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Conformal Control Tools for Statistical Manifolds and for γ-Manifolds

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Abstract: The theory of statistical manifolds w.r.t. a conformal structure is reviewed in a creative manner and developed. By analogy, the γ-manifolds are introduced. New conformal invariant tools are defined. A necessary condition for the f-conformal equivalence of γ-manifolds is found, extending that for the α-conformal equivalence for statistical manifolds. Certain examples of these new defined geometrical objects are given in the theory of Iinformation.

Keywords: γ-manifolds; statistical manifolds; dual connections; conformal connections; Fisher metric; control tools; Weyl manifolds; f-conformal equivalence

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1. Introduction

The notion of statistical manifold was defined by S. Amari in [1], as a Riemannian manifold (M, g) endowed with a torsion-free affine connection ∇ such that

$$(\nabla_Z g)(X, Y) = (\nabla_Y g)(X, Z) \quad (1)$$

Since then, a huge literature ensued, where various techniques and notions from differential geometry were applied in information geometry, via statistics (see, for example, [2–6]). In [7], we included an expository part, devoted to reviewing, clarifying and extending the classical framework of statistical manifolds and of their dual connections ∇^1 and ∇^2, which satisfy the identity

$$Xg(Y, Z) - g(\nabla^1_X Y, Z) - g(Y, \nabla^2_X Z) = 0.$$

The notions, techniques and results from conformal geometry have widely been used in the study of statistical manifolds. The α-conformal equivalence and its relevance for the α-connections were considered in [8,9]. Conformal transformations of the Fisher metric for exponential families were studied in [10], with application to sequential estimation. In [11], 1-conformally equivalent statistical manifolds are characterized. Topological properties of some five-dimensional compact, conformally flat statistical manifolds are found in [12]. Conformal submersions with horizontal distribution and associated statistical structures were defined and characterized in [13].

Several generalizations of the conformal geometry of statistical manifolds were defined and studied, such as the conformal-projective geometry [14–19] and the geometry of semi-Weyl manifolds in [20].

In this paper, we review in a creative manner and develop the theory of statistical manifolds w.r.t. a conformal structure, together with a natural generalization of them, the γ-manifolds. New conformal invariant tools are defined, and control objects are highlighted, at both the affine differential and the conformal levels. We construct some examples of these new geometrical objects and point out the connection with the theory of information geometry.

In Section 2, we recall definitions and properties of the main invariants related to statistical manifolds and dual connections, especially referring to [7]. We prove Theorem 1, which provides a characterization of the statistical structures on a given semi-Riemannian manifold.

In Section 3, we define the γ-manifolds, which, for a fixed prescribed γ, satisfy

$$(\nabla_Z g)(X,Y) + (\nabla_Y g)(X,Z) = \gamma(X,Y,Z) \tag{2}$$

or

$$(\nabla_Z g)(X,Y) - (\nabla_Y g)(X,Z) = \gamma(X,Y,Z), \tag{3}$$

instead of (1). The γ-manifolds simultaneously provide a generalization, an extension and an analogue of statistical manifolds, and, at the same time, a generalization of the semi-Weyl manifolds from [18] and of the statistical manifolds "with torsion" from [21,22]. The cubic form γ acts as a control on the parallelism of the semi-Riemannian metric g, in a quite twisted way, which cannot be derived directly from the behavior of the vector fields along curves. We give here a few properties of these new manifolds, for the use of later sections only, and postpone their systematic study for a further paper. In particular, a characterization of γ-manifolds is proven, which provides hints about how to construct examples.

In Section 4, we consider a Weyl structure $\hat{g}$ on (M,g) and construct statistical structures and γ-structures, depending on conformal invariants. As a byproduct, several methods for constructing γ-manifolds are highlighted, including one involving f-connections.

Section 5 contains a two-fold generalization of the notion of α-conformal equivalence, which is called f-conformal equivalence: firstly, we consider as the control function f instead of number α; secondly, instead of statistical manifolds, we work with γ-manifolds. We find a necessary condition for the f-conformal equivalence (Theorem 4); we prove two corollaries and a characterization for f-conformal equivalence for γ-manifolds (the Theorem 5), which extend similar results from [9].

Section 6 is devoted to examples of γ-manifolds. Detailed formulas are provided in dimension two.

2. Tool Box Remainder: Dual Connections and Statistical Manifolds

We begin this section by recalling some definitions and results form our paper [7]. Consider a semi-Riemannian manifold (M,g) with the Levi–Civita connection ∇^0. Denote by $\mathcal{F}(M), \mathcal{C}(M), \mathcal{C}_s(M)$ the sets of smooth real valued functions on M, of affine connections and of symmetric (i.e., torsion-free) connections on M, respectively (w.r.t. the canonical structure of $\mathcal{F}(M)$-module). For a connection $\nabla \in \mathcal{C}(M)$, there exists a unique $A \in \mathcal{T}_2^1(M)$, such that $\nabla = \nabla^0 + A$. Its torsion tensor T^∇ is given by the formula

$$T^\nabla(X,Y) = A(X,Y) - A(Y,X).$$

The affine differential control A measures how much a connection differs from ∇^0. We define [7]

$$\mathcal{C}_m(M,g) := \{\nabla \in \mathcal{C}(M) \mid (\nabla_Z g)(X,Y) = 0\},$$

$$\mathcal{C}_c(M,g) := \{\nabla \in \mathcal{C}(M) \mid (\nabla_Z g)(X,Y) = (\nabla_Y g)(X,Z)\},$$

$$\mathcal{C}_{sc}(M,g) := \mathcal{C}_s(M) \cap \mathcal{C}_c(M,g).$$

Then we derive ([7]) $\mathcal{C}_s(M) \cap \mathcal{C}_m(M,g) = \{\nabla^0\}$ and

$$\begin{aligned}
\mathcal{C}_m(M,g) &= \{\nabla^0 + A \mid g(A(X,Y),Z) + g(A(X,Z),Y) = 0\},\\
\mathcal{C}_c(M,g) &= \{\nabla^0 + A \mid g(A(X,Y),Z) + g(A(X,Z),Y) =\\
&= g(A(Z,Y),X) + g(A(Z,X),Y)\},\\
\mathcal{C}_{sc}(M,g) &= \{\nabla^0 + A \mid g(A(X,Y),Z) = g(A(Z,Y),X)\ ,\ A(X,Y) = A(Y,X)\}.
\end{aligned}$$

Remark 1 ([7]).

(i) *The transformation $\Phi : \mathcal{C}(M) \to \mathcal{C}(M)$ assigns to every $\nabla \in \mathcal{C}(M)$ its dual connection $\nabla^* := \Phi(\nabla)$, by the formula ([1,2])*

$$g(X, \nabla^*_Z Y) = Zg(X,Y) - g(\nabla_Z X, Y) \quad . \tag{4}$$

We see that $\nabla^* g = -\nabla g$. In particular, ∇ satisfies (1) if ∇^* satisfies (1).

(ii) *Let us consider $\nabla = \nabla^0 + A$. Denote by A', $'A \in \mathcal{T}^1_2(M)$ the adjoint operators, through the formula*

$$g(X, A'(Z,Y)) = g(Y, A(Z,X))\ \ ,\ \ g(X, {}'A(Y,Z)) = g(Y, A(X,Z)).$$

We have $\nabla^* := \nabla^0 + A^*$, where $A^* := -A'$.

(iii) *To any pair of conjugate connections (∇, ∇^*), we associate a 1-parameter family of f-connections $\{\nabla^{(f)}\}_{f\in\mathcal{F}(M)}$, called the connections, by*

$$\nabla^{(f)} := \frac{1+f}{2}\nabla + \frac{1-f}{2}\nabla^*.$$

Then $(\nabla^{(-f)}, \nabla^{(f)})$ are dual w.r.t. to g. The function f acts as a differential control tool over the set of connections. We obtain $\nabla^{(f)} = \nabla^0 + A^f$, with

$$A^f := \frac{1+f}{2}A - \frac{1-f}{2}A' \in \mathcal{T}^1_2(M).$$

As a particular case, we obtain the α-connections $\{\nabla^\alpha\}_{\alpha\in\mathbb{R}}$, which are more general than the classical ones ([2,23]), which, in addition, are symmetric.

Consider the semi-Riemannian manifold (M,g), its Levi–Civita connection ∇^0, ∇ and ∇^* dual connections. Denote by T^∇ the torsion tensor field of ∇, defined by

$$T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

A triple (M,g,∇) is a statistical manifold (or a statistical manifold with torsion) if $\nabla \in \mathcal{C}_{sc}(M,g)$ (or $\nabla \in \mathcal{C}_c(M,g)$). We denote also (M,g,∇,∇^*) instead of (M,g,∇).

We remark that the affine and metric properties of $\mathcal{C}_{sc}(M)$ w.r.t. ∇^0 are the main object of study of the theory of statistical manifolds (with or without torsion).

Remark 2 ([7]). *We describe five (equivalent) characterizations of a statistical manifold (M,g,∇):*

(I) *By relation (1) and $T^\nabla = 0$.*

(II) *Through symmetric $A \in \mathcal{T}^1_2(M)$, with*

$$\nabla = \nabla^0 + A\ \ ,\ \ g(A(X,Z),Y) = g(A(Y,Z),X).$$

(III) *Through $T^\nabla = 0$ and $B \in \mathcal{T}^0_3(M)$ such that*

$$(\nabla_X g)(Y,Z) = B(X,Y,Z)\ \ ,\ \ B(X,Y,Z) = B(Y,X,Z).$$

(IV) Through ∇^* in (4), with $T^{\nabla} = T^{\nabla^*} = 0$.
(V) Through ∇^* in (4), with both A and A' symmetric.

The theory of statistical manifolds with symmetric dual connections was generalized for dual connections with torsion, in the sense of Kurose and Matsuzoe [21,22,24] (apud [21]). These statistical manifolds were denoted generically with the acronym SMAT. They satisfy

$$(\nabla_Z g)(X,Y) - (\nabla_Y g)(X,Z) = -g(T(Z,Y),X). \tag{5}$$

Contrary to the common sense belief, the SMAT are not the only statistical-like structures involving connections with torsion, as their denomination might suggest. For example, in [7], we defined (another) nine new similar families of generalized statistical manifolds (with torsion), denoted SMAT$_i$, for $i = \overline{1,9}$. Perhaps a better formulation might be "the SMAT is a particular family of statistical manifolds with torsion".

Remark 3. *Let (M,g) be a semi-Riemannian manifold, a and b a (1,2)-type and a (0,3)-type tensor fields on M, respectively. Suppose a is skew-symmetric and $b(X,Y,Z) = b(X,Z,Y)$, for all vector fields X,Y,Z on M. Then, it is well-known that there exists a unique connection ∇ on M, such that $T^{\nabla} = a$ and $(\nabla_X g)(Y,Z) = b(X,Y,Z)$. This property, together with the relation (5), suggests the following result, which completely characterizes the SMAT structures on (M,g) and also completes the previous characterization of $\mathcal{C}_c(M,g)$.*

Theorem 1. *Let (M,g) be a semi-Riemannian manifold and α a skew-symmetric (1,2)-type tensor field on M. Then there exists a (not necessarily unique) connection ∇ on M, such that $T^{\nabla} = \alpha$ and relation (5) holds. These connections are completely determined by*

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - \\ & -g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]) + \\ & -\frac{1}{2}g(X,\alpha(Y,Z)) + \frac{1}{2}g(Y,\alpha(Z,X)) + \frac{1}{2}g(Z,\alpha(X,Y)) + \\ & +\frac{1}{2}\{\lambda(X,Y,Z) - \lambda(Y,Z,X) - \lambda(Z,X,Y)\}, \end{aligned} \tag{6}$$

where λ is an arbitrary (0,3)-type tensor field on M, satisfying $\lambda(X,Y,Z) = \lambda(X,Z,Y)$ and

$$\lambda(X,Y,Z) - g(\alpha(Z,Y),X) = \lambda(Y,X,Z) - g(\alpha(Z,X),Y),$$

for all vector fields X,Y,Z on M. Moreover,

$$(\nabla_Z g)(X,Y) + (\nabla_Y g)(X,Z) = \lambda(X,Y,Z)\ .$$

The theorem is a consequence of the following lemma, where we replaced $\beta(X,Y,Z) := -g(\alpha(Z,Y),X)$.

Lemma 1. *Let (M,g) be a semi-Riemannian manifold, α a skew-symmetric (1,2)-type tensor field and β a (0,3)-type tensor field on M, such that $\beta(X,Y,Z) = -\beta(X,Z,Y)$, for all vector fields X,Y,Z on M. Then there exists a (not necessarily unique) connection ∇ on M, such that $T^{\nabla} = \alpha$ and*

$$(\nabla_Z g)(X,Y) - (\nabla_Y g)(X,Z) = \beta(X,Y,Z). \tag{7}$$

These connections are completely determined by

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - \\ & -g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]) - \end{aligned} \tag{8}$$

$$-g(X,\alpha(Y,Z))+g(Y,\alpha(Z,X))+g(Z,\alpha(X,Y))+$$
$$+\beta(X,Y,Z)+\frac{1}{2}\{\lambda(X,Y,Z)-\lambda(Y,Z,X)-\lambda(Z,X,Y)\},$$

where λ is an arbitrary (0,3)-type tensor fields on M, satisfying $\lambda(X,Y,Z)=\lambda(X,Z,Y)$, $\lambda(X,Y,Z)+\beta(X,Y,Z)=\lambda(Y,X,Z)+\beta(Y,X,Z)$, for all vector fields X,Y,Z on M. Moreover,

$$(\nabla_Z g)(X,Y)+(\nabla_Y g)(X,Z)=\lambda(X,Y,Z). \tag{9}$$

Proof. The proof is standard, by analogy with the proof of the classical result quoted in the Remark 3.

The uniqueness is as follows: Suppose there exist two connections ∇^1 and ∇^2, which satisfy the relation (8). Subtracting the two relations, we obtain $g(\nabla^1_X Y-\nabla^2_X Y,Z)=0$, for all vector fields X,Y,Z, which proves that $\nabla^1=\nabla^2$.

The existence is as follows: Let λ be an arbitrary (0,3)-type tensor fields on M, satisfying $\lambda(X,Y,Z)=\lambda(X,Z,Y)$ and $\lambda(X,Y,Z)+\beta(X,Y,Z)=\lambda(Y,X,Z)+\beta(Y,X,Z)$, for all vector fields X,Y,Z. We define, formally, a mathematical function $\nabla:\mathcal{X}(M)\times\mathcal{X}(M)\to\mathcal{X}(M)$, by (8). Direct checking shows ∇ has the properties of a connection in $\mathcal{C}(M)$. Moreover, $g(T^\nabla(X,Y),Z)=g(\alpha(X,Y),Z)$, so $T^\nabla=\alpha$. It follows also that (7) and (9) hold true. □

Corollary 1. *Let (M,g) be a semi-Riemannian manifold. Then there exists a (not necessarily unique) symmetric connection ∇ on M, such that*

$$(\nabla_Z g)(X,Y)-(\nabla_Y g)(X,Z)=0. \tag{10}$$

Moreover, such connections are completely determined by

$$2g(\nabla_X Y,Z)=Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)- \tag{11}$$
$$-g(X,[Y,Z])+g(Y,[Z,X])+g(Z,[X,Y])+$$
$$+\frac{1}{2}\{\lambda(X,Y,Z)-\lambda(Y,Z,X)-\lambda(Z,X,Y)\},$$

where λ is an arbitrary (0,3)-type tensor field on M, satisfying $\lambda(X,Y,Z)=\lambda(X,Z,Y)$, $\lambda(X,Y,Z)=\lambda(Y,X,Z)$, for all vector fields X,Y,Z on M. Moreover, λ verifies also the relation (9).

Remark 4.

(i) *Formula (11) shows that*

$$4g(\nabla_X Y-\nabla^0_X Y,Z)=\lambda(X,Y,Z)-\lambda(Y,Z,X)-\lambda(Z,X,Y)$$

and must be compared with the previous characterization of $\mathcal{C}_c(M)$ and with the Remark 2, (III).

(ii) *Using Lemma 1, Theorem 1 can be generalized, by replacing Formula (5) in the hypothesis with*

$$(\nabla_Z g)(X,Y)-(\nabla_Y g)(X,Z)=g(E(Z,Y),X),$$

for any fixed arbitrary skew-symmetric (1,2)-tensor field E. For $E=-T^\nabla$, we recover Theorem 1.

(iii) *Formula (8) is written in a more "symmetric" form also as*

$$2g(\nabla_X Y,Z)=Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)-$$
$$-g(X,[Y,Z])+g(Y,[Z,X])+g(Z,[X,Y])-$$
$$-g(X,\alpha(Y,Z))+g(Y,\alpha(Z,X))+g(Z,\alpha(X,Y))+$$

$$+\frac{1}{2}\{\beta(X,Y,Z)-\beta(Y,Z,X)-\beta(Z,X,Y)\}+$$
$$+\frac{1}{2}\{\lambda(X,Y,Z)-\lambda(Y,Z,X)-\lambda(Z,X,Y)\}.$$

3. Variations on the Same Theme: γ-Manifolds

Let (M,g) be a semi-Riemannian manifold and γ a (0,3)-tensor field on M, satisfying $\gamma(X,Y,Z)=\gamma(X,Z,Y)$. Denote $\mathcal{C}_{+c}(M,g,\gamma):=\{\nabla\in\mathcal{C}(M)\mid\nabla\text{ satisfies }(2)\}$ and $\mathcal{C}_{+sc}(M,g,\gamma):=\mathcal{C}_{+c}(M,g,\gamma)\cap\mathcal{C}_s(M)$.

Definition 1. *A triple (M,g,∇) is called $(+\gamma)$-manifold without torsion, or $(+\gamma)$- manifold (with torsion) if $\nabla\in\mathcal{C}_{+sc}(M,g,\gamma)$, or $\nabla\in\mathcal{C}_{+c}(M,g,\gamma)$, respectively.*

If (M,g,∇) is a $(+\gamma)$-manifold, then (M,g,∇^*) is a $(+(-\gamma))$-manifold, too. Moreover, $\mathcal{C}_{+c}(M,g,0)\cap\mathcal{C}_c(M,g)=\mathcal{C}_m(M,g)$ and $\mathcal{C}_{+sc}(M,g,0)\cap\mathcal{C}_{sc}(M,g)=\{\nabla^0\}$. We shall improve these obvious properties in Theorem 3.

Let (M,g) be a semi-Riemannian manifold and γ a (0,3)-tensor field on M, satisfying $\gamma(X,Y,Z)=-\gamma(X,Z,Y)$. Denote $\mathcal{C}_{-c}(M,g,\gamma):=\{\nabla\in\mathcal{C}(M)\mid\nabla\text{ whichsatisfies }(3)\}$ and $\mathcal{C}_{-sc}(M,g,\gamma):=\mathcal{C}_{-c}(M,g,\gamma)\cap\mathcal{C}_s(M)$.

Definition 2. *A triple (M,g,∇) is called $(-\gamma)$-manifold without torsion, or $(-\gamma)$- manifold (with torsion) if $\nabla\in\mathcal{C}_{-sc}(M,g,\gamma)$, or $\nabla\in\mathcal{C}_{-c}(M,g,\gamma)$, respectively.*

The $(+\gamma)$-manifolds and the $(-\gamma)$-manifolds are called, shortly, γ-manifolds.

Lemma 1 provides examples of $(-\gamma)$-manifolds with and without torsion. In particular, for $\gamma(X,Y,Z):=0$, we obtain the statistical manifolds (with and without torsion); for $\gamma(X,Y,Z):=g(X,T(Y,Z))$, we recover the SMATs; with minor differences, we may adapt this remark for other SMAT$_i$s (see [7]).

Remark 5.

(i) *Every semi-Riemannian manifold is a $(-\gamma_1)$-manifold and a $(+\gamma_2)$-manifold, once we fix some arbitrary connection: the (unprescribed) γ_1 and γ_2 can directly be derived from the metric and the connection. This remark does not make the notion of γ-manifold useless: the key property for the previous definitions is the fact that γ must be a priori prescribed, and hence it imposes strong constraints on the structure of the manifold (but which are, however, weaker than the parallelism of the metric or than the property settled by relation (1)). From a "dynamic" viewpoint, we may interpret γ as a (differential) control tool over the set of the semi-Riemannian metrics on M and/or the set $\mathcal{C}(M)$.*

(ii) *Apparently, the centro-affine properties of $\mathcal{C}_{+sc}(M,g,\gamma)$, $\mathcal{C}_{-sc}(M,g,\gamma)$, $\mathcal{C}_{+c}(M,g,\gamma)$ and $\mathcal{C}_{-c}(M,g,\gamma)$ w.r.t. ∇^0 (together with the metric properties) are similar, formally, to those from the theory of statistical manifolds (with or without torsion). For example, the α-connections from Remark 1, (iii) can be adapted for γ-manifolds, accordingly. However, the deep geometric and statistical properties are quite different in their essence. In what follows, this paragraph depicts some few of the former, and we postpone a detailed study to a further paper.*

(iii) *We point out here an interesting analogy: from the very beginning, the theory of statistical manifolds was determined by the properties of some specific cubic forms. Starting with the definition of statistical manifolds by Lauritzen [25], the cubic form (the "skewness" tensor field) $(X,Y,Z)\rightsquigarrow(\nabla_Z g)(X,Y)$ was fundamental in describing the symmetries of the models. Our cubic forms are related to it, but only as its symmetric and skew-symmetric parts, respectively. In this sense, through ϵ and the γs, we impose weaker but, at the same time, nuanced and calibrated geometric hypothesis. However, we cannot, for the moment, associate precise statistical interpretation of these objects.*

The following lemma is the counterpart of Lemma 1, with a similar proof, which is skipped.

Lemma 2. *Let (M, g) be a semi-Riemannian manifold, α a skew-symmetric (1,2)-type tensor field and γ a (0,3)-type tensor field on M, such that $\gamma(X, Y, Z) = \gamma(X, Z, Y)$, for all vector fields X, Y, Z on M. Then there exists a (not necessarily unique) connection ∇ on M, such that $T^{\nabla} = \alpha$ and*

$$(\nabla_Z g)(X, Y) + (\nabla_Y g)(X, Z) = \gamma(X, Y, Z). \tag{12}$$

Moreover, such connections are completely determined by

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - \\ & -g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]) - \\ & -g(X, \alpha(Y, Z)) + g(Y, \alpha(Z, X)) + g(Z, \alpha(X, Y)) + \\ & +\frac{1}{2}\{\beta(X, Y, Z) - \beta(Y, Z, X) - \beta(Z, X, Y)\} + \\ & +\frac{1}{2}\{\gamma(X, Y, Z) - \gamma(Y, Z, X) - \gamma(Z, X, Y)\}, \end{aligned} \tag{13}$$

where β is an arbitrary (0,3)-type tensor fields on M, satisfying $\beta(X, Y, Z) = -\beta(X, Z, Y)$, $\gamma(X, Y, Z) + \beta(X, Y, Z) = \gamma(Y, X, Z) + \beta(Y, X, Z)$, for all vector fields X, Y, Z on M. Moreover,

$$(\nabla_Z g)(X, Y) - (\nabla_Y g)(X, Z) = \beta(X, Y, Z). \tag{14}$$

Theorem 2. *Given the semi-Riemannian manifold (M, g) and γ a (0,3)-tensor field on M, satisfying $\gamma(X, Y, Z) = \gamma(X, Z, Y)$, the following three (equivalent) characterizations of a γ-manifold (M, g, ∇) hold true:*

(I) *By relation (2) and $T^{\nabla} = 0$.*

(II) *Through $A \in \mathcal{T}_2^1(M)$ such that $\nabla = \nabla^0 + A$, with*

$$-g(A(X, Z), Y) - g(A(X, Y), Z) = 2g(A(Y, Z), X) + \gamma(X, Y, Z)\ ,\ A(X, Y) = A(Y, X).$$

(III) *Through $T^{\nabla} = 0$ and $B \in \mathcal{T}_3^0(M)$ such that*

$$(\nabla_X g)(Y, Z) = B(X, Y, Z)\ \ ,\ \ B(Z, X, Y) + B(Y, X, Z) = \gamma(X, Y, Z).$$

The previous result is quite similar to that quoted in the Remark 2 and we omit its proof.

Theorem 3. *Let (M, g) be a semi-Riemannian manifold. Then $\mathcal{C}_{+c}(M, g, 0) = \mathcal{C}_m(M, g)$ and $\mathcal{C}_{+sc}(M, g, 0) = \{\nabla^0\}$.*

Proof. Suppose $(\nabla_Z g)(X, Y) + (\nabla_Y g)(X, Z) = 0$. Then

$$\begin{aligned} (\nabla_Z g)(X, Y) &= -(\nabla_Y g)(X, Z) = -(\nabla_Y g)(Z, X) = \\ &= (\nabla_X g)(Z, Y) = (\nabla_Y g)(X, Z) = -(\nabla_Z g)(X, Y). \end{aligned}$$

It follows that $(\nabla_Z g) = 0$. Moreover, the case when ∇ is symmetric is analogous. □

Remark 6. *Formula (13) shows that*

$$\begin{aligned} 4g(\nabla_X Y - \nabla_X^0 Y, Z) = {} & -2g(X, \alpha(Y, Z)) + 2g(Y, \alpha(Z, X)) + 2g(Z, \alpha(X, Y)) + \\ & +\beta(X, Y, Z) - \beta(Y, Z, X) - \beta(Z, X, Y) + \end{aligned}$$

$$+\gamma(X,Y,Z)-\gamma(Y,Z,X)-\gamma(Z,X,Y)=$$
$$=-2g(X,\alpha(Y,Z))+2g(Y,\alpha(Z,X))+2g(Z,\alpha(X,Y))+$$
$$+\beta(X,Y,Z)+\gamma(X,Y,Z)-\gamma(Y,Z,X)-\gamma(Z,X,Y)$$

and must be compared with the previous characterization of $\mathcal{C}_{+sc}(M,g)$ and with Theorem 2, (III). It can be used to construct examples of $(+\gamma)$-structures (with or without torsion).

Proposition 1. *Let $\epsilon=\pm1$, a $(sgn(\epsilon)\gamma)$-manifold (M,g,∇) and ∇^* the dual of ∇. Then the triple (M,g,∇^*) is a $(sgn(\epsilon)(-\gamma))$-manifold.*

Proposition 2. *Let $\epsilon=\pm1$, a function $f\in\mathcal{F}(M)$ and a $(sgn(\epsilon)\gamma)$-manifold (M,g,∇). Then the triple $(M,g,\nabla^{(f)})$ is a $(sgn(\epsilon)(f\gamma))$-manifold.*

4. Statistical Structures and γ-Structures on Weyl Manifolds

Let g be a semi-Riemannian metric on M and $\hat{g}=\{e^u g|u\in\mathcal{F}(M)\}$ be the conformal (equivalence) class defined by g. All the metrics in $\hat{g}$ have the same index (as g).

Proposition 3. *The conformal class $\hat{g}$ is convex.*

Proof. Let $h^{(1)}$, $h^{(0)}\in\hat{g}$, $h^{(i)}=e^{u^{(i)}}g$ and $t\in[0,1]$. Then $t\,h^{(1)}+(1-t)h^{(0)}=e^u g$, where $u:=ln\{te^{u^{(1)}}+(1-t)e^{u^{(0)}}\}$. □

This result can be compared with the well-known property that the set of semi-Riemannian metrics of index ν on M is convex if, and only if, $\nu=0$ or $\nu=n$.

We fix a one-form $w\in\Lambda^1(M)$ and define $W:\mathcal{F}(M)\to\Lambda^1(M)$, $W(u):=w-du$. Alternatively, W may be viewed as an invariant of the conformal manifold $(M,\hat{g})$, acting as an operator $W:\hat{g}\to\Lambda^1(M)$, $W(e^u g):=w-du$. The triple $(M,\hat{g},W)$, denoted also $(M,\hat{g},w)$, is called a Weyl manifold [26]. Both u and w act as (conformal) control tools associated to the conformal structure.

A linear connection $\nabla\in\mathcal{C}(M)$ is compatible with the Weyl structure W if $\nabla g+w\otimes g=0$. We denote the set of the connections compatible with the Weyl structure by $\mathcal{C}_w(M,g)$; this is an affine submodule of $\mathcal{C}(M)$. In particular, $\mathcal{C}_0(M,g)=\mathcal{C}_m(M,g)$ and contains all the Levi–Civita connections associated to the metrics in $\hat{g}$. A short calculation proves that

$$\mathcal{C}_w(M,g)\cap\mathcal{C}_c(M,g)=\mathcal{C}_0(M,g). \tag{15}$$

The next remark gathers some known results in a new formalism.

Remark 7. *Let T a skew-symmetric (1,2)-tensor field on (M,g). Then there exists a unique connection $\nabla\in\mathcal{C}_w(M,g)$ with torsion T.*

(i) *Let $u\in\mathcal{F}(M)$, $h:=e^u g$ and $W_h:=W(h)=w-du$. Then $\nabla h+W_h\otimes h=0$, i.e., the property of ∇ is invariant under conformal changes of the metric g.*
(ii) *Denote $\nabla=\nabla^0+A$; then $w(X)Z=A(X,Z)+A'(X,Z)$.*
(iii) *The torsion tensor field satisfies*

$$T(X,Y)=w(X)Y-w(Y)X-A'(X,Y)+A'(Y,X).$$

(iv) *The tensor field T acts as a control on $\mathcal{C}_w(M,g)$. When we vary it, in order to find triples (M,g,∇) which are statistical manifolds, we find that examples become scarce. Indeed, Formula (15) shows that*

$$\mathcal{C}_w(M,g)\cap\mathcal{C}_c(M,g)\cap\mathcal{C}_s(M)=\{\nabla^0\},$$

so there exists a unique symmetric connection, compatible with the Weyl structure and such that (M,g,∇) is a statistical manifold, namely the Levi–Civita connection of g.

Proposition 4. *Let $\epsilon = \pm 1$ and (M, g, ∇) a $(sgn(\epsilon)\gamma)$-manifold, with*

$$(\nabla_Z g)(X,Y) + \epsilon(\nabla_Y g)(X,Z) = \gamma(X,Y,Z)$$

and $u \in \mathcal{F}(M)$. Then $(M, e^u g, \nabla)$ is a $(sgn(\epsilon)\tilde{\gamma})$-manifold, where

$$\tilde{\gamma}(X,Y,Z) = e^u\{\gamma(X,Y,Z) + du(Z)g(X,Y) + \epsilon du(Y)g(X,Z)\}\ . \tag{16}$$

Corollary 2. *Let $\epsilon = \pm 1$, (M, g) a semi-Riemannian manifold, $\nabla \in \mathcal{C}(M)$, a nowhere vanishing $u \in \mathcal{F}(M)$ and*

$$\gamma(X,Y,Z) = \frac{e^u}{1-e^u}\{du(Z)g(X,Y) + \epsilon du(Y)g(X,Z)\}\ .$$

Then (M, g, ∇) is a $(sgn(\epsilon)\gamma)$-manifold if $(M, e^u g, \nabla)$ is a $(sgn(\epsilon)\gamma)$-manifold.

Corollary 3. *Let $\epsilon = \pm 1$, (M, g), be a semi-Riemannian manifold, $\nabla \in \mathcal{C}(M)$, $u \in \mathcal{F}(M)$ and*

$$\gamma(X,Y,Z) = -w(Z)g(X,Y) - \epsilon w(Y)g(X,Z)\ .$$

If (M, g, ∇) is a $(sgn(\epsilon)\gamma)$-manifold, then $(M, e^u g, \nabla)$ is a $(sgn(\epsilon)\tilde{\gamma})$-manifold, where

$$\tilde{\gamma}(X,Y,Z) = -e^u\{W(e^u g)(Z)g(X,Y) + \epsilon W(e^u g)(Y)g(X,Z)\}\ .$$

Corollary 4. *Let (M, g) be a semi-Riemannian manifold, $\nabla \in \mathcal{C}(M)$, $u, f \in \mathcal{F}(M)$. Consider $\nabla^{*,u}$ the dual of ∇ w.r.t. the metric $e^u g$, with the notation $\nabla^{*,0} = \nabla^*$. Denote $\nabla^{(f),u}$ the f-connection associated to ∇ in $(M, e^u g)$, with the notation $\nabla^{(f),0} = \nabla^{(f)}$. Then:*

(i) $\nabla^{*,u}_X Y = \nabla^*_X Y + du(X)Y$;

(ii) $\nabla^{(f),u}_X Y = \nabla^{(f)}_X Y + \frac{1-f}{2} du(X)Y$.

(iii) If, moreover, for a fixed $\epsilon = \pm 1$, (M, g, ∇) is a $(sgn(\epsilon)\gamma)$-manifold, then $(M, e^u g, \nabla^{(f)})$ is a $(sgn(\epsilon)\hat{\gamma})$-manifold, where

$$\hat{\gamma}(X,Y,Z) = e^u\{f\gamma(X,Y,Z) + du(Z)g(X,Y) + du(Y)g(X,Z)\}.$$

In particular, for $f := u$, the previous correspondence $u \rightsquigarrow \hat{\gamma}$ is one to one.

5. f-Conformal Equivalence of γ-Manifolds

Consider $\epsilon = \pm 1$, the functions $u, f \in \mathcal{F}(M)$, (M, g, ∇) a $(sgn(\epsilon)\gamma)$-manifold and $(M, e^u g, \tilde{\nabla})$ a $(sgn(\epsilon)\tilde{\gamma})$-manifold.

Definition 3. *The manifolds (M, g, ∇) and $(M, e^u g, \tilde{\nabla})$ are called f-conformal equivalent if*

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{1-f}{2}\{du(X)Y + du(Y)X\} - \frac{1+f}{2} g(X,Y) grad\, u \tag{17}$$

This notion generalizes the well-known α-conformal equivalence of statistical manifolds, where α is a real number, the connections are symmetric, $\epsilon = -1$ and $\gamma = \tilde{\gamma} = 0$ (see, for example [9]).

Theorem 4. *With the previous notations, let (M, g, ∇) be a $(sgn(\epsilon)\gamma)$-manifold and $(M, e^u g, \tilde{\nabla})$ be a $(sgn(\epsilon)\tilde{\gamma})$-manifold. A necessary condition for being f-conformal equivalent is*

$$e^{-u}\tilde{\gamma}(X,Y,Z) - \gamma(X,Y,Z) = \tag{18}$$

$$= f(1+\epsilon)\{du(Z)g(X,Y) + du(X)g(Y,Z) + du(Y)g(Z,X)\}.$$

Proof. By hypothesis,

$$(\nabla_Z g)(X,Y) + \epsilon(\nabla_Y g)(X,Z) = \gamma(X,Y,Z) \tag{19}$$

$$(\tilde{\nabla}_Z e^u g)(X,Y) + \epsilon(\tilde{\nabla}_Y e^u g)(X,Z) = \tilde{\gamma}(X,Y,Z) \tag{20}$$

Relation (20) rewrites successively

$$e^u\{(\tilde{\nabla}_Z g)(X,Y) + \epsilon(\tilde{\nabla}_Y g)(X,Z)\}+$$

$$+e^u\{du(Z)g(X,Y) + \epsilon du(Y)g(X,Z)\} = \tilde{\gamma}(X,Y,Z)\ ,$$

$$(\tilde{\nabla}_Z g)(X,Y) + \epsilon(\tilde{\nabla}_Y g)(X,Z) = \tag{21}$$

$$= e^{-u}\tilde{\gamma}(X,Y,Z) - du(Z)g(X,Y) - \epsilon du(Y)g(X,Z)\ .$$

On another hand, we calculate

$$(\tilde{\nabla}_Z g)(X,Y) = Zg(X,Y) - g(\tilde{\nabla}_Z X,Y) - g(X,\tilde{\nabla}_Z Y) =$$

$$= Zg(X,Y) - g(\nabla_Z X,Y) - g(X,\nabla_Z Y) -$$

$$-\frac{1-f}{2}g(du(Z)X,Y) - \frac{1-f}{2}g(du(X)Z,Y) + \frac{1+f}{2}g(Z,X)du(Y) -$$

$$-\frac{1-f}{2}g(du(Z)Y,X) - \frac{1-f}{2}g(du(Y)Z,X) + \frac{1+f}{2}g(Z,Y)du(X) =$$

$$= (\nabla_Z g)(X,Y) - (1-f)du(Z)g(Y,X) + f du(Y)g(Z,X) + f du(X)g(Z,Y)\ .$$

We replace it in (21) and it follows that

$$e^{-u}\tilde{\gamma}(X,Y,Z) - du(Z)g(X,Y) - \epsilon du(Y)g(X,Z) = \gamma(X,Y,Z) -$$

$$-(1-f)du(Z)g(Y,X) + f du(Y)g(Z,X) + f du(X)g(Z,Y) +$$

$$+\epsilon\{-(1-f)du(Y)g(Z,X) + f du(Z)g(Y,X) + f du(X)g(Z,Y)\}$$

and from it we deduce Formula (18). □

Remark 8. *With the notations and in the hypothesis of the previous theorem, we have the following particular cases:*

(i) *If $\epsilon = -1$, then $\tilde{\gamma}(X,Y,Z) = e^u\gamma(X,Y,Z)$.*
(ii) *If $\epsilon = -1$ and $\tilde{\gamma}(X,Y,Z) = \gamma(X,Y,Z) = 0$, we see that (18) is identically satisfied. This is, in fact, a known result for statistical manifolds ([9]).*
(iii) *If $\epsilon = 1$, then*

$$e^{-u}\tilde{\gamma}(X,Y,Z) - \gamma(X,Y,Z) = 2f\{du(Z)g(X,Y) + du(X)g(Y,Z) + du(Y)g(Z,X)\}.$$

This fact confirms our claim that the geometries of the $(-\gamma)$-manifolds and of the $(+\gamma)$-manifolds have significant and important different features.

Applying Propositions 1 and 2, we obtain the following two corollaries, which generalize similar claims in [9], true for statistical manifolds.

Corollary 5. *With the previous notations, let (M,g,∇) be a $(sgn(\epsilon)\gamma)$-manifold and ∇^* the dual of ∇; let $(M,e^u g,\tilde{\nabla})$ be a $(sgn(\epsilon)\tilde{\gamma})$-manifold and $\tilde{\nabla}^*$ the dual of $\tilde{\nabla}$. Suppose (M,g,∇) and $(M,e^u g,\tilde{\nabla})$ are f-conformal equivalent.*

Then, the $(sgn(\epsilon)(-\gamma))$-manifold (M,g,∇^*) and the $(sgn(\epsilon)(-\tilde{\gamma}))$-manifold $(M,e^u g,\tilde{\nabla}^*)$ are also $(-f)$-conformal equivalent.

Corollary 6. *With the previous notations, let (M, g, ∇) be a $(sgn(\epsilon)\gamma)$-manifold and $(M, e^u g, \tilde{\nabla})$ be a $(sgn(\epsilon)\tilde{\gamma})$-manifold. Then they are $(+1)$-conformal equivalent if, and only if, the $(sgn(\epsilon)(f\gamma))$-manifold $(M, g, \nabla^{(f)})$ and the $(sgn(\epsilon)(f\tilde{\gamma}))$-manifold $(M, e^u g, \tilde{\nabla}^{(f)})$ are f-conformal equivalent.*

Theorem 5. *Let (M, g, ∇) be a $(sgn(\epsilon)\gamma)$-manifold and ∇^0 the Levi-Civita connection of g. Denote $A := \nabla - \nabla^0$. Fix $u, f \in \mathcal{F}(M)$. Let $(M, e^u g, \tilde{\nabla})$ be a $(sgn(\epsilon)\tilde{\gamma})$-manifold and $\tilde{\nabla}^0$ the Levi–Civita connection of $e^u g$. Denote $\tilde{A} := \tilde{\nabla} - \tilde{\nabla}^0$. Suppose, moreover, the relation (18) holds. Then the manifolds (M, g, ∇) and $(M, e^u g, \tilde{\nabla})$ are f-conformal equivalent if, and only if,*

$$\tilde{A}(X, Y) = A(X, Y) - \frac{f}{2}\{du(X)Y + du(Y)X + g(X, Y)gradu\} . \tag{22}$$

Proof. One knows that

$$2\tilde{\nabla}^0_X Y = 2\nabla^0_X Y + du(Y)X + du(X)Y - g(X, Y)gradu . \tag{23}$$

In (17), we replace ∇ and $\tilde{\nabla}$ in terms of ∇^0, $\tilde{\nabla}^0$, A and $\tilde{A}$. We obtain

$$\tilde{\nabla}^0_X Y + \tilde{A}(X, Y) = \nabla^0_X Y + A(X, Y) + \frac{1-f}{2}\{du(Y)X + du(X)Y\} - \frac{1+f}{2}g(X, Y)gradu$$

We use relation (23) and replace $\tilde{\nabla}^0$. The previous formula becomes (22). □

Remark 9. *Theorem 5 generalizes the main result from [9], which was proven in the particular case of statistical Riemannian manifolds (i.e., for $\epsilon = -1$, $\gamma = \tilde{\gamma} = 0$ and g Riemannian metric). It provides a framework for the construction of pairs of f-conformal equivalent γ-manifolds, starting from the Levi–Civita connections ∇^0 and $\tilde{\nabla}^0$, the functions u and f, the tensor fields A and $\tilde{A}$ and the cubic forms γ and $\tilde{\gamma}$, subject to the compatibility constraints (18) and (22).*

6. Examples

In what follows, we use the notations from ([2], Chapters 2 and 3) and [7,23].

Let us take two positive integers n and m. Consider M an m-dimensional differentiable manifold and a family of probability distributions $p : \mathbb{R}^n \times M \to \mathbb{R}$, with $p = p(x, \xi)$, $p(x, \xi) > 0$ and $\int p(x, \xi)dx = 1$. All the following integrals are supposed to be (correctly) defined on $\mathbb{R}^n$. Let $f : \mathbb{R}^n \times M \to \mathbb{R}$ be an arbitrary function. We consider $\xi \to E_\xi[f]$ a function from M to $\mathbb{R}$, where

$$E_\xi[f] := \int f(x, \xi)\, p(x, \xi)dx .$$

Denote $\xi = (\xi^1, \dots \xi^m)$ the local coordinates on M and the log-likelihood function by $l = l(x, \xi) : \mathbb{R}^n \times M \to \mathbb{R}$, where $l(x, \xi) := ln\, p(x, \xi)$. We consider the Gibbs entropy function $S = S_\xi : M \to \mathbb{R}$, given by $E_\xi[-l]$, i.e.,

$$S_\xi := -\int l(x, \xi)\, p(x, \xi)dx$$

and the Fisher Riemannian metric, given by the $m \times m$-matrix $G(\xi) := (g_{ij}(\xi))_{i,j=\overline{1,m}}$, defined by

$$g_{ij}(\xi) := \int \partial_i l(x, \xi)\, \partial_j l(x, \xi)\, p(x, \xi)dx .$$

Here, and in the following, we denote $\partial_i l := \frac{\partial l}{\partial \xi^i}$. One knows ([2]) that

$$g_{ij}(\xi) = E_\xi[\partial_i l\, \partial_j l] = -E_\xi[\partial_{ij} l] = \int \frac{1}{p(x, \xi)}\partial_i p(x, \xi)\partial_j p(x, \xi)dx .$$

The Christoffel coefficients are written as

$$(\Gamma^{(0)}_{ij,k})_\xi = E_\xi[(\partial_i\partial_j l + \frac{1}{2}\,\partial_i l\ \partial_j l\ )\ \partial_k l]\ .$$

In order to not complicate the formulas, we shall try to skip the variable ξ.

With the previous notations, let us consider another connection ∇ on M, such that $\nabla = \nabla^0 + A$, where $A \in \mathcal{T}^1_2(M)$ is fixed and arbitrary. We denote $A_{ij,k} := g(A(\partial_i, \partial_j), \partial_k)$ and $\Gamma_{ij,k} := \Gamma^{(0)}_{ij,k} + A_{ij,k}$. In what follows, we shall particularize A, in order to find examples of γ-manifolds.

Example 1. *The normal distribution is usually associated with a hyperbolic space. The density of a normal family is*

$$p(x,\xi) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, x \in \mathbf{R},$$

with parameter(s) $\xi = (\xi^1, \xi^2) = (\mu, \sigma) \in \mathbf{R} \times (0, \infty)$. The matrix of the Fisher (i.e., Fisher–Rao) Riemannian metric is given by

$$(g_{ij}) = \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2} \end{pmatrix}.$$

The Christoffel symbols of the Levi–Civita connection are given by

$$(\Gamma^0)^1_{ij} = \begin{pmatrix} 0 & -\frac{1}{\sigma} \\ -\frac{1}{\sigma} & 0 \end{pmatrix}\ ,\ (\Gamma^0)^2_{ij} = \begin{pmatrix} \frac{1}{2\sigma} & 0 \\ 0 & -\frac{1}{\sigma} \end{pmatrix}.$$

In the following, we determine examples of connections $\nabla \in \mathcal{C}_{+c}(M, g, \gamma)$, for certain particular $(+\gamma)-$ structures.

(a) We consider

$$\gamma(X, Y, Z) = f[Z(u)g(X, Y) + Y(u)g(X, Z)], f = \frac{e^u}{1 - e^u}, u \in \mathcal{F}(M),$$

where u is non-vanishing. (M, g, ∇) is $(+\gamma)$ -manifold iff $(M, e^u g, \nabla)$ is $(+\gamma)$-manifold.

If $\nabla = \nabla^0 + A$, formula (2) leads to

$$-A^l_{ki}g_{lj} - A^l_{kj}g_{li} - A^l_{jk}g_{li} - A^l_{ji}g_{lk} = f[u_k g_{ij} + u_j g_{ik}],$$

where $u_i = \frac{\partial u}{\partial x^i}$.

In dimension 2, one obtains

$$A^1_{11} = -\frac{1}{2}fu_1, A^2_{22} = -\frac{1}{2}fu_2, A^1_{21} = -\frac{1}{2}fu_2, A^2_{12} = -\frac{1}{2}fu_1,$$

$$2A^2_{11} + A^1_{12} = 0, A^1_{22} + 2A^2_{21} = 0.$$

In particular, for $A^1_{22} = fu_1$, $A^2_{11} = fu_2$, we obtain the following components of the torsion tensor associated to ∇:

$$(T^1_{ij}) = \begin{pmatrix} 0 & -\frac{3}{2}fu_2 \\ \frac{3}{2}fu_2 & 0 \end{pmatrix}, (T^2_{ij}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

(b) We consider

$$\gamma(X, Y, Z) = -[\omega(Z)g(X, Y) + \omega(Y)g(X, Z)],$$

$\omega \in \Lambda^1(M)$.

If $\nabla = \nabla^0 + A$, then from (2) one obtains

$$-A^l_{ki}g_{lj} - A^l_{kj}g_{li} - A^l_{jk}g_{li} - A^l_{ji}g_{lk} = -(\omega_k g_{ij} + \omega_j g_{ik}),$$

where $\omega = \omega_i x^i$.

If (M, g, ∇) is $(+\gamma)$-manifold, then $(M, e^u g, \nabla)$ is $(+\tilde{\gamma})$-manifold, where

$$\tilde{\gamma}(X, Y, Z) = -e^u[W(e^u g)(Z)g(X, Y) + W(e^u g)(Y)g(X, Z)]$$

and $W(e^u g) = \omega - du, \omega$ and u, being the conformal tools associated to the conformal structure.

In dimension 2, one obtains

$$A^1_{11} = \frac{1}{2}\omega_1, A^2_{22} = \frac{1}{2}\omega_2, A^1_{21} = \frac{1}{2}\omega_2, A^2_{12} = \frac{1}{2}\omega_1,$$

$$2A^2_{11} + A^1_{12} = 0, A^1_{22} + 2A^2_{21} = 0.$$

where $\omega = \omega_1 dx^1 + \omega_2 dx^2$.

In particular, for $A^2_{21} = \omega_1, A^1_{12} = \omega_2$, we obtain the following components of the torsion tensor associated to ∇:

$$(T^1_{ij}) = \begin{pmatrix} 0 & \frac{1}{2}\omega_2 \\ -\frac{1}{2}\omega_2 & 0 \end{pmatrix}, (T^2_{ij}) = \begin{pmatrix} 0 & -\frac{1}{2}\omega_1 \\ \frac{1}{2}\omega_1 & 0 \end{pmatrix}.$$

Example 2. *We consider the manifold (M, g), where the metric is $g_{ij} = \delta_{ij}$. Therefore, $(\Gamma^0)^k_{ij} = 0$, $\Gamma^k_{ij} = A^k_{ij}$. Let us determine examples of connections $\nabla \in \mathcal{C}_{+c}(M, g, \gamma)$ for certain particular $(+\gamma)-$ structures.*

(a) We consider $\gamma(X, Y, Z) = f[Z(u)g(X, Y) + Y(u)g(X, Z)]$, $f = \frac{e^u}{1-e^u}$ and the $u \in \mathcal{F}(M)$ nowhere vanishing function. Then (M, g, ∇) is the $(+\gamma)$-manifold if $(M, e^u g, \nabla)$ is the $(+\gamma)$-manifold.

In dimension 2, one obtains

$$\Gamma^1_{11} = \Gamma^2_{12} = -\frac{1}{2}fu_1, \Gamma^2_{22} = \Gamma^1_{21} = -\frac{1}{2}fu_2, \Gamma^1_{22} + \Gamma^2_{21} = \Gamma^2_{11} + \Gamma^1_{12} = 0,$$

where $u_i = \frac{\partial u}{\partial x^i}$.

In particular, for $\Gamma^1_{22} = \Gamma^2_{11} = \Gamma^2_{21} = \Gamma^1_{12} = 0$, we obtain the components of the torsion tensor associated to ∇ :

$$(T^1_{ij}) = \begin{pmatrix} 0 & \frac{1}{2}fu_2 \\ -\frac{1}{2}fu_2 & 0 \end{pmatrix}, (T^2_{ij}) = \begin{pmatrix} 0 & -\frac{1}{2}fu_1 \\ \frac{1}{2}fu_1 & 0 \end{pmatrix}.$$

(b) We consider $\gamma(X, Y, Z) = -[\omega(Z)g(X, Y) + \omega(Y)g(X, Z)]$, where $\omega \in \Lambda^1(M)$.

If (M, g, ∇) is $(+\gamma)$-manifold, then $(M, e^u g, \nabla)$ is $(+\tilde{\gamma})$-manifold, where

$$\tilde{\gamma}(X, Y, Z) = -e^u[W(e^u g)(Z)g(X, Y) + W(e^u g)(Y)g(X, Z)],$$

$W(e^u g) = \omega - du, \omega$,$u$ being the conformal tools associated to the conformal structure.

In dimension 2, one obtains

$$\Gamma^1_{11} = \Gamma^2_{12} = \frac{1}{2}\omega_1, \Gamma^2_{22} = \Gamma^1_{21} = \frac{1}{2}\omega_2, \Gamma^1_{22} + \Gamma^2_{21} = \Gamma^2_{11} + \Gamma^1_{12} = 0,$$

where $\omega = \omega_1 dx^1 + \omega_2 dx^1$.

In particular, for $\Gamma^1_{22} = -\Gamma^2_{21} = \frac{1}{2}\omega_1, \Gamma^2_{11} = -\Gamma^1_{12} = \frac{1}{2}\omega_2$, *we obtain the components of the torsion tensor associated to* ∇ :

$$(T^1_{ij}) = \begin{pmatrix} 0 & -\omega_2 \\ \omega_2 & 0 \end{pmatrix}, (T^2_{ij}) = \begin{pmatrix} 0 & \omega_1 \\ -\omega_1 & 0 \end{pmatrix}.$$

Remark 10.

(i) *In Examples 1 and 2, an infinite family of* γ*-structures is defined on* $M := \mathbb{R} \times (0, \infty)$, *or on an arbitrary M, respectively. To each such structure, one associates a connection* ∇, *via the tensor field A, which measures "how far"* ∇ *is from the Levi–Civita connection* ∇^0 *of the Fisher metric. The geodesics determined from* ∇^0 *are (locally) distance-minimizing curves between points in M, i.e., between normal PDFs. By analogy, auto-parallel curves of* ∇ *may connect points of M by other paths, whose spread flow might be controlled through the* γ*-structure parameters.*

(ii) *The Fisher metric in Example 1 is the geometrical counterpart of the Fisher information, used in parameter estimation, measuring the quantity of information about the parameter(s) of the system ([27,28] for details). Its curvature, and especially the scalar curvature, can distinguish different values of the parameter. A parameter variation is measurable along the geodesics also, but, in Example 1, the geodesics depend on only one of the two parameters (the* σ*). Instead, the auto-parallel curves of a connection* ∇, *determined by the* γ*-structure, may depend on both parameters* μ *and* σ*. However, we must point out that finding (exact parameterizations of) the auto-parallel curves of* ∇ *may be just as difficult as finding the geodesics.*

7. Discussions

(i) The paper dealt with notions and results intended to understand geometrical objects, such as statistical manifolds and dual connections, beyond the classical setting. The new notion of γ-manifold seems appropriate and potentially useful. Another generalization that seems promising is the f-conformal equivalence of γ-manifolds, which provides new conformal-like control tools for the covariant derivation of a semi-Riemannian metric.

(ii) From now on, several research directions are open: the detailed study of the geometry of γ-manifolds per se; the statistical relevance of the new conformal invariants; specific statistical applications for the new examples of geometric structures introduced here; and the optimization results on the space of the control tensors, which appear in Lemmas 1 and 2.

(iii) Beyond the formal theory we developed in this paper, we must point out, at a speculative level, two guiding ideas which led us toward this subject: firstly, the deep, hidden conformal nature of the Universe, which reveals itself, from time to time, in various physical theories, in particular in statistical mechanics ([29]); secondly, the more prosaic, but also important, conformal patterns arising in recent machine learning theories, as the metric tools are no more enough ([30,31]; with—apparently—no connection with the "conformal inference prediction"). It would not be a surprise if these two ideas eventually converge.

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References

1. Amari, S. *Differential-Geometrical Methods in Statistics*; Lecture Notes in Statistics; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 1985.
2. Amari, S.; Nagaoka, H. *Methods of Information Geometry*; Translations of Mathematical Monographs; American Mathematical Society: Providence, RI, USA, 2000; Volume 191.
3. Amari, S. *Information Geometry and Its Applications*; Springer: Berlin/Heidelberg, Germany, 2016; Volume 194.
4. Ay, N.; Jurgen, J.; Le, H.V.; Schwachhöfer, L. *Information Geometry*; Ergebnisse der Mathematik und ihrer Grenzgebiete, 64; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 2017.
5. Calin, O.; Udriste, C. *Geometric Modeling in Probability and Statistics*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 2014.
6. Nielsen, F. An Elementary Introduction to Information Geometry. *Entropy* **2020**, *22*, 1100. [CrossRef] [PubMed]
7. Hirică, I.E.; Pripoae, C.L.; Pripoae, G.T.; Preda, V. Affine differential control tools for statistical manifolds. *Mathematics* **2021**, *9*, 1654. [CrossRef]
8. Addad, K.; Ouakkas, S. On the α-connections and the α-conformal equivalence on statistical manifolds. *Arab J. Math. Sci.* 2021, *ahead-of-print*. [CrossRef]
9. Uohashi, K. α-Connections and a Symmetric Cubic Form on a Riemannian Manifold. *Entropy* **2017**, *19*, 344. [CrossRef]
10. Kumon, M.; Takemura, A.; Takeuchi, K. Conformal geometry of statistical manifold with application to sequential estimation. *Seq. Anal.* **2011**, *30*, 308–337. [CrossRef]
11. Kurose, T. On the divergences of 1-conformally flat statistical manifolds. *Tohoku Math. J.* **1994**, *46*, 427–433. [CrossRef]
12. Mikeš, J.; Stepanova, E. A five-dimensional Riemannian manifold with an irreducible SO(3)-structure as a model of abstract statistical manifold. *Ann. Glob. Anal. Geom.* **2014**, *45*, 111–128. [CrossRef]
13. Mahesh, T.V.; Moosath, K.S. Affine and conformal submersions with horizontal distribution and statistical manifolds. *Balkan J. Geom. Appl.* **2021**, *26*, 34–45.
14. Kurose, T. Conformal-Projective Geometry of Statistical Manifolds. *Interdiscip. Inf. Sci.* **2002**, *8*, 89–100. [CrossRef]
15. Mahesh, T.V.; Moosath, K.S. Immersions into Statistical Manifolds. *arXiv* **2020**, arXiv:1803.02538v3.
16. Matsuzoe, H. On realization of conformally-projectively flat statistical manifolds and the divergences. *Hokkaido Math. J.* **1998**, *27*, 409–421. [CrossRef]
17. Matsuzoe, H. Geometry of contrast functions and conformal geometry. *Hiroshima Math. J.* **1999**, *29*, 175–191. [CrossRef]
18. Matsuzoe, H. Traceless cubic forms on statistical manifolds and Tchebychev geometry. In *PDEs, Submanifolds and Affine Differential Geometry*; Banach Center Publ., Institute of Mathematics, Polish Academy of Sciences: Warsawa, Poland, 2005; Volume 69, pp. 179–187.
19. Morales, P.A.; Rosas, F.E. A generalization of the maximum entropy principle for curved statistical manifolds. *arXiv* **2021**, arXiv:2105.07953v1.
20. Matsuzoe, H. Geometry of semi-Weyl manifolds and Weyl manifolds. *Kyushu J. Math.* **2001**, *55*, 107–117. [CrossRef]
21. Matsuzoe, H. Information geometry of Bayesian statistics. *AIP Conf. Proc.* **2015**, *279*, 1641.
22. Matsuzoe, H. Statistical manifolds and affine differential geometry. In *Probabilistic Approach to Geometry*; Advanced Studies in Pure Mathematics; Mathematical Society of Japan: Tokyo, Japan, 2010; Volume 57, pp. 303–321.
23. Matsuzoe, H.; Takeuchi, J.; Amari, S. Equiaffine structures on statistical manifolds and Bayesian statistics. *Diff. Geom. Appl.* **2006**, *24*, 567–578. [CrossRef]
24. Kurose, T. *Statistical Manifolds Admitting Torsion. Geometry and Something*; Fukuoka University: Fukuoka-shi, Japan, 2007. (In Japanese)
25. Lauritzen, S.L. Statistical manifolds. In *Differential Geometry in Statistical Inferences*; IMS Lecture Notes Monograph Series 10; Institute of Mathematical Statistics: Hayward, CA, USA, 1987; pp. 163–216.
26. Hirică, I.E.; Nicolescu, L. On Weyl structures. *Rend. Circ. Mat. Palermo.* **2004**, *53*, 390–400. [CrossRef]
27. Nielsen, F. Cramér-Rao Lower Bound and Information Geometry. In *Connected at Infinity II: A Selection of Mathematics by Indians*; Bhatia, R., Rajan, C.S., Singh, A.I., Eds.; Hindustan Book Agency: New Delhi, India, 2013; pp.18–37.
28. Rao, C.R. Information and the accuracy attainable in the estimation of statistical parameters. In *Breakthroughs in Statistics I*; Kotz, S., Johnson, N.L., Eds.; Springer: New York, NY, USA, 1992; pp. 235–247.
29. Cardy, J. *Scaling and Renormalization in Statistical Physics*; Cambridge Lecture Notes in Physics; Cambridge University Press: Cambridge, UK, 1996.
30. Ross, B.L.; Cresswell, J.C. Tractable Density Estimation on Learned Manifolds with Conformal Embedding Flows. In *Advances in Neural Information Processing Systems*; Ranzato, M., Beygelzimer, A., Nguyen, K., Liang, P.S., Vaughan, J.W., Dauphin, Y., Eds.; MIT Press: Cambridge, CA, USA, 2021.
31. Sha, F.; Saul, L.K. Analysis and Extension of Spectral Methods for Nonlinear Dimensionality Reduction. In Proceedings of the ICML '05: Proceedings of the 22nd International Conference on Machine Learning, Bonn, Germany, 7–11 August 2015; De Raedt, L., Wrobel, S., Eds.; ACM Press: New York, NY, USA, 2005; pp. 784–791.