}$ ,  $\beta = 1$ , and  $c = -1$ , the generalized Whittaker distribution is reduced to Lévy distribution [29].
- (b) If  $\alpha = 1$ ,  $\beta = 1$ , and  $c = -1$ , the generalized Whittaker distribution is reduced to the inverse exponential distribution.
- (c) If  $2\alpha = v$ ,  $\beta = 1$ ,  $\lambda = \frac{1}{2}$ , and  $c = -1$ , the generalized Whittaker distribution is reduced to the inverse chi-square distribution.

#### 4.7. Type-2 Gumbel Distribution

If  $X \sim \text{GWD}(1, 1, \gamma, \lambda, c)$ , then  $X$  follows a Type-2 Gumbel distribution [30], as,

$$f(x) = e^{-\frac{\gamma}{x}} |c| \lambda e^{-\lambda x^c} \frac{x^{c-1}}{W_{0, \frac{1}{2}}(\gamma)}.$$

From Corollary 2 (I), we acquire  $W_{0, \frac{1}{2}}(\gamma) = e^{-\frac{\gamma}{2}}$ .

Therefore,  $f(x) = |c| \lambda x^{c-1} e^{-\lambda x^c}$ .

Note: If  $\alpha = 1$ ,  $\beta = 1$ , and  $c = -2$ , then the generalized Whittaker distribution is reduced to inverse Rayleigh distribution [31].

### 5. Estimation

In this section, we focus on the estimation of the parameters  $\alpha$ ,  $\beta$ ,  $\lambda$ , and  $\gamma$  of the Whittaker distribution. We introduce the estimation of parameters via the moments' and the maximum likelihood methods, denoted by MME and MLE, respectively.

Let  $X_1, X_2, \dots, X_n$  be i.i.d.  $WD(\alpha, \beta, \gamma, \lambda)$ , then we have the following estimates.

5.1. Method of Moments

$$\hat{\lambda}_{MM} = \hat{\alpha}_{MM} \frac{\sqrt{\hat{\gamma}_{MM}}}{\bar{X}} \frac{W_{\frac{\hat{\beta}_{MM}-\hat{\alpha}_{MM}-1}{2}, \frac{\hat{\alpha}_{MM}+\hat{\beta}_{MM}}{2}}(\hat{\gamma}_{MM})}{W_{\frac{\hat{\beta}_{MM}-\hat{\alpha}_{MM}}{2}, \frac{\hat{\alpha}_{MM}+\hat{\beta}_{MM}-1}{2}}(\hat{\gamma}_{MM})}, \tag{9}$$

where  $\hat{\alpha}_{MM}, \hat{\beta}_{MM}$ , and  $\hat{\gamma}_{MM}$  can be calculated by solving the following system of non-linear equations:

$$S^2 = \bar{X}^2 \left( 1 + \frac{1}{\hat{\alpha}_{MM}} \right) \frac{W_{\frac{\hat{\beta}_{MM}-\hat{\alpha}_{MM}}{2}, \frac{\hat{\alpha}_{MM}+\hat{\beta}_{MM}-1}{2}}(\hat{\gamma}_{MM}) W_{\frac{\hat{\beta}_{MM}-\hat{\alpha}_{MM}-1}{2}, \frac{\hat{\alpha}_{MM}+\hat{\beta}_{MM}+1}{2}}(\hat{\gamma}_{MM})}{\left[ W_{\frac{\hat{\beta}_{MM}-\hat{\alpha}_{MM}-1}{2}, \frac{\hat{\alpha}_{MM}+\hat{\beta}_{MM}}{2}}(\hat{\gamma}_{MM}) \right]^2} - \bar{X}^2 \tag{10}$$

$$\frac{\sum_{i=1}^n X_i^3}{n\bar{X}^3} = \frac{\Gamma(\hat{\alpha}_{MM} + 3)}{\hat{\alpha}_{MM}^2 \Gamma(\hat{\alpha}_{MM} + 1)} \frac{\left[ W_{\frac{\hat{\beta}_{MM}-\hat{\alpha}_{MM}}{2}, \frac{\hat{\alpha}_{MM}+\hat{\beta}_{MM}-1}{2}}(\hat{\gamma}_{MM}) \right]^2 W_{\frac{\hat{\beta}_{MM}-\hat{\alpha}_{MM}-3}{2}, \frac{\hat{\alpha}_{MM}+\hat{\beta}_{MM}+1}{2}}(\hat{\gamma}_{MM})}{\left[ W_{\frac{\hat{\beta}_{MM}-\hat{\alpha}_{MM}-1}{2}, \frac{\hat{\alpha}_{MM}+\hat{\beta}_{MM}}{2}}(\hat{\gamma}_{MM}) \right]^3} \tag{11}$$

In addition,

$$\frac{\sum_{i=1}^n X_i^4}{n\bar{X}^4} = \frac{\Gamma(\hat{\alpha}_{MM} + 4)}{\hat{\alpha}_{MM}^3 \Gamma(\hat{\alpha}_{MM} + 1)} \frac{\left[ W_{\frac{\hat{\beta}_{MM}-\hat{\alpha}_{MM}}{2}, \frac{\hat{\alpha}_{MM}+\hat{\beta}_{MM}-1}{2}}(\hat{\gamma}_{MM}) \right]^3 W_{\frac{\hat{\beta}_{MM}-\hat{\alpha}_{MM}-2}{2}, \frac{\hat{\alpha}_{MM}+\hat{\beta}_{MM}+3}{2}}(\hat{\gamma}_{MM})}{\left[ W_{\frac{\hat{\beta}_{MM}-\hat{\alpha}_{MM}-1}{2}, \frac{\hat{\alpha}_{MM}+\hat{\beta}_{MM}}{2}}(\hat{\gamma}_{MM}) \right]^4}, \tag{12}$$

where  $\bar{X}$  is the sample mean, and  $S^2$  is the sample variance.

5.2. Maximum Likelihood Estimates

The log-likelihood function is given by,

$$\ln L = n\alpha \ln \lambda - n \left( \frac{\alpha+\beta}{2} - 1 \right) \ln \gamma - n \ln \Gamma(\alpha) - n \ln W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma) - \lambda \sum_{i=1}^n x_i - n \frac{\gamma}{2} + (\alpha - 1) \sum_{i=1}^n \ln x_i + (\beta - 1) \sum_{i=1}^n \ln(\gamma + \lambda x_i).$$

The MLE of  $\alpha, \beta, \gamma$ , and  $\lambda$  can be obtained by solving numerically the equations  $\frac{\partial \ln L}{\partial \alpha} = 0, \frac{\partial \ln L}{\partial \beta} = 0, \frac{\partial \ln L}{\partial \gamma} = 0$ , and  $\frac{\partial \ln L}{\partial \lambda} = 0$ , respectively:

$$\frac{\partial \ln L}{\partial \lambda} = \frac{n\alpha}{\lambda} - \sum_{i=1}^n x_i + (\beta - 1) \sum_{i=1}^n \frac{x_i}{\gamma + \lambda x_i}, \tag{13}$$

$$\frac{\partial \ln L}{\partial \gamma} = -\frac{n \left( \frac{\alpha+\beta}{2} - 1 \right)}{\gamma} - \frac{n}{2} + (\beta - 1) \sum_{i=1}^n \frac{1}{\gamma + \lambda x_i} - \frac{n}{2} \left( 1 - \frac{\beta - \alpha}{\gamma} \right) + \frac{n}{\gamma} \frac{W_{\frac{\beta-\alpha}{2}+1, \frac{\alpha+\beta-1}{2}}(\gamma)}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)}, \tag{14}$$

$$\frac{\partial \ln L}{\partial \alpha} = n \ln \lambda - n \ln \gamma + \sum_{i=1}^n \ln x_i - \frac{n \left[ \frac{\gamma \frac{\alpha+\beta}{2} e^{-\frac{\gamma}{2}}}{\Gamma(\alpha)} \int_0^\infty e^{-\gamma t} t^{\alpha-1} (1+t)^{\beta-1} \log(t) dt \right]}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)}. \tag{15}$$

where,

$$\frac{\partial}{\partial \alpha} W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma) = \frac{\log(\gamma)}{2} W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma) - \psi(\alpha) W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma) + \frac{\gamma \frac{\alpha+\beta}{2} e^{-\frac{\gamma}{2}}}{\Gamma(\alpha)} \int_0^\infty e^{-\gamma t} t^{\alpha-1} (1+t)^{\beta-1} \log(t) dt,$$

In addition,

$$\frac{\partial \ln L}{\partial \beta} = -n \ln \gamma + \sum_{i=1}^n \ln(\gamma + \lambda x_i) - \frac{n \left[ \frac{\gamma^{\frac{\alpha+\beta}{2}} e^{-\frac{\gamma}{2}}}{\Gamma(\alpha)} \int_0^\infty e^{-\gamma t} t^{\alpha-1} (1+t)^{\beta-1} \log(1+t) dt \right]}{W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma)}, \tag{16}$$

where,

$$\frac{\partial}{\partial \beta} W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma) = \frac{\log(\gamma)}{2} W_{\frac{\beta-\alpha}{2}, \frac{\alpha+\beta-1}{2}}(\gamma) + \frac{\gamma^{\frac{\alpha+\beta}{2}} e^{-\frac{\gamma}{2}}}{\Gamma(\alpha)} \int_0^\infty e^{-\gamma t} t^{\alpha-1} (1+t)^{\beta-1} \log(1+t) dt.$$

and  $\psi(\cdot)$  is the di-gamma function.

### 6. Validations

In this section, we consider the application of *GWD* and *WD* to four different data sets to study the behavior of the parameters  $c, \alpha, \beta, \lambda$ , and  $\gamma$ . Mathematica software is adopted to find the parameter estimates (The computer codes are provided in Appendix B). The four data sets are described below.

Data set 1: The data set consisted of third-party motor insurance data for Sweden (1977), described and analyzed in [32].

Data set 2: The data set represented an uncensored data set corresponding to remission times (in months) of a random sample of 128 patients of bladder cancer reported in [33].

Data set 3: It represented the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported in [34].

Data set 4: It represented waiting times (in minutes) before service of 100 bank customers as discussed [35].

Table 1 lists the model evaluation statistics for fitting the *GWD* and the *WD* together with some other distributions. The best model is shown in bold.

Table 2 presents the values of the Vuong test for *WD* against normal, Johnson  $S_L$ , and log-normal distributions for all data sets.

The variance covariance matrix  $I(\theta)^{-1}$  of the MLEs under the Whittaker distribution for data set 1 are computed as,

$$\begin{pmatrix} 93.8653 & -227.9330 & 0.0029 & -5.3054 \\ -227.9330 & 555.0520 & 0.0154 & 13.0219 \\ 0.0029 & 0.0154 & 0.0163 & 0.0004 \\ -5.3054 & 13.0219 & 0.0004 & 0.3127 \end{pmatrix}$$

Thus, the 95% confidence intervals for  $\alpha, \beta, \gamma$ , and  $\lambda$  were (0, 21.1722), (42.6242, 134.978), (39.733, 40.2333), and (3.6761, 5.8683), respectively. Since  $\alpha$  is always nonnegative, the lower bound of the confidence interval of  $\alpha$  was set to zero.

For *GWD* (*WD*), the value of Pearson’s chi-square test was  $\chi^2 = 1.758$  (2.1865) with  $p$ -value 0.1849 (0.3351), which implies that *GWD* (*WD*) fits the data well. Regarding the maximum values of the log-likelihood, and the minimum values of the AIC and BIC for *GWD*, and *WD*, the results of both models are nearly identical. From the result of fitting  $GWD(\alpha, \beta, \gamma, \lambda, c)$ , the estimated value is  $c = 1.03$ . Therefore, we perform an LRT for the hypothesis  $H_0 : c = 1$  against  $H_0 : c \neq 1$ . The value of the test statistic is  $\chi^2 = 0.18$  with a  $p$ -value of 0.671, providing  $c = 1$ . This implies that we can reduce the model *GWD* to the model *WD* with fewer parameters. Figure 5 illustrates the estimated density of *WD* for data set 1.

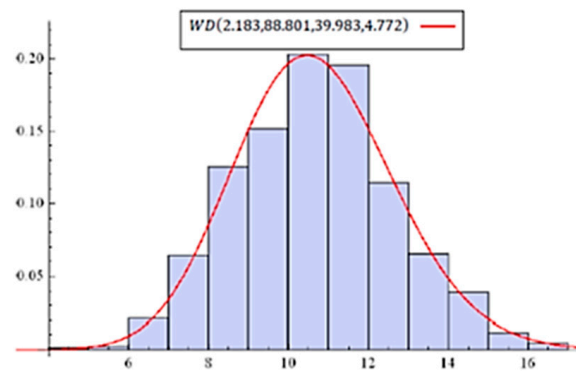
The *WD* distribution is better than the log-normal distribution in terms of log-likelihood, AIC, and BIC. The value of the Vuong test for *WD* versus log-normal is  $V = 2.873$  ( $p$ -value = 0.0041), which implies that the *WD* yielded a significantly better fit than the log-normal distribution. While the *WD* distribution outperforms the normal distribution in terms of log-likelihood and AIC, this was not the case in terms of BIC. The value of the Vuong test for the *WD* distribution versus the normal distribution is  $V = 0.8716$  ( $p$ -value = 0.3834), indicating no significant difference between the models.

**Table 1.** Estimation results for all data sets.

	Data Set 1		Data Set 2		Data Set 3		Data Set 4	
Model	MLE Parameters	Log(L)	MLE Parameters	Log(L)	MLE Parameters	Log(L)	MLE Parameters	Log(L)
GWD	$\alpha = 3.6368$ $\beta = 91.5863$ $\lambda = 4.3822$ $\gamma = 46.4312$ $c = 1.0304$	-3768.62	$\alpha = 0.9679$ $\beta = -3.5285$ $\lambda = 0.0051$ $\gamma = 0.1665$ $c = 1.0921$	-400.762	$\alpha = 3.2583$ $\beta = 1.0196$ $\lambda = 1.8831$ $\gamma = 134.651$ $c = 0.9701$	-94.2268	$\alpha = 32.4000$ $\beta = -29.696$ $\lambda = 0.2028$ $\gamma = 0.0054$ $c = 0.9707$	-316.930
WD	$\alpha = 2.1829$ $\beta = 88.8009$ $\lambda = 4.7722$ $\gamma = 39.9831$	-3768.71	$\alpha = 1.0959$ $\beta = -4.8971$ $\lambda = 0.0033$ $\gamma = 0.1030$	-400.815	$\alpha = 3.0834$ $\beta = 1.0000$ $\lambda = 1.7438$ $\gamma = 2.9426$	-94.2291	$\alpha = 31.5464$ $\beta = -28.9529$ $\lambda = 0.1775$ $\gamma = 0.0050$	-316.931
Log-normal	$\mu = 2.3506$ $\sigma = 0.1899$	-3788.58	$\mu = 1.5109$ $\sigma = 1.2819$	-406.802	$\mu = 0.3991$ $\sigma = 0.629$	-97.5222	$\mu = 2.0211$ $\sigma = 0.7801$	-319.174
Normal	$\mu = 10.6786$ $\sigma = 1.9757$	-3773.45	$\mu = 8.5688$ $\sigma = 10.5191$	-482.883	$\mu = 1.7682$ $\sigma = 1.0273$	-104.106	$\mu = 9.877$ $\sigma = 7.2007$	-339.312
Johnson $S_L$			$\delta = 1.1633$ $\gamma = -2.2532$ $\tau = -1.3757$	-406.447	$\delta = 1.5878$ $\gamma = -1.9284$ $\lambda = 0.7481$ $\tau = 0.3849$	-92.9256	$\delta = 1.2306$ $\gamma = 2.6058$ $\lambda = 78.0372$ $\tau = -0.5843$	-317.086
Generalized Gamma	$\alpha = 31.9737$ $\lambda = 0.1925$ $c = 0.8918$	-3774.44	$\alpha = 1.6997$ $\lambda = 0.2410$ $c = 0.4266$	-401.66	$\alpha = 3.1607$ $\lambda = 0.5208$ $c = 0.9701$	-94.2268	$\alpha = 3.7429$ $\lambda = 0.1178$ $c = 0.4788$	-317.225
Weibull	$\lambda = 11.5022$ $k = 5.8106$	-3818.69	$\lambda = 8.2303$ $k = 0.9227$	-402.191	$\lambda = 1.996$ $k = 1.8254$	-95.790	$\lambda = 10.9553$ $k = 1.4585$	-318.731
Gamma	$\alpha = 28.5388$ $\lambda = 2.6725$	-3773.27	$\alpha = 0.9155$ $\lambda = 0.1068$	-402.624	$\alpha = 3.0834$ $\lambda = 1.7438$	-94.2291	$\alpha = 2.0088$ $\lambda = 0.2034$	-317.3
Exponential			$\lambda = 0.1167$	-402.96				
Generalized Lindley	$\alpha = 28.0971$ $\lambda = 2.6938$ $\gamma = 5.4342$	-3773.13	$\alpha = 0.3594$ $\lambda = 0.1455$ $\gamma = 1.1499$	-407.868	$\lambda = 1.8248$ $\alpha = 2.3367$ $\gamma = 14.7453$	-94.0893	$\lambda = 0.2300$ $\alpha = 1.4949$ $\gamma = 0.8024$	-317.836
Lindley			$\lambda = 0.2129$	-417.924				

**Table 2.** Vuong test of WD vs. some models for all data sets.

	Data Set 1		Data Set 2		Data Set 3		Data Set 4	
WD vs	Test statistic	p-value	Test statistic	p-value	Test statistic	p-value	Test statistic	p-value
Log-normal	2.8733	0.0041	1.4188	0.1559	0.3568	0.7212	0.1284	0.8979
Normal	0.8716	0.3834	4.4671	<0.001	2.0183	0.0436	3.5893	0.0003
Johnson			(Johnson $S_L$ ) 2.3096	0.0209	(Johnson $S_U$ ) -1.0613	0.2885	(Johnson $S_B$ ) 0.3735	0.7088



**Figure 5.** Estimated density of  $WD(\alpha, \beta, \gamma, \lambda)$  for data set 1.

The generalized gamma distribution and Weibull distributions are special cases of the generalized Whittaker distribution. Therefore, there is no need to fit these distributions. The gamma and generalized Lindley distributions are special cases of the Whittaker distribution at  $\beta = 1$  and  $\beta = 2$ , respectively. It is clear that  $\hat{\beta} = 88.8$  in the *WD* is neither one nor two. For the gamma distribution, the value of LRT is  $\chi^2 = 9.12$  ( $p$ -value = 0.0104). Then,  $\beta \neq 1$ . The same conclusion is obtained under the generalized Lindley,  $\beta \neq 2$ , where the value of the test statistic is  $\chi^2 = 8.84$  ( $p$ -value = 0.003).

The variance covariance matrix  $I(\theta)^{-1}$  of the MLEs under the Whittaker distribution for data set 2 was computed as,

$$\begin{pmatrix} 0.0238 & 0.1390 & -2.65 \times 10^{-6} & 0.0002 \\ 0.1390 & 5.9668 & -0.00004 & 0.0051 \\ -2.65 \times 10^{-6} & -0.00004 & 8.32 \times 10^{-6} & 2.16 \times 10^{-7} \\ 0.0002 & 0.0051 & 2.16 \times 10^{-7} & 4.86 \times 10^{-6} \end{pmatrix}$$

Thus, the 95% confidence intervals of  $\alpha, \beta, \gamma$ , and  $\lambda$  are (0.7937, 1.3981), (−9.684, −0.1094), (0.09735, 0.1087), and (0, 0.0076), respectively. The lower bound of the confidence interval of  $\lambda$  is negative. Since  $\lambda$  is always non-negative, zero is considered the lower bound.

The results of Pearson’s chi-square test for *GWD* (*WD*) yield  $\chi^2 = 12.0676$  (12.3476) with a  $p$ -value 0.0984 (0.1944), which implies that *GWD* (*WD*) fits the data well. The result of *GWD* shows that the estimated value of  $c$  is 1.09, and the other parameters are nearly the same as those of *WD*. Therefore, we perform the LRT for hypotheses  $H_0 : c = 1$  and  $H_0 : c \neq 1$ . The value of the test statistic is  $\chi^2 = 0.106$  ( $p$ -value = 0.7447). This implies that we can reduce *GWD* to *WD*, which has fewer parameters. Figure 6 illustrates the estimated density of *WD* for data set 2.

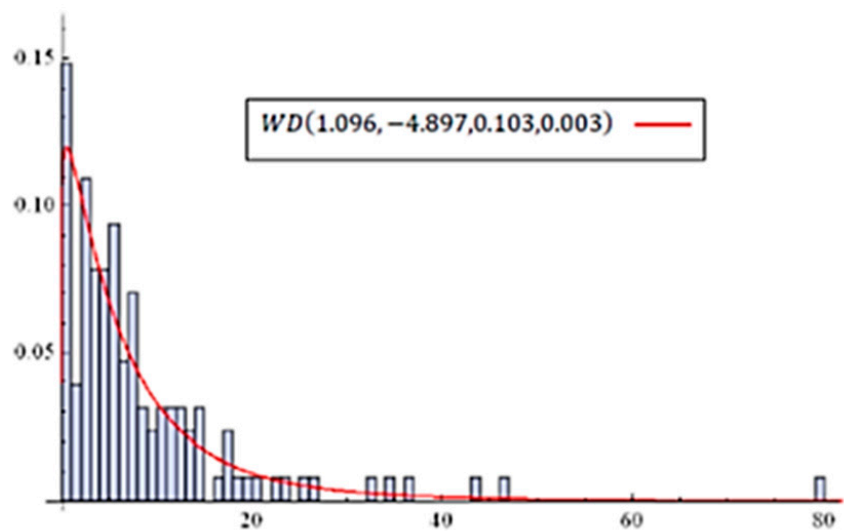


Figure 6. Estimated density of  $WD(\alpha, \beta, \gamma, \lambda)$  for data set 2.

According to the Vuong test, we can conclude that *WD* fits the data better than the normal distribution and Johnson’s  $S_L$  with ( $V = 4.47$  with  $p$ -value  $< 0.001$ ) and ( $V = 2.31$  with  $p$ -value = 0.021), respectively. There is no significant difference between the *WD* distribution and the log-normal distribution with a  $p$ -value of 0.156.

The generalized gamma and Weibull distributions are special cases of *GWD*, and thus are eliminated from the comparison. Using the LRT to check where the data had been drawn from an exponential distribution, the value of the test statistic becomes  $\chi^2 = 4.29$  ( $p$ -value = 0.2318). This implies that the exponential distribution fits the data.

The variance covariance matrix  $I(\theta)^{-1}$  of the MLEs under the Whittaker distribution for data set 3 is computed as follows.

$$\begin{pmatrix} 3.0194 & -14.3655 & 0.0022 & -2.1450 \\ -14.3655 & 74.2163 & -0.0114 & 11.7791 \\ 0.0022 & -0.0114 & 0.0095 & -0.0018 \\ -2.1450 & 11.7791 & -0.0018 & 1.9596 \end{pmatrix}$$

Thus, the 95% confidence intervals of  $\alpha, \beta, \gamma,$  and  $\lambda$  were  $(0, 6.4892), (-15.8852, 17.8852), (2.7513, 3.1339),$  and  $(0, 4.4875),$  respectively. The lower bounds of the confidence intervals of  $\alpha$  and  $\lambda$  are negative; and as both  $\alpha$  and  $\lambda$  are always non-negative, zero is used as lower bound.

We fit the models *WD* and *GWD* to data on survival times. The results of Pearson’s chi-square test for *GWD* (*WD*) yield a value  $\chi^2 = 3.766$  (3.8114) with *p*-value 0.1521 (0.2826), which implies that *GWD* (*WD*) fits the data well. The results (in Table 1) of *GWD* show that the estimated value of *c* is 0.97, and the other parameters are nearly identical to those of *WD*. Using the LRT, the value of the test statistic is  $\chi^2 = 0.005$  (*p*-value = 0.946). This implies that we can reduce the model *GWD* to *WD*, which has fewer parameters. Figure 7 illustrates the estimated density of *WD* for data set 3.

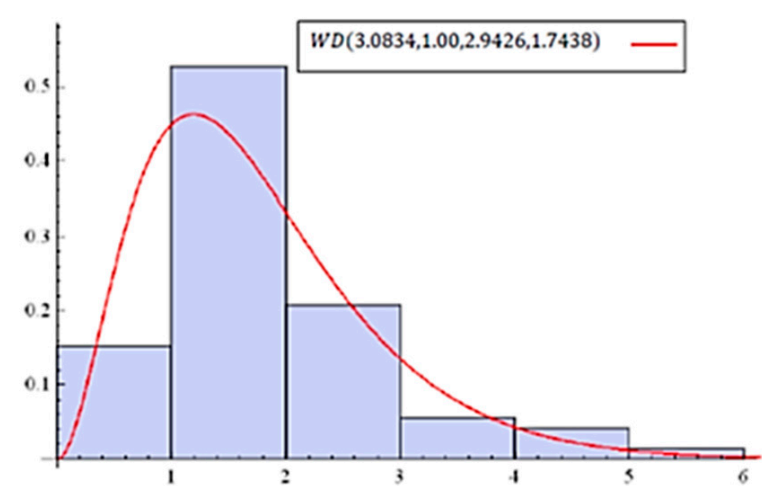


Figure 7. Estimated density of  $WD(\alpha, \beta, \gamma, \lambda)$  for data set 3.

Using the Vuong test (in Table 2), there is no significant difference between the fitted model using the *WD* distribution and all remaining tested distributions.

The generalized gamma and Weibull distributions are a special case of *GWD*, and thus are eliminated from the comparison. From the results of fitting *WD*, we note that the estimated value of  $\beta$  is one, and the other fitting measures of  $\alpha$  and  $\lambda$  are identical to those of  $Gamma(\alpha, \lambda)$ . Therefore, we perform the LRT ( $H_0 : \beta = 1$ ) with a *p*-value of one, with providing  $\beta = 1$ . The generalized Lindley is a special case of the *WD* when  $\beta$  is two. Without testing, we know that the gamma distribution yields a better fit than the generalized Lindley for this data set.

The variance covariance matrix  $I(\theta)^{-1}$  of the MLEs under the Whittaker distribution for data set 4 is computed as follows.

$$\begin{pmatrix} 1442.85 & -1461.15 & -0.0001 & -0.9952 \\ -1461.15 & 1479.8 & 0.0001 & 1.0190 \\ -0.0001 & 0.0001 & 1.78 \times 10^{-8} & -1.67 \times 10^{-9} \\ -0.9952 & 1.0190 & -1.67 \times 10^{-9} & 0.0019 \end{pmatrix}$$



Thus, the 95% confidence intervals for  $\alpha, \beta, \gamma,$  and  $\lambda$  are  $(0, 106.00), (-104.35, 46.4446), (0.0047, 0.0053),$  and  $(0.0922, 0.2627),$  respectively. The lower bound of the confidence interval of  $\alpha$  is negative. As  $\alpha$  is non-negative, zero is used as lower bound.

We fit the models *WD* and *GWD* to data of waiting times. The results of Pearson’s chi-square test for *GWD* (*WD*) yield a value of  $\chi^2 = 10.9955$  (11.0327) with a  $p$ -value 0.202 (0.2735), which implies that *GWD* (*WD*) fits the data well. The result (in Table 1) for *GWD* shows that the estimated value of  $c$  is 0.97. Using the LRT for  $c = 1,$  the value of the test statistic is  $\chi^2 = 0.002$  ( $p$ -value = 0.964). This implies that we can reduce the model *GWD* to *WD,* which has fewer parameters. Figure 8 illustrates the estimated density of *WD* for data set 4.

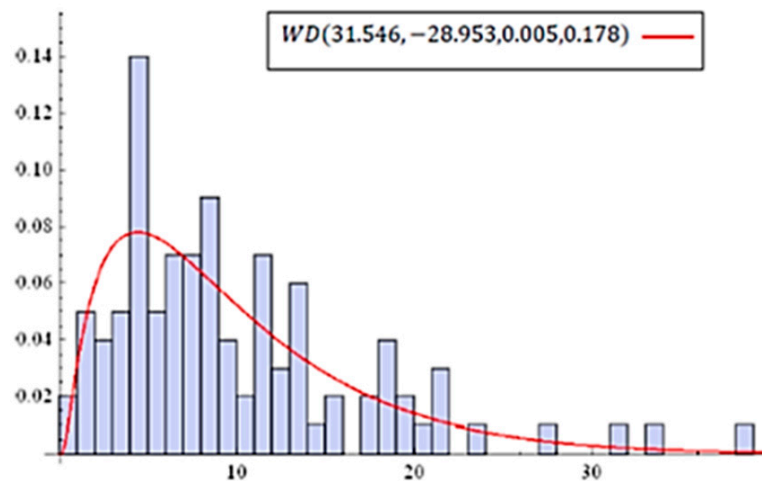


Figure 8. Estimated density of  $WD(\alpha, \beta, \gamma, \lambda)$  for data set 4.

According to the Vuong test, we can conclude that *WD* yields a better fit than the normal with ( $V = 4.47$  with  $p$ -value  $< 0.001$ ), and there is no significant difference between the fitted model using the *WD* distribution, and the log-normal and Johnson’s  $S_B$  distributions. The generalized gamma and Weibull distributions are special cases of *GWD,* and thus are eliminated from the comparison. Using the LRT, the value of the test statistic of  $H_0 : \beta = 2$  was  $\chi^2 = 1.81$  ( $p$ -value = 0.179). This implies that the generalized Lindley distribution fits the data. Finally, the value of the test statistic of the LRT for  $H_0 : \beta = 1$  is  $\chi^2 = 0.738$  ( $p$ -value = 0.691). This implies that the gamma distribution fits the data as well.

### 7. Conclusions

Based on Whittaker function a modified life distribution that is called Whittaker distribution has been introduced. The Whittaker distribution is a generalization of many well-known continuous distributions such as gamma, exponential, chi-square, generalized Lindley, Lindley, beta prime, and Lomax distributions. Furthermore, we have defined the generalized Whittaker distribution that also has the generalized gamma, half-normal, Weibull, half-t, half-Cauchy, Nakagami, Rayleigh, Dagum, Lévy, type-2 Gumbel, inverse gamma, inverse chi-square, inverse exponential, inverse Weibull, and inverse-Rayleigh distributions as special cases. We Validated the proposed distributions on four real data sets. The results of Pearson’s chi-square test confirmed that the Generalized Whittaker distribution and the Whittaker distribution fit the data well for all the four data sets. In some applications, the vuong test showed that the Whittaker distribution has significantly better fit than the normal, log-normal and Jhonson distributions.

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supervision, A.A.A.; project administration, S.A.; funding acquisition, Y.A.T. All authors have read and agreed to the published version of the manuscript.

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**Data Availability Statement:** Data set 1: The third-party motor insurance data for Sweden for the year 1977 described and analyzed by Hallin and Ingenbleek (1983) in The Swedish automobile portfolio in 1977 [DATA TYPE: the logarithm of the total number which had at least one claim]. The data can be found in [www.math.uni.wroc.pl/~dabr/R/motorins.html](http://www.math.uni.wroc.pl/~dabr/R/motorins.html) (1 March 2022). Data set 2: The data set represents an uncensored data set corresponding to remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang (2003) in Statistical Methods for Survival Data Analysis [DATA TYPE: the remission times (in months)]. The data can be found in <http://www.naturalspublishing.com/files/published/j9wsil53h390x8.pdf> (1 March 2022). Data set 3: The data are given in Bjerkedal (1960) in Acquisition of resistance in guinea pigs infected with different doses of virulent tubercle bacilli [DATA TYPE: the survival times (in days)]. The data can be found in <https://rivista-statistica.unibo.it/article/viewFile/6282/6061> (1 March 2022). Data set 4: The data represent the waiting times (in minutes) before service of 100 bank customers as discussed by Ghitany et al. (2008). The data can be found in <http://biomedicine.imedpub.com/on-generalized-lindley-distribution-and-its-applications-to-model-lifetime-data-from-biomedical-science-and-engineering.pdf> (1 March 2022).

**Conflicts of Interest:** The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

## Appendix A

The code for calculating the 25th percentiles of the Whittaker distribution:

(Code file)

```
Clear[a1,a2,b1,b2,x,f,g,cf]
a1 = 10.5; a2 = 1.25; b = 3.5; v = 2.5;
f[x_]:= Exp[-v/2]/Gamma[a1] b^a1/v^((a1 + a2)/2 - 1)*(x^(a1 - 1) (v + b*x)^(a2 - 1))/WhittakerW[(a2 - a1)/2,(a1 + a2 - 1)/2,v] Exp[-x*b]
cf[u_]:= Exp[-v/2]/Gamma[a1] b^a1/v^((a1 + a2)/2 - 1)*1/WhittakerW[(a2 - a1)/2,(a1 + a2 - 1)/2,v] NIntegrate[x^(a1 - 1) (v + b*x)^(a2 - 1) Exp[-x*b],{x,0,u}]
w1 = FindRoot[Exp[-v/2]/Gamma[a1] b^a1/v^((a1 + a2)/2 - 1)*1/WhittakerW[(a2 - a1)/2,(a1 + a2 - 1)/2,v] NIntegrate[x^(a1 - 1) (v + b*x)^(a2 - 1) Exp[-x*b],{x,0,u}]-0.25,{u,0.12}]
q1 = u /. w1
cf[q1]
{u -> 2.38469}
2.38469
0.25
```

## Appendix B

The computer codes for the maximum likelihood estimation, the Young test and for the variance covariance matrix  $I(\theta)^{-1}$  of the MLEs under the Whittaker distribution for data set 2:

(Code file)

```
y = {0.08,0.52,0.22,0.82,0.62,0.39,0.96,0.19,0.66,0.4,0.26,0.31,0.73,3.48,6.97,25.74,2.54,5.32,14.83,1.05,4.26,17.14,4.34,19.13,6.54,3.36,6.94,13.29,2.46,5.17,10.06,0.9,4.23,7.63,1.35,7.93,3.25,12.03,8.65,13.11,2.26,5.09,9.74,32.15,4.18,7.62,46.12,5.62,18.1,6.25,2.02,22.69,0.2,5.06,9.47,26.31,3.88,7.59,16.62,2.83,11.64,4.4,12.02,6.76,2.09,4.98,13.8,0.51,3.82,10.34,36.66,2.75,11.25,3.02,11.98,4.51,2.07,4.87,9.02,0.5,3.7,7.32,34.26,2.69,5.41,79.05,5.71,1.76,8.53,6.93,8.66,0.4,3.64,7.28,14.77,
```

```

2.69,5.41,17.12,2.87,11.79,4.5,20.28,12.63,23.63,3.57,7.26,14.76,2.64,5.34,10.75,1.26,7.87,1.46,8.37,
3.36,5.49,2.23,7.09,14.24,0.81,5.32,10.66,43.01,4.33,17.36,5.85,2.02,12.07];
n = 128;
w1 = FindRoot[{c*Sum[Log[y[[i]]], {i, 1, n}] - (n*((1/2)*Derivative[0, 1, 0][WhittakerW][(a2 - a1)/2, (1/2)*(a1 + a2 - 1), v] -
(1/2)*Derivative[1, 0, 0][WhittakerW][(a2 - a1)/2, (1/2)*(a1 + a2 - 1), v]))/WhittakerW[(a2 - a1)/2, (1/2)*(a1 + a2 - 1), v] - n*PolyGamma[0, a1] + n*Log[b] - (1/2)*n*Log[v],
Sum[Log[b*y[[i]]^c + v], {i, 1, n}] - (n*((1/2)*Derivative[0, 1, 0][WhittakerW][(a2 - a1)/2,
(1/2)*(a1 + a2 - 1), v] +
(1/2)*Derivative[1, 0, 0][WhittakerW][(a2 - a1)/2, (1/2)*(a1 + a2 - 1), v]))/WhittakerW[(a2 - a1)/2, (1/2)*(a1 + a2 - 1), v] - (1/2)*n*Log[v],
(a2 - 1)*Sum[1/(b*y[[i]]^c + v), {i, 1, n}] - (n*((1/2 - (a2 - a1)/(2*v))*WhittakerW[(a2 - a1)/2, (1/2)*(a1 + a2 - 1), v] - WhittakerW[(a2 - a1)/2 + 1, (1/2)*(a1 + a2 - 1),
v]/v)]/WhittakerW[(a2 - a1)/2, (1/2)*(a1 + a2 - 1), v] - (n*((a1 + a2)/2 - 1))/v - n/2,
(a2 - 1)*Sum[y[[i]]^c/(b*y[[i]]^c + v), {i, 1, n}] - Sum[y[[i]]^c, {i, 1, n}] + (a1*n)/b && a1 > 0
&& v > 0 && b > 0}, {{a1, 1.08}, {a2, -4.57}, {b, 0.005}, {v, 0.15}}]
{a1 - > 1.09587, a2 - > -4.89705, b - > 0.00331532, v - > 0.103035}
Clear[a1, a2, v, b, c]
w = FindRoot[{n*Log[b] - (1/2)*n*Log[v] - n*PolyGamma[0, a1] + c*Sum[Log[y[[i]]],
{i, 1, n}] - (n*((1/2)*Derivative[0, 1, 0][WhittakerW][(1/2)*(-a1 + a2), (1/2)*(-1 + a1 +
a2), v] - (1/2)*Derivative[1, 0, 0][WhittakerW][(1/2)*(-a1 + a2), (1/2)*(-1 + a1 +
a2), v]))/ WhittakerW[(1/2)*(-a1 + a2), (1/2)*(-1 + a1 + a2), v], (-1/2)*n*Log[v] +
Sum[Log[v + b*y[[i]]^c], {i, 1, n}] - (n*((1/2)*Derivative[0, 1, 0][WhittakerW][(1/2)*(-a1 +
a2), (1/2)*(-1 + a1 + a2), v] + (1/2)*Derivative[1, 0, 0][WhittakerW][(1/2)*(-a1 + a2),
(1/2)*(-1 + a1 + a2), v]))/ WhittakerW[(1/2)*(-a1 + a2), (1/2)*(-1 + a1 + a2), v], -(n/2) -
((-1 + (a1 + a2)/2)*n)/v + (-1 + a2)*Sum[1/(v + b*y[[i]]^c), {i, 1, n}] - (n*((1/2 - (a1 +
a2)/(2*v))*WhittakerW[(1/2)*(-a1 + a2), (1/2)*(-1 + a1 + a2), v] - WhittakerW[1 +
(1/2)*(-a1 + a2), (1/2)*(-1 + a1 + a2), v]/v)]/ WhittakerW[(1/2)*(-a1 + a2), (1/2)*(-1 +
a1 + a2), v], (a1*n)/b - Sum[y[[i]]^c, {i, 1, n}] + (-1 + a2)*Sum[y[[i]]^c/(v + b*y[[i]]^c),
{i, 1, n}], n/c + a1*Sum[Log[y[[i]]], {i, 1, n}] - b*Sum[Log[y[[i]]]*y[[i]]^c, {i, 1, n}] + (-1 +
a2)*Sum[(b*Log[y[[i]]]*y[[i]]^c)/(v + b*y[[i]]^c), {i, 1, n}] && a1 > 0 && v > 0 && b > 0}, {{a1,
0.98}, {a2, -3.47}, {b, 0.005}, {v, 0.15}, {c, 1.5}}]
{a1 - > 0.967908, a2 - > -3.52849, b - > 0.00510853, v - > 0.166534, c - > 1.09214}
"Vuong test"
k = 4;
a1 = 1.09587;
a2 = -4.897054;
b = 0.0033153184;
v = 0.103034;
f[x_] := (Exp[-v/2]/Gamma[a1])*(b^a1/v^((a1 + a2)/2 - 1))*((x^(a1 - 1)*(v + b*x)^(a2 -
1))/WhittakerW[(a2 - a1)/2, (a1 + a2 - 1)/2, v])*Exp[(-x)*b]
"Log-normal"
Clear[u, se]
wln = FindRoot[{- (n/se) - Sum[-(( -u + Log[y[[i]]])^2/se^3), {i, 1, n}], -Sum[-(( -u +
Log[y[[i]]])/se^2), {i, 1, n}], {{u, 2.3}, {se, 0.2}}]
se = se /. wln;
u = u /. wln;
fln[x_] := (1/(x*se*Sqrt[2*Pi]))*Exp[-((Log[x] - u)^2/(2*se^2))];
lln = (-n)*Log[Sqrt[2*Pi]] - Sum[Log[y[[i]]], {i, 1, n}] - n*Log[se] - Sum[(Log[y[[i]]] -
u)^2/(2*se^2), {i, 1, n}]
lrln = Sum[Log[fy[[i]]/fln[y[[i]]], {i, 1, n}] - (k - 2);
wnln = (1/n)*Sum[Log[fy[[i]]/fln[y[[i]]]^2, {i, 1, n}] - ((1/n)*Sum[Log[fy[[i]]/fln[y[[i]]],
{i, 1, n}])^2;
vln = lrln/Sqrt[n*wnln]

```

```

2*(1 - CDF[NormalDistribution[], Abs[vln]])
"Normal"
Clear[sn, un, w1, vnor, lnr, wnnor, fn, lnr]
w1 = FindRoot[{- (n/sn) + Sum[(-un + y[[i]])^2, {i, 1, n}]/sn^3, -(Sum[-2*(-un +
y[[i]]), {i, 1, n}]/(2*sn^2))}, {{un, 1}, {sn, 1}}]
un = un /. w1;
sn = sn /. w1;
fn[x_]:= (1/Sqrt[2*Pi*sn^2])*Exp[-((x - un)^2/(2*sn^2))]
lnr = (- (n/2))*Log[2*Pi] - n*Log[sn] - Sum[(y[[i]] - un)^2, {i, 1, n}]/(2*sn^2)
lnr = Sum[Log[f[y[[i]]]/fn[y[[i]]], {i, 1, n}] - (k - 2);
wnnr = (1/n)*Sum[Log[f[y[[i]]]/fn[y[[i]]]^2, {i, 1, n}] - ((1/n)*Sum[Log[f[y[[i]]]/
fn[y[[i]]], {i, 1, n}])^2;
vnr = lnr/Sqrt[n*wnnr]
2*(1 - CDF[NormalDistribution[], Abs[vnr]])
"Johnson"
Clear[bj, tj, cj]
vjo = -2.25315;
cj = 1.1633;
tj = -1.37569;
fj[x_]:= (cj/Sqrt[2*Pi])*(1/(x - tj))*Exp[(-2^(-1))*(vjo + cj*Log[x - tj])^2]
lnj = n*Log[cj/Sqrt[2*Pi]] - Sum[Log[y[[i]] - tj], {i, 1, n}] - (1/2)*Sum[(vjo + cj*Lo
g[y[[i]] - tj])^2, {i, 1, n}]
lrj = Sum[Log[f[y[[i]]]/fj[y[[i]]], {i, 1, n}] - (k - 3);
wnj = (1/n)*Sum[Log[f[y[[i]]]/fj[y[[i]]]^2, {i, 1, n}] - ((1/n)*Sum[Log[f[y[[i]]]/f
j[y[[i]]], {i, 1, n}])^2;
vj1 = lrj/Sqrt[n*wnj]
2*(1 - CDF[NormalDistribution[], Abs[vj1]])
Vuong test0e
Log-normal
{u - > 1.51087, se - > 1.28189}
-406.802
1.41883
0.155948
Normal
{un - > 8.56875, sn - > 10.5191}
-482.833
4.46717
7.92596 × 10-6
Johnson
-406.447
2.30962
0.0209092
"CI"
a11 = n*((v^((a1 + a2)/2)*Exp[-v/2])/Gamma[a1])*(NIntegrate[Exp[(-v)*t]^t^(a1 -
1)*(1 + t)^(a2 - 1)*Log[t]^2, {t, 0, Infinity}]/WhittakerW[(a2 - a1)/2, (a1 + a2 - 1)/2, v]) -
n*((v^((a1 + a2)/2)*Exp[-v/2])/Gamma[a1])*(NIntegrate[Exp[(-v)*t]^t^(a1 - 1)*(1 + t)^(a2
- 1)*Log[t], {t, 0, Infinity}]/WhittakerW[(a2 - a1)/2, (a1 + a2 - 1)/2, v])^2;
a12 = n*((v^((a1 + a2)/2)*Exp[-v/2])/Gamma[a1])*(NIntegrate[Exp[(-v)*t]^t^(a1 -
1)*(1 + t)^(a2 - 1)*Log[t]*Log[1 + t], {t, 0, Infinity}]/WhittakerW[(a2 - a1)/2, (a1 + a2 -
1)/2, v]) - n*((v^((a1 + a2)/2)/Gamma[a1])*(Exp[-v/2]/WhittakerW[(a2 - a1)/2, (a1 +
a2 - 1)/2, v]))^2*NIntegrate[Exp[(-v)*t]^t^(a1 - 1)*(1 + t)^(a2 - 1)*Log[t], {t, 0, Infinity}]*
NIntegrate[Exp[(-v)*t]^t^(a1 - 1)*(1 + t)^(a2 - 1)*Log[1 + t], {t, 0, Infinity}];
a13 = -(-(n/v) - n*(a2 - v)*(v^((a1 + a2)/2 - 1)*Exp[-v/2])/Gamma[a1])*(NInteg
rate[Exp[(-v)*t]^t^(a1 - 1)*(1 + t)^(a2 - 1)*Log[t], {t, 0, Infinity}]/

```

```

WhittakerW[(a2 - a1)/2, (a1 + a2 - 1)/2, v]) + n*((v^((a1 + a2)/2)*Exp[-v/2])/Gamma
a[a1])*(NIntegrate[Exp[(-v)*t]*t^a1*(1 + t)^(a2 - 1)*Log[t], {t, 0, Infinity}]/ WhittakerW[(a2
- a1)/2, (a1 + a2 - 1)/2, v]) - n*((v^((a1 + a2)/2 - 1)*Exp[-v/2])/Gamma[a1])*Whittaker
W[(a2 - a1)/2 + 1, (a1 + a2 - 1)/2, v]^*
(NIntegrate[Exp[(-v)*t]*t^(a1 - 1)*(1 + t)^(a2 - 1)*Log[t], {t, 0, Infinity}]/Whittaker
W[(a2 - a1)/2, (a1 + a2 - 1)/2, v]^2));
a14 = -(n/b);
a22 = n*((v^((a1 + a2)/2)*Exp[-v/2])/Gamma[a1])*(NIntegrate[Exp[(-v)*t]*t^(a1 -
1)*(1 + t)^(a2 - 1)*Log[1 + t]^2, {t, 0, Infinity}]/WhittakerW[(a2 - a1)/2, (a1 + a2 - 1)/2, v])
- n*((v^((a1 + a2)/2)*Exp[-v/2])/Gamma[a1])*(NIntegrate[Exp[(-v)*t]*t^(a1 - 1)*(1 +
t)^(a2 - 1)*Log[1 + t], {t, 0, Infinity}]/WhittakerW[(a2 - a1)/2, (a1 + a2 - 1)/2, v])^2;
a23 = -(-(n/v) + Sum[1/(v + b*y[[i]]), {i, 1, n}]) + n*((v^((a1 + a2)/2)*Exp[-v/2])/Gamma
a[a1])*(NIntegrate[Exp[(-v)*t]*t^a1*(1 + t)^(a2 - 1)*Log[1 + t], {t, 0, Infinity}]/WhittakerW[(a2
- a1)/2, (a1 + a2 - 1)/2, v]) + n*(1 - a2/v)*((v^((a1 + a2)/2)*Exp[-v/2])/Gamma[a1])*(N
Integrate[Exp[(-v)*t]*t^(a1 - 1)*(1 + t)^(a2 - 1)*Log[1 + t], {t, 0, Infinity}]/WhittakerW[(a2
- a1)/2, (a1 + a2 - 1)/2, v]) - (n/v)*((v^((a1 + a2)/2)*Exp[-v/2])/Gamma[a1])*Whittaker
W[(a2 - a1)/2 + 1, (a1 + a2 - 1)/2, v]^* (NIntegrate[Exp[(-v)*t]*t^(a1 - 1)*(1 + t)^(a2 -
1)*Log[1 + t], {t, 0, Infinity}]/WhittakerW[(a2 - a1)/2, (a1 + a2 - 1)/2, v]^2));
a24 = -(n/b - Sum[v/((v + b*y[[i]])*b), {i, 1, n}]);
a33 = -((n/v^2)*(a1 - 1) + (n/v^2)*(v/2 - (a2 - a1)/2 - 2)*(WhittakerW[(a2 - a1)/2
+ 1, (a1 + a2 - 1)/2, v]/WhittakerW[(a2 - a1)/2, (a1 + a2 - 1)/2, v]) -
(n/v^2)*(WhittakerW[(a2 - a1)/2 + 2, (a1 + a2 - 1)/2, v]/WhittakerW[(a2 - a1)/2,
(a1 + a2 - 1)/2, v]) - (a2 - 1)*Sum[1/(v + b*y[[i]])^2, {i, 1, n}]);
a34 = (a2 - 1)*Sum[y[[i]]/(v + b*y[[i]])^2, {i, 1, n}];
a44 = -(-(n*(a1 + a2 - 1))/b^2) + (-1 + a2)*Sum[(y[[i]]*v)/(b*(v + b*y[[i]])^2), {i, 1,
n}] + (-1 + a2)*Sum[v/(b^2*(v + b*y[[i]])), {i, 1, n}];
beta = {{a11, a12, a13, a14}, {a12, a22, a23, a24}, {a13, a23, a33, a34}, {a14, a24, a34, a44}};
inbeta = Inverse[beta];
MatrixForm[inbeta]
CI
({{0.0237651, 0.139018, -2.65017 × 10-6, 0.00020543},
{0.139018, 5.96683, -0.0000375354, 0.00511258},
{-2.65017 × 10-6, -0.0000375354, 8.32295 × 10-6, 2.16077 × 10-7},
{0.00020543, 0.00511258, 2.16077 × 10-7, 4.85993 × 10-6}})
"a1"
Sqrt[inbeta[[1,1]]]
"p-value"
2*(1 - CDF[NormalDistribution[], a1/Sqrt[inbeta[[1,1]]]])
a1 + 1.96*Sqrt[inbeta[[1,1]]]
a1 - 1.96*Sqrt[inbeta[[1,1]]]
"a2"
Sqrt[inbeta[[2,2]]]
"p-value"
2*(1 - CDF[NormalDistribution[], Abs[a2/Sqrt[inbeta[[2,2]]]])]
a2 + 1.96*Sqrt[inbeta[[2,2]]]
a2 - 1.96*Sqrt[inbeta[[2,2]]]
"v"
Sqrt[inbeta[[3,3]]]
"p-value"
2*(1 - CDF[NormalDistribution[], v/Sqrt[inbeta[[3,3]]]])
v + 1.96*Sqrt[inbeta[[3,3]]]
v - 1.96*Sqrt[inbeta[[3,3]]]
"b"
Sqrt[inbeta[[4,4]]]

```

“p-value”  
 $2 \cdot (1 - \text{CDF}[\text{NormalDistribution}[], b/\text{Sqrt}[\text{inbeta}[[4,4]]]])$   
 $b + 1.96 \cdot \text{Sqrt}[\text{inbeta}[[4,4]]]$   
 $b - 1.96 \cdot \text{Sqrt}[\text{inbeta}[[4,4]]]$   
 a1  
 0.154159  
 p-value  
 $1.16995 \times 10^{-12}$   
 1.39805  
 0.793748  
 a2  
 2.44271  
 p-value  
 0.0449863  
 −0.10939  
 −9.68481  
 v  
 0.00288495  
 p-value  
 0.  
 0.108655  
 0.0973455  
 b  
 0.00220453  
 p-value  
 0.134414  
 0.00762087  
 −0.00102087

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