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An Iterative Algorithm to Approximate Fixed Points of Non-Linear Operators with an Application

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Abstract: In this article, we study the JF iterative algorithm to approximate the fixed points of a non-linear operator that satisfies condition (E) in uniformly convex Banach spaces. Further, some weak and strong convergence results are presented for the same operator using the JF iterative algorithm. We also demonstrate that the JF iterative algorithm is weakly w^2 $\mathcal{G}$-stable with respect to almost contractions. In connection with our results, we provide some illustrative numerical examples to show that the JF iterative algorithm converges faster than some well-known iterative algorithms. Finally, we apply the JF iterative algorithm to estimate the solution of a functional non-linear integral equation. The results of the present manuscript generalize and extend the results in existing literature and will draw the attention of researchers.

Keywords: iterative algorithms; non-linear operator (E); almost contraction; fixed points; Banach space; non-linear integral equation

MSC: 47H09; 47H10; 54H25; 47H30

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1. Introduction

Fixed point theory has an eminent position in pure and applied mathematics because it has a variety of applications in different fields within mathematics, such as differential and integral equations, variational inequalities, approximation theory, etc. The application of fixed point results is not merely confined to mathematics, but is also relevant in other fields, such as statistics, computer sciences, chemical sciences, physical sciences, economics, biological sciences, medical sciences, engineering, game theory, etc. (see, e.g., [1,2]). It is a domain that is of great interest in two research directions: the first is to find progressively wider classes of mappings and conditions under which the existence of fixed points can be proved; the second is to define iterative algorithms for the approximation of the fixed points of these mappings, as it is not always an easy task to approximate the fixed points using direct methods.

The fundamental result in metric fixed point theory is the Banach contraction principle, which was first introduced in the literature in 1922. This result provides the guarantee of the existence and uniqueness of the fixed point of a contraction mapping in a complete metric space. It not only demonstrates the existence and uniqueness of a fixed point, but also allows the Picard iterative algorithm to converge to that fixed point. Further, on account of its simplicity, utility and applicability, the Banach contraction principle has become an extremely well-known tool in solving existence problems in numerous branches of mathematical analysis. As such, several authors have improved, extended and generalized the Banach contraction principle. One of the most important generalizations of the Banach contraction principle was produced by Berinde [3] in 2003. He defined almost contraction mapping as follows.

A self-mapping $\mathcal{G}$ defined on a non-empty subset $\mathcal{S}$ of a Banach space $\mathcal{B}$ is called an almost contraction when constants $\delta \in (0,1)$ and $L \geq 0$ exist in such a way that:

$$\|\mathcal{G}x - \mathcal{G}y\| \leq \delta\|x-y\| + L\|y - \mathcal{G}x\|, \quad \forall x, y \in \mathcal{S}. \tag{1}$$

It is worth mentioning here that condition (1) only ensures the existence of a fixed point of an almost contraction (see, [3]). For the uniqueness of the fixed point of an almost contraction, he proved the following result.

Theorem 1 ([3])**.** *Let $(\mathcal{B}, d)$ be a complete metric space and $\mathcal{G} : \mathcal{B} \to \mathcal{B}$ be an almost contraction (1). When constants $\delta \in (0,1)$ and $L \geq 0$ exist in such a way that:*

$$d(\mathcal{G}x, \mathcal{G}y) \leq \delta d(x,y) + Ld(x, \mathcal{G}x), \quad \forall x, y \in \mathcal{S}. \tag{2}$$

Then $\mathcal{G}$ has a unique fixed point, i.e., t, in $\mathcal{B}$.

Berinde has also shown that almost contractions include the classes of Kannan [4], Chatterjea [5] and Zamfirescu [6] mappings.

A widely studied extension of contraction mappings is the class of non-expansive mappings, which is natural and vast due to isometry and metric projections. A self-mapping $\mathcal{G}$ defined on a non-empty subset $\mathcal{S}$ of a Banach space $\mathcal{B}$ is said to be non-expansive when:

$$\|\mathcal{G}x - \mathcal{G}y\| \leq \|x-y\|, \ \forall x, y \in \mathcal{S}.$$

The fixed point theory for non-expansive mappings has a variety of applications in convex feasibility problems, convex optimization problems, monotone inequality problems, image restorations, etc. Due to its applicability, a large number of eminent researchers have generalized and extended this theory to the large classes of non-linear mappings. One of the most important generalizations of non-expansive mappings was produced by Garcia-Falset et al. [7] in 2011, which is defined as follows.

Definition 1 ([7])**.** *Let $\mathcal{S}$ be a non-empty subset of a Banach space $\mathcal{B}$ and $\mu \geq 1$. An operator $\mathcal{G} : \mathcal{S} \to \mathcal{B}$ is said to satisfy condition (E_μ) when:*

$$\|x - \mathcal{G}y\| \leq \mu\|x - \mathcal{G}x\| + \|x-y\|, \quad \forall x, y \in \mathcal{S}. \tag{3}$$

Moreover, $\mathcal{G}$ is said to satisfy condition (E) when $\mathcal{G}$ satisfies condition (E_μ) with $\mu \geq 1$.

It can be easily seen that when $\mathcal{G} : \mathcal{S} \to \mathcal{B}$ is a non-expansive mapping, it satisfies condition (E_μ) with $\mu = 1$. It is worth mentioning here that the class of operators that satisfy condition (E) properly includes the classes of Hardy and Rogers mappings [8], mappings satisfying Suzuki's condition (C) [9], generalized α non-expansive mappings [10] and generalized α–Reich–Suzuki non-expansive mappings [11].

In many instances, it is not possible to find the exact solution of fixed point problems. Therefore, iterative algorithms are used to approximate the solutions of the fixed point problems. Thus, a large number of iterative algorithms have been introduced and studied for the approximation of solutions to fixed point problems (see, e.g., [12–22], etc).

Very recently, Ali et al. [23] introduced a new iterative algorithm called the JF iterative algorithm, which is defined as follows.

Let $\mathcal{S}$ be a non-empty closed and convex subset of a Banach space $\mathcal{B}$ and let $\mathcal{G} : \mathcal{S} \to \mathcal{S}$ be the mapping. Then, the sequence $\{\tau_n\}$ is generated by an initial point τ_0 and defined by:

$$\begin{cases} \tau_{n+1} = \mathcal{G}((1-\mu_n)\sigma_n + \mu_n \mathcal{G}\sigma_n), \\ \sigma_n = \mathcal{G}\xi_n, \\ \xi_n = \mathcal{G}((1-\theta_n)\tau_n + \theta_n \mathcal{G}\tau_n), \ n \in \mathbb{Z}^+, \end{cases} \tag{4}$$

where $\{\mu_n\}$ and $\{\theta_n\}$ are control sequences in $(0,1)$ and $\mathbb{Z}^+$ denotes the set of non-negative integers. They pointed that the JF iterative algorithm is independent from all other iterative algorithms that have been previously defined in the literature. They have produced some weak and strong convergence results for Hardy and Rogers generalized non-expansive mappings using the JF iterative algorithm in uniformly convex Banach spaces. They have also numerically shown that the JF iterative algorithm converges to the fixed point of Hardy and Rogers generalized non-expansive mappings faster than some other remarkable iterative algorithms.

On the other hand, many scientific and engineering problems are presented in the form of non-linear integral equations. The class of initial and boundary value problems can be transformed to Fredholm or Volterra non-linear integral equations. The solution of non-linear integral equations exists locally and has blow-up phenomena (see, [24,25]). In Section 5, we apply the JF iterative method to approximate the solution of a non-linear integral equation in the setting of a Banach space.

Inspired by the above study, we aim to prove that the JF iterative algorithm is weakly w^2 $\mathcal{G}$- stable with respect to almost contractions in the current manuscript. Further, we present some weak and strong convergence results for the operators that satisfy condition (E) using the JF iterative algorithm in uniformly convex Banach spaces. We numerically show that the JF iterative algorithm converges to the fixed point of the operators that satisfy condition (E) faster than Mann, Ishikawa, Noor, SP, S and Picard-S iterative algorithms. Finally, we approximate the solution of a mixed Volterra–Fredholm functional non-linear integral equation. The results of the present manuscript generalize and extend the results in existing literature, particularly those of [20,23].

2. Preliminaries

The aim of this section is to set out some lemmas and definitions that are used in this paper.

Lemma 1 ([26])**.** *Let $\{u_n\}$ and $\{\epsilon_n\}$ be sequences in $\mathbb{R}_+$ that satisfy the following inequality:*

$$u_{n+1} \leq (1 - v_n)u_n + \epsilon_n,$$

where $v_n \in (0,1)$ for all $n \in \mathbb{Z}^+$ with $\sum_{n=0}^{\infty} v_n = \infty$. When $\lim_{n\to\infty} \frac{\epsilon_n}{v_n} = 0$, then $\lim_{n\to\infty} u_n = 0$.

Definition 2 ([27])**.** *Let $\mathcal{B}$ be a Banach space and $\{\tau_n\}$ be a weakly convergent sequence to $x \in \mathcal{B}$, then $\mathcal{B}$ satisfies Opial's property when:*

$$\lim_{n\to\infty} \inf \|\tau_n - x\| < \lim_{n\to\infty} \inf \|\tau_n - y\|$$

holds for all $y \in \mathcal{B}$ with $y \neq x$.

Example 1. *All Hilbert spaces and ℓ^p $(1 < p < \infty)$ spaces satisfy Opial's property, while $L^p[0,2\pi]$ $(1 < p \neq 2)$ spaces do not satisfy Opial's property.*

Definition 3 ([28])**.** *An operator $\mathcal{G} : \mathcal{S} \to \mathcal{S}$ satisfies condition (I) when a non-decreasing function $\psi : [0,\infty) \to [0,\infty)$ exists with $\psi(0) = 0$ and $\psi(y) > 0, \forall y > 0$, such that $\|y - \mathcal{G}y\| \geq \psi(d(y, F(\mathcal{G})))$ and $\forall y \in \mathcal{S}$, where $F(\mathcal{G}) = \{t \in \mathcal{S} : \mathcal{G}t = t\}$ and $d(y, F(\mathcal{G})) = \inf\{\|y - t\| : t \in F(\mathcal{G})\}$.*

Definition 4 ([29])**.** *Let $\mathcal{S}$ be a non-empty subset of a Banach space $\mathcal{B}$. The two sequences $\{\tau_n\}$ and $\{t_n\}$ in $\mathcal{S}$ are said be equivalent when:*

$$\lim_{n\to\infty} \|\tau_n - t_n\| = 0.$$

Definition 5 ([30]). *Let $\mathcal{S}$ be a non-empty subset of a Banach space $\mathcal{B}$, let $\mathcal{G} : \mathcal{S} \to \mathcal{S}$ be a mapping with at least one fixed point, i.e., t, and let $\{\tau_n\}$ be a sequence defined by:*

$$\begin{cases} \tau_0 \in \mathcal{S}, \\ \tau_{n+1} = h(\mathcal{G}, \tau_n),\ n \in \mathbb{Z}^+, \end{cases}$$

where h is a function of $\mathcal{G}$ and τ_n. Assume that the sequence $\{\tau_n\}$ converges to a fixed point of $\mathcal{G}$ and $\{t_n\}$ is an equivalent sequence of $\{\tau_n\}$ in $\mathcal{S}$. When:

$$\lim_{n\to\infty} \|t_{n+1} - h(\mathcal{G}, t_n)\| = 0 \quad \Longrightarrow \lim_{n\to\infty} t_n = t,$$

then the iterative sequence $\{\tau_n\}$ is called weakly w^2-stable with respect to $\mathcal{G}$.

Definition 6. *Let $\mathcal{S}$ be a non-empty, closed and convex subset of $\mathcal{B}$. Let $\{\tau_n\}$ be a bounded sequence in $\mathcal{B}$ and for $x \in \mathcal{S}$:*

$$r(x, \{\tau_n\}) = \lim_{n\to\infty} \sup \|\tau_n - x\|.$$

The asymptotic radius and asymptotic center of $\{\tau_n\}$ relative to $\mathcal{S}$ are defined, respectively, by:

$$r(\mathcal{S}, \{\tau_n\}) = \inf\{r(x, \{\tau_n\}) : x \in \mathcal{S}\}.$$

$$A(\mathcal{S}, \{\tau_n\}) = \{x \in \mathcal{S} : r(x, \{\tau_n\}) = r(\mathcal{S}, \{\tau_n\})\}.$$

When $\mathcal{B}$ is a uniformly convex Banach space, then the set $A(\mathcal{S}, \{\tau_n\})$ is a singleton.

Lemma 2 ([31]). *Assume $\mathcal{B}$ is a uniformly convex Banach space and $0 < a \le s_n \le b < 1, \forall n \ge 1$. Let $\{\tau_n\}$ and $\{\sigma_n\}$ be two sequences in $\mathcal{B}$ that satisfy $\lim_{n\to\infty} \sup \|\tau_n\| \le w$, $\lim_{n\to\infty} \sup \|\sigma_n\| \le w$ and $\lim_{n\to\infty} \|s_n\tau_n + (1 - s_n)\sigma_n\| = w$ holds for $w \ge 0$. Then, $\lim_{n\to\infty} \|\tau_n - \sigma_n\| = 0$.*

Lemma 3 ([7]). *Let $\mathcal{S}$ be a non-empty, closed and convex subset of a uniformly convex Banach space $\mathcal{B}$ that satisfies Opial's property. Let $\mathcal{G} : \mathcal{S} \to \mathcal{B}$ be an operator that satisfies condition (E). When the sequence $\{\tau_n\}$ converges weakly to t and $\lim_{n\to\infty} \|\tau_n - \mathcal{G}\tau_n\| = 0$, then $t \in F(\mathcal{G})$.*

3. Weak w^2-Stability of the JF Iterative Algorithm

The purpose of this section is to prove the convergence and stability results for the JF iterative algorithm with respect to almost contractions in an arbitrary Banach space. The following theorem shows the convergence and stability of the iterative algorithm (4) for almost contractions.

Theorem 2. *Let $\mathcal{S}$ be a non-empty, closed and convex subset of a Banach space $\mathcal{B}$ and let $\mathcal{G} : \mathcal{S} \to \mathcal{S}$ be an almost contraction that satisfies inequality (2). Then, the sequence $\{\tau_n\}$ defined by the iterative algorithm (4) converges to a unique fixed point of $\mathcal{G}$. Moreover, the iterative sequence $\{\tau_n\}$ is weakly w^2-stable with respect to the almost contraction.*

Proof. Since $\mathcal{G}$ is an almost contraction that satisfies inequality (2), a constant $\beta \in [0, 1)$ exists in such a way that for all $x \in \mathcal{S}$ and $t \in F(\mathcal{G}) = \{t \in \mathcal{S} : \mathcal{G}t = t\}$:

$$\|\mathcal{G}x - \mathcal{G}t\| = \|\mathcal{G}x - t\| \le \beta\|x - t\|.$$

Using iterative algorithm (4), we obtain:

$$\begin{aligned} \|\xi_n - t\| &= \|\mathcal{G}((1-\theta_n)\tau_n + \theta_n\mathcal{G}\tau_n) - t\| \\ &\leq \beta\|(1-\theta_n)\tau_n + \theta_n\mathcal{G}\tau_n - t\| \\ &\leq \beta((1-\theta_n)\|\tau_n - t\| + \theta_n\|\mathcal{G}\tau_n - t\|) \\ &\leq \beta((1-\theta_n)\|\tau_n - t\| + \beta\theta_n\|\tau_n - t\|) \\ &= \beta(1-(1-\beta)\theta_n)\|\tau_n - t\|. \end{aligned}$$

Since $0 \leq \beta < 1$ and $\theta_n \in (0,1)$ and using the fact that $0 < (1-(1-\beta)\theta_n) \leq 1$, we obtain:

$$\|\xi_n - t\| \leq \beta\|\tau_n - t\|. \tag{5}$$

Using Equation (5), we obtain:

$$\begin{aligned} \|\sigma_n - t\| &= \|\mathcal{G}\xi_n - t\| \\ &\leq \beta\|\xi_n - t\| \\ &\leq \beta^2\|\tau_n - t\|. \end{aligned} \tag{6}$$

Using Equation (6), we obtain:

$$\begin{aligned} \|\tau_{n+1} - t\| &= \|\mathcal{G}((1-\mu_n)\sigma_n + \mu_n\mathcal{G}\sigma_n) - t\| \\ &\leq \beta\|(1-\mu_n)\sigma_n + \mu_n\mathcal{G}\sigma_n - t\| \\ &\leq \beta((1-\mu_n)\|\sigma_n - t\| + \mu_n\|\mathcal{G}\sigma_n - t\|) \\ &\leq \beta((1-\mu_n)\|\sigma_n - t\| + \beta\mu_n\|\sigma_n - t\|) \\ &\leq \beta(1-(1-\beta)\mu_n)\|\sigma_n - t\| \\ &\leq \beta^3\|\tau_n - t\|. \end{aligned} \tag{7}$$

Inductively, we then obtain:

$$\|\tau_{n+1} - t\| \leq \beta^{3(n+1)}\|\tau_0 - t\|. \tag{8}$$

Since $0 \leq \beta < 1$, it can be concluded that $\{\tau_n\}$ converges to t.

Now, we aim to prove the stability of the iterative algorithm (4). Let $\{t_n\}$ be an equivalent sequence of $\{\tau_n\}$ in $\mathcal{S}$, let the sequence that is defined by the iterative algorithm (4) be $\tau_{n+1} = h(\mathcal{G}, \tau_n)$ and assume $\epsilon_n = \|t_{n+1} - h(\mathcal{G}, t_n)\|$, $n \in \mathbb{Z}^+$. Now, we can show that $\lim_{n\to\infty} \epsilon_n = 0 \implies \lim_{n\to\infty} t_n = t$.

Let $\lim_{n\to\infty} \epsilon_n = 0$. Then, using the iterative algorithm (4), we obtain:

$$\begin{aligned} \|t_{n+1} - t\| &\leq \|t_{n+1} - h(\mathcal{G}, t_n)\| + \|h(\mathcal{G}, t_n) - t\| \\ &= \epsilon_n + \|h(\mathcal{G}, t_n) - t\| \\ &\leq \epsilon_n + \beta^3(1-(1-\beta)\mu_n)\|t_n - t\|. \end{aligned}$$

By defining $u_n = \|t_n - t\|$ and $v_n = (1-\beta)\mu_n \in (0,1)$, then:

$$u_{n+1} \leq \beta^3(1-v_n)u_n + \epsilon_n.$$

Since $\lim_{n\to\infty} \epsilon_n = 0$, then $\frac{\epsilon_n}{v_n} \to 0$ as $n \to \infty$. Thus, according to Lemma 1, $\lim_{n\to\infty} u_n = 0$, i.e., $\lim_{n\to\infty} t_n = t$. Thus, the iterative sequence that is defined by the algorithm (4) is weakly w^2-stable with respect to the almost contraction. □

4. Convergence Results for the Non-linear Operator (E)

The purpose of this section is to prove convergence results for the operator that satisfies condition (E) in uniformly convex Banach spaces. First, we aim to prove the following fruitful lemmas that helped us to obtain the these results. Throughout this section, it is assumed that $\mathcal{S}$ is a non-empty, closed and convex subset of a uniformly convex Banach space $\mathcal{B}$, $\mathcal{G} : \mathcal{S} \to \mathcal{S}$ is an operator that satisfies condition (E) and $F(\mathcal{G}) = \{t \in \mathcal{S} : \mathcal{G}t = t\}$.

Lemma 4. *Assume that $F(\mathcal{G}) \neq \emptyset$ and let $\{\tau_n\}$ be a sequence that is developed by the iterative algorithm (4), then $\lim_{n\to\infty} \|\tau_n - t\|$ exists for all $t \in F(\mathcal{G})$.*

Proof. As the operator $\mathcal{G}$ satisfies condition (E) and $F(\mathcal{G}) \neq \emptyset$, for $t \in F(\mathcal{G})$, we obtain:

$$\|\mathcal{G}\tau_n - t\| \leq \|\tau_n - t\|,$$

for all $\tau_n \in \mathcal{S}$. Using the iterative algorithm (4), we obtain:

$$\begin{aligned} \|\xi_n - t\| &= \|\mathcal{G}((1-\theta_n)\tau_n + \theta_n\mathcal{G}\tau_n) - t\| \\ &\leq \|(1-\theta_n)\tau_n + \theta_n\mathcal{G}\tau_n - t\| \\ &\leq (1-\theta_n)\|\tau_n - t\| + \theta_n\|\mathcal{G}\tau_n - t\| \\ &\leq (1-\theta_n)\|\tau_n - t\| + \theta_n\|\tau_n - t\| \\ &\leq \|\tau_n - t\|. \end{aligned} \tag{9}$$

Using Equation (9), we obtain:

$$\begin{aligned} \|\sigma_n - t\| &= \|\mathcal{G}\xi_n - t\| \\ &\leq \|\xi_n - t\| \\ &\leq \|\tau_n - t\|. \end{aligned} \tag{10}$$

Using Equation (10), we obtain:

$$\begin{aligned} \|\tau_{n+1} - t\| &= \|\mathcal{G}((1-\mu_n)\sigma_n + \mu_n\mathcal{G}\sigma_n) - t\| \\ &\leq \|(1-\mu_n)\sigma_n + \mu_n\mathcal{G}\sigma_n - t\| \\ &\leq (1-\mu_n)\|\sigma_n - t\| + \mu_n\|\mathcal{G}\sigma_n - t\| \\ &\leq (1-\mu_n)\|\sigma_n - t\| + \mu_n\|\sigma_n - t\| \\ &\leq (1-\mu_n)\|\tau_n - t\| + \mu_n\|\tau_n - t\| \\ &= \|\tau_n - t\|. \end{aligned} \tag{11}$$

This shows that the sequence $\{\|\tau_n - t\|\}$ is non-increasing and bounded below $\forall t \in F(\mathcal{G})$. Thus, $\lim_{n\to\infty} \|\tau_n - t\|$ exists. □

Lemma 5. *Let $\{\tau_n\}$ be a sequence that is developed by the iterative algorithm (4). Then, $F(\mathcal{G}) \neq \emptyset$ when, and only when, $\{\tau_n\}$ is bounded and $\lim_{n\to\infty} \|\tau_n - \mathcal{G}\tau_n\| = 0$.*

Proof. Presume that $F(\mathcal{G}) \neq \emptyset$ and $t \in F(\mathcal{G})$. Then, $\lim_{n\to\infty} \|\tau_n - t\|$ exists according to Lemma 4 and $\{\tau_n\}$ is bounded. Presume that:

$$\lim_{n\to\infty} \|\tau_n - t\| = \alpha. \tag{12}$$

From Equations (9), (10) and (12), we obtain:

$$\lim_{n\to\infty} \sup \|\xi_n - t\| \leq \lim_{n\to\infty} \sup \|\tau_n - t\| \leq \alpha. \tag{13}$$

$$\lim_{n\to\infty} \sup \|\sigma_n - t\| \leq \lim_{n\to\infty} \sup \|\tau_n - t\| \leq \alpha. \tag{14}$$

Since $\mathcal{G}$ satisfies condition (E), we obtain:

$$\|\mathcal{G}\tau_n - t\| \leq \|\tau_n - t\|$$

$$\iff \lim_{n\to\infty} \sup \|\mathcal{G}\tau_n - t\| \leq \lim_{n\to\infty} \sup \|\tau_n - t\| \leq \alpha. \tag{15}$$

Now:

$$\begin{aligned} \|\tau_{n+1} - t\| &= \|\mathcal{G}((1-\mu_n)\sigma_n + \mu_n \mathcal{G}\sigma_n) - t\| \\ &\leq \|(1-\mu_n)\sigma_n + \mu_n \mathcal{G}\sigma_n - t\| \\ &\leq (1-\mu_n)\|\sigma_n - t\| + \mu_n\|\mathcal{G}\sigma_n - t\| \\ &\leq (1-\mu_n)\|\sigma_n - t\| + \mu_n\|\sigma_n - t\| \\ &= \|\sigma_n - t\|. \end{aligned}$$

Taking $\lim_{n\to\infty}$ on both sides, we obtain:

$$\alpha = \lim_{n\to\infty}\inf\|\tau_{n+1} - t\| \leq \lim_{n\to\infty}\inf\|\sigma_n - t\|. \tag{16}$$

So, it follows from (14) and (16) that:

$$\begin{aligned} \alpha \leq \lim_{n\to\infty}\inf\|\sigma_n - t\| &\leq \lim_{n\to\infty}\sup\|\sigma_n - t\| \leq \alpha \\ \Longleftrightarrow \lim_{n\to\infty}\|\sigma_n - t\| &= \alpha. \end{aligned} \tag{17}$$

Additionally:

$$\begin{aligned} \|\sigma_n - t\| &= \|\mathcal{G}\xi_n - t\| \\ &\leq \|\xi_n - t\|. \end{aligned}$$

Taking $\lim_{n\to\infty}$ on both sides, we obtain:

$$\alpha = \lim_{n\to\infty}\inf\|\sigma_n - t\| \leq \lim_{n\to\infty}\inf\|\xi_n - t\|. \tag{18}$$

So, it follows from (13) and (18) that:

$$\begin{aligned} \alpha \leq \lim_{n\to\infty}\inf\|\xi_n - t\| &\leq \lim_{n\to\infty}\sup\|\xi_n - t\| \leq \alpha \\ \Longleftrightarrow \lim_{n\to\infty}\|\xi_n - t\| &= \alpha. \end{aligned} \tag{19}$$

Thus:

$$\begin{aligned} \alpha &= \lim_{n\to\infty}\|\xi_n - t\| = \lim_{n\to\infty}\|\mathcal{G}((1-\theta_n)\tau_n + \theta_n\mathcal{G}\tau_n) - t\| \\ &\leq \lim_{n\to\infty}\|(1-\theta_n)\tau_n + \theta_n\mathcal{G}\tau_n - t\| \\ &= \lim_{n\to\infty}\|(1-\theta_n)(\tau_n - t) + \theta_n(\mathcal{G}\tau_n - t)\| \\ &\leq \lim_{n\to\infty}((1-\theta_n)\|\tau_n - t\| + \theta_n\|\mathcal{G}\tau_n - t\|) \\ &\leq \lim_{n\to\infty}((1-\theta_n)\|\tau_n - t\| + \theta_n\|\tau_n - t\|) \\ &\leq \alpha. \end{aligned}$$

Hence:

$$\lim_{n\to\infty}\|(1-\theta_n)(\tau_n - t) + \theta_n(\mathcal{G}\tau_n - t)\| = \alpha. \tag{20}$$

From (13), (15) and (20) and using Lemma 2, we obtain:

$$\lim_{n\to\infty}\|\tau_n - \mathcal{G}\tau_n\| = 0.$$

Conversely, assume that $\{\tau_n\}$ is bounded and $\lim_{n\to\infty} \|\tau_n - \mathcal{G}\tau_n\| = 0$. Let $t \in A(\mathcal{S}, \{\tau_n\})$, then we obtain:

$$\begin{aligned} r(\mathcal{G}t, \{\tau_n\}) &= \lim_{n\to\infty} \sup \|\tau_n - \mathcal{G}t\| \\ &\leq \lim_{n\to\infty} \sup(\|\tau_n - t\| + \mu\|\mathcal{G}\tau_n - \tau_n\|) \\ &= \lim_{n\to\infty} \sup \|\tau_n - t\| \\ &= r(t, \{\tau_n\}) = r(\mathcal{S}, \{\tau_n\}). \end{aligned}$$

This implies that $\mathcal{G}t \in A(\mathcal{S}, \{\tau_n\})$. Since $\mathcal{B}$ is uniformly convex, $A(\mathcal{S}, \{\tau_n\})$ is a singleton, which implies that $\mathcal{G}t = t$. □

Now, we aim to prove the following weak convergence theorem for the operators that satisfy condition (E) using the iterative algorithm (4).

Theorem 3. *Presume that $F(\mathcal{G}) \neq \emptyset$ and $\mathcal{B}$ satisfies Opial's property, then the sequence $\{\tau_n\}$ that is defined by the iterative algorithm (4) converges weakly to a fixed point of the operator $\mathcal{G}$.*

Proof. In Lemma 4, we demonstrated that $\lim_{n\to\infty} \|\tau_n - t\|$ exists. Now, we have to show that $\{\tau_n\}$ has a unique weak subsequential limit in $F(\mathcal{G})$. Let t and q be two weak limits of $\{\tau_{n_j}\}$ and $\{\tau_{n_k}\}$, respectively, where $\{\tau_{n_j}\}$ and $\{\tau_{n_k}\}$ are subsequences of $\{\tau_n\}$. According to Lemma 5, $\lim_{n\to\infty} \|\tau_n - \mathcal{G}\tau_n\| = 0$ and therefore, using Lemma 3, $t \in F(\mathcal{G})$ and similarly, $q \in F(\mathcal{G})$.

Now, our aim is to show that $t = q$. When $t \neq q$, then using Opial's property, we obtain:

$$\begin{aligned} \lim_{n\to\infty} \|\tau_n - t\| &= \lim_{j\to\infty} \|\tau_{n_j} - t\| \\ &< \lim_{j\to\infty} \|\tau_{n_j} - q\| \\ &= \lim_{n\to\infty} \|\tau_n - q\| \\ &= \lim_{k\to\infty} \|\tau_{n_k} - q\| \\ &< \lim_{k\to\infty} \|\tau_{n_k} - t\| \\ &= \lim_{n\to\infty} \|\tau_n - t\|. \end{aligned}$$

which is not possible and hence, $t = q$. It can be deduced that $\{\tau_n\}$ converges weakly to $t \in F(\mathcal{G})$. □

Theorem 4. *The sequence $\{\tau_n\}$ that is defined by the iterative algorithm (4) converges strongly to $t \in F(\mathcal{G})$ when, and only when, $\lim_{n\to\infty} \inf d(\tau_n, F(\mathcal{G})) = 0$, where $d(\tau_n, F(\mathcal{G})) = \inf\{\|\tau_n - t\| : t \in F(\mathcal{G})\}$.*

Proof. The first part is trivial. Now, we aim to prove the converse part. Presume that $\lim_{n\to\infty} \inf d(\tau_n, F(\mathcal{G})) = 0$. According to Lemma 4, $\lim_{n\to\infty} \|\tau_n - t\|$ exists for all $t \in F(\mathcal{G})$; therefore, it can be hypothesized that $\lim_{n\to\infty} d(\tau_n, F(\mathcal{G})) = 0$.

Now, our claim is that $\{\tau_n\}$ is a Cauchy sequence in $\mathcal{S}$. Since $\lim_{n\to\infty} d(\tau_n, F(\mathcal{G})) = 0$ for a given $\eta > 0$, $M \in \mathbb{N}$ exists in such a way that for all $n \geq M$:

$$\begin{aligned} d(\tau_n, F(\mathcal{G})) &< \frac{\eta}{2} \\ \Longrightarrow \inf\{\|\tau_n - t\| : t \in F(\mathcal{G})\} &< \frac{\eta}{2}. \end{aligned}$$

In particular, $\inf\{\|\tau_M - t\| : t \in F(\mathcal{G})\} < \frac{\eta}{2}$. Therefore, $t \in F(\mathcal{G})$ exists in such a way that:

$$\|\tau_M - t\| < \frac{\eta}{2}.$$

Now, for $m, n \geq M$:

$$\begin{aligned} \|\tau_{n+m} - \tau_n\| &\leq \|\tau_{n+m} - t\| + \|\tau_n - t\| \\ &\leq \|\tau_M - t\| + \|\tau_M - t\| \\ &= 2\|\tau_M - t\| < \eta. \end{aligned}$$

This implies that $\{\tau_n\}$ is a Cauchy sequence in $\mathcal{S}$, so there is an element $\ell \in \mathcal{S}$ such that $\lim_{n\to\infty} \tau_n = \ell$. Since $\lim_{n\to\infty} d(\tau_n, F(\mathcal{G})) = 0$, it follows that $d(\ell, F(\mathcal{G})) = 0$ and thus, we obtain $\ell \in F(\mathcal{G})$. □

We now aim to prove the following strong convergence result by applying condition (I).

Theorem 5. *Assume that $F(\mathcal{G}) \neq \emptyset$ and the operator $\mathcal{G}$ satisfies condition (I). Then, the sequence $\{\tau_n\}$ that is defined by the iterative algorithm (4) converges strongly to a fixed point of $\mathcal{G}$.*

Proof. We demonstrated in Lemma 5 that:

$$\lim_{n\to\infty} \|\tau_n - \mathcal{G}\tau_n\| = 0. \tag{21}$$

By applying condition (I) and Equation (21), we obtain:

$$\begin{aligned} 0 \leq \lim_{n\to\infty} \psi(d(\tau_n, F(\mathcal{G}))) &\leq \lim_{n\to\infty} \|\tau_n - \mathcal{G}\tau_n\| = 0 \\ \Longrightarrow \lim_{n\to\infty} \psi(d(\tau_n, F(\mathcal{G}))) &= 0. \end{aligned}$$

It then follows that:

$$\lim_{n\to\infty} (d(\tau_n, F(\mathcal{G}))) = 0.$$

Hence, using Theorem 4, the sequence $\{\tau_n\}$ converges strongly to a fixed point of $\mathcal{G}$. □

Now, we present the following example to support Theorem 5.

Example 2. *Let $\mathcal{B} = \mathbb{R}$ be a Banach space with respect to the usual norm and $\mathcal{S} = [-2, \infty)$ be a non-empty, closed and convex subset of $\mathcal{B}$. Let $\mathcal{G} : \mathcal{S} \to \mathcal{S}$ be an operator that is defined by:*

$$\mathcal{G}(x) = \begin{cases} \frac{x}{4}, & \text{if } x \in [-2, \frac{1}{2}], \\ \frac{x}{5}, & \text{if } x \in (\frac{1}{2}, \infty). \end{cases}$$

Since $\mathcal{G}$ is discontinuous at $x = \frac{1}{2}$ and we know that every non-expansive mapping is continuous, it follows that $\mathcal{G}$ is not a non-expansive mapping. Now, we verify that $\mathcal{G}$ satisfies condition (E). For this, the following cases arise:

Case-I. When $x, y \in [-2, \frac{1}{2}]$, then we obtain:

$$\begin{aligned} \|x - \mathcal{G}y\| &= \left\|x - \frac{y}{4}\right\| \\ &= \left\|x - \frac{x}{4} + \frac{x}{4} - \frac{y}{4}\right\| \\ &\leq \left\|x - \frac{x}{4}\right\| + \frac{1}{4}\left\|x - y\right\| \end{aligned}$$

$$\leq \frac{16}{15}\left\|x-\mathcal{G}x\right\|+\|x-y\|.$$

Case-II. When $x, y \in (\frac{1}{2}, \infty)$, then we obtain:

$$\begin{aligned}\|x-\mathcal{G}y\| &= \left\|x-\frac{y}{5}\right\| \\ &= \left\|x-\frac{x}{5}+\frac{x}{5}-\frac{y}{5}\right\| \\ &\leq \left\|x-\frac{x}{5}\right\|+\frac{1}{5}\left\|x-y\right\| \\ &\leq \frac{16}{15}\left\|x-\mathcal{G}x\right\|+\|x-y\|.\end{aligned}$$

Case-III. When $x \in [-2, \frac{1}{2}]$ and $y \in (\frac{1}{2}, \infty)$, then we obtain:

$$\begin{aligned}\|x-\mathcal{G}y\| &= \left\|x-\frac{y}{5}\right\| \\ &= \left\|x-\frac{x}{5}+\frac{x}{5}-\frac{y}{5}\right\| \\ &\leq \left\|x-\frac{x}{5}\right\|+\frac{1}{5}\left\|x-y\right\| \\ &\leq \left\|x-\frac{x}{4}+\frac{x}{4}-\frac{x}{5}\right\|+\|x-y\| \\ &= \left\|\left(x-\frac{x}{4}\right)+\frac{1}{15}\left(x-\frac{x}{4}\right)\right\|+\|x-y\| \\ &= \frac{16}{15}\left\|x-\frac{x}{4}\right\|+\|x-y\| \\ &= \frac{16}{15}\left\|x-\mathcal{G}x\right\|+\|x-y\|.\end{aligned}$$

Case-IV. When $x \in (\frac{1}{2}, \infty)$ and $y \in [-2, \frac{1}{2}]$, then we obtain:

$$\begin{aligned}\|x-\mathcal{G}y\| &= \left\|x-\frac{y}{4}\right\| \\ &= \left\|x-\frac{x}{4}+\frac{x}{4}-\frac{y}{4}\right\| \\ &\leq \left\|x-\frac{x}{4}\right\|+\frac{1}{4}\left\|x-y\right\| \\ &\leq \left\|x-\frac{x}{5}+\frac{x}{5}-\frac{x}{4}\right\|+\|x-y\| \\ &= \left\|\left(x-\frac{x}{5}\right)-\frac{1}{16}\left(x-\frac{x}{5}\right)\right\|+\|x-y\| \\ &= \frac{15}{16}\left\|x-\frac{x}{5}\right\|+\|x-y\| \\ &\leq \frac{16}{15}\left\|x-\mathcal{G}x\right\|+\|x-y\|.\end{aligned}$$

Hence, for all of the above cases, $\mathcal{G}$ satisfies condition (E) with $\mu = \frac{16}{15}$ and $\mathcal{G}$ has a fixed point $t = 0$. Thus, $F(\mathcal{G}) = \{0\} \neq \emptyset$. Now, we consider a function $\psi(x) = \frac{x}{3}$, where

$x \in (0, \infty)$, which is non-decreasing and satisfies $\psi(0) = 0$ and $\psi(x) > 0$ for all $x \in (0, \infty)$. Now:

$$\begin{aligned} d(x, F(\mathcal{G})) &= \inf_{t \in F(\mathcal{G})} \|x - t\| \\ &= \inf \|x - 0\| \\ &= \inf \|x\| \\ &= \begin{cases} 0, & \text{if } x \in [-2, \frac{1}{2}], \\ \frac{1}{2}, & \text{if } x \in (\frac{1}{2}, \infty). \end{cases} \\ \implies \psi(d(x, F(\mathcal{G}))) &= \begin{cases} 0, & \text{if } x \in [-2, \frac{1}{2}], \\ \frac{1}{6}, & \text{if } x \in (\frac{1}{2}, \infty). \end{cases} \end{aligned}$$

Now, we have the following cases:

Case-I. When $x \in [-2, \frac{1}{2}]$, then we obtain:

$$\|x - \mathcal{G}x\| = \left\| x - \frac{x}{4} \right\| = \frac{3}{4}\|x\| \geq 0 = \psi(d(x, F(\mathcal{G}))).$$

Case-II. When $x \in (\frac{1}{2}, \infty)$, then we obtain:

$$\|x - \mathcal{G}x\| = \left\| x - \frac{x}{5} \right\| = \frac{4}{5}\|x\| \geq \frac{1}{6} = \psi(d(x, F(\mathcal{G}))).$$

Hence, from both the cases, we obtain:

$$\|x - \mathcal{G}x\| \geq \psi(d(x, F(\mathcal{G}))).$$

Thus, the operator $\mathcal{G}$ satisfies condition (I). Now, all of the assumptions of Theorem 5 are satisfied. Hence, using Theorem 5, the sequence that is defined by the JF iterative algorithm converges strongly to the fixed point $t = 0$ of $\mathcal{G}$.

Now, we extend the following example to compare the rate of convergence of the JF iterative algorithm to some other well-known iterative algorithms for operators that satisfy condition (E).

Example 3. *Let $\mathcal{B} = \mathbb{R}$ be a Banach space with respect to the usual norm and let $\mathcal{S} = [-1, 1]$ be a subset of $\mathcal{B}$. Let $\mathcal{G} : \mathcal{S} \to \mathcal{S}$ be defined by:*

$$\mathcal{G}(x) = \begin{cases} -x, & \text{if } x \in [0, \frac{3}{4}) \bigcup (\frac{3}{4}, 1], \\ \frac{1}{2}\sin x, & \text{if } x \in [-1, 0), \\ 0, & \text{if } x = \frac{3}{4}. \end{cases}$$

It can easily be seen that the operator $\mathcal{G}$ satisfies condition (E) with $\mu = 4$.

Now, we choose control sequences $\mu_n = 0.22$, $\theta_n = 0.65$ and $\eta_n = 0.95$ for all $n \in \mathbb{Z}^+$ with the initial estimate of $\tau_0 = 0.5$ to numerically compare the rate of convergence of remarkable iterative algorithms.

Using MATLAB 2015a, we demonstrate that the proposed iterative algorithm (4) converges to the fixed point $t = 0$ of the operator $\mathcal{G}$ faster than Mann, Ishikawa, S, Picard-S, Noor and SP iterative algorithms, which can easily be seen in Table 1 and Figure 1.

Table 1. A comparison of the rate of convergence of well-known iterative algorithms.

Iter.	Mann	Ishikawa	S	Picard-S	Noor	SP	JF
1	0.500000	0.500000	0.500000	0.500000	0.500000	0.500000	0.500000
2	0.280000	0.423000	−0.357000	0.357000	0.287150	0.110880	0.084000
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
9	0.004836	0.131200	0.033772	0.033772	0.005917	0.000003	0.000000
10	0.002708	0.110995	−0.024113	0.024113	0.003398	0.000001	0.000000
11	0.001517	0.093902	0.017217	0.017217	0.001951	0.000000	0.000000
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
25	0.000000	0.009034	0.000154	0.000154	0.000001	0.000000	0.000000
26	0.000000	0.007642	−0.000110	0.000110	0.000000	0.000000	0.000000
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
41	0.000000	0.000622	0.000001	0.000001	0.000000	0.000000	0.000000
42	0.000000	0.000526	−0.000001	0.000001	0.000000	0.000000	0.000000
43	0.000000	0.000445	0.000000	0.000000	0.000000	0.000000	0.000000
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
84	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

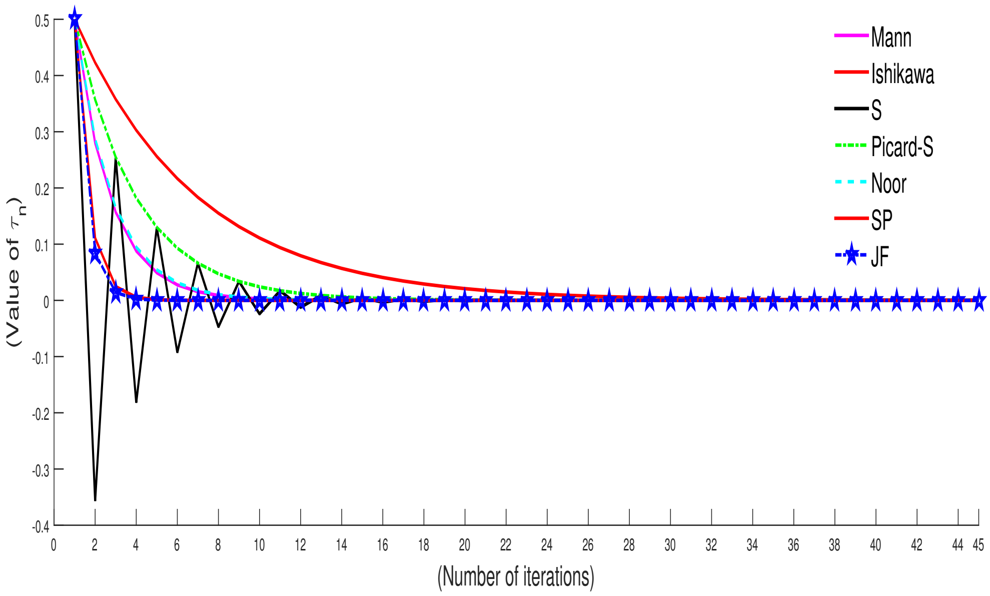

Figure 1. Graphical representation of the rate of convergence of well known iterative algorithms.

5. Application

The purpose of this section is to estimate the solution of a mixed Volterra–Fredholm functional non-linear integral equation using the iterative algorithm (4).

We considered the following non-linear integral equation (see [32]):

$$x(z) = T\left(z, x(z), \int_{c_1}^{z_1} \dots \int_{c_n}^{z_n} K(z, s, x(s))ds, \int_{c_1}^{d_1} \dots \int_{c_n}^{d_n} H(z, s, x(s))ds\right), \tag{22}$$

where $[c_1, d_1] \times ... \times [c_n, d_n]$ is an interval in $\mathbb{R}^n$, $z = (z_1, z_2 ... z_n)$, $s = (s_1, s_2, ... s_n) \in [c_1, d_1] \times ... \times [c_n, d_n]$, $K, H : [c_1, d_1] \times ... \times [c_n, d_n] \times [c_1, d_1] \times ... \times [c_n, d_n] \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $T : [c_1, d_1] \times ... \times [c_n, d_n] \times \mathbb{R}^3 \to \mathbb{R}$.

Assume that the following prerequisites are satisfied:
(D_1) $K, H \in C([c_1, d_1] \times ... \times [c_n, d_n] \times [c_1, d_1] \times ... \times [c_n, d_n] \times \mathbb{R})$;
(D_2) $T \in C([c_1, d_1] \times ... \times [c_n, d_n] \times \mathbb{R}^3)$;
(D_3) constants $\alpha, \beta, \gamma \geq 0$ exist in such a way that:

$$|T(z, u_1, u_2, u_3) - T(z, v_1, v_2, v_3)| \leq \alpha|u_1 - v_1| + \beta|u_2 - v_2| + \gamma|u_3 - v_3|,$$

for all $z \in [c_1, d_1] \times ... \times [c_n, d_n]$, $u_i, v_i \in \mathbb{R}$, $i = 1, 2, 3$;
(D_4) constants $L_K \geq 0$ and $L_H \geq 0$ exist in such a way that:

$$|K(z, s, u) - K(z, s, v)| \leq L_K|u - v|,$$

$$|H(z, s, u) - H(z, s, v)| \leq L_H|u - v|,$$

for all $z, s \in [c_1, d_1] \times ... \times [c_n, d_n]$, *and* $u, v \in \mathbb{R}$;
(D_5) $\alpha + (\beta L_K + \gamma L_H)(d_1 - c_1)...(d_n - c_n) < 1$.

Using the solution to problem (22), we obtain a function $x_* \in C([c_1, d_1] \times ... \times [c_n, d_n])$.

The following existence result for problem (22) was proved by Crăciun and Şerban [32].

Theorem 6. *Assume that prerequisites $(D_1) - (D_5)$ are satisfied. Then, problem (22) has a unique solution of $x_* \in C([c_1, d_1] \times ... \times [c_n, d_n])$.*

We now demonstrate the main result of this section using the iterative algorithm (4).

Theorem 7. *Let $\mathcal{B} = C([c_1, d_1] \times ... \times [c_n, d_n], \|.\|)$ be a Banach space with Chebyshev's norm. Let $\{\tau_n\}$ be a sequence that is defined by the iterative algorithm (4) for the operator $\mathcal{G} : \mathcal{B} \to \mathcal{B}$, which is defined as:*

$$\mathcal{G}x(z) = T\left(z, x(z), \int_{c_1}^{z_1} ... \int_{c_n}^{z_n} K(z, s, x(s))ds, \int_{c_1}^{d_1} ... \int_{c_n}^{d_n} H(z, s, x(s))ds\right), \tag{23}$$

where T, K and H are defined as above. Assume that prerequisites $(D_1) - (D_5)$ are satisfied. Then, the iterative algorithm (4) converges to the unique solution, i.e., $x_* \in C([c_1, d_1] \times ... \times [c_n, d_n])$ of problem (22).

Proof. In Theorem 6, we saw that problem (22) has a unique solution, so let us assume that x_* is the fixed point of $\mathcal{G}$. Now, we aim to show that the sequence $\{\tau_n\}$ that is defined by the JF iterative algorithm (4) converges to the solution of problem (22), i.e., x_*. First, we need to show that the operator $\mathcal{G}$ that is defined in (23) is an almost contraction.

Presume that the prerequisites $(D_1)-(D_4)$ are satisfied. Then:

$$
\begin{aligned}
\|\mathcal{G}x-\mathcal{G}x_*\| &= \|\mathcal{G}x-x_*\| = |\mathcal{G}x(z)-\mathcal{G}x_*(z)| \\
&= \Bigg| T\Big(z, x(z), \int_{c_1}^{z_1}\ldots\int_{c_n}^{z_n} K(z,s,x(s))ds, \int_{c_1}^{d_1}\ldots\int_{c_n}^{d_n} H(z,s,x(s))ds\Big) \\
&- T\Big(z, x_*(z), \int_{c_1}^{z_1}\ldots\int_{c_n}^{z_n} K(z,s,x_*(s))ds, \int_{c_1}^{d_1}\ldots\int_{c_n}^{d_n} H(z,s,x_*(s))ds\Big)\Bigg| \\
&\le \alpha\big|x(z)-x_*(z)\big| \\
&+ \beta\Big|\int_{c_1}^{z_1}\ldots\int_{c_n}^{z_n} K(z,s,x(s))ds - \int_{c_1}^{z_1}\ldots\int_{c_n}^{z_n} K(z,s,x_*(s))ds\Big| \\
&+ \gamma\Big|\int_{c_1}^{d_1}\ldots\int_{c_n}^{d_n} H(z,s,x(s))ds - \int_{c_1}^{d_1}\ldots\int_{c_n}^{d_n} H(z,s,x_*(s))ds\Big| \\
&\le \alpha\big|x(z)-x_*(z)\big| \\
&+ \beta\int_{c_1}^{z_1}\ldots\int_{c_n}^{z_n}\Big|K(z,s,x(s))-K(z,s,x_*(s))\Big|ds \\
&+ \gamma\int_{c_1}^{d_1}\ldots\int_{c_n}^{d_n}\Big|H(z,s,x(s))-H(z,s,x_*(s))\Big|ds \\
&\le \alpha\big|x(z)-x_*(z)\big| + \beta\int_{c_1}^{z_1}\ldots\int_{c_n}^{z_n} L_K\Big|x(s)-x_*(s)\Big|ds \\
&+ \gamma\int_{c_1}^{d_1}\ldots\int_{c_n}^{d_n} L_H\Big|x(s)-x_*(s)\Big|ds \\
&\le \alpha\big\|x-x_*\big\| + \beta\int_{c_1}^{z_1}\ldots\int_{c_n}^{z_n} L_K\big\|x-x_*\big\|ds \\
&+ \gamma\int_{c_1}^{d_1}\ldots\int_{c_n}^{d_n} L_H\big\|x-x_*\big\|ds \\
&= \alpha\|x-x_*\| + \beta L_K(z_1-c_1)\ldots(z_n-c_n)\|x-x_*\| \\
&+ \gamma L_H(d_1-c_1)\ldots(d_n-a_n)\|x-x_*\|.
\end{aligned}
$$

$$
\Longrightarrow \|\mathcal{G}x-\mathcal{G}x_*\| \le [\alpha+(\beta L_K+\gamma L_H)(d_1-c_1)\ldots(d_n-c_n)]\|x-x_*\|. \tag{24}
$$

By using condition (D_5) and defining $\delta := \alpha+(\beta L_K+\gamma L_H)(d_1-c_1)\ldots(d_n-c_n) < 1$, then for any $L \ge 0$ Equation (24) becomes:

$$
\|\mathcal{G}x-\mathcal{G}x_*\| \le \delta\|x-x_*\| + L\|x-\mathcal{G}x\|.
$$

This shows that $\mathcal{G}$ is an almost contraction. Hence, using Theorem 2, the sequence $\{\tau_n\}$ that is defined by the JF iterative algorithm (4) converges to the solution of problem (22). This completes the proof. □

6. Conclusions

The purpose of this manuscript was to study a well-known and effective iterative algorithm to approximate the fixed points of non-linear operators that satisfy condition (E) within the contest of Banach spaces. It is well known that the class of operators that satisfy condition (E) includes the classes of mappings that satisfy Suzuki's condition (C), Hardy and Rogers mappings, generalized α non-expansive mappings, Reich–Suzuki generalized non-expansive mappings, etc. Therefore, the results of the present manuscript generalize and extend the relevant results in existing literature (see, for example, [8,9,18,20,23]). It is also shown here that the JF iterative algorithm is weakly w^2-stable with respect to almost contractions. The JF iterative algorithm can be successfully implemented to approximate the solutions of non-linear integral equations. Thus, the results of the current manuscript are very useful and interesting.

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