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Article

Controllability of Second Order Functional Random Differential Equations with Delay

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Abstract: In this article, we study some existence and controllability results for two classes of second order functional differential equations with delay and random effects. To begin, we employ a random fixed point theorem with a stochastic domain to demonstrate the existence of mild random solutions. Next, we prove that our problems are controllable. Finally, an example is given to validate the theory part.

Keywords: random fixed point; functional differential equation; state-dependent delay; cosine and sine family; mild solution; controllability; finite delay

MSC: 34G20; 34K20; 34K30; 93B05

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1. Introduction

Throughout the development of emerging control theory, the controllability of differential equation problems has played a major role. It typically indicates that the set of permissible controls may be used to direct a dynamical system from an arbitrary initial state to the intended terminal one. The qualitative properties of control systems have a particular importance in control theory. The controllability of linear and nonlinear systems described by ordinary differential equations in finite-dimensional space has received a great deal of attention. Numerous researchers have expanded the notion to infinite-dimensional systems with bounded operators in Banach spaces, see [1–4]. The authors of [5] demonstrated how to transform the controllability problem into a fixed point problem. We suggest the papers [6,7] for more details. In [8–11], the authors explored a wide range of functional differential equations and inclusions and suggested various controllability results. Dilao et al. [12] considered the controllability of a class of integrodifferential evolution equations.

In several instances, treating second-order abstract differential equations directly without always converting them to first-order systems is preferable. The theory of strongly continuous cosine families is a valuable tool for studying second-order problems. We shall use some of the fundamental concepts of cosine family theory [13]. In [14,15], the authors provided adequate criteria for controllability of second-order systems in Banach spaces for deterministic and stochastic systems utilizing alternative fixed point theorems and strongly continuous cosine family with nonlinearity meeting Lipschitz condition.

As natural generalizations of deterministic differential equations, random differential equations emerge in a wide range of applications and have been studied by numerous mathematicians, the reader is referred to the papers [16–18] for more details. The nature of a dynamic system is determined by the precision of the knowledge we have about the system's characteristics. A deterministic dynamical system emerges when information about a dynamic system is exact. However, plenty of the relevant data for the identification and assessment of dynamic system characteristics is erroneous, unclear, or ambiguous. In other terms, determining the parameters of a dynamical system is fraught with uncertainty. When we have statistical understanding about the characteristics of a dynamic system, that is, when the knowledge is probable, the standard technique in mathematical modeling of such systems is to employ random or stochastic differential equations.

Controllability is an essential topic in control theory and engineering because it is closely related to pole assignment, quadratic optimal control, observer design and structural decomposition, among other things. Several authors, including Benchohra et al. [19–22], Balachandran et al. [23], Mophou et al. [24], Wang et al. [25], have written extensively in recent years about the problem of controllability for various types of differential equations, neutral functional differential equations, integrodifferential equations differential inclusions and impulsive differential inclusions in Banach spaces.

In [26], Balachandran and Sakthivel considered the following integrodifferential system:

$$\begin{aligned} &x'(t) = Ax(t) + (Bu)(t) + f\left(t, x(t), \int_0^t g(t,s,x(s))\mathrm{d}s\right), \\ &x(0) = x_0, \quad t \in J = [0,b], \end{aligned}$$

where the state $x(\cdot)$ takes values in a Banach space X with the norm $\|\cdot\|$ and the control function $u(\cdot)$ is given in $L^2(J,U)$, a Banach space of admissible control functions, with U as a Banach space. Here, A is the infinitesimal generator of a strongly continuous semigroup $T(t), t \geqslant 0$ in the Banach space X and $g\colon \Delta \times X \to X, f\colon J \times X \times X \to X$ are given functions and B is a bounded linear operator from U into X. Here $\Delta = \{(t,s)\colon 0 \leqslant s \leqslant t \leqslant b\}$. The authors employed a fixed-point theorem due to Schaefer.

In [27], Yan investigated the controllability of the following fractional-order partial neutral functional integrodifferential inclusions with infinite delay in Banach spaces:

$$\begin{aligned} &{}^cD^q[x(t) - g(t,x_t)] \in Ax(t) + (Bu)(t) + F\left(t, x_t, \int_0^t h(t,s,x_s)ds\right), \quad t \in J = [0,b] \\ &x(t) = \phi(t), \quad t \in (-\infty, 0] \end{aligned}$$

where the unknown $x(\cdot)$ takes values in Banach space X with norm $\|\cdot\|, {}^cD^q$ is the Caputo fractional derivative of order $0 < q < 1, A$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $\{T(t), t \geq 0\}$ in X. The authors established sufficient conditions for the controllability for the problem in Banach spaces by relying on analytic semigroups and fractional powers of closed operators and nonlinear alternative of Leray-Schauder type for multivalued maps due to D. O'Regan.

As a continuation of the studies in the preceding publications and in order to expand the controllability results to more problems, in this paper, we consider the following functional differential equation with delay and random effect:

$$\begin{cases} x''(\vartheta,\delta) = \mathcal{Z}_1 x(\vartheta,\delta) + \psi(\vartheta, x_\vartheta(\cdot,\delta),\delta) + \mathcal{Z}_2\mathfrak{f}(\vartheta,\delta), \text{ a.e. } \vartheta \in \Theta := [0,\kappa], \\ x(\vartheta,\delta) = \varpi_1(\vartheta,\delta);\ \vartheta \in (-\infty,0], \\ x'(0,\delta) = \varpi_2(\delta), \end{cases} \tag{1}$$

where (Ψ, F, P) is a complete probability space, $\psi : \Theta \times \mathfrak{D} \times \Psi \to \Xi$, $\varpi_1 \in \mathfrak{D} \times \Psi$ are given functions, $\mathcal{Z}_1 : D(\mathcal{Z}_1) \subset \Xi \to \Xi$ is the infinitesimal generator of of a strongly continuous cosine family of bounded linear operators $(\mathcal{S}_1(\vartheta))_{\vartheta\in\mathbb{R}}$ on Ξ, $\mathfrak{D}$ is the phase space, and $(\Xi, |\cdot|)$ is a real Banach space. The control function $\mathfrak{f}(\cdot,\delta)$ is given in $L^2(\Theta,\Omega)$, a Banach space of admissible control functions with Ω as a Banach space, and $\mathcal{Z}_2$ is a bounded linear operator from Ω into Ξ.

We denote by $x_{\vartheta}(\cdot, \delta)$ the element of $\mathfrak{D} \times \Psi$ given by $x_{\vartheta}(\iota, \delta) = x(\vartheta + \iota, \delta), \iota \in (-\infty, 0]$. Here $x_{\vartheta}(\cdot, \delta)$ represents the history of the state from time $-\infty$, up to the present time ϑ, we assume that the histories $x_{\vartheta}(\cdot, \delta)$ belong to some abstract phases $\mathfrak{D}$.

Next, we consider the following random problem

$$\begin{cases} x^{''}(\vartheta, \delta) = \mathcal{Z}_1 x(\vartheta, \delta) + \psi(\vartheta, x_{\xi(\vartheta, x_{\vartheta})}(\cdot, \delta), \delta) + \mathcal{Z}_2 \mathfrak{f}(\vartheta, \delta); \quad \text{a.e. } \vartheta \in \Theta, \\ x(\vartheta, \delta) = \varpi_1(\vartheta, \delta); \quad \vartheta \in (-\infty, 0], \\ x^{'}(0, \delta) = \varpi_2(\delta), \end{cases} \tag{2}$$

where $\psi : \Theta \times \mathfrak{D} \times \Psi \to \Xi$, $\varpi_1 \in \mathfrak{D} \times \Psi$ are given random functions, $\mathcal{Z}_1 : D(\mathcal{Z}_1) \subset \Xi \to \Xi$ is as in problem (1), $\mathfrak{D}$ is the phase space, $\xi : \Theta \times \mathfrak{D} \to (-\infty, \kappa]$, and $(\Xi, |\cdot|)$ is a real Banach space. We based our arguments for the main results on Schauder's fixed theorem [28] and random fixed point theorem combined with the family of cosine operators.

Cosine function theory is connected to abstract linear second order differential equations in much the same way as semigroup theory of bounded linear operators is connected to first order partial differential equations, and both are interesting due to their simplicity and clarity. We suggest the papers [13,29] for fundamental principles and applications of this theory. The following is how this paper is structured. Section 2 provides some preliminary results. Sections 3 and 4 are devoted to our main results in the cases of infinite fixed delay, and state-dependent delay, respectively. The last part includes an instructive example.

2. Preliminaries

In this section, we will go over some of the notations, definitions, and theorems that will be employed all through the paper. Let $\Theta := [0, \kappa]$, $\kappa > 0$ and consider the Banach space $\mathfrak{D}(\Xi)$ of bounded linear operators from Ξ into Ξ, with the norm

$$\|G\|_{\mathfrak{D}(\Xi)} = \sup_{\|x\|=1} \|G(x)\|.$$

Let $\mathcal{C} := C(I, \Xi)$ be the Banach space of continuous functions $x : \Theta \to \Xi$ with the norm

$$\|x\|_{\mathcal{C}} = \sup_{\vartheta \in \Theta} |x(\vartheta)|.$$

We will adopt an axiomatic definition of the phase space $\mathfrak{D}$ presented in [30] and adhere to the terminology employed in [31]. Then, Let $(\mathfrak{D}, \|\cdot\|_{\mathfrak{D}})$ be a seminormed linear space of functions mapping $(-\infty, 0]$ into Ξ, and verifying the following:

(A_1) If x: $(-\infty, \kappa) \to \Xi, \kappa > 0$, is continuous on Θ and $x_0 \in \mathfrak{D}$, then for every $\vartheta \in \Theta$ the requirements that follows are met.

(a) $x_{\vartheta} \in \mathfrak{D}$;

(b) There exists a positive constant ρ such that $|x(\vartheta)| \leq \rho \|x_{\vartheta}\|_{\mathfrak{D}}$;

(c) There exist two functions $\gamma(\cdot), \sigma(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of x with γ continuous and bounded, and σ locally bounded where:

$$\|x_{\vartheta}\|_{\mathfrak{D}} \leq \gamma(\vartheta) \sup\{ |x(\varrho)| : 0 \leq \varrho \leq \vartheta\} + \sigma(\vartheta) \|x_0\|_{\mathfrak{D}}.$$

(A_2) For the function x in (A_1), x_{ϑ} is a $\mathfrak{D}-$valued continuous function on Θ.

(A_3) The space $\mathfrak{D}$ is complete.

Set

$$\zeta = \sup\{\gamma(\vartheta) : \vartheta \in \Theta\}, \; \textit{and } \sigma = \sup\{\sigma(\vartheta) : \vartheta \in \Theta\}.$$

Remark 1. *We have*

1. *(b) is equivalent to* $|\varpi_1(0)| \leq \rho \|\varpi_1\|_{\mathfrak{D}}$ *for every* $\varpi_1 \in \mathfrak{D}$.
2. *Since* $\| \cdot \|_{\mathfrak{D}}$ *is a seminorm, two elements* $\varpi_1, \varkappa \in \mathfrak{D}$ *can satisfy* $\|\varpi_1 - \varkappa\|_{\mathfrak{D}} = 0$ *without* $\varpi_1(\iota) = \varkappa(\iota)$ *for all* $\iota \leq 0$.
3. *For all* $\varpi_1, \varkappa \in \mathfrak{D}$ *where* $\|\varpi_1 - \varkappa\|_{\mathfrak{D}} = 0$, *we have* $\varpi_1(0) = \varkappa(0)$.

Consider the space

$$\Lambda := \{x : (-\infty, \kappa] : x|_{(-\infty,0]} \in \mathfrak{D} \text{ and } x|_{\Theta} \in \mathcal{C}\}.$$

Let $\|x\|_{\Lambda}$ be the seminorm in Λ given by

$$\|x\|_{\Lambda} = \|\varpi_1\|_{\mathfrak{D}} + \|x\|_{\mathcal{C}}.$$

Definition 1. *A family* $\{\mathcal{S}_1(\vartheta) : \vartheta \in \mathbb{R}\}$ *of bounded linear operators in the Banach space* Ξ *is strongly continuous cosine family if*

- $\mathcal{S}_1(0) = I$ *(I is the identity operator);*
- $\mathcal{S}_1(\vartheta)\mathfrak{y}$ *is strongly continuous in* ϑ *on* $\mathbb{R}$ *for each fixed* $\mathfrak{y} \in \Xi$;
- $\mathcal{S}_1(\vartheta + \varrho) + \mathcal{S}_1(\vartheta - \varrho) = 2\mathcal{S}_1(\vartheta)\mathcal{S}_1(\varrho)$ *for all* $\vartheta, \varrho \in \mathbb{R}$.

Let $\{\mathcal{S}_1(\vartheta) : \vartheta \in \mathbb{R}\}$ be a strongly continuous cosine family in Ξ. Define the linked sine family $\{\mathcal{S}_2(\vartheta) : \vartheta \in \mathbb{R}\}$ by

$$\mathcal{S}_2(\vartheta)\mathfrak{y} = \int_0^{\vartheta} \mathcal{S}_1(\varrho)\mathfrak{y}\, d\varrho, \;\; \mathfrak{y} \in \Xi, \;\; \vartheta \in \mathbb{R}.$$

We define the infinitesimal generator $\mathcal{Z}_1 : \Xi \to \Xi$ of the cosine family $\{\mathcal{S}_1(\vartheta) : \vartheta \in \mathbb{R}\}$ by

$$\mathcal{Z}_1\mathfrak{y} = \frac{d^2}{d\vartheta^2}\mathcal{S}_1(\vartheta)\mathfrak{y}|_{\vartheta=0}, \; \mathfrak{y} \in D(\mathcal{Z}_1),$$

where

$$D(\mathcal{Z}_1) = \{\mathfrak{y} \in \Xi : \mathcal{S}_1(\cdot)\mathfrak{y} \in C^2(\mathbb{R}, \Xi)\}.$$

Definition 2. *A map* $\psi : \Theta \times \mathfrak{D} \times \Psi \to \Xi$ *is said to be random Carathéodory if*

(i) $\vartheta \to \psi(\vartheta, x, \delta)$ *is measurable for all* $x \in \mathfrak{D}$ *and for all* $\delta \in \Psi$;
(ii) $x \to \psi(\vartheta, x, \delta)$ *is continuous for almost each* $\vartheta \in \Theta$, *and for all* $\delta \in \Psi$;
(iii) $\delta \to \psi(\vartheta, x, \delta)$ *is measurable for all* $x \in \mathfrak{D}$, *and for most each* $\vartheta \in \Theta$.

Let Ξ be a separable Banach space with the Borel σ-algebra $\mathfrak{D}_{\Xi}$. The map $\mathfrak{p} : \Psi \longrightarrow \Xi$ is a random variable in Ξ if for each $\Upsilon \in \mathfrak{D}_{\Xi}, \mathfrak{p}^{-1}(\Upsilon) \in F$, $G : \Psi \times \Xi \longrightarrow \Xi$ is a random operator if $G(\cdot, \mathfrak{p})$ is measurable for each $\mathfrak{p} \in \Xi$, expressed as $G(\delta, \mathfrak{p}) = G(\delta)\mathfrak{p}$.

Definition 3 ([32])**.** *Let* $\tilde{G}$ *be a mapping from* Ψ *into* 2^{Ξ}. *A mapping* $G : \{(\delta, \mathfrak{p}) : \delta \in \Psi \wedge \mathfrak{p} \in \tilde{G}(\delta)\} \longrightarrow \Xi$ *is a random operator with stochastic domain* $\tilde{G}$ *if for all closed* $\Upsilon_1 \subseteq \Xi, \{\delta \in \Psi : \tilde{G}(\delta) \cap \Upsilon_1 \neq \emptyset\} \in F)$ *and for all open* $\Upsilon_2 \subseteq \Xi$ *and all* $\mathfrak{p} \in \Xi, \{\delta \in \Psi : \mathfrak{p} \in \tilde{G}(\delta) \wedge G(\delta, \mathfrak{p}) \in \Upsilon_2\} \in F$. *G is continuous if every* $G(\delta)$ *is continuous. A mapping* $\mathfrak{p} : \Psi \longrightarrow \Xi$ *is a random fixed point of G if for all* $\delta \in \Psi$, $\mathfrak{p}(\delta) \in \tilde{G}(\delta)$ *and* $G(\delta)\mathfrak{p}(\delta) = \mathfrak{p}(\delta)$ *and* $\mathfrak{p}$ *is measurable if for all open* $\Upsilon_2 \subseteq \Xi, \{\delta \in \Psi : \mathfrak{p}(\delta) \in \Upsilon_2\} \in F$.

Lemma 1 ([32])**.** *Let* $\tilde{G} : \Psi \longrightarrow 2^{\Xi}$ *be measurable with* $\tilde{G}(\delta)$ *closed, convex and solid (i.e., int* $\tilde{G}(\delta) \neq \emptyset$*) for all* $\delta \in \Psi$. *We suppose that there exists measurable* $\mathfrak{p}_0 : \Psi \longrightarrow \Xi$ *with* $\mathfrak{p}_0 \in$ *int* $\tilde{G}(\delta)$ *for all* $\delta \in \Psi$. *Let G be a continuous random operator with stochastic domain* $\tilde{G}$ *such that for every* $\delta \in \Psi, \{\mathfrak{p} \in \tilde{G}(\delta) : G(\delta)\mathfrak{p} = \mathfrak{p}\} \neq \emptyset$. *Then G has a stochastic fixed point.*

The mapping $\mathfrak{p}$ of $\Theta \times \Psi$ into Ξ is a stochastic process if for each $\vartheta \in \Theta$, the function $\mathfrak{p}(\vartheta, \cdot)$ is measurable. Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 4 ([33]). *Let χ be a Banach space and Ψ_χ the bounded subsets of Ξ. The Kuratowski measure of noncompactness is the map $\mu : \Psi_\chi \to [0, \infty)$ given by*

$$\mu(\mathrm{Y}) = \inf\{\epsilon > 0 : \mathrm{Y} \subseteq \cup_{i=1}^{n} \mathrm{Y}_i \text{ and } diam(\mathrm{Y}_i) \leq \epsilon\}; \text{ here } \mathrm{Y} \in \Psi_\chi,$$

and verifies the properties:

(a) $\mu(\mathrm{Y}) = 0 \Longleftrightarrow \overline{\mathrm{Y}}$ *is compact (Y is relatively compact);*
(b) $\mu(\mathrm{Y}) = \mu(\overline{\mathrm{Y}})$;
(c) $\tilde{\mathrm{Y}} \subset \mathrm{Y} \Longrightarrow \mu(\tilde{\mathrm{Y}}) \leq \mu(\mathrm{Y})$;
(d) $\mu(\tilde{\mathrm{Y}} + \mathrm{Y}) \leq \mu(\tilde{\mathrm{Y}}) + \mu(\mathrm{Y})$;
(e) $\mu(\epsilon \mathrm{Y}) = |\epsilon| \mu(\mathrm{Y}); \epsilon \in \mathbb{R}$;
(f) $\mu(conv\mathrm{Y}) = \mu(B)$.

Lemma 2 ([34]). *If $\mathfrak{g} \subset C(\Theta, \Xi)$ is bounded and equicontinuous, then $\mu(\mathfrak{g}(\vartheta))$ is continuous on Θ and*

$$\mu\left(\left\{\int_\Theta \mathfrak{y}(\varrho)d\varrho : \mathfrak{y} \in \mathfrak{g}\right\}\right) \leq \int_\Theta \mu(\mathfrak{g}(\varrho))d\varrho,$$

where $\mathfrak{g}(\varrho) = \{\mathfrak{y}(\varrho) : \mathfrak{y} \in \mathfrak{g}\}$, $\vartheta \in \Theta$, and μ is the Kuratowski measure of noncompactnes on the space Ξ.

3. Controllability Results for the Constant Delay Case

Definition 5. *The problem (1) is controllable on the interval $(-\infty, \kappa]$, if for every final state $x^1(\delta)$, there exists a control $\mathfrak{f}(\cdot, \delta)$ in $L^2(\Theta, \Omega)$, such that the solution $x(\vartheta, \delta)$ of (1) verifies $x(\kappa, \delta) = x^1(\delta)$.*

Definition 6. *A stochastic process $x : (-\infty, \kappa] \times \Psi \to \Xi$ is a random mild solution of problem (1) if $x(\vartheta, \delta) = \varpi_1(\vartheta, \delta)$; $\vartheta \in (-\infty, 0]$, $x'(0, \delta) = \varpi_2(\delta)$ and the restriction of $x(\cdot, \delta)$ to the interval Θ is continuous and verifies:*

$$\begin{aligned} x(\vartheta, \delta) &= \mathcal{S}_1(\vartheta)\varpi_1(0, \delta) + \mathcal{S}_2(\vartheta)\varpi_2(\delta) + \int_0^\vartheta \mathcal{S}_1(\vartheta - \varrho)\psi(\varrho, x_\varrho(\cdot, \delta), \delta)d\varrho \\ &\quad + \int_0^\vartheta \mathcal{S}_1(\vartheta - \varrho)\mathcal{Z}_2\mathfrak{f}(\vartheta, \delta)d\varrho \end{aligned}$$

Let

$$\sigma = \sup\left\{\|\mathcal{S}_1(\vartheta)\|_{\mathfrak{D}(\Xi)} : \vartheta \geq 0\right\} \text{ and } \sigma' = \sup\left\{\|\mathcal{S}_2(\vartheta)\|_{\mathfrak{D}(\Xi)} : \vartheta \geq 0\right\}.$$

We will need to introduce the following hypotheses:

(H_1) $\mathcal{S}_1(\vartheta)$ is compact for $\vartheta > 0$,
(H_2) The function $\psi : \Theta \times \mathfrak{D} \times \Psi \to \Xi$ is random Carathéodory,
(H_3) There exist functions $\varkappa : \Theta \times \Psi \to \mathbb{R}^+$ and $p : \Theta \times \Psi \to \mathbb{R}^+$ such that for each $\delta \in \Psi$, $\varkappa(\cdot, \delta)$ is continuous nondecreasing and $p(\cdot, \delta)$ integrable with:

$$|\psi(\vartheta, \mathfrak{f}, \delta)| \leq p(\vartheta, \delta)\, \varkappa(\|\mathfrak{f}\|_{\mathfrak{D}}, \delta) \text{ for a.e. } \vartheta \in \Theta \text{ and each } \mathfrak{f} \in \mathfrak{D},$$

(H_4) There exists a random function $Q : \Psi \longrightarrow \mathbb{R}^+ \backslash \{0\}$ where:

$$\sigma(1+\kappa\sigma\zeta)(\|\varpi_1\|_{\mathfrak{D}} + \varkappa(D,\delta)\|p\|_{L^1}) + \kappa\sigma\zeta\left\|x^1\right\| + \sigma^{'}(1+\kappa\sigma\zeta)|\varpi_2| \leq Q(\delta)$$

where

$$D := \zeta Q(\delta) + \sigma\|\varpi_1\|_{\mathfrak{D}},$$

(H_5) The linear operator $\mathcal{K} : L^2(\Theta, \Omega) \to \Xi$ given by:

$$\mathcal{K}\mathfrak{f} = \int_0^{\kappa} \mathcal{S}_1(\kappa - \varrho)\mathcal{Z}_2\mathfrak{f}(\varrho, \delta)d\varrho$$

has a pseudo-inverse operator $\mathcal{K}^{-1}$ in $L^2(\Theta, \Omega)/ker\mathcal{K}$ and there exists a positive constant ζ such that $\left\|\mathcal{Z}_2\mathcal{K}^{-1}\right\| \leq \zeta$,

(H_6) For each $\delta \in \Psi, \varpi_1(\cdot, \delta)$ is continuous and for each $\vartheta, \varpi_1(\vartheta, \cdot)$ is measurable, and for each $\delta \in \Psi$, $\varpi_2(\delta)$ is measurable.

Theorem 1. *If (H_1)–(H_6) are satisfied, then the problem (1) is controllable on Θ.*

Proof. Define the control:

$$\mathfrak{f}(\vartheta, \delta) = \mathcal{K}^{-1}\left(x^1(\delta) - \mathcal{S}_1(\kappa)\varpi_1(0, \delta) - \mathcal{S}_2(\kappa)\varpi_2(\delta) - \int_0^{\kappa} \mathcal{S}_1(\kappa - \varrho)\psi(\varrho, x_\varrho(\cdot, \delta), \delta)d\varrho\right).$$

We define the operator $\mathfrak{T} : \Psi \times \Lambda \longrightarrow \Lambda$ by: $(\mathfrak{T}(\delta)x)(\vartheta) = \varpi_1(\vartheta, \delta)$, if $\vartheta \in (-\infty, 0]$, and for $\vartheta \in \Theta$:

$$(\mathfrak{T}(\delta)x)(\vartheta) = \mathcal{S}_1(\vartheta)\varpi_1(0,\delta) + \mathcal{S}_2(\vartheta)\varpi_2(\delta) + \int_0^{\vartheta} \mathcal{S}_1(\vartheta-\varrho)\ \psi(\varrho, x_\varrho(\cdot,\delta),\delta),\delta)d\varrho + \int_0^{\vartheta} \mathcal{S}_1(\vartheta-\varrho)\mathcal{Z}_2\mathcal{K}^{-1}$$
$$\times \left(x^1(\delta) - \mathcal{S}_1(\kappa)\varpi_1(0,\delta) - \mathcal{S}_2(\kappa)\varpi_2(\delta) - \int_0^{\kappa} \mathcal{S}_1(\kappa-\varepsilon)\psi(\varepsilon, x_\varepsilon(\cdot,\delta),\delta)d\varepsilon\right)d\varrho. \quad (3)$$

Using (H_5), we will demonstrate that $\mathfrak{T}$ has a fixed point $x(\vartheta, \delta)$ which is a mild solution of (1). This implies that the problem (1) is controllable on Θ. Further, we prove that $\mathfrak{T}(\cdot)$ is a random operator. For that, we demonstrate that for any $x \in \Lambda$, $\mathfrak{T}(\cdot)(x) : \Psi \longrightarrow \Lambda$ is a random variable. Then we demonstrate that $\mathfrak{T}(\cdot)(x) : \Psi \longrightarrow \Lambda$ is measurable. As the mapping $\psi(\vartheta, x, \cdot)$, $\vartheta \in \Theta, x \in \Lambda$ is measurable by assumption (H_2) and (H_6). Let $D : \Psi \longrightarrow 2^{\Lambda}$ be given by:

$$D(\delta) = \{x \in \Lambda : \|x\|_{\Lambda} \leq Q(\delta)\}.$$

$D(\delta)$ is bounded, closed, convex and solid for all $\delta \in \Psi$. Then D is measurable by Lemma 17 in [35]. Let $\delta \in \Psi$ be fixed, then for any $x \in D(\delta)$ and by (A_1), we obtain:

$$\begin{aligned} \|x_\varrho\|_{\mathfrak{D}} &\leq \gamma(\varrho)|x(\varrho)| + \sigma(\varrho)\|x_0\|_{\mathfrak{D}} \\ &\leq \zeta_\kappa|x(\varrho)| + \sigma_\kappa\|\varpi_1\|_{\mathfrak{D}}, \end{aligned}$$

and by (H_3) and (H_4), we have

$$\begin{aligned}
|(\mathfrak{T}(\delta)x)(\vartheta)| &\leq \sigma\|\varpi_1\|_{\mathfrak{D}} + \sigma^{'}|\varpi_2| + \sigma\int_0^{\vartheta}|\psi(\varrho,x_\varrho,\delta)|d\varrho \\
&\quad +\sigma\zeta\int_0^{\vartheta}\left|x^1(\delta)\right| + \sigma\|\varpi_1\|_{\mathfrak{D}} + \sigma^{'}|\varpi_2|d\varrho \\
&\quad +\sigma\zeta\int_0^{\vartheta}\int_0^{\kappa}\|\mathcal{S}_1(\varepsilon-\varrho)\||\psi(\varepsilon,x_\varepsilon,\delta)|\,d\varepsilon d\varrho \\
&\leq \sigma\|\varpi_1\|_{\mathfrak{D}} + \sigma^{'}|\varpi_2| + \sigma\int_0^{\kappa}p(\varrho,\delta)\,\varkappa(\|x_\varrho\|_{\mathfrak{D}},\delta)\,d\varrho \\
&\quad +\kappa\sigma\zeta\left|x^1(\delta)\right| + \kappa\sigma^2\zeta\|\varpi_1\|_{\mathfrak{D}} + \kappa\sigma\sigma^{'}\zeta|\varpi_2| \\
&\quad +\kappa\sigma^2\zeta\int_0^{\kappa}p(\varepsilon,\delta)\,\varkappa(\|x_\varepsilon\|_{\mathfrak{D}},\delta)\,d\varepsilon \\
&\leq \sigma(1+\kappa\sigma\zeta)\|\varpi_1\|_{\mathfrak{D}} + \kappa\sigma\zeta\left|x^1(\delta)\right| + \sigma^{'}(1+\kappa\sigma\zeta)|\varpi_2| \\
&\quad +\sigma(1+\kappa\sigma\zeta)\int_0^{\kappa}p(\varrho,\delta)\,\varkappa\big(\|x_\varrho\|_{\mathfrak{D}},\delta\big)\,d\varrho \\
&\leq \sigma(1+\kappa\sigma\zeta)\left(\|\varpi_1\|_{\mathfrak{D}} + \varkappa(D_\kappa,\delta)\int_0^{\kappa}p(\varrho,\delta)\;d\varrho\right) \\
&\quad +\kappa\sigma\zeta\left\|x^1(\delta)\right\| + \sigma^{'}(1+\kappa\sigma\zeta)|\varpi_2|.
\end{aligned}$$

Set

$$D_\kappa := \zeta_\kappa Q(\delta) + \sigma_\kappa\|\varpi_1\|_{\mathfrak{D}}.$$

Then, we have

$$\begin{aligned}
|(\mathfrak{T}(\delta)x)(\vartheta)| \leq{}& \sigma(1+\kappa\sigma\zeta)\left(\|\varpi_1\|_{\mathfrak{D}} + \varkappa(D_\kappa,\delta)\int_0^{\kappa}p(\varrho,\delta)\;d\varrho\right) \\
&+\kappa\sigma\zeta\left|x^1(\delta)\right| + \sigma^{'}|\varpi_2|(1+\kappa\sigma\zeta).
\end{aligned}$$

Thus

$$\begin{aligned}
\|(\mathfrak{T}(\delta)x)\|_\Lambda &\leq \sigma(1+\kappa\sigma\zeta)(\|\varpi_1\|_{\mathfrak{D}} + \varkappa(D_\kappa,\delta)\|p\|_{L^1}) \\
&\quad +\kappa\sigma\zeta\left|x^1(\delta)\right| + \sigma^{'}(1+\kappa\sigma\zeta)|\varpi_2| \\
&\leq Q(\delta).
\end{aligned}$$

Thus, we deduce that $\mathfrak{T}$ is a random operator with stochastic domain D and $\mathfrak{T}(\delta):D(\delta)\longrightarrow D(\delta)$ for each $\delta\in\Psi$. □

Claim 1: $\mathfrak{T}$ is continuous.
Let x^n be a sequence where $x^n\longrightarrow x$ in Y. Then

$$\begin{aligned}
&|(\mathfrak{T}(\delta)x^n)(\vartheta) - (\mathfrak{T}(\delta)x)(\vartheta)| \\
\leq{}& \sigma\int_0^{\vartheta}|\psi(\varrho,x^n_\varrho,\delta) - \psi(\varrho,x_\varrho,\delta)|\,d\varrho \\
&+\zeta\sigma\int_0^{\vartheta}\int_0^{\kappa}\|\mathcal{S}_1(\kappa-\varepsilon)\||\psi(\varepsilon,x^n_\varepsilon,\delta) - \psi(\varepsilon,x_\varepsilon,\delta)|d\varepsilon d\varrho \\
\leq{}& \sigma\int_0^{\vartheta}|\psi(\varrho,x^n_\varrho,\delta) - \psi(\varrho,x_\varrho,\delta)|\,d\varrho \\
&+\kappa\sigma^2\zeta\int_0^{\kappa}|\psi(\varepsilon,x^n_\varepsilon,\delta) - \psi(\varepsilon,x_\varepsilon,\delta)|d\varepsilon \\
\leq{}& \sigma(1+\kappa\sigma\zeta)\int_0^{\kappa}|\psi(\varepsilon,x^n_\varepsilon,\delta) - \psi(\varepsilon,x_\varepsilon,\delta)|d\varepsilon.
\end{aligned}$$

As $\psi(\varrho,\cdot,\delta)$ is continuous, we obtain

$$\|\psi(\cdot,x^n_\cdot,\delta)-\psi(\cdot,x_\cdot,\delta)\|_{L^1}\to 0 \text{ as } n\to+\infty.$$

Thus $\mathfrak{T}$ is continuous.
Claim 2: We demonstrate that for every $\delta\in\Psi$, $\{x\in D(\delta):\mathfrak{T}(\delta)x=x\}\neq\emptyset$.
We apply Schauder's theorem.

(a) $\mathfrak{T}$ maps bounded sets into equicontinuous sets in $D(\delta)$.
Let $\varepsilon_1,\varepsilon_2\in[0,\kappa]$ with $\varepsilon_2>\varepsilon_1$, $D(\delta)$ be a bounded set as in Claim 2, and $x\in D(\delta)$. Then

$$\begin{aligned}
&|(\mathfrak{T}(\delta)x)(\varepsilon_2)-(\mathfrak{T}(\delta)x)(\varepsilon_1)|\\
\le\;&\|\mathcal{S}_1(\varepsilon_2)-\mathcal{S}_1(\varepsilon_1)\|_{\mathfrak{D}(\Xi)}\|\varpi_1\|_{\mathfrak{D}}+\|\mathcal{S}_2(\varepsilon_2)-\mathcal{S}_2(\varepsilon_1)\|_{\mathfrak{D}(\Xi)}|\varpi_2|\\
&+\int_0^{\varepsilon_1}\|\mathcal{S}_1(\varepsilon_2-\varrho)-\mathcal{S}_1(\varepsilon_1-\varrho)\|_{\mathfrak{D}(\Xi)}\big|\psi(\varrho,x_\varrho,\delta)\big|\,d\varrho\\
&+\int_{\varepsilon_1}^{\varepsilon_2}\|C(\varepsilon_2-\varrho)\|_{\mathfrak{D}(\Xi)}\big|\psi(\varrho,x_\varrho,\delta)\,\big|d\varrho\\
&+\zeta\int_0^{\varepsilon_1}\|\mathcal{S}_1(\varepsilon_2-\varrho)-\mathcal{S}_1(\varepsilon_1-\varrho)\|_{\mathfrak{D}(\Xi)}\\
&\times\Big[\big|x^1(\delta)\big|+\|\mathcal{S}_1(\kappa)\|_{\mathfrak{D}(\Xi)}\|\varpi_1\|_{\mathfrak{D}}+\|\mathcal{S}_2(\kappa)\|_{\mathfrak{D}(\Xi)}|\varpi_2|\Big]d\varrho\\
&+\zeta\int_0^{\varepsilon_1}\|\mathcal{S}_1(\varepsilon_2-\varrho)-\mathcal{S}_1(\varepsilon_1-\varrho)\|_{\mathfrak{D}(\Xi)}\int_0^{\kappa}\|\mathcal{S}_1(\kappa-\varepsilon)\|_{\mathfrak{D}(\Xi)}|\psi(\varepsilon,x_\varepsilon(\cdot,\delta),\delta)|d\varepsilon d\varrho\\
&+\zeta\int_{\varepsilon_1}^{\varepsilon_2}\|C(\varepsilon_2-\varrho)\|_{\mathfrak{D}(\Xi)}\Big[\big|x^1(\delta)\big|+\|\mathcal{S}_1(\kappa)\|_{\mathfrak{D}(\Xi)}\|\varpi_1\|_{\mathfrak{D}}+\|\mathcal{S}_2(\kappa)\|_{\mathfrak{D}(\Xi)}|\varpi_2|\Big]d\varrho\\
&+\zeta\int_{\varepsilon_1}^{\varepsilon_2}\|C(\varepsilon_2-\varrho)\|_{\mathfrak{D}(\Xi)}\int_0^{\kappa}\|\mathcal{S}_1(\kappa-\varepsilon)\|_{\mathfrak{D}(\Xi)}|\psi(\varepsilon,x_\varepsilon(\cdot,\delta),\delta)|d\varepsilon d\varrho\\
\le\;&\|\mathcal{S}_1(\varepsilon_2-\varrho)-\mathcal{S}_1(\varepsilon_1-\varrho)\|_{\mathfrak{D}(\Xi)}\|\varpi_1\|_{\mathfrak{D}}+\|\mathcal{S}_2(\varepsilon_2)-\mathcal{S}_2(\varepsilon_1)\|_{\mathfrak{D}(\Xi)}|\varpi_2|\\
&+\varkappa(D_\kappa,\delta)\int_0^{\varepsilon_1}\|\mathcal{S}_1(\varepsilon_2-\varrho)-\mathcal{S}_1(\varepsilon_1-\varrho)\|_{\mathfrak{D}(\Xi)}p(\varrho,\delta)d\varrho\\
&+\sigma\varkappa(D_\kappa,\delta)\int_{\varepsilon_1}^{\varepsilon_2}p(\varrho,\delta)d\varrho\\
&+\zeta\int_0^{\varepsilon_1}\|\mathcal{S}_1(\varepsilon_2-\varrho)-\mathcal{S}_1(\varepsilon_1-\varrho)\|_{\mathfrak{D}(\Xi)}\\
&\times\Big[\big|x^1(\delta)\big|+\|\mathcal{S}_1(\kappa)\|_{\mathfrak{D}(\Xi)}\|\varpi_1\|_{\mathfrak{D}}+\|\mathcal{S}_2(\kappa)\|_{\mathfrak{D}(\Xi)}|\varpi_2|\Big]d\varrho\\
&+\zeta\sigma\varkappa(D_\kappa,\delta)\int_0^{\varepsilon_1}\|\mathcal{S}_1(\varepsilon_2-\varrho)-\mathcal{S}_1(\varepsilon_1-\varrho)\|_{\mathfrak{D}(\Xi)}\int_0^{\kappa}p(\varepsilon,\delta)d\varepsilon d\varrho\\
&+\zeta\sigma\int_{\varepsilon_1}^{\varepsilon_2}(\big|x^1(\delta)\big|+\|\mathcal{S}_1(\kappa)\|_{\mathfrak{D}(\Xi)}\|\varpi_1\|_{\mathfrak{D}}+\|\mathcal{S}_2(\kappa)\|_{\mathfrak{D}(\Xi)}|\varpi_2|\\
&+\sigma\varkappa(D_\kappa,\delta)\int_0^{\kappa}p(\varepsilon,\delta)d\varepsilon)d\varrho.
\end{aligned}$$

The right-hand of the above inequality tends to zero as $\varepsilon_2-\varepsilon_1\to 0$, since $\mathcal{S}_1(\vartheta),\mathcal{S}_2(\vartheta)$ are compact for $\vartheta>0$, and strongly continuous, then we obtain the continuity in the uniform operator topology (see [13,36]).

(b) Let $\vartheta\in[0,\kappa]$ be fixed and let $x\in D(\delta)$. From assumptions (H_3), (H_5) and since $\mathcal{S}_1(\vartheta)$ is compact, the set

$$\left\{\int_0^{\vartheta}\mathcal{S}_1(\vartheta-\varrho)\psi(\varrho,x_\varrho(\cdot,\delta),\delta)d\varrho+\int_0^{\vartheta}\mathcal{S}_1(\vartheta-\varrho)\mathcal{Z}_2\mathfrak{f}(\vartheta,\delta)d\varrho\right\}$$

is precompact in Ξ, and the set

$$\left\{ \mathcal{S}_1(\vartheta)\varpi_1(0,\delta) + \mathcal{S}_2(\vartheta)\varpi_2(\delta) + \int_0^\vartheta \mathcal{S}_1(\vartheta-\varrho)\psi(\varrho, x_\varrho(\cdot,\delta),\delta)d\varrho + \int_0^\vartheta \mathcal{S}_1(\vartheta-\varrho)\mathcal{Z}_2\mathfrak{f}(\vartheta,\delta)d\varrho \right\}$$

is precompact in Ξ. Thus, $\mathfrak{T}(\delta) : D(\delta) \to D(\delta)$ is continuous and compact. Schauder's theorem implies that $\mathfrak{T}(\delta)$ has a fixed point $x(\delta)$ in $D(\delta)$. Since $\bigcap_{\delta\in\Psi} D(\delta) \neq \emptyset$, and a measurable selector of int$D$ exists, then by Lemma 1, we conclude that $\mathfrak{T}$ has a stochastic fixed point $x^*(\delta)$, which is a random mild solution of (1).

4. Controllability Results for the State-Dependent Delay Case

In this section we give our main controllability result for problem (2).

Definition 7. *A stochastic process $x : (-\infty, \kappa] \times \Psi \to \Xi$ is said to be a random mild solution of problem (2) if $x(\vartheta,\delta) = \varpi_1(\vartheta,\delta)$; $\vartheta \in (-\infty, 0]$, $x'(0,\delta) = \varpi_2(\delta)$ and the restriction of $x(\cdot,\delta)$ to the interval Θ is continuous and verifies equation:*

$$\begin{aligned} x(\vartheta,\delta) &= \mathcal{S}_1(\vartheta)\varpi_1(0,\delta) + \mathcal{S}_2(\vartheta)\varpi_2(\delta) + \int_0^\vartheta \mathcal{S}_1(\vartheta-\varrho)\psi(\varrho, x_{\xi(\varrho,x_\varrho)}(\cdot,\delta),\delta)d\varrho \\ &\quad + \int_0^\vartheta \mathcal{S}_1(\vartheta-\varrho)\mathcal{Z}_2\mathfrak{f}(\vartheta,\delta)d\varrho. \end{aligned}$$

Set

$$\mathcal{Q}(\xi^-) = \{\xi(\varrho,\varpi_2) : (\varrho,\varpi_2) \in \Theta \times \mathfrak{D}, \xi(\varrho,\varpi_2) \leq 0\}.$$

Suppose that $\xi : \Theta \times \mathfrak{D} \to (-\infty, \kappa]$ is continuous. And,

(H_{ϖ_1}) The function $\vartheta \to \varpi_{1\vartheta}$ is continuous from $\mathcal{Q}(\xi^-)$ into $\mathfrak{D}$ and there exists a continuous and bounded function $\gamma^{\varpi_1} : \mathcal{Q}(\xi^-) \to (0,\infty)$ where

$$\|\varpi_{1\vartheta}\|_{\mathfrak{D}} \leq \gamma^{\varpi_1}(\vartheta)\|\varpi_1\|_{\mathfrak{D}} \quad \text{for every } \vartheta \in \mathcal{Q}(\xi^-).$$

Remark 2 ([31])**.** *The hypothesis (H_{ϖ_1}) is satisfied by continuous and bounded functions.*

Lemma 3 ([37])**.** *If $x : (-\infty,\kappa] \to \Xi$ is a function such that $x_0 = \varpi_1$, then*

$$\|x_\varrho\|_{\mathfrak{D}} \leq (\sigma_\kappa + \gamma^{\varpi_1})\|\varpi_1\|_{\mathfrak{D}} + \zeta_\kappa \sup\{|x(\iota)|; \iota \in [0, max\{0,\varrho\}]\},\ \varrho \in \mathcal{Q}(\xi^-) \cup \Theta,$$

where $\gamma^{\varpi_1} = \sup\limits_{\vartheta\in\mathcal{Q}(\xi^-)} \gamma^{\varpi_1}(\vartheta)$.

We consider now the hypotheses:

(H_1') $\mathcal{S}_1(\vartheta)$ is compact for $\vartheta > 0$ in Ξ.

(H_2') The function $\psi : \Theta \times \mathfrak{D} \times \Psi \to \Xi$ is random Carathéodory.

(H_3') There exist a function $\varkappa : \Theta \times \Psi \to \mathbb{R}^+$ and $p : \Theta \times \Psi \to \mathbb{R}^+$ such that for each $\delta \in \Psi$, $\varkappa(\cdot,\delta)$ is a continuous nondecreasing function and $p(\cdot,\delta)$ integrable with:

$$|\psi(\vartheta,\mathfrak{f},\delta)| \leq p(\vartheta,\delta)\ \varkappa(\|\mathfrak{f}\|_{\mathfrak{D}},\delta) \text{ for a.e. } \vartheta \in \Theta \text{ and each } \mathfrak{f} \in \mathfrak{D}.$$

(H_4') There exists a function $\beta : \Theta \times \Psi \longrightarrow \mathbb{R}^+$ with $\beta(\cdot,\delta) \in L^1(\Theta,\mathbb{R}^+)$ for each $\delta \in \Psi$ such that for any bounded $B \subseteq \Xi$.

$$\mu(\psi(\vartheta,B,\delta)) \leq \beta(\vartheta,\delta)\mu(B).$$

(H_5') There exists a random function $Q : \Psi \longrightarrow \mathbb{R}^+ \backslash \{0\}$ where:

$$\sigma(1+\kappa\sigma\lambda)\Big(\|\varpi_1\|_{\mathfrak{D}} + \varkappa((\sigma_\kappa+\gamma^{\varpi_1})\|\varpi_1\|_{\mathfrak{D}} + \zeta_\kappa Q(\delta),\delta)\int_0^\kappa p(\varrho,\delta)\ d\varrho\Big)$$

$$+\kappa\sigma\lambda\Big\|x^1(\delta)\Big\| + \sigma'(1+\kappa\sigma\lambda)|\varpi_2| \le Q(\delta).$$

(H_6') The linear operator $\mathcal{K} : L^2(\Theta,\Omega) \to \Xi$ defined by:

$$\mathcal{K}\mathfrak{f} = \int_0^\kappa \mathcal{S}_1(\kappa-\varrho)\mathcal{Z}_2\mathfrak{f}(\varrho,\delta)d\varrho$$

has a pseudo-inverse operator $\mathcal{K}^{-1}$ which takes values in $L^2(\Theta,\Omega)/ker\mathcal{K}$ and there exists a positive constant λ such that $\|\mathcal{Z}_2\mathcal{K}^{-1}\| \le \lambda$.

(H_7') For each $\delta \in \Psi, \varpi_1(\cdot,\delta)$ is continuous and for each $\vartheta, \varpi_1(\vartheta,\cdot)$ is measurable and for each $\delta \in \Psi, \varpi_2(\delta)$ is measurable.

Theorem 2. *Assume that $(H_1') - (H_7')$ and (H_{ϖ_1}) hold. If*

$$\sigma(1+\sigma\lambda\kappa)\int_0^\kappa \beta(\varrho)\zeta(\varrho)d\varrho < 1, \tag{4}$$

then the random problem (2) is controllable on Θ.

Proof. Using (H_6), we define the control:

$$\mathfrak{f}(\vartheta,\delta) = \mathcal{K}^{-1}\Big(x^1(\delta) - \mathcal{S}_1(\kappa)\varpi_1(0,\delta) - \mathcal{S}_2(\kappa)\varpi_2(\delta)$$

$$-\int_0^\kappa \mathcal{S}_1(\kappa-\varrho)\psi(\varrho, x_{\xi(\varrho,x_\varrho)},\delta)d\varrho\Big).$$

We define the operator $\mathfrak{T} : \Psi \times \Lambda \longrightarrow \Lambda$ by: $(\mathfrak{T}(\delta)x)(\vartheta) = \varpi_1(\vartheta,\delta)$, if $\vartheta \in (-\infty,0]$, and for $\vartheta \in \Theta$ by:

$$(\mathfrak{T}(\delta)x)(\vartheta) = \mathcal{S}_1(\vartheta)\varpi_1(0,\delta) + \mathcal{S}_2(\vartheta)\varpi_2(\delta)$$
$$+\int_0^\vartheta \mathcal{S}_1(\vartheta-\varrho)\ \psi(\varrho, x_{\xi(\varrho,x_\varrho)},\delta),\delta)d\varrho + \int_0^\vartheta \mathcal{S}_1(\vartheta-\varrho)\mathcal{Z}_2\mathcal{K}^{-1}$$
$$\times\Big(x^1(\delta) - \mathcal{S}_1(\kappa)\varpi_1(0,\delta) - \mathcal{S}_2(\kappa)\varpi_2(\delta) - \int_0^\kappa \mathcal{S}_1(\kappa-\varepsilon)\psi(\varepsilon, x_{\xi(\varepsilon,x_\varepsilon)},\delta)d\varepsilon\Big)d\varrho. \tag{5}$$

Proving that $\mathfrak{T}(\cdot)$ has a fixed point $x(\vartheta,\delta)$ and that (2) is controllable. Further, we demonstrate that $\mathfrak{T}(\cdot)$ is a random operator by showing that for any $x \in \Lambda$, $\mathfrak{T}(\cdot)(x)$: $\Psi \longrightarrow \Lambda$ is a random variable. Also, we show that $\mathfrak{T}(\cdot)(x) : \Psi \longrightarrow \Lambda$ is measurable, as a mapping $\psi(\vartheta,x,\cdot)$, $\vartheta \in \Theta, x \in \Lambda$ is measurable by (H_2') and (H_6'). Let $D : \Psi \longrightarrow 2^\Lambda$ be given by:

$$D(\delta) = \{x \in \Lambda : \|x\|_\Lambda \le Q(\delta)\}.$$

$D(\delta)$ is bounded, closed, convex and solid for all $\delta \in \Psi$. Then D is measurable. Let $\delta \in \Psi$ be fixed. If $x \in D(\delta)$, then

$$\Big\|x_{\xi(\vartheta,x_\vartheta)}\Big\|_{\mathfrak{D}} = (\sigma_\kappa + L^{\varpi_1})\|\varpi_1\|_{\mathfrak{D}} + \zeta_\kappa Q(\delta),$$

and for each $x \in D(\delta)$, from (H_3') and (H_5'), for each $\vartheta \in \Theta$, we have

$$\begin{aligned}
|(\mathfrak{T}(\delta)x)(\vartheta)| &\le \sigma\|\varpi_1\|_{\mathfrak{D}} + \sigma'|\varpi_2| + \sigma\int_0^{\vartheta} |\psi(\varrho, x_{\xi(\varrho,x_\varrho)}, \delta)|d\varrho \\
&\quad + \sigma\lambda\int_0^{\vartheta}\left|x^1(\delta)\right| + \sigma\|\varpi_1\|_{\mathfrak{D}} + \sigma'|\varpi_2|d\varrho \\
&\quad + \sigma\lambda\int_0^{\vartheta}\int_0^{\kappa}\|\mathcal{S}_1(\varepsilon-\varrho)\|\left|\psi(\varepsilon, x_{\xi(\varepsilon,x_\varepsilon)}, \delta)\right| d\varepsilon d\varrho \\
&\le \sigma\|\varpi_1\|_{\mathfrak{D}} + \sigma'|\varpi_2| + \sigma\int_0^{\kappa} p(\varrho,\delta)\,\varkappa\Big(\|x_{\xi(\varrho,x_\varrho)}\|_{\mathfrak{D}}, \delta\Big)\,d\varrho \\
&\quad + \kappa\sigma\lambda\left|x^1(\delta)\right| + \kappa\sigma^2\zeta\|\varpi_1\|_{\mathfrak{D}} + \kappa\sigma\sigma'\zeta|\varpi_2| \\
&\quad + \kappa\sigma^2\lambda\int_0^{\kappa} p(\varepsilon,\delta)\,\varkappa\Big(\|x_{\xi(\varepsilon,x_\varepsilon)}\|_{\mathfrak{D}}, \delta\Big)\,d\varepsilon \\
&\le \sigma(1+\kappa\sigma\lambda)\|\varpi_1\|_{\mathfrak{D}} + \kappa\sigma\lambda\left|x^1(\delta)\right| + \sigma'(1+\kappa\sigma\lambda)|\varpi_2| \\
&\quad + \sigma(1+\kappa\sigma\lambda)\int_0^{\kappa} p(\varrho,\delta)\,\varkappa\Big(\|x_{\xi(\varrho,x_\varrho)}\|_{\mathfrak{D}}, \delta\Big)\,d\varrho \\
&\le \sigma(1+\kappa\sigma\lambda) \\
&\quad \times\left(\|\varpi_1\|_{\mathfrak{D}} + \varkappa((\sigma_\kappa + \gamma^{\varpi_1})\|\varpi_1\|_{\mathfrak{D}} + \zeta_\kappa Q(\delta), \delta)\int_0^{\kappa} p(\varrho,\delta)\,d\varrho\right) \\
&\quad + \kappa\sigma\lambda\left|x^1(\delta)\right| + \sigma'(1+\kappa\sigma\lambda)|\varpi_2|.
\end{aligned}$$

Thus, $\mathfrak{T}$ is a random operator with stochastic domain D and $\mathfrak{T}(\delta) : D(\delta) \to D(\delta)$ for each $\delta \in \Psi$. □

Claim 1: $\mathfrak{T}$ is continuous.
Let x^n be a sequence such that $x^n \longrightarrow x$ in Λ. Then

$$\begin{aligned}
&|(\mathfrak{T}(\delta)x^n)(\vartheta) - (\mathfrak{T}(\delta)x)(\vartheta)| \\
\le\ & \sigma\int_0^{\vartheta} |\psi(\varrho, x^n_{\xi(\varrho,x^n_\varrho)}, \delta) - \psi(\varrho, x_{\xi(\varrho,x_\varrho)}, \delta)|\,d\varrho \\
& + \lambda\sigma\int_0^{\vartheta}\int_0^{\kappa}\|\mathcal{S}_1(\kappa-\varepsilon)\||\psi(\varepsilon, x^n_{\xi(\varrho,x^n_\varrho)}, \delta) - \psi(\varepsilon, x_{\xi(\varrho,x_\varrho)}, \delta)|d\varepsilon d\varrho \\
\le\ & \sigma\int_0^{\vartheta} |\psi(\varrho, x^n_{\xi(\varrho,x^n_\varrho)}, \delta) - \psi(\varrho, x_{\xi(\varrho,x_\varrho)}, \delta)|\,d\varrho \\
& + \kappa\sigma^2\lambda\int_0^{\kappa} |\psi(\varrho, x^n_{\xi(\varrho,x^n_\varrho)}, \delta) - \psi(\varrho, x_{\xi(\varrho,x_\varrho)}, \delta)|\,d\varrho \\
\le\ & \sigma(1+\kappa\sigma\lambda)\int_0^{\kappa} |\psi(\varrho, x^n_{\xi(\varrho,x^n_\varrho)}, \delta) - \psi(\varrho, x_{\xi(\varrho,x_\varrho)}, \delta)|\,d\varrho.
\end{aligned}$$

As $\psi(\varrho, \cdot, \delta)$ is continuous, then $\|(\mathfrak{T}(\delta)x^n)(\vartheta) - (\mathfrak{T}(\delta)x)(\vartheta)\|_\Lambda \to 0$ as $n \to +\infty$. Thus $\mathfrak{T}$ is continuous.

Claim 2: We demonstrate that for every $\delta \in \Psi, \{x \in D(\delta) : \mathfrak{T}(\delta)x = x\} \neq \emptyset$ by employing Mönch fixed point theorem [38,39].

(a) $\mathfrak{T}$ maps bounded sets into equicontinuous sets in $D(\delta)$.
Let $\varepsilon_1, \varepsilon_2 \in [0,\kappa]$ with $\varepsilon_2 > \varepsilon_1$, $D(\delta)$ be a bounded set, and $x \in D(\delta)$. Then

$$
\begin{aligned}
& |(\mathfrak{T}(\delta)x)(\varepsilon_2) - (\mathfrak{T}(\delta)x)(\varepsilon_1)| \\
\leq\ & \|\mathcal{S}_1(\varepsilon_2) - \mathcal{S}_1(\varepsilon_1)\|_{\mathfrak{D}(\Xi)}\|\varpi_1\|_{\mathfrak{D}} + \|\mathcal{S}_2(\varepsilon_2) - \mathcal{S}_2(\varepsilon_1)\|_{\mathfrak{D}(\Xi)}|\varpi_2| \\
& + \int_0^{\varepsilon_1} \|\mathcal{S}_1(\varepsilon_2 - \varrho) - \mathcal{S}_1(\varepsilon_1 - \varrho)\|_{\mathfrak{D}(\Xi)} \psi(\varrho, x_{\xi(\varrho,x_\varrho)}, \delta) d\varrho \\
& + \int_{\varepsilon_1}^{\varepsilon_2} \|\mathcal{S}_1(\varepsilon_2 - \varrho)\|_{\mathfrak{D}(\Xi)} \psi(\varrho, x_{\xi(\varrho,x_\varrho)}, \delta) d\varrho \\
& +\lambda \int_0^{\varepsilon_1} \|\mathcal{S}_1(\varepsilon_2 - \varrho) - \mathcal{S}_1(\varepsilon_1 - \varrho)\|_{\mathfrak{D}(\Xi)} \left[\left|x^1(\delta)\right| + \|\mathcal{S}_1(\kappa)\| |\varpi_1(0,\delta)|\right] d\varrho \\
& +\lambda \int_0^{\varepsilon_1} \|\mathcal{S}_1(\varepsilon_2 - \varrho) - \mathcal{S}_1(\varepsilon_1 - \varrho)\|_{\mathfrak{D}(\Xi)} \int_0^{\kappa} \|\mathcal{S}_1(\kappa - \varepsilon)\|_{\mathfrak{D}(\Xi)} \left|\psi(\varepsilon, x_{\xi(\varrho,x_\varrho)}, \delta)\right| d\varepsilon d\varrho \\
& +\lambda \int_{\varepsilon_1}^{\varepsilon_2} \|\mathcal{S}_1(\varepsilon_2 - \varrho)\|_{\mathfrak{D}(\Xi)} \left[\left|x^1(\delta)\right| + \|\mathcal{S}_1(\kappa)\| |\varpi_1(0,\delta)|\right] d\varrho \\
& +\lambda \int_{\varepsilon_1}^{\varepsilon_2} \|\mathcal{S}_1(\varepsilon_2 - \varrho)\|_{\mathfrak{D}(\Xi)} \int_0^{\kappa} \|\mathcal{S}_1(\kappa - \varepsilon)\|_{B(\Xi)} \left|\psi(\varepsilon, x_{\xi(\varepsilon,x_\varepsilon)}, \delta)\right| d\varepsilon d\varrho.
\end{aligned}
$$

Thus

$$
\begin{aligned}
& |(\mathfrak{T}(\delta)x)(\varepsilon_2) - (\mathfrak{T}(\delta)x)(\varepsilon_1)| \\
\leq\ & |\mathcal{S}_1(\varepsilon_2) - \mathcal{S}_1(\varepsilon_1)| \|\varpi_1\|_{\mathfrak{D}} + \|\mathcal{S}_2(\varepsilon_2) - \mathcal{S}_2(\varepsilon_1)\|_{\mathfrak{D}(\Xi)}|\varpi_2| \\
& + \int_0^{\varepsilon_1} \|\mathcal{S}_1(\varepsilon_2 - \varrho) - \mathcal{S}_1(\varepsilon_1 - \varrho)\|_{\mathfrak{D}(\Xi)} \psi(\varrho, x_{\xi(\varrho,x_\varrho)}, \delta) d\varrho \\
& + \int_{\varepsilon_1}^{\varepsilon_2} \|\mathcal{S}_1(\varepsilon_2 - \varrho)\|_{\mathfrak{D}(\Xi)} \psi(\varrho, x_{\xi(\varrho,x_\varrho)}, \delta) d\varrho \\
& +\lambda \int_0^{\varepsilon_1} \|\mathcal{S}_1(\varepsilon_2 - \varrho) - \mathcal{S}_1(\varepsilon_1 - \varrho)\|_{\mathfrak{D}(\Xi)} \left[\left\|x^1(\delta)\right\| + \|\mathcal{S}_1(\kappa)\|_{\mathfrak{D}(\Xi)} |\varpi_1(0,\delta)|\right] d\varrho \\
& +\lambda \int_0^{\varepsilon_1} \|\mathcal{S}_1(\varepsilon_2 - \varrho) - \mathcal{S}_1(\varepsilon_1 - \varrho)\|_{\mathfrak{D}(\Xi)} \varkappa\big((\sigma_\kappa + \gamma^{\varpi_1})\|\varpi_1\|_{\mathfrak{D}} + \zeta_\kappa Q(\delta)\big) \\
& \times \int_0^{\kappa} p(\varepsilon,\delta) d\varepsilon d\varrho \\
& +\lambda\sigma \int_{\varepsilon_1}^{\varepsilon_2} \left\|x^1\right\| + \|\mathcal{S}_1(\kappa)\|_{\mathfrak{D}(\Xi)} |\varpi_1(0,\delta)| + \sigma\varkappa\big((\sigma_\kappa + \gamma^{\varpi_1})\|\varpi_1\|_{\mathfrak{D}} + \zeta_\kappa Q(\delta)\big) \\
& \times \int_0^{\kappa} p(\varepsilon,\delta) d\varepsilon d\varrho.
\end{aligned}
$$

Hence

$$
\begin{aligned}
& |(\mathfrak{T}(\delta)x)(\varepsilon_2) - (\mathfrak{T}(\delta)x)(\varepsilon_1)| \\
\leq\ & \|\mathcal{S}_1(\varepsilon_2) - \mathcal{S}_1(\varepsilon_1)\|_{\mathfrak{D}(\Xi)}\|\varpi_1\|_{\mathfrak{D}} + \|\mathcal{S}_2(\varepsilon_2) - \mathcal{S}_2(\varepsilon_1)\|_{\mathfrak{D}(\Xi)}|\varpi_2| \\
& +\varkappa\big((\sigma_\kappa + \gamma^{\varpi_1})\|\varpi_1\|_{\mathfrak{D}} + \zeta_\kappa Q(\delta)\big) \int_0^{\varepsilon_1} \|\mathcal{S}_1(\varepsilon_2 - \varrho) - \mathcal{S}_1(\varepsilon_1 - \varrho)\|_{\mathfrak{D}(\Xi)} p(\varrho,\delta) d\varrho \\
& +\sigma\varkappa\big((\sigma_\kappa + \gamma^{\varpi_1})\|\varpi_1\|_{\mathfrak{D}} + \zeta_\kappa Q(\delta),\delta\big) \int_{\varepsilon_1}^{\varepsilon_2} p(\varrho,\delta) d\varrho \\
& +\lambda \int_0^{\varepsilon_1} \|\mathcal{S}_1(\varepsilon_2 - \varrho) - \mathcal{S}_1(\varepsilon_1 - \varrho)\|_{\mathfrak{D}(\Xi)} \left[\left\|x^1(\delta)\right\| + \|\mathcal{S}_1(\kappa)\|_{\mathfrak{D}(\Xi)} |\varpi_1(0,\delta)|\right] d\varrho \\
& + \int_0^{\varepsilon_1} \|\mathcal{S}_1(\varepsilon_2 - \varrho) - \mathcal{S}_1(\varepsilon_1 - \varrho)\|_{\mathfrak{D}(\Xi)} \varkappa\big((\sigma_\kappa + \gamma^{\varpi_1})\|\varpi_1\|_{\mathfrak{D}} + \zeta_\kappa Q(\delta)\big) \\
& \times \int_0^{\kappa} p(\varepsilon,\delta) d\varepsilon d\varrho \\
& +\lambda\sigma \int_{\varepsilon_1}^{\varepsilon_2} \left\|x^1(\delta)\right\| + \|\mathcal{S}_1(\kappa)\|_{\mathfrak{D}(\Xi)} |\varpi_1(0,\delta)| \\
& +\sigma\varkappa\big((\sigma_\kappa + \gamma^{\varpi_1})\|\varpi_1\|_{\mathfrak{D}} + \zeta_\kappa Q(\delta)\big) \int_0^{\kappa} p(\varepsilon,\delta) d\varepsilon d\varrho.
\end{aligned}
$$

The right-hand of the above inequality tends to zero as $\varepsilon_2 - \varepsilon_1 \to 0$, since $\mathcal{S}_1(\vartheta)$, and $\mathcal{S}_2(\vartheta)$ are a strongly continuous operator and the compactness of $\mathcal{S}_1(\vartheta)$ and $\mathcal{S}_2(\vartheta)$ for $\vartheta > 0$, thus the continuity in the uniform operator topology.

Further, let $\delta \in \Psi$ be fixed.

(b) Let Φ be a subset of $D(\delta)$ where $\Phi \subset \overline{conv}(\mathfrak{T}(\Phi) \cup \{0\})$. Φ is bounded and equicontinuous, thus the function $\vartheta \to v(\vartheta) = \mu(\Phi(\vartheta))$ is continuous on $(-\infty, \kappa]$. By $(H4)$, Lemma 2 and the properties of the measure μ we have for each $\vartheta \in (-\infty, \kappa]$

$$\begin{aligned}
v(\vartheta) &\leq \mu(\mathfrak{T}(\Phi))(\vartheta) \cup \{0\}) \\
&\leq \mu(\mathfrak{T}(\Phi(\vartheta)) \\
&\leq \mu(\mathcal{S}_1(\vartheta)\varpi_1(0,\delta)) + \mu(\mathcal{S}_2(\vartheta)\varpi_2(\delta)) \\
&\quad + \mu\left(\int_0^\vartheta \mathcal{S}_1(\vartheta - \varrho)\, \psi(\varrho, x_{\xi(\varrho, x_\varrho)}, \delta), \delta) d\varrho\right) \\
&\quad + \sigma\lambda \int_0^\vartheta \mu\Big(x^1(\delta) - \mathcal{S}_1(\kappa)\varpi_1(0,\delta) - \mathcal{S}_2(\kappa)\varpi_2(\delta)\Big) \\
&\quad + \mu\left(\int_0^\kappa \mathcal{S}_1(\kappa - \varepsilon)\psi(\varepsilon, x_{\xi(\varepsilon, x_\varepsilon)}, \delta) d\varepsilon\right) d\varrho \\
&\leq \sigma \int_0^\vartheta \mu\left(\psi(\varrho, x_{\xi(\varrho, x_\varrho)}, \delta), \delta)\right) d\varrho \\
&\quad + \sigma\lambda \int_0^\vartheta \int_0^\kappa \mu\Big(\mathcal{S}_1(\kappa - \varepsilon)\psi(\varepsilon, x_{\xi(\varepsilon, x_\varepsilon)}, \delta)\Big) d\varepsilon d\varrho \\
&\leq \sigma \int_0^\vartheta \beta(\varrho)\mu(\Big\{x_{\xi(\varrho, x_\varrho)} : x \in \Phi\Big\}) d\varrho \\
&\quad + \sigma\lambda \int_0^\vartheta \int_0^\kappa \mu\Big(\mathcal{S}_1(\kappa - \varepsilon)\psi(\varepsilon, x_{\xi(\varepsilon, x_\varepsilon)}, \delta)\Big) d\varepsilon d\varrho
\end{aligned}$$

$$\begin{aligned}
&\leq \sigma \int_0^\vartheta \beta(\varrho)\zeta(\varrho) \sup_{0 \leq \varepsilon \leq \varrho} \mu(\Phi(\varepsilon)) d\varrho + \sigma^2\lambda \int_0^\vartheta \int_0^\kappa \mu\Big(\psi(\varepsilon, x_{\xi(\varepsilon, x_\varepsilon)}, \delta)\Big) d\varepsilon d\varrho \\
&\leq \sigma \int_0^\vartheta \beta(\varrho)\zeta(\varrho)\mu(\Phi(\varrho)) d\varrho + \sigma^2\lambda\kappa \int_0^\kappa \beta(\varepsilon)\mu(\Big\{x_{\xi(\varepsilon, x_\varepsilon)} : x \in \Phi\Big\}) d\varepsilon
\end{aligned}$$

$$\begin{aligned}
&\leq \sigma \int_0^\vartheta v(\varrho)\, \beta(\varrho)\zeta(\varrho) d\varrho + \sigma^2\lambda\kappa \int_0^\kappa \beta(\varepsilon)\zeta(\varepsilon)\mu(\Phi(\varepsilon)) d\varepsilon \\
&= \sigma \int_0^\vartheta \beta(\varrho)\zeta(\varrho)v(\varrho) d\varrho + \sigma^2\lambda\kappa \int_0^\kappa \beta(\varepsilon)\zeta(\varepsilon)v(\varepsilon) d\varepsilon. \\
&\leq \sigma \int_0^\kappa \beta(\varrho)\zeta(\varrho)v(\varrho) d\varrho + \sigma^2\lambda\kappa \int_0^\kappa \beta(\varepsilon)\zeta(\varepsilon)v(\varepsilon) d\varepsilon. \\
&\leq \sigma(1 + \sigma\lambda\kappa) \int_0^\kappa \beta(\varrho)\zeta(\varrho)v(\varrho) d\varrho. \\
&\leq \sigma(1 + \sigma\lambda\kappa) \int_0^\kappa \beta(\varrho)\zeta(\varrho) \sup_{0 \leq \varepsilon \leq \varrho} v(\varepsilon) d\varrho \\
&\leq \sigma(1 + \sigma\lambda\kappa)\|v\|_\infty \int_0^\kappa \beta(\varrho)\zeta(\varrho) d\varrho.
\end{aligned}$$

Thus

$$\|v\|_\infty \leq \sigma(1 + \sigma\lambda\kappa)\|v\|_\infty \int_0^\kappa \beta(\varrho)\zeta(\varrho) d\varrho.$$

Then

$$\|v\|_\infty \left(1 - \sigma(1 + \sigma\lambda\kappa) \int_0^\kappa \beta(\varrho)\zeta(\varrho) d\varrho\right) \le 0.$$

Consequently, $\|v\|_\infty = 0$, thus $v(\vartheta) = 0$ for each $\vartheta \in \Theta$, and then $\Phi(\vartheta)$ is relatively compact in Ξ. As a result of the Ascoli-Arzelà theorem, Φ is relatively compact in $D(\delta)$. By Mönch fixed point theorem, we deduce that $\mathfrak{T}$ has a fixed point $x(\delta) \in D(\delta)$. As $\bigcap_{\delta\in\Psi} D(\delta) \neq \emptyset$, and a measurable selector of int$D$ exists, by Lemma 1, $\mathfrak{T}$ has a stochastic fixed point $x^*(\delta)$, which is a mild solution of (2).

5. An Example

Consider the problem:

$$\frac{\partial^2}{\partial\vartheta^2}\mathfrak{w}(\vartheta,\mathfrak{y},\delta) = \frac{\partial^2}{\partial\mathfrak{y}^2}\mathfrak{w}(\vartheta,\mathfrak{y},\delta)$$

$$+ \psi(\vartheta,\mathfrak{w}(\vartheta,\mathfrak{y},\delta),\delta) + \mathcal{Z}_2\mathfrak{f}(\vartheta,\delta)\ \mathfrak{y} \in [0,\pi];\ \vartheta \in \Theta = [0,\kappa], \tag{6}$$

$$\mathfrak{w}(\vartheta,0,\delta) = \mathfrak{w}(\vartheta,\pi,\delta) = 0;\ \vartheta \in [0,\kappa], \delta \in \Psi, \tag{7}$$

$$\mathfrak{w}(\vartheta,\mathfrak{y},\delta) = \varpi_1(\vartheta,\delta), \frac{\partial}{\partial\vartheta}\mathfrak{w}(0,\mathfrak{y},\delta) = \varpi_2(\mathfrak{y},\delta);\ \vartheta \in (-\infty,0],\ \delta \in \Psi, \tag{8}$$

where $\psi : \Theta \times \mathbb{R} \times \Psi \longrightarrow \mathbb{R}$ is a given function. Let $\Xi = L^2[0,\pi]$, and $\mathcal{Z}_1 : \Xi \longrightarrow \Xi$ given by $\mathcal{Z}_1\varpi = \varpi''$ with domain $D(\mathcal{Z}_1) = \{\varpi \in \Xi; \varpi, \varpi''$ are absolutely continuous, $\varpi'' \in \Xi, \varpi(0) = \varpi(\pi) = 0\}$.

The operator $\mathcal{Z}_1$ is the infinitesimal generator of a strongly continuous cosine function $(\mathcal{S}_1(\vartheta))_{\vartheta\in\mathbb{R}}$ on Ξ. Furthermore, $\mathcal{Z}_1$ has discrete spectrum, the eigenvalues are $-n^2, n \in IN$ with corresponding normalized eigenvectors

$$\mathfrak{w}_n(\varepsilon) := \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(n\varepsilon),$$

and

(a) $\{\mathfrak{w}_n : n \in IN\}$ is an orthonormal basis of Ξ,

(b) If $x \in \Xi$, then $\mathcal{Z}_1 x = -\sum_{n=1}^{\infty} n^2\langle x, \mathfrak{w}_n\rangle\mathfrak{w}_n$,

(c) For $x \in \Xi, \mathcal{S}_1(\vartheta)x = \sum_{n=1}^{\infty} cos(nt)\langle x, \mathfrak{w}_n\rangle\mathfrak{w}_n$, and the associated sine family is

$$\mathcal{S}_2(\vartheta)x = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n}\langle x, \mathfrak{w}_n\rangle\mathfrak{w}_n.$$

Consequently, $\mathcal{S}_2(\vartheta)$ is compact for all $\vartheta > 0$ and

$$\|\mathcal{S}_1(\vartheta)\| = \|\mathcal{S}_2(\vartheta)\| \le 1, \text{ for all } \vartheta \ge 0.$$

(d) If we denote the group of translations on Ξ by

$$\bar{\Phi}(\vartheta)x(\mathfrak{w},\delta) = \tilde{x}(\mathfrak{w} + \vartheta,\delta),$$

where $\tilde{x}$ is the extension of x with period 2π, then

$$\mathcal{S}_1(\vartheta) = \frac{1}{2}(\bar{\Phi}(\vartheta) + \bar{\Phi}(-\vartheta)); \mathcal{Z}_1 = D,$$

where D is the infinitesimal generator of the group on

$$X = \{x(\cdot,\delta) \in H^1(0,\pi) : x(0,\delta) = x(\pi,\delta) = 0\}.$$

Suppose that $\mathcal{Z}_2$ is a bounded linear operator from Ω into Ξ and the linear operator $\mathcal{K} : L^2(\Theta, \Omega) \to \Xi$ given by:

$$\mathcal{K}\mathfrak{f} = \int_0^{\kappa} \mathcal{S}_1(\kappa - \varrho)\mathcal{Z}_2\mathfrak{f}(\varrho, \delta)d\varrho,$$

has an inverse operator $\mathcal{K}^{-1}$ in $L^2(\Theta, \Omega)/ker\mathcal{K}$. We deduce that (1) is an abstract formulation of (6)–(8). If $(H_1) - (H_6)$ are met. By Theorem 1, we conclude that (6)–(8) is controllable.

6. Conclusions

In this work, we have presented some existence and controllability results for two classes of second order functional differential equations with delay and random effects. Our arguments are based on a random fixed point theorem with a stochastic domain. Next, we prove that our problems are controllable. On the other hand, we have given an illustrative example which indicates the applicability of this study. Some of the results in this direction constitute our future research plan. More work can be done by changing and generalizing the conditions, the functional spaces, or even extend the study to some fractional differential problems.

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