mathematics

Article

Satellites of Functors for Nonassociative Algebras with Metagroup Relations

Sergey Victor Ludkowski

Department of Applied Mathematics, MIREA—Russian Technological University, av. Vernadsky 78, 119454 Moscow, Russia; sludkowski@mail.ru

Abstract: The article is devoted to non-associative algebras with metagroup relations and modules over them. Their functors are studied. Satellites of functors are scrutinized. An exactness of satellite sequences and diagrams is investigated.

Keywords: non-associative algebra; satellites of functors; module; cohomology; metagroup

MSC: Primary 18E25; 18B40; Secondary 16D70; 18G60; 17A60; 03C90

Citation: Ludkowski, S.V. Satellites of Functors for Nonassociative Algebras with Metagroup Relations. *Mathematics* **2022**, *10*, 1169. https://doi.org/10.3390/math10071169

Academic Editor: Alexander Felshtyn

Received: 11 March 2022
Accepted: 1 April 2022
Published: 4 April 2022

1. Introduction

Structure and functors of associative algebras are very important and have found wide-spread application (see, for example, Refs. [1–5] and references therein). This is tightly related with their cohomology theory. Certainly, a great amount of attention is paid to algebras with groups identities. It is worth mentioning that functors and satellites in conjunction with cohomology theory of associative algebras were investigated by Cartan, Eilenber, Hochschild, and other authors [6–9], but it is not applicable to non-associative algebras.

On the other hand, non-associative algebras with some identities in them, such as Cayley–Dickson algebras and their generalizations, compose a great part in algebra. Moreover, they obtained many-sided applications in physics, noncommutative geometry, quantum field theory, PDEs, and other sciences (see [10–24] and references therein). Other actual non-associative structures and their applications are described in [25–27]. For example, the Klein–Gordon hyperbolic PDE of the second order with constant coefficients was solved by Dirac with the help of complexified quaternions [28]. Cayley–Dickson algebras were used for decompositions of higher order PDEs into lower order PDEs that permitted to integrate and analyze them subsequently [18,29,30]. PDEs or their systems frequently possess groups of their symmetries [9]. These gave rise to group algebras over the complex field $\mathbf{C}$ in conjunction with Cayley–Dickson algebras leading to extensions that are more general metagroup algebras. This leads to operator algebras over Cayley–Dickson algebras, and they also induce the metagroup algebras. It is necessary to note that, besides algebras over the real field $\mathbf{R}$ or the complex field $\mathbf{C}$, there are such algebras over other fields. The latter are important in non-Archimedean quantum mechanics and quantum field theory. Then, analysis of PDEs and operators over Cayley–Dickson algebras induce generalized Cayley–Dickson algebras or metagroup algebras, which act on function modules.

A remarkable fact was outlined in the 20th century that a noncommutative geometry exists, if there exists a corresponding quasi-group [31–33]. On the other hand, metagroups are quasigroups with weak relations.

Previously, examples of non-associative algebras, modules and homological complexes with metagroup relations were given in [15,16,34,35]. Cohomology theory on them began to be studied in [15]. These algebras also are related with Hopf and quasi-Hopf algebras. For digital Hopf spaces, cohomologies were investigated in [36].

Smashed and twisted wreath products of metagroups or groups were studied in [17]. It allowed to construct ample families of metagroups even starting from groups. It also demonstrates that metagroups appear naturally in algebra. That is, metagroup algebras compose an enormous class of non-associative algebras.

This article is devoted to functors and satellites for non-associative algebras. These non-associative algebras are with mild relations induced from metagroup structures. Modules over non-associative metagroup algebras are investigated in the framework of categories and functors on them in Section 2. Necessary Definitions 1–5 and notations in Remark 1 are provided. Exactness and additivity of functors and their sequences are investigated in Propositions 1–7. Their relations with structure of modules over non-associative algebras with metagroup relations are studied. Examples 1–3 of categories and functors are given. Satellites for modules over non-associative algebras with metagroup relations are investigated in Section 3. Additivity of morphism related with functors is studied in Propositions 8 and 9. An exactness of satellite sequences and diagrams is scrutinized in Theorems 1–3 (see also formulas and diagrams (1)–(54)).

Auxiliary necessary definitions and notations are provided in Appendix A (A1)–(A21) (they also are contained in [15,16,35]).

All main results of this paper are obtained for the first time. They can be used for further studies of non-associative algebras and their modules, their categories and functors, cohomologies, algebraic geometry, PDEs, their applications in the sciences, etc.

2. Functors for Categories with Metagroup Relations

Remark 1. *Let $\mathcal{T}$ be an associative commutative unital ring, and let ${}^jG$ and ${}^j_rG$ be metagroups, ${}^jA = \mathcal{T}[{}^jG]$ and ${}^j_rA = \mathcal{T}[{}^j_rG]$ be metagroup algebras over $\mathcal{T}$, ${}^jB$ be a unital smashly ${}^jG$-graded ${}^jA$-algebra, ${}^jX$ be a left ${}^jG$-graded ${}^jB$-module (or right ${}^jB$-module, or $({}^jB,{}^j_rB)$-bimodule) $j \in {}_\mu\Omega$, where ${}_\mu\Omega$ is a class or a set, where ${}^iA$ is embedded into ${}^iB$ as ${}^iA1_{{}^iB}$, and ${}^iG$ is embedded into ${}^iA$ as ${}^iG1_{{}^iB}$. For the sake of brevity, "smashly" may be omitted. For left modules ${}^jX$, ${}^iX$ with $w_{j,i} = (s,\gamma_{j,i})$, $\gamma_{j,i} = \gamma \in \{({}^jG,{}^iG),\ ({}^jA,{}^iA),\ ({}^jB,{}^iB)\}$ by $Hom_{l,w_{j,i}}({}^jX,{}^iX)$ will be denoted the family of all left $\mathcal{T}$-linear homomorphisms, which are γ-epigeneric if $s = \mathsf{eg}$, γ-exact if $s = \mathsf{e}$, γ-generic if $s = \mathsf{g}$, where $h(x+y) = h(x)+h(y)$, $h(bx) = h'(b)h(x)$ for each $h \in Hom_{l,w_{j,i}}({}^jX,{}^iX)$, $b \in {}^jB$, x and y in ${}^jX$, where $h' \in Hom_l({}^jB,{}^iB)$ is a left $\mathcal{T}$-homomorphism associated with h, $h' : {}^jB \to {}^iB$, where, as usual, $h'|_{{}^jA1_{{}^jB}} : {}^jA1_{{}^jB} \to {}^iA1_{{}^iB}$, $h'|_{{}^jG1_{{}^jB}} : {}^jG1_{{}^jB} \to {}^iG1_{{}^iB}$. It is naturally assumed that a $({}^jB,{}^iB)$-epigeneric (or exact, or generic) homomorphism is also $({}^jA,{}^iA)$-epigeneric (or exact, or generic correspondingly); a $({}^jA,{}^iA)$-epigeneric (or exact, or generic) homomorphism is also $({}^jG,{}^iG)$-epigeneric (or exact, or generic correspondingly).*

For right modules $Hom_{r,w_{j,i}}({}^jX,{}^iX)$ will be used similarly. For $({}^jG,{}^j_rG)$-graded $({}^jB,{}^j_rB)$-bimodules ${}^jX$ with $w_{j,i} = (v,\beta_{j,i})$, $v = (s,{}_rs)$, $\beta_{j,i} = \beta = (\gamma_{j,i},{}_r\gamma_{j,i})$, $\gamma_{j,i} \in \{({}^jG,{}^iG),({}^jA,{}^iA),$ $({}^jB,{}^iB)\}$, ${}_r\gamma_{j,i} \in \{({}^j_rG,{}^i_rG),\ ({}^j_rA,{}^i_rA),\ ({}^j_rB,{}^i_rB)\}$, s and ${}_rs$ in $\{\mathsf{eg},\mathsf{e},\mathsf{g}\}$, let $Hom_{w_{j,i}}({}^jX,{}^iX) =$ $\{f : {}^jX \to {}^iX |\ f \in Hom_{l,(s,\gamma_{j,i})}({}^jX,{}^iX) \& f \in Hom_{r,({}_rs,{}_r\gamma_{j,i})}({}^jX,{}^iX)\}$.

By $Ob({}_\mu\mathcal{M}) = \{{}^jM = ({}^jG,{}^jB,{}^jX) : j \in {}_\mu\Omega\}$ (or $Ob(\mathcal{M}_\nu)$ or $Ob({}_\mu\mathcal{M}_\nu)$) will be denoted a family of all ${}^jG$-graded left ${}^jB$-modules ${}^jX$ for $\mu = \{({}^jG,\mathcal{T},{}^jB) : j \in \Omega\}$ (or similarly for right modules, or bimodules), where $\Omega = {}_\mu\Omega$. Let

$$ {}^{s,\tau}_{\mu}\mathcal{M} = (Ob({}_\mu\mathcal{M}), Hom_{l,w_{i,p}}({}^iM,{}^pM) : i \in {}_\mu\Omega, p \in {}_\mu\Omega, w_{i,p} = (s,{}^\tau\gamma_{i,p})) $$

denote a category over $\mathcal{T}$, where ${}^jM \in Ob({}_\mu\mathcal{M}) =: Ob({}^{s,\tau}_{\mu}\mathcal{M})$, ${}^jM = ({}^jG,{}^jB,{}^jX)$, where ${}^p_i\mathbf{f} \in Hom_{l,w_{i,p}}({}^iM,{}^pM)$ will also be written in place of ${}^p_if \in Hom_{l,w_{i,p}}({}^iX,{}^pX)$ due to embeddings and Conditions (14)–(19) provided in Definition 2 in [35] and above, where ${}^p_i\mathbf{f}$ is also used as a shortening of $({}^p_if',{}^p_if)$, where ${}^1\gamma_{j,i} = ({}^jG,{}^iG)$, ${}^2\gamma_{j,i} = ({}^jA,{}^iA)$, ${}^3\gamma_{j,i} = ({}^jB,{}^iB)$, $\tau \in \{1,2,3\}$, $s \in \{\mathsf{eg},\mathsf{e},\mathsf{g}\}$; s and τ are fixed. This ${}^p_i\mathbf{f}$ will also be called a left $w_{i,p}$-homomorphism.

A sequence

$$ \cdots \to {}^{i_1}M \xrightarrow[{}^{i}_{i_1}\mathbf{f}]{} {}^iM \xrightarrow[{}^{i_2}_{i}\mathbf{f}]{} {}^{i_2}M \to \ldots $$

is exact by the definition if and only if a sequence

$$\cdots \to {}^{i_1}X \xrightarrow[{}^{i}_{i_1}f]{} {}^{i}X \xrightarrow[{}^{i_2}_{i}f]{} {}^{i_2}X \to \dots$$

is exact. If ${}^jM = ({}^jG, {}^jB, 0)$*, then it will also be written shortly* ${}^jM_0$*. If* ${}^p_i\mathbf{f} = ({}^p_if', 0)$*, then it will also be written for brevity* ${}^p_i\mathbf{f}_0$.

Definition 1. *Let* $T = (T_1, T_2, T_3, T_4)$ *be a functor such that* $T({}^jM) \in Ob({}^{s,\tau}_{\zeta}\mathcal{M})$ *for each* ${}^jM \in Ob({}^{s,\tau}_{\mu}\mathcal{M})$*, where* $k = k(T, j) \in {}_{\zeta}\Omega$*,* $\zeta \ni (T_1({}^jG), T_2(\mathcal{T}), T_3({}^jB))$ *for each* $j \in {}_{\mu}\Omega$*,* $T_1({}^jG) = {}^kG$ *is a metagroup,* $T_2(\mathcal{T}) = \mathcal{T}_1$ *is an associative commutative unital ring,* $T_2 : \mathcal{T} \to \mathcal{T}_1$ *is a ring homomorphism,* $T_3({}^jB) = {}^kB$*,* $T_4({}^jX) = {}^kX$ *(see Remark 1). For each* $h \in Hom_{l,w_{j,p}}({}^jX, {}^pX)$*, let* $T_4(h) \in Hom_{l,T(w_{j,p})}(T_4({}^jX), T_4({}^pX))$*,* $T_4(h') \in Hom_{l,T(w_{j,p})}(T_3({}^jB), T_3({}^pB))$*,* $T(w) = (s, T(\gamma))$ *for* $w = (s, \gamma)$*, where* $T(\gamma) = (T_1({}^jG), T_1({}^pG))$ *if* $\gamma_{j,p} = \gamma = ({}^jG, {}^pG)$*,* $T(\gamma) = (T_2(\mathcal{T})[T_1({}^jG)], T_2(\mathcal{T})[T_1({}^pG)])$ *if* $\gamma = ({}^jA, {}^pA)$*,* $T(\gamma) = (T_3({}^jB), T_3({}^pB))$ *if* $\gamma = ({}^jB, {}^pB)$*, where* $T_1(h) : T_1({}^jG) \to T_1({}^pG)$ *is induced by the restriction of* $T_3(h)$ *on* $T_1({}^jG)1_{T_3({}^jB)}$*,* $T_4(h') = T_3(h) : T_3({}^jB) \to T_3({}^pB)$ *is a homomorphism of algebras, where* $T_4(h(bx)) = (T_3(h)(T_3(b)))(T_4(h)(T_4(x)))$*,* $T_1(h(c)) = T_1(h)(T_1(c))$ *for each* $c \in {}^jG$*,* $T_3(h(\epsilon a)) = (T_2(\epsilon))(T_3(h)(T_3(a)))$ *for each* $\epsilon \in \mathcal{T}$ *and* $a \in {}^jA$*,* $T_3(h(ab)) = (T_3(h)(T_3(a)))(T_3(h)(T_3(b)))$ *for each* $a \in {}^jB$ *and* $b \in {}^jB$*. Notice that, if* h *is an identity homomorphism, then* $T(h)$ *is an identity homomorphism,* $T(0) = 0$*. The functor* T *is called additive if it satisfies:*

$$T_4({}^{i_1}_if + {}^{i_1}_ig) = T_4({}^{i_1}_if) + T_4({}^{i_1}_ig)$$

for each ${}^{i_1}_i\mathbf{f}$ *and* ${}^{i_1}_i\mathbf{g}$ *in* $Hom_{l,w_{i,i_1}}({}^iM, {}^{i_1}M)$ *with* $T_3({}^{i_1}_if') = T_3({}^{i_1}_ig')$.

Assume that, if $\mathbf{h} \in Hom_{l,w_{i,p}}({}^iM, {}^pM)$*,* $\mathbf{f} \in Hom_{l,w_{p,j}}({}^pM, {}^jM)$*, then* $T(\mathbf{f} \circ \mathbf{h}) = T(\mathbf{f}) \circ T(\mathbf{h})$*. If these conditions are satisfied, then* T *is called a s-covariant functor from the category* ${}^{s,\tau}_{\mu}\mathcal{M}$ *over* $\mathcal{T}$ *into the category* ${}^{s,\tau}_{\zeta}\mathcal{M}$ *over* $\mathcal{T}_1$*. If* s *is specified, it may be shortly called a covariant functor.*

If $T(\mathbf{f} \circ \mathbf{h}) = T(\mathbf{h}) \circ T(\mathbf{f})$ *for each* $\mathbf{f}$ *and* $\mathbf{h}$ *as above, then it is said that* T *is a contravariant functor.*

Similarly, functors are defined for the category $\mathcal{M}^{s,\tau}_{\mu}$ *of right modules and for the category* ${}^{v,\rho}_{\mu}\mathcal{M}_v$ *of bimodules, where* $v = (s, {}_rs)$*,* $\rho = (\tau, {}_r\tau)$.

Let $\mathcal{T}$*,* $\mathcal{T}_1$*,* $\mathcal{T}_2$ *be commutative associative unital rings and* ${}^{s,\tau}_{\mu}\mathcal{M}$*,* ${}^{s,\tau}_{\mu_1}\mathcal{M}$*,* ${}^{s,\tau}_{\mu_2}\mathcal{M}$ *be the categories of* ${}^iG$*-graded left* ${}^iB$*-modules over the rings* $\mathcal{T}$*,* $\mathcal{T}_1$*,* $\mathcal{T}_2$ *respectively, for* i *in* ${}_{\mu}\Omega$*,* ${}_{\mu_1}\Omega$*,* ${}_{\mu_2}\Omega$*, respectively, where* $s \in \{\mathsf{eg}, \mathsf{e}, \mathsf{g}\}$*,* $\tau \in \{1, 2, 3\}$*,* s *and* τ *are fixed. Let for each* ${}^iM \in Ob({}^{s,\tau}_{\mu_1}\mathcal{M})$*,* ${}^jM \in Ob({}^{s,\tau}_{\mu_2}\mathcal{M})$ *there be posed* $T({}^iM, {}^jM) \in Ob({}^{s,\tau}_{\mu}\mathcal{M})$*, to each* ${}^{i_1}_i\mathbf{f} = \mathbf{f} \in Hom_{l,w_{i,i_1}}({}^iM, {}^{i_1}M)$*,* ${}^{j}_{j_1}\mathbf{h} = \mathbf{h} \in Hom_{l,w_{j_1,j}}({}^{j_1}M, {}^jM)$*, where* ${}^{i_1}M \in Ob({}^{s,\tau}_{\mu_1}\mathcal{M})$*,* ${}^{j_1}M \in Ob({}^{s,\tau}_{\mu_2}\mathcal{M})$*,* $T = (T_1, T_2, T_3, T_4)$*,* $T_1({}^iG, {}^jG) = {}^kG$ *is a metagroup,* $T_2(\mathcal{T}_1, \mathcal{T}_2) = \mathcal{T}$ *is the commutative associative unital ring,* $T_3({}^iB, {}^jB) = {}^kB$ *is a* ${}^kG$*-graded* ${}^kA$*-algebra,* $T_4({}^iX, {}^jX) = {}^kX$ *is a* ${}^kG$*-graded left* ${}^kB$*-module, there are posed homomorphisms* $T(\mathbf{f}, {}^jM) : T({}^iM, {}^jM) \to T({}^{i_1}M, {}^jM)$ *and* $T({}^iM, \mathbf{h}) : T({}^iM, {}^jM) \to T({}^iM, {}^{j_1}M)$*, such that*

$$T(\mathbf{f}, {}^jM) \in Hom_{l,T(w_{i,i_1}, {}^jM)}(T({}^iM, {}^jM), T({}^{i_1}M, {}^jM)),$$

$$T({}^iM, \mathbf{h}) \in Hom_{l,T({}^iM, w_{j_1,j})}(T({}^iM, {}^jM), T({}^iM, {}^{j_1}M)),$$

where $T(\gamma, {}^jM) = (T_1({}^iG, {}^jG), T_1({}^pG, {}^jG))$ *if* $\gamma_{i,p} = \gamma = ({}^iG, {}^pG)$*,*

$$T(\gamma, {}^jM) = (T_2(\mathcal{T}_1, \mathcal{T}_2)[T_1({}^iG, {}^jG)], T_2(\mathcal{T}_1, \mathcal{T}_2)[T_1({}^pG, {}^jG)]) if \gamma = ({}^iA, {}^pA),$$

$$T(\gamma, {}^jM) = (T_3({}^iB, {}^jB), T_3({}^pB, {}^jB)) \ if \ \gamma = ({}^iB, {}^pB), similarly$$

$$T({}^iM, \gamma) = (T_1({}^iG, {}^pG), T_1({}^iG, {}^jG)) \ if \ \gamma_{j,p} = \gamma = ({}^jG, {}^pG),$$

$$T({}^iM, \gamma) = (T_2(\mathcal{T}_1, \mathcal{T}_2)[T_1({}^iG, {}^pG)], T_2(\mathcal{T}_1, \mathcal{T}_2)[T_1({}^iG, {}^jG)]) \ if \ \gamma = ({}^jA, {}^pA),$$

$$T({}^iM, \gamma) = (T_3({}^iB, {}^pB), T_3({}^iB, {}^jB)) if \gamma = ({}^jB, {}^pB),$$

$$T(w, {}^jM) = (s, T(\gamma, {}^jM)) \text{ and } T({}^iM, \gamma) = (s, T({}^iM, \gamma)) \text{ for } w = (s, \gamma).$$

Note that, if $\mathbf{f} = \mathbf{id}_{{}^iM}$, $\mathbf{h} = \mathbf{id}_{{}^jM}$, *then* $T(\mathbf{f}, {}^jM) = \mathbf{id}_{T({}^iM, {}^jM)}$, $T({}^iM, \mathbf{h}) = \mathbf{id}_{T({}^iM, {}^jM)}$.

Assume also that $T({}^{i_2}_{i_1}\mathbf{f} \circ {}^{i_1}_{i}\mathbf{f}, {}^jM) = T({}^{i_2}_{i_1}\mathbf{f}, {}^jM) \circ T({}^{i_1}_{i}\mathbf{f}, {}^jM)$ *and* $T({}^iM, {}^{j}_{j_1}\mathbf{h} \circ {}^{j_1}_{j_2}\mathbf{h}) = T({}^iM, {}^{j_1}_{j_2}\mathbf{h}) \circ T({}^iM, {}^{j}_{j_1}\mathbf{h})$; *there is the commutative diagram:*

$$\begin{array}{ccc} T({}^iM, {}^jM) & \xrightarrow[T({}^{i_1}_{i}\mathbf{f}, {}^jM)]{} & T({}^{i_1}M, {}^jM) \\ T({}^iM, {}^{j}_{j_1}\mathbf{h}) \downarrow & & \downarrow T({}^{i_1}M, {}^{j}_{j_1}\mathbf{h}) \\ T({}^iM, {}^{j_1}M) & \xrightarrow[T({}^{i_1}_{i}\mathbf{f}, {}^{j_1}M)]{} & T({}^{i_1}M, {}^{j_1}M). \end{array}$$

Then, it is said that T is a functor of two arguments covariant in the first and contravariant in the second argument. If we fix ${}^jM$, *then* $T(\cdot, {}^jM)$ *will be a covariant functor; if we fix* ${}^iM$, *then* $T({}^iM, \cdot)$ *will be a contravariant functor. We shall consider additive functors:*

$$\begin{aligned} &T_4({}^{i_1}_{i}f + {}^{i_1}_{i}g, {}^jX) = T_4({}^{i_1}_{i}f, {}^jX) + T_4({}^{i_1}_{i}g, {}^jX) \text{ and} \\ &T_4({}^iX, {}^{j}_{j_1}h + {}^{j}_{j_1}q) = T_4({}^iX, {}^{j}_{j_1}h) + T_4({}^iX, {}^{j}_{j_1}q) \end{aligned}$$

for each ${}^{i_1}_{i}\mathbf{f}$ *and* ${}^{i_1}_{i}\mathbf{g}$ *in* $Hom_{l,w_{i,i_1}}({}^iM, {}^{i_1}M)$ *with* $T_3({}^{i_1}_{i}f', {}^jX) = T_3({}^{i_1}_{i}g', {}^jX)$, ${}^{j}_{j_1}\mathbf{h}$ *and* ${}^{j}_{j_1}\mathbf{q}$ *in* $Hom_{l,w_{j,j_1}}({}^jM, {}^{j_1}M)$ *with* $T_3({}^iX, {}^{j}_{j_1}h') = T_3({}^iX, {}^{j}_{j_1}q')$. *In particular, if* ${}^iX = 0$ *(or* ${}^jX = 0$*), then* $T_4({}^iX, {}^jX) = 0$. *Notice that* $T_4({}^{i_1}_{i}0, {}^jX) = 0$, $T({}^iX, {}^{j}_{j_1}0) = 0$, *where* ${}^{i_1}_{i}0$ *and* ${}^{j}_{j_1}0$ *denote zero homomorphisms.*

Henceforward, additive functors are considered if some other is not specified.

Proposition 1. *Assume that sequences of homomorphisms*

$$\begin{array}{l} {}^{i_1}X \xrightarrow[{}^{i}_{i_1}f]{} {}^iX \xrightarrow[{}^{i_1}_{i}f]{} {}^{i_1}X \text{ and} \\ {}^{j_1}X \xrightarrow[{}^{j}_{j_1}h]{} {}^jX \xrightarrow[{}^{j_1}_{j}h]{} {}^{j_1}X \end{array}$$

with $i_1 \in \Lambda_i$, $j_1 \in \Lambda_j$, $card(\Lambda_i) < \aleph_0$, $card(\Lambda_j) < \aleph_0$, *induce representations of the left modules* ${}^iX$ *and* ${}^jX$ *as direct sums, where* ${}^{i_1}G = {}^iG$, ${}^{i_1}B = {}^iB$, ${}^{i_1}\mathcal{T} = {}^i\mathcal{T}$ *for each* $i_1 \in \Lambda_i$, ${}^{j_1}G = {}^jG$, ${}^{j_1}B = {}^jB$, ${}^{j_1}\mathcal{T} = {}^j\mathcal{T}$ *for each* $j_1 \in \Lambda_j$. *Then, sequences of homomorphisms*

$$T_4({}^{i_1}X, {}^{j_1}X) \xrightarrow[T_4({}^{i}_{i_1}f, {}^{j_1}_{j}h)]{} T_4({}^iX, {}^jX) \xrightarrow[T_4({}^{i_1}_{i}f, {}^{j}_{j_1}h)]{} T_4({}^{i_1}X, {}^{j_1}X)$$

induce a representation of $T_4({}^iX, {}^jX)$ *as a direct sum.*

Proof. Since $T({}^{i'_1}_{i'}\mathbf{f}, {}^{j'}_{j'_1}\mathbf{h}) \circ T({}^{i}_{i_1}\mathbf{f}, {}^{j_1}_{j}\mathbf{h}) = T({}^{i'_1}_{i'}\mathbf{f} \circ {}^{i}_{i_1}\mathbf{f}, {}^{j_1}_{j}\mathbf{h} \circ {}^{j'}_{j'_1}\mathbf{h})$, then the composition $T({}^{i'_1}_{i}\mathbf{f}, {}^{j}_{j'_1}\mathbf{h}) \circ T({}^{i}_{i_1}\mathbf{f}, {}^{j_1}_{j}\mathbf{h})$ is the identity map if $(i_1, j_1) = (i'_1, j'_1)$; otherwise, it is zero. The sum

$$\sum_{i_1, j_1} T_4({}^{i}_{i_1}f, {}^{j_1}_{j}h) \circ T_4({}^{i_1}_{i}f, {}^{j}_{j_1}h) =$$

$$\sum_{i_1, j_1} T_4({}^{i}_{i_1}f \circ {}^{i_1}_{i}f, {}^{j}_{j_1}h \circ {}^{j_1}_{j}h) = T_4(\sum_{i_1} {}^{i}_{i_1}f \circ {}^{i_1}_{i}f, \sum_{j_1} {}^{j}_{j_1}h \circ {}^{j_1}_{j}h)$$

is the identity map such that $T_3(\sum_{i_1} {}^{i}_{i_1}f' \circ {}^{i_1}_{i}f', \sum_{j_1} {}^{j}_{j_1}h' \circ {}^{j_1}_{j}h')$ and $T_4(\sum_{i_1} {}^{i}_{i_1}f \circ {}^{i_1}_{i}f, \sum_{j_1} {}^{j}_{j_1}h \circ {}^{j_1}_{j}h)$ are the identity maps of the corresponding algebra $T_3({}^iB, {}^jB)$ and module $T_4({}^iX, {}^jX)$, respectively.

This implies that the family of homomorphisms $\{T_4({}^{i}_{i_1}f, {}^{j_1}_{j}h), T_4({}^{i_1}_{i}f, {}^{j}_{j_1}h)\}$ provides a representation of the module $T_4({}^iX, {}^jX)$ as the direct sum. □

Corollary 1. *For each split, exact sequences*

$$0 \to {}^{i_1}X \to {}^{i}X \to {}^{i_2}X \to 0;$$
$$0 \to {}^{j_1}X \to {}^{j}X \to {}^{j_2}X \to 0$$

the sequences

$$0 \to T_4({}^{i_1}X, {}^jX) \to T_4({}^iX, {}^jX) \to T_4({}^{i_2}X, {}^jX) \to 0;$$
$$0 \to T_4({}^iX, {}^{j_1}X) \to T_4({}^iX, {}^jX) \to T_4({}^iX, {}^{j_2}X) \to 0$$

are also split and exact.

Definition 2. *Assume that ${}^1T$ and ${}^2T$ are two functors covariant in ${}^iM$ and contravariant in ${}^jM$ with ${}^1T_k = {}^2T_k$ for each $k \in \{1,2,3\}$. Assume also that there are homomorphisms $\mathbf{p}({}^iM, {}^jM) \in Hom_{l,{}^1T(w_{i,j})}({}^1T({}^iM, {}^jM), {}^2T({}^iM, {}^jM))$ such that, for each ${}^{i_1}_{i}\mathbf{f} \in Hom_{l,w_{i,i_1}}({}^iM, {}^{i_1}M)$ and ${}^{j}_{j_1}\mathbf{h} \in Hom_{l,w_{j_1,j}}({}^{j_1}M, {}^jM)$, there exists the commutative diagram*

$$\begin{array}{ccc} {}^1T({}^iM, {}^jM) & \xrightarrow[\mathbf{p}({}^iM,{}^jM)]{\hspace{6em}} & {}^2T({}^iM, {}^jM) \\ {}^1T({}^{i_1}_{i}\mathbf{f}, {}^{j}_{j_1}\mathbf{h}) \downarrow & & \downarrow {}^2T({}^{i_1}_{i}\mathbf{f}, {}^{j}_{j_1}\mathbf{h}) \\ {}^1T({}^{i_1}M, {}^{j_1}M) & \xrightarrow[\mathbf{p}({}^{i_1}M,{}^{j_1}M)]{\hspace{6em}} & {}^2T({}^{i_1}M, {}^{j_1}M), \end{array}$$

where $w_{i,i_1} = (v, \gamma_{i,i_1})$ for each i, i_1, where v is fixed.

Then, $\mathbf{p} : {}^1T \to {}^2T$ is called a natural v-map of the functor ${}^1T$ into the functor ${}^2T$. Moreover, if each map $\mathbf{p}({}^iM, {}^jM)$ is an isomorphism of ${}^1T({}^iM, {}^jM)$ onto ${}^2T({}^iM, {}^jM)$, then $\mathbf{p}$ is called a natural v-equivalence or v-isomorphism. If v is provided, it can be shortened to natural map (natural equivalence, natural isomorphism, respectively).

Example 1. *Let*

$$Ob({}_\mu\check{\mathcal{M}}) := \{{}^iM \in Ob({}_\mu\mathcal{M}) : \ \forall i \in {}_\mu\Omega, {}^iM = (G, B, {}^iX)\},$$

where G, $\mathcal{T}$ and B are fixed, with families of homomorphisms

$$\check{H}om_{l,1}({}^iX, {}^jX) = \{\pi \in Hom_{l,w_{i,j}}({}^iX, {}^jX) : \ \pi' = id_B\}.$$

A category with these restrictions will be denoted by ${}_\mu\check{\mathcal{M}}_1$. In this case, $\pi' = id_B$, so it can be omitted for shortening the notation, while $w_{i,j}$ corresponds to $s = \mathsf{e}$ and $\tau = 3$, since G, $\mathcal{T}$, B are fixed. Therefore, it is possible to consider $\check{H}om_{l,1}({}^iX, {}^jX)$ as an additive group such that

$$(\pi + \xi)(x + y) = (\pi(x) + \xi(x)) + (\pi(y) + \xi(y))$$

for each π and ξ in $\check{H}om_{l,1}({}^iX, {}^jX)$, x and y in ${}^iX$. Let $\xi \in \check{H}om_{l,1}({}^iX, {}^jX)$, where ${}^iM$ and ${}^jM$ belong to $Ob({}_\mu\check{\mathcal{M}})$. For each ${}^{i}_{i_1}f \in \check{H}om_{l,1}({}^{i_1}X, {}^iX)$ and ${}^{j_1}_{j}f \in \check{H}om_{l,1}({}^jX, {}^{j_1}X)$, let

$$\check{H}om_{l,1}({}^{i}_{i_1}f, {}^{j_1}_{j}f) : \check{H}om_{l,1}({}^iX, {}^jX) \to \check{H}om_{l,1}({}^{i_1}X, {}^{j_1}X) \textit{ be such that}$$
$$\check{H}om_{l,1}({}^{i}_{i_1}f, {}^{j_1}_{j}f)\xi = {}^{j_1}_{j}f \circ \xi \circ {}^{i}_{i_1}f.$$

Therefore, the pair $(\check{H}om_{l,1}({}^iX, {}^jX), \check{H}om_{l,1}({}^{i}_{i_1}f, {}^{j_1}_{j}f))$ composes an additive functor contravariant in ${}^iX$ and covariant in ${}^jX$ on ${}_\mu\check{\mathcal{M}}_1$.

If, for $Ob({}_\mu\check{\mathcal{M}})$, families of homomorphisms $\check{H}om_{l,w_{i,j}}({}^iX, {}^jX) = \{\pi \in Hom_{l,w_{i,j}}({}^iX, {}^jX) : \ \pi' : B \to B\}$ are considered, then it gives a category ${}^{s,\tau}_{\mu}\check{\mathcal{M}}$, where $s \in \{\mathsf{eg}, \mathsf{e}, \mathsf{g}\}$,

$\tau \in \{1,2,3\}$. Evidently, ${}_{\mu}\check{\mathcal{M}}_1$ is a subcategory in ${}^{s,\tau}_{\mu}\check{\mathcal{M}}$ and the latter is a subcategory in ${}^{s,\tau}_{\mu}\mathcal{M}$.

Example 2. *On the category* ${}^{s,\tau}_{\mu}\mathcal{M}$*, let* $Hom({}^{i}_{i_1}\mathbf{f}, {}^{j_1}_{j}\mathbf{f})\mathbf{h} = {}^{j_1}_{j}\mathbf{f} \circ \mathbf{h} \circ {}^{i}_{i_1}\mathbf{f}$ *for each* ${}^{i}_{i_1}\mathbf{f} \in Hom_{l,w_{i_1,i}}({}^{i_1}M, {}^{i}M)$ *and* ${}^{j_1}_{j}\mathbf{f} \in Hom_{l,w_{j,j_1}}({}^{j}M, {}^{j_1}M)$ *and* $\mathbf{h} \in Hom_{l,w_{i,j}}({}^{i}M, {}^{j}M)$*, where* ${}^{i}M$ *and* ${}^{j}M$ *are in* $Ob({}^{s,\tau}_{\mu}\mathcal{M})$*, where*

$Hom({}^{i}_{i_1}\mathbf{f}, {}^{j_1}_{j}\mathbf{f}) : Hom_{l,w_{i,j}}({}^{i}M, {}^{j}M) \to Hom_{l,w_{i_1,j_1}}({}^{i_1}M, {}^{j_1}M)$*. Notice that* $({}^{j_1}_{j}f \circ h \circ {}^{i}_{i_1}f)' = {}^{j_1}_{j}f' \circ h' \circ {}^{i}_{i_1}f'$*. Therefore,* $(Hom_{l,w_{i,j}}({}^{i}M, {}^{j}M), Hom({}^{i}_{i_1}\mathbf{f}, {}^{j_1}_{j}\mathbf{f}))$ *provides an additive functor contravariant in* ${}^{i}M$ *and covariant in* ${}^{j}M$ *on* ${}^{s,\tau}_{\mu}\mathcal{M}$.

Example 3. *Let* ${}_{\mu}\check{\mathcal{M}}_{\nu,1}$ *be a subcategory of* ${}^{v,\rho}_{\mu}\mathcal{M}_{\nu}$ *for fixed* G, $\mathcal{T}$ *and* B*, that is,* ${}^{i}G = G$, ${}^{i}\mathcal{T} = \mathcal{T}$, ${}^{i}B = B$ *for each* $i \in {}_{\mu}\Omega_{\nu}$*, with homomorphisms* $\check{H}om_1({}^{i}X, {}^{j}X) = \{\pi \in Hom_w({}^{i}X, {}^{j}X) : \pi' = id_B\}$*. A G-smashed tensor product* ${}^{i}X \bigotimes_B {}^{j}X$ *is provided by Definition 7 in [35] for each* ${}^{i}M$ *and* ${}^{j}M$ *in* $Ob({}_{\mu}\check{\mathcal{M}}_{\nu,1})$*. For any* ${}^{i_1}_{i}f \in \check{H}om_1({}^{i}X, {}^{i_1}X)$ *and* ${}^{j_1}_{j}h \in \check{H}om_1({}^{j}X, {}^{j_1}X)$*, it will be put* $({}^{i_1}_{i}f \otimes {}^{j_1}_{j}h)(x \otimes y) = ({}^{i_1}_{i}f(x)) \otimes ({}^{j_1}_{j}h(y))$ *for each* $x \in {}^{i}X$ *and* $y \in {}^{j}X$*. Therefore, there exists a functor* ${}^{\otimes}T$ *defined by* ${}^{\otimes}T({}^{i}M, {}^{j}M) = (G, B, {}^{i}X \bigotimes_B {}^{j}X)$ *and* ${}^{\otimes}T_4({}^{i_1}_{i}f, {}^{j_1}_{j}h) = {}^{i_1}_{i}f \otimes {}^{j_1}_{j}h$ *with* $T_3(id_B, id_B) = id_B$*. Hence, it satisfies*

$$
\begin{array}{l}
{}^{\otimes}T_4({}^{i_1}_{i}f + {}^{i_1}_{i}g, {}^{j}X) = {}^{\otimes}T_4({}^{i_1}_{i}f, {}^{j}X) + {}^{\otimes}T_4({}^{i_1}_{i}g, {}^{j}X) \text{ and} \\
{}^{\otimes}T_4({}^{i}X, {}^{j_1}_{j}h + {}^{j_1}_{j}q) = {}^{\otimes}T_4({}^{i}X, {}^{j_1}_{j}h) + {}^{\otimes}T_4({}^{i}X, {}^{j_1}_{j}q)
\end{array}
$$

for each ${}^{i_1}_{i}f$ *and* ${}^{i_1}_{i}g$ *in* $\check{H}om_1({}^{i}X, {}^{i_1}X)$*,* ${}^{j_1}_{j}h$ *and* ${}^{j_1}_{j}q$ *in* $\check{H}om_1({}^{j}X, {}^{j_1}X)$*. Thus,* ${}^{\otimes}T$ *is the covariant functor in two arguments.*

Definition 3. *Assume that* $T({}^{i}M, {}^{j}M)$ *is a functor covariant in* ${}^{i}M$ *and contravariant in* ${}^{j}M$*, where* ${}^{i}M$ *and* ${}^{j}M$ *belong to* $Ob({}^{s,\tau}_{\mu}\mathcal{M})$*. Assume that, for exact sequences,*

$$
\begin{array}{l}
{}^{i_1}M \to {}^{i}M \to {}^{i_2}M \text{ and} \\
{}^{j_1}M \to {}^{j}M \to {}^{j_2}M
\end{array}
$$

with left w-homomorphisms (i.e., ${}^{p}_{k}\mathbf{f} \in Hom_{l,w_{k,p}}({}^{k}M, {}^{p}M)$ *for each* p*,* k*, where* $w_{k,p} = (s, \gamma_{k,p})$*,* $s \in \{\mathsf{eg}, \mathsf{e}, \mathsf{g}\}$*), the sequences*

$$
\begin{array}{l}
T({}^{i_1}M, {}^{j}M) \to T({}^{i}M, {}^{j}M) \to T({}^{i_2}M, {}^{j}M) \text{ and} \\
T({}^{i}M, {}^{j_2}M) \to T({}^{i}M, {}^{j}M) \to T({}^{i}M, {}^{j_1}M)
\end{array}
$$

are also exact with left $T(w_{k,p}, {}^{j}M)$*- and* $T({}^{i}M, w_{k,p})$*-homomorphisms, respectively, for each* p *and* k*. Then, the functor* T *is called w-exact (or shortly exact).*

Proposition 2. *The functor* $T({}^{i}M, {}^{j}M)$ *covariant in* ${}^{i}M$ *and contravariant in* ${}^{j}M$ *is w-exact in the category* ${}^{s,3}_{\mu}\mathcal{M}$ *with* $w_{k,p} = (s, \gamma_{k,p})$*,* $s = \mathsf{eg}$ *for each corresponding* k *and* p*, if and only if, for each exact sequences,*

$$
\begin{array}{l}
{}^{i'_1}M_0 \to {}^{i_1}M \to {}^{i}M \to {}^{i_2}M \to {}^{i'_2}M_0 \text{ and} \\
{}^{j'_1}M_0 \to {}^{j_1}M \to {}^{j}M \to {}^{j_2}M \to {}^{j'_2}M_0
\end{array}
$$

with left $w_{k,p}$*-homomorphisms for each corresponding* k *and* p*, the following sequences*

$$
\begin{array}{l}
T({}^{i'_1}M_0, {}^{j}M) \to T({}^{i_1}M, {}^{j}M) \to T({}^{i}M, {}^{j}M) \to T({}^{i_2}M, {}^{j}M) \to T({}^{i'_2}M_0, {}^{j}M) \text{ and} \\
T({}^{i}M, {}^{j'_2}M_0) \to T({}^{i}M, {}^{j_2}M) \to T({}^{i}M, {}^{j}M) \to T({}^{i}M, {}^{j_1}M) \to T({}^{i}M, {}^{j'_1}M_0)
\end{array}
$$

are exact with left $T(w_{k,p}, {}^{j}M)$*- and* $T({}^{i}M, w_{k,p})$*-homomorphisms, respectively, for each corresponding* p *and* k*.*

Proof. From Definition 3, the necessity follows. For proving the sufficiency, we consider any exact sequence

$$^{i_1}M \xrightarrow[\ ^{i}_{i_1}\mathbf{f}\ ]{} {}^{i}M \xrightarrow[\ ^{i_2}_{i}\mathbf{f}\ ]{} {}^{i_2}M.$$

We put $^{i_1}N = Ker(^{i}_{i_1}f)$, $^{i}N = Ker(^{i_2}_{i}f)$, $^{i_2}N = {}^{i_2}_{i}f(^{i}X)$, where $^{i}M = (^{i}G, {}^{i}B, {}^{i}X)$, $^{i}K = (^{i}G, {}^{i}B, {}^{i}N)$. Since $s = \mathsf{eg}$, they induce exact sequences with $\gamma_{k,p}$-epigeneric homomorphisms (for the corresponding k, p)

$$0 \to {}^{i_1}N \to {}^{i_1}X \to {}^{i}N \to 0,$$
$$0 \to {}^{i}N \to {}^{i}X \to {}^{i_2}N \to 0 \textit{ and}$$
$$0 \to {}^{i_2}N \to {}^{i_2}X \to {}^{i_2}X/{}^{i_2}N \to 0,$$

where the quotient ^{i_2}G-graded left ^{i_2}B-module $^{i_2}X/{}^{i_2}N$ exists, since ^{i_2}X is the commutative group relative to the addition operation, while the homomorphism $^{i_2}_{i}f$ is γ_{i,i_2}-epigeneric, $\tau = 3$, such that $^{i_2}_{i}f'(^{i}B) = {}^{i_2}B$. By the conditions of this proposition, this implies the exactness of the sequences

$$T(^{i_1}M, {}^{j}M) \to T(^{i}K, {}^{j}M) \to T(^{i}K_0, {}^{j}M),$$
$$T(^{i'_2}K_0, {}^{j}M) \to T(^{i_2}K, {}^{j}M) \to T(^{i_2}M, {}^{j}M) \textit{ and}$$
$$T(^{i}K, {}^{j}M) \to T(^{i}M, {}^{j}M) \to T(^{i_2}K, {}^{j}M)$$

with the $T(\gamma_{k,p}, {}^{j}M)$-homomorphisms (for the corresponding k, p), since the functor T maps γ into $T(\gamma)$ by Definition 1.

From the exactness of these sequences, it follows that the sequence

$$T(^{i_1}M, {}^{j}M) \to T(^{i}M, {}^{j}M) \to T(^{i_2}M, {}^{j}M)$$

is exact with the $T(\gamma_{k,p}, {}^{j}M)$-homomorphisms (for the corresponding k, p). A similar proof is in the second argument ^{j}M. □

Definition 4. *Let*

$$^{i'_1}M_0 \to {}^{i_1}M \to {}^{i}M \to {}^{i_2}M \to {}^{i'_2}M_0 \textit{ and}$$
$$^{j'_1}M_0 \to {}^{j_1}M \to {}^{j}M \to {}^{j_2}M \to {}^{j'_2}M_0$$

be exact sequences with left $w_{k,p}$-homomorphisms for each corresponding k and p, where $s \in \{\mathsf{eg}, \mathsf{e}, \mathsf{g}\}$, $w_{k,p} = (s, \gamma_{k,p})$, ^{k}M and ^{p}M belong to $Ob(^{s,\tau}_{\mu}\mathcal{M})$. A functor T will be called w-half exact, if there are exact sequences

$$T(^{i_1}M, {}^{j}M) \to T(^{i}M, {}^{j}M) \to T(^{i_2}M, {}^{j}M) \textit{ and}$$
$$T(^{i}M, {}^{j_2}M) \to T(^{i}M, {}^{j}M) \to T(^{i}M, {}^{j_1}M)$$

with left $(s, T(\gamma_{k,p}, {}^{j}M))$- and $(s, T(^{i}M, \gamma_{k,p}))$-homomorphisms, respectively (for the corresponding k, p).

The functor T is called w-exact on the right if there exist exact sequences

$$T(^{i_1}M, {}^{j}M) \to T(^{i}M, {}^{j}M) \to T(^{i_2}M, {}^{j}M) \to T(^{i'_2}M_0, {}^{j}M) \textit{ and}$$
$$T(^{i}M, {}^{j_2}M) \to T(^{i}M, {}^{j}M) \to T(^{i}M, {}^{j_1}M) \to T(^{i}M, {}^{j'_1}M_0)$$

with left $(s, T(\gamma_{k,p}, {}^{j}M))$- and $(s, T(^{i}M, \gamma_{k,p}))$-homomorphisms, respectively (for the corresponding k, p).

Symmetrically, the functor T is called w-exact on the left, if there exist exact sequences

$$T(^{i'_1}M_0, {}^{j}M) \to T(^{i_1}M, {}^{j}M) \to T(^{i}M, {}^{j}M) \to T(^{i_2}M, {}^{j}M) \textit{ and}$$
$$T(^{i}M, {}^{j'_2}M_0) \to T(^{i}M, {}^{j_2}M) \to T(^{i}M, {}^{j}M) \to T(^{i}M, {}^{j_1}M)$$

with left $(s, T(\gamma_{k,p}, {}^{j}M))$*- and* $(s, T({}^{i}M, \gamma_{k,p}))$*-homomorphisms, respectively (for the corresponding k, p).*

Similar definitions are for the categories $\mathcal{M}_{\nu}^{s,\tau}$ *and* ${}_{\mu}^{v,\rho}\mathcal{M}_{\nu}$*.*

Proposition 3. *The following conditions are equivalent:*

(i) *the functor T is w-exact on the right, where* $w = (s, \gamma)$*;*

(ii) *for each exact sequence,*

$$
\begin{aligned}
&{}^{i_1}M \to {}^{i}M \to {}^{i_2}M \to {}^{i'_2}M_0 \text{ and} \\
&{}^{j'_1}M_0 \to {}^{j_1}M \to {}^{j}M \to {}^{j_2}M
\end{aligned}
$$

with left $w_{k,p}$*-homomorphisms (for each corresponding k and p) in* ${}_{\mu}^{s,\tau}\mathcal{M}$ *with* $s \in \{\mathtt{eg}, \mathtt{e}\}$ *there exist exact sequences*

$$T({}^{i_1}M, {}^{j}M) \to T({}^{i}M, {}^{j}M) \to T({}^{i_2}M, {}^{j}M) \to T({}^{i'_2}M_0, {}^{j}M) \text{ and}$$

$$T({}^{i}M, {}^{j_2}M) \to T({}^{i}M, {}^{j}M) \to T({}^{i}M, {}^{j_1}M) \to T({}^{i}M, {}^{j'_1}M_0)$$

with left $T(w_{k,p}, {}^{j}M)$*- and* $T({}^{i}M, w_{k,p})$*-homomorphisms, respectively, for each corresponding p and k;*

(iii) *moreover, in the subcategory* ${}_{\mu}\check{\mathcal{M}}_1$ *(ii) is equivalent to: for each exact sequence*

$${}^{i_1}X \to {}^{i}X \to {}^{i_2}X \to 0 \text{ and}$$

$$0 \to {}^{j_1}X \to {}^{j}X \to {}^{j_2}X$$

with left w-homomorphisms there exists the exact sequence

$$T_4({}^{i_1}X, {}^{j}X) \oplus T_4({}^{i}X, {}^{j_2}X) \xrightarrow[h]{} T_4({}^{i}X, {}^{j}X) \to T_4({}^{i_2}X, {}^{j_1}X) \to 0 \text{ with}$$

$$h = T_4({}_{i_1}^{i}f, {}^{j}X) \oplus T_4({}^{i}X, {}_{j}^{j_2}f).$$

Proof. $(i) \Rightarrow (ii)$. Let ${}^{i}N = Ker({}_{i_1}^{i}f)$, ${}^{i_1}N = Im({}_{i_1}^{i}f)$, where ${}_{i_1}^{i}\mathbf{f} \in Hom_{l,w_{i_1,i}}({}^{i_1}M, {}^{i}M)$, ${}^{i}M = ({}^{i}G, {}^{i}B, {}^{i}X)$, ${}^{i}K = ({}^{i}G, {}^{i}B, {}^{i}N)$. Hence, there exist exact sequences

$$0 \to {}^{i}N \to {}^{i_1}X \to {}^{i_1}N \to 0 \text{ and}$$

$$0 \to {}^{i_1}N \to {}^{i}X \to {}^{i_2}X \to 0$$

with $w_{k,p}$-homomorphisms for the corresponding k, p. Therefore, there are exact sequences

$$T({}^{i_1}M, {}^{j}M) \to T({}^{i_1}K, {}^{j}M) \to T({}^{i'_1}K_0, {}^{j}M) \text{ and}$$

$$T({}^{i_1}K, {}^{j}M) \to T({}^{i}M, {}^{j}M) \to T({}^{i_2}M, {}^{j}M) \to T({}^{i'_2}M_0, {}^{j}M)$$

with left $w_{k,p}$-homomorphisms for the corresponding k, p. This implies that the sequence

$$T({}^{i_1}M, {}^{j}M) \to T({}^{i}M, {}^{j}M) \to T({}^{i_2}M, {}^{j}M) \to T({}^{i'_2}M_0, {}^{j}M)$$

is exact with left $T(w_{k,p}, {}^{j}M)$-homomorphisms for the corresponding k, p. A similar proof is in the second argument.

$(ii) \Rightarrow (i)$. It is evident from Definition 4.

$(ii) \Rightarrow (iii)$ in the subcategory ${}_{\mu}\check{\mathcal{M}}_1$, where G, $\mathcal{T}$, B are fixed. Since in this case ${}^{i}M = (G, \mathcal{T}, B, {}^{i}X)$, we consider ${}^{i}X$. At first, we take the following commutative diagram with left w-homomorphisms and exact rows and columns:

$$\begin{array}{ccccc}
 & & {}^{i_1}X & \xrightarrow[{}^{j_3}_{i_1}f]{} & {}^{j_3}X \to 0 \\
 & & {}^{i}_{i_1}f \downarrow & & \downarrow {}^{j_2}_{j_3}f \\
{}^{i_2}X & \xrightarrow[{}^{i}_{i_2}f]{} & {}^{i}X & \xrightarrow[{}^{j_2}_{i}f]{} & {}^{j_2}X \to 0 \\
{}^{j_4}_{i_2}f \downarrow & & \downarrow {}^{j_1}_{i}f & & \downarrow {}^{j}_{j_2}f \\
{}^{j_4}X & \xrightarrow[{}^{j_1}_{j_4}f]{} & {}^{j_1}X & \xrightarrow[{}^{j}_{j_1}f]{} & {}^{j}X \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0.
\end{array}$$

This implies that the sequence

$$ {}^{i_1}X \oplus {}^{i_2}X \xrightarrow[{}^{i}_{i_1}f \oplus {}^{i}_{i_2}f]{} {}^{i}X \xrightarrow[{}^{j}_{i}f]{} {}^{j}X \to 0$$

is exact with left w-homomorphisms, since $({}^{i}_{i_1}f \oplus {}^{i}_{i_2}f)(x_1 + x_2) = {}^{i}_{i_1}f(x_1) \oplus {}^{i}_{i_2}f(x_2)$ for each $x_1 \in {}^{i_1}X$ and $x_2 \in {}^{i_2}X$, ${}^{j}_{j_1}f \circ {}^{j_1}_{i}f = {}^{j}_{j_2}f \circ {}^{j_2}_{i}f$, where ${}^{j}_{i}f = {}^{j}_{j_1}f \circ {}^{j_1}_{i}f$. Note that $Im({}^{i}_{i_1}f \oplus {}^{i}_{i_2}f) \subset Ker({}^{j}_{i}f)$.

Let $y \in Ker({}^{j}_{i}f)$; then, ${}^{j_1}_{i}f(y) \in Ker({}^{j}_{j_1}f) = Im({}^{j_1}_{j_4}f)$, hence there exists $x_4 \in {}^{j_4}X$ such that ${}^{j}_{j_4}f(x_4) = {}^{j_1}_{i}f(y)$. Notice that the w-homomorphism ${}^{j_4}_{i_2}f$ is epimorphic; consequently, there exists $x_2 \in {}^{i_2}X$ such that ${}^{j_4}_{i_2}f(x_2) = x_4$, hence ${}^{j_1}_{i}f({}^{i}_{i_2}f(x_2)) = {}^{j_1}_{j_4}f({}^{j_4}_{i_2}f(x_2)) = {}^{j_1}_{j_4}f(x_4)$. This implies that ${}^{i}_{i_2}f(x_2) - y \in Ker({}^{j_1}_{i}f)$, where $Ker({}^{j_1}_{i}f) = Im({}^{i}_{i_1}f)$. Therefore, there exists $x_1 \in {}^{i_1}X$ such that ${}^{i}_{i_2}f(x_2) - y = -{}^{i}_{i_1}f(x_1)$; consequently, $y = {}^{i}_{i_1}f(x_1) + {}^{i}_{i_2}f(x_2)$. Thus, $Ker({}^{j}_{i}f) \subset Im({}^{i}_{i_1}f \oplus {}^{i}_{i_2}f)$ and, consequently, $Ker({}^{j}_{i}f) = Im({}^{i}_{i_1}f \oplus {}^{i}_{i_2}f)$.

On the other hand, there is the commutative diagram with exact rows and columns and $T_4(w, {}^{p}X)$ and $T_4({}^{k}X, w)$ homomorphisms for the corresponding p and k:

$$\begin{array}{ccccc}
T_4({}^{i_1}X, {}^{j_2}X) & \xrightarrow[T_4({}^{i}_{i_1}f, {}^{j_2}X)]{} & T_4({}^{i}X, {}^{j_2}X) & \xrightarrow[T_4({}^{i_2}_{i}f, {}^{j_2}X)]{} & T_4({}^{i_2}X, {}^{j_2}X) \to 0 \\
T_4({}^{i_1}X, {}^{j_2}_{j}f) \downarrow & & T_4({}^{i}X, {}^{j_2}_{j}f) \downarrow & & T_4({}^{i_2}X, {}^{j_2}_{j}f) \downarrow \\
T_4({}^{i_1}X, {}^{j}X) & \xrightarrow[T_4({}^{i}_{i_1}f, {}^{j}X)]{} & T_4({}^{i}X, {}^{j}X) & \xrightarrow[T_4({}^{i_2}_{i}f, {}^{j}X)]{} & T_4({}^{i_2}X, {}^{j}X) \to 0 \\
T_4({}^{i_1}X, {}^{j}_{j_1}f) \downarrow & & T_4({}^{i}X, {}^{j}_{j_1}f) \downarrow & & T_4({}^{i_2}X, {}^{j}_{j_1}f) \downarrow \\
T_4({}^{i_1}X, {}^{j_1}X) & \xrightarrow[T_4({}^{i}_{i_1}f, {}^{j_1}X)]{} & T_4({}^{i}X, {}^{j_1}X) & \xrightarrow[T_4({}^{i_2}_{i}f, {}^{j_1}X)]{} & T_4({}^{i_2}X, {}^{j_1}X) \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0.
\end{array}$$

From the last three diagrams and the proof above, the implication $(ii) \Rightarrow (iii)$ in the subcategory ${}_{\mu}\check{\mathcal{M}}_1$ follows.

$(iii) \Rightarrow (ii)$ in the subcategory ${}_{\mu}\check{\mathcal{M}}_1$. Applying (iii) in two cases ${}^{j_2}X = 0$ and ${}^{j_1}X = {}^{j}X$, ${}^{i_1}X = 0$ and ${}^{i}X = {}^{i_2}X$, one gets (ii). □

Symmetrically to Proposition 3, the following proposition for functors w-exact on the left is formulated and proved.

Proposition 4. *The following conditions are equivalent:*

(i) *the functor T is w-exact on the left, where $w = (s, \gamma)$;*

(ii) *for each exact sequence*
${}^{i'_1}M_0 \to {}^{i_1}M \to {}^{i}M \to {}^{i_2}M$ *and* ${}^{j_1}M \to {}^{j}M \to {}^{j_2}M \to {}^{j'_2}M_0$
with left $w_{k,p}$-homomorphisms (for each corresponding k and p) in ${}^{s,\tau}_{\mu}\mathcal{M}$ with $s \in \{\mathsf{eg}, \mathsf{e}\}$ there exist exact sequences

$$T({}^{i_1'}M_0, {}^jM) \to T({}^{i_1}M, {}^jM) \to T({}^iM, {}^jM) \to T({}^{i_2}M, {}^jM) \text{ and}$$

$$T({}^iM, {}^{j_2'}M_0) \to T({}^iM, {}^{j_2}M) \to T({}^iM, {}^jM) \to T({}^iM, {}^{j_1}M)$$

with left $T(w_{k,p}, {}^jM)$- and $T({}^iM, w_{k,p})$-homomorphisms, respectively, for each corresponding p and k;

(iii) *moreover, in the subcategory ${}_\mu\check{\mathcal{M}}_1$ (ii) is equivalent to: for each exact sequence*

$$0 \to {}^{i_1}X \to {}^iX \to {}^{i_2}X \text{ and}$$
$${}^{j_1}X \to {}^jX \to {}^{j_2}X \to 0$$

with left w-homomorphisms, there exists the exact sequence

$$0 \to T_4({}^{i_1}X, {}^{j_2}X) \to T_4({}^iX, {}^jX) \xrightarrow[\mathbf{g}]{} T_4({}^{i_2}X, {}^jX) \oplus T_4({}^iX, {}^{j_1}X)$$

with $\mathbf{g} = T_4({}^{i_2}_{i}f, {}^jX) \oplus T_4({}^iX, {}^{j_1}_{j}f)$.

Proposition 5. *On the category ${}^{s,\tau}_{\mu}\mathcal{M}$ with $s \in \{\mathrm{eg}, \mathrm{e}\}$, the functor $Hom_{l,w}$ is exact on the left.*

Proof. Choose any exact sequence with left $w_{k,p}$-homomorphisms for each corresponding k and p:

$${}^{i_1'}M_0 \to {}^{i_1}M \xrightarrow[{}^{i}_{i_1}\mathbf{f}]{} {}^iM \xrightarrow[{}^{i_2}_{i}\mathbf{f}]{} {}^{i_2}M \to {}^{i_2'}M_0,$$

where ${}^{i_1}M$, ${}^iM$ and ${}^{i_2}M$ belong to

$$Ob({}^{s,\tau}_{\mu}\mathcal{M}).$$

This induces the sequence

$$Hom_{l,w_{i_2',j}}({}^{i_2'}M_0, {}^jM) \to Hom_{l,w_{i_2,j}}({}^{i_2}M, {}^jM) \xrightarrow[Hom({}^{i_2}_{i}\mathbf{f}, {}^jM)]{}$$
$$Hom_{l,w_{i,j}}({}^iM, {}^jM) \xrightarrow[Hom({}^{i}_{i_1}\mathbf{f}, {}^jM)]{} Hom_{l,w_{i_1,j}}({}^{i_1}M, {}^jM) \to Hom_{l,w_{i_1',j}}({}^{i_1'}M_0, {}^jM)$$

for each ${}^jM \in Ob({}^{s,\tau}_{\mu}\mathcal{M})$ (see Example 2). Therefore, $Hom({}^{i}_{i_1}f, {}^jX) \circ Hom({}^{i_2}_{i}f, {}^jX) = 0$; consequently, the homomorphism $Hom({}^{i_2}_{i}\mathbf{f}, {}^jM)$ induces a homomorphism $\mathbf{q} = (q', q)$ with

$$q : Hom_{l,w_{i_2,j}}({}^{i_2}X, {}^jX) \to Ker(Hom({}^{i}_{i_1}f, {}^jX)).$$

Let $\mathbf{h} = (h', h)$ be a homomorphism such that $h : Ker(Hom({}^{i}_{i_1}f, {}^jX)) \to Hom_{l,w_{i_2,j}}({}^{i_2}X, {}^jX)$ with $h({}^{j}_{i}f(x_2)) = {}^{j}_{i}f(x)$ for each ${}^{j}_{i}f \in Hom_{l,w_{i,j}}({}^iX, {}^jX) \cap Ker(Hom({}^{i}_{i_1}f, {}^jX))$ and for each $x_2 \in {}^{i_2}X$ with $x \in {}^iX$ satisfying ${}^{i_2}_{i}f(x) = x_2$, since ${}^{j}_{i}f \circ {}^{i}_{i_1}f = 0$ and ${}^{j}_{i}f(x)$ only depends on x_2. This implies that $q \circ h = id$ and $h \circ q = id$. Thus, q is the isomorphism. The exactness on the left in the second argument is proved similarly. □

Proposition 6. *In the subcategory ${}_\mu\check{\mathcal{M}}_1$, the functor $\bigotimes_B$ of the smashed G-graded tensor product over B is exact on the right.*

Proof. Take any exact sequence

$$0 \to {}^{i_1}X \xrightarrow[{}^{i}_{i_1}f]{} {}^iX \xrightarrow[{}^{i_2}_{i}f]{} {}^{i_2}X \to 0$$

with w-homomorphisms ${}^{i}_{i_1}f$ and ${}^{i_2}_{i}f$, where ${}^{i_1}M$, ${}^iM$, ${}^jM$ and ${}^{i_2}M$ belong to $Ob({}_\mu\check{\mathcal{M}}_1$. We consider the sequence

$${}^{i_1}X\bigotimes_B {}^jX \xrightarrow[h]{} {}^iX\bigotimes_B {}^jX \xrightarrow[q]{} {}^{i_2}X\bigotimes_B {}^jX \to 0,$$

where $h = {}^{i}_{i_1}f \bigotimes id_{{}^jX}$, $q = {}^{i_2}_{i}f \bigotimes id_{{}^jX}$. One gets that $q \circ h = 0$. Therefore, the homomorphism q induces a homomorphism $f : Coker(h) \to {}^{i_2}X \bigotimes_B {}^jX$. For each $x_2 \in {}^{i_2}X$ and $y \in {}^jX$, there exists $x \in {}^iX$ such that ${}^{i_2}_{i}f(x) = x_2$. Let $p(x_2, y)$ denote an image in $Coker(h)$ of the element $x \otimes y$. Evidently, $p(x_2, y)$ has the same value for all $x \in ({}^{i_2}_{i}f)^{-1}(x_2)$. This map $p(x_2, y)$ satisfies the following conditions:

$$p(x_{2,g_2}b_{g_1}, y_g) = t_3(g_2, g_1, g)p(x_{2,g_2}, b_{g_1}y_g),$$
$$p(cx_2, y) = cp(x_2, y) \text{ and } p(x_2, yc) = p(x_2, y)c$$

for each $c \in N(B)$, $x_2 \in {}^{i_2}X$, $y \in {}^jX$, g, g_1 and g_2 in G (see also Definition 7 in [35]). Let z be a homomorphism $z : {}^{i_2}X \bigotimes_B {}^jB \to Coker(h)$ such that $z(x_2 \otimes y) = p(x_2, y)$. This means that $z \circ f$ and $f \circ z$ are identities; consequently, f is the isomorphism. □

Definition 5. *Assume that X is a G-graded left B-module and for each G-graded left B-modules Y and ${}^1Y$ and homomorphisms $f : X \to {}^1Y$ and $g : Y \to {}^1Y$, where f and g are B-epigeneric, $g(Y) = {}^1Y$, there exists a homomorphism $h : X \to Y$ with $f = g \circ h$. Then, the module X is called projective.*

If, for the G-graded left B-module X, for each G-graded left B-modules Y and ${}^1Y$ with an injective B-epigeneric homomorphism $g : {}^1Y \to Y$, for each B-epigeneric homomorphism $f : {}^1Y \to X$, there exists a B-epigeneric homomorphism $h : Y \to X$ such that $g \circ h = f$; then, the G-graded left B-module X is called injective.

Proposition 7. *The G-graded left B-module X is projective (or injective) if and only if the functor $T({}^jM) = Hom_{l,w}(M, {}^jM)$ (or $Q({}^jM) = Hom_{l,w}({}^jM, M)$, respectively) is exact in the category ${}^{s,\tau}_{\mu}\check{\mathcal{M}}$ with $w = (s, \gamma)$, $s = \mathsf{e}$, $\gamma = (B, B)$, where $M = (G, B, X)$, ${}^jM = (G, B, {}^jX)$, $\tau = 3$.*

Proof. The functor T is exact on the left by Proposition 5. Therefore, it is exact, in the category ${}^{s,\tau}_{\mu}\check{\mathcal{M}}$ with $w = (s, \gamma)$, $s = \mathsf{e}$, $\gamma = (B, B)$, $\tau = 3$, if and only if, for each B-exact epimorphism ${}^{j_1}_{j}f : {}^jX \to {}^{j_1}X$, the map $Hom_{l,w}(X, {}^jX) \to Hom_{l,w}(X, {}^{j_1}X)$ is also a B-exact epimorphism.

In view of Proposition 3, the functor Q is exact on the right. Then, Q is exact if and only if for each injective B-epigeneric homomorphism ${}^{j}_{j_1}f : {}^{j_1}X \to {}^jX$ the map $Hom_{l,w}({}^jX, X) \to Hom_{l,w}({}^{j_1}X, X)$ is an injective B-epigeneric homomorphism. □

3. Satellites for Modules over Nonassociative Algebras with Metagroup Relations

Remark 2. *In the category ${}^{s,\tau}_{\mu}\check{\mathcal{M}}$ with $s \in \{\mathsf{eg}, \mathsf{e}\}$, let a diagram*

$$\begin{array}{ccccccccc} {}^{i'_1}M_0 & \to & {}^{i_1}M & \xrightarrow[{}^{i}_{i_1}\mathbf{f}]{} & {}^iM & \xrightarrow[{}^{i_2}_{i}\mathbf{f}]{} & {}^{i_2}M & \to & {}^{i'_2}M_0 \\ & & & & & & \downarrow {}^{j_2}_{i_2}\mathbf{f} & & \\ {}^{j'_1}M_0 & \to & {}^{j_1}M & \xrightarrow[{}^{j}_{j_1}\mathbf{f}]{} & {}^jM & \xrightarrow[{}^{j_2}_{j}\mathbf{f}]{} & {}^{j_2}M & \to & {}^{j'_2}M_0 \end{array}$$

be with exact rows in subcategories ${}_{\mu_1}\check{\mathcal{M}}_1$ for the upper row and ${}_{\mu_2}\check{\mathcal{M}}_1$ for the lower row with $\mu_1 \cup \mu_2 \subset \mu$ and a projective G-graded left B-module ${}^iX$, where ${}^{j_m}_{i_m}f' = {}^{j}_{i}f'$ for each m. That is, there is a diagram with exact rows

$$
\begin{array}{ccccccccc}
0 \to {}^{i_1}X & \xrightarrow[{}^{i}_{i_1}f]{} & {}^{i}X & \xrightarrow[{}^{i_2}_{i}f]{} & {}^{i_2}X \to 0 \\
 & & & & {}^{j_2}_{i_2}f \downarrow \\
0 \to {}^{j_1}X & \xrightarrow[{}^{j}_{j_1}f]{} & {}^{j}X & \xrightarrow[{}^{j_2}_{j}f]{} & {}^{j_2}X \to 0.
\end{array}
$$

This implies that there exists a homomorphism ${}^{j}_{i}f : {}^{i}X \to {}^{j}X$ *such that* ${}^{j_2}_{j}f \circ {}^{j}_{i}f = {}^{j_2}_{i_2}f \circ {}^{i_2}_{i}f$. *The homomorphism* ${}^{j}_{i}f$ *induces a homomorphism* ${}^{j_1}_{i_1}\mathbf{f} : {}^{i_1}M \to {}^{j_1}M$ *satisfying* ${}^{j}_{j_1}\mathbf{f} \circ {}^{j_1}_{i_1}\mathbf{f} = {}^{j}_{i}\mathbf{f} \circ {}^{i}_{i_1}\mathbf{f}$, *where* ${}^{m}_{n}\mathbf{f}$ *is the shortening of* $({}^{m}_{n}f', {}^{m}_{n}f)$ *for each* m, n.

Let T *be a covariant additive functor on* ${}^{s,\tau}_{\mu}\check{\mathcal{M}}$. *Then, the diagram*

$$
\begin{array}{ccc}
T({}^{i_1}M) & \xrightarrow[T({}^{i}_{i_1}\mathbf{f})]{} & T({}^{i}M) \\
T({}^{j_1}_{i_1}\mathbf{f}) \downarrow & & T({}^{j}_{i}\mathbf{f}) \downarrow \\
T({}^{j_1}M) & \xrightarrow[T({}^{j}_{j_1}\mathbf{f})]{} & T({}^{j}M)
\end{array}
$$

is commutative. That is, the diagrams

$$
\begin{array}{ccc}
T_4({}^{i_1}X) & \xrightarrow[T_4({}^{i}_{i_1}f)]{} & T_4({}^{i}X) \\
T_4({}^{j_1}_{i_1}f) \downarrow & & T_4({}^{j}_{i}f) \downarrow \\
T_4({}^{j_1}X) & \xrightarrow[T_4({}^{j}_{j_1}f)]{} & T_4({}^{j}X) \text{ and}
\end{array}
$$

$$
\begin{array}{ccc}
T_3({}^{i}B) & \xrightarrow[id_{T_3({}^{i}B)}]{} & T_3({}^{i}B) \\
T_3({}^{j}_{i}f') \downarrow & & T_3({}^{j}_{i}f') \downarrow \\
T_3({}^{j}B) & \xrightarrow[id_{T_3({}^{j}B)}]{} & T_3({}^{j}B)
\end{array}
$$

are commutative. Therefore, the homomorphism $T({}^{j_1}_{i_1}\mathbf{f})$ *induces a homomorphism denoted by* $\theta_1({}^{j_2}_{i_2}\mathbf{f})$ *from* ${}^{i_1}N := (T_1(G), T_3(B), Ker(T_4({}^{i}_{i_1}f)))$ *into* ${}^{j_1}N := (T_1(G), T_3(B), Ker(T_4({}^{j}_{j_1}f)))$ *such that* $\theta_1({}^{j_2}_{i_2}f) : Ker(T_4({}^{i}_{i_1}f)) \to Ker(T_4({}^{j}_{j_1}f))$, *where* $(\theta_1({}^{j_2}_{i_2}\mathbf{f}))' = \theta_1({}^{j_2}_{i_2}f')$.

If a functor T *is contravariant, then directions of all arrows change on inverse arrows in the latter diagram and there exists a homomorphism denoted by* $\theta^1({}^{j_2}_{i_2}\mathbf{f})$ *from* ${}^{j_1}K := (T_1(G), T_3(B), Coker(T_4({}^{j}_{j_1}f)))$ *into* ${}^{i_1}K := (T_1(G), T_3(B), Coker\ (T_4({}^{i}_{i_1}f)))$ *such that* $\theta^1({}^{j_2}_{i_2}f) : Coker(T_4({}^{j}_{j_1}f)) \to Coker(T_4({}^{i}_{i_1}f))$, *where* $(\theta^1({}^{j_2}_{i_2}\mathbf{f}))' = \theta^1({}^{j_2}_{i_2}f')$.

Proposition 8. *Assume that the conditions of Remark 2 are satisfied. Then, the homomorphisms* $\theta_1({}^{j_2}_{i_2}\mathbf{f})$ *and* $\theta^1({}^{j_2}_{i_2}\mathbf{f})$ *for the category* ${}^{s,\tau}_{\mu}\check{\mathcal{M}}$ *are independent of a choice of* ${}^{j}_{i}\mathbf{f}$ *satisfying*

$$ {}^{j_2}_{j}\mathbf{f} \circ {}^{j}_{i}\mathbf{f} = {}^{j_2}_{i_2}\mathbf{f} \circ {}^{i_2}_{i}\mathbf{f} \text{ such that} \tag{1} $$

$$ \theta_1({}^{j_2}_{i_2}f + {}^{j_2}_{i_2}\tilde{f}) = \theta_1({}^{j_2}_{i_2}f) + \theta_1({}^{j_2}_{i_2}\tilde{f}), \tag{2} $$

$$ \theta^1({}^{j_2}_{i_2}f + {}^{j_2}_{i_2}\tilde{f}) = \theta^1({}^{j_2}_{i_2}f) + \theta^1({}^{j_2}_{i_2}\tilde{f}) \tag{3} $$

are additive for homomorphisms of the corresponding modules ${}^kX$, ${}^nX$ *for each* ${}^n_k\mathbf{f}$ *and* ${}^n_k\tilde{\mathbf{f}}$ *in* $Hom_{l,w}({}^kM, {}^nM)$ *for the corresponding n, k. Moreover, for the following diagram:*

$$\begin{array}{ccccccccc}
{}^{i_1'}M_0 & \to & {}^{i_1}M & \xrightarrow[{}^{i}_{i_1}\mathbf{f}]{} & {}^{i}M & \xrightarrow[{}^{i_2}_{i}\mathbf{f}]{} & {}^{i_2}M & \to & {}^{i_2'}M_0 \\
 & & & & & & {}^{j_2}_{i_2}\mathbf{f}\downarrow & & \\
{}^{j_1'}M_0 & \to & {}^{j_1}M & \xrightarrow[{}^{j}_{j_1}\mathbf{f}]{} & {}^{j}M & \xrightarrow[{}^{j_2}_{j}\mathbf{f}]{} & {}^{j_2}M & \to & {}^{j_2'}M_0 \\
 & & & & & & {}^{p_2}_{j_2}\mathbf{f}\downarrow & & \\
{}^{p_1'}M_0 & \to & {}^{p_1}M & \xrightarrow[{}^{p}_{p_1}\mathbf{f}]{} & {}^{p}M & \xrightarrow[{}^{p_2}_{p}\mathbf{f}]{} & {}^{p_2}M & \to & {}^{p_2'}M_0
\end{array} \tag{4}$$

with exact rows in the subcategories ${}_{\mu_k}\check{\mathcal{M}}_1$ *with* $\mu_1 \cup \mu_2 \cup \mu_3 \subset \mu$, $k \in \{1,2,3\}$ *correspondingly, with* ${}^{j_m}_{i_m}f' = {}^{j}_{i}f'$ *and* ${}^{p_m}_{j_m}f' = {}^{p}_{j}f'$ *for each m and projective G-graded left B-modules* ${}^iX$ *and* ${}^jX$, θ_1 *and* θ^1 *are transitive:*

$$\theta_1({}^{j_3}_{j_2}\mathbf{f} \circ {}^{j_2}_{i_2}\mathbf{f}) = \theta_1({}^{j_3}_{j_2}\mathbf{f}) \circ \theta_1({}^{j_2}_{i_2}\mathbf{f}), \tag{5}$$

$$\theta^1({}^{j_3}_{j_2}\mathbf{f} \circ {}^{j_2}_{i_2}\mathbf{f}) = \theta^1({}^{j_2}_{i_2}\mathbf{f}) \circ \theta^1({}^{j_3}_{j_2}\mathbf{f}). \tag{6}$$

Proof. The first diagram in Remark 2 has the exact lower row. Therefore, for ${}^j_i\mathbf{f}$ and ${}^j_i\mathbf{h}$ in $Hom_{l,w}({}^iM, {}^jM)$ such that ${}^{j_2}_{j}\mathbf{f} \circ {}^{j}_{i}\mathbf{h} = {}^{j_2}_{i_2}\mathbf{f} \circ {}^{i_2}_{i}\mathbf{f}$ and ${}^j_i\mathbf{f}$, satisfying the conditions of this proposition, one gets ${}^j_ih = {}^j_if + {}^{j}_{j_1}f \circ {}^{j_1}_{i}g$, where ${}^{j_1}_{i}\mathbf{g} \in Hom_{l,w}({}^iM, {}^{j_1}M)$. For ${}^{j_1}_{i_1}\mathbf{f}$ and ${}^{j_1}_{i_1}\mathbf{h}$ in $Hom_{l,w}({}^{i_1}M, {}^{j_1}M)$ such that ${}^{j}_{j_1}\mathbf{f} \circ {}^{j_1}_{i_1}\mathbf{f} = {}^{j}_{i}\mathbf{f} \circ {}^{i}_{i_1}\mathbf{f}$ and ${}^{j}_{j_1}\mathbf{f} \circ {}^{j_1}_{i_1}\mathbf{h} = {}^{j}_{i}\mathbf{h} \circ {}^{i}_{i_1}\mathbf{f}$, we infer that ${}^{j_1}_{i_1}h = {}^{j_1}_{i_1}f + {}^{j_1}_{i}g \circ {}^{i}_{i_1}f$. Therefore, $T_4({}^{j_1}_{i_1}h) = T_4({}^{j_1}_{i_1}f) + T_4({}^{j_1}_{i}g) \circ T_4({}^{i}_{i_1}f)$; consequently, $T_4({}^{j_1}_{i_1}h)x = T_4({}^{j_1}_{i_1}f)x$ for each $x \in Ker({}^{i}_{i_1}f)$. This implies that the homomorphism $\theta_1({}^{j_2}_{i_2}\mathbf{f})$ is the same for all ${}^j_i\mathbf{f}$ satisfying Condition (1).

If the functor T is contravariant, then $T_4({}^{j_1}_{i_1}h) = T_4({}^{j_1}_{i_1}f) + T_4({}^{i}_{i_1}f) \circ T_4({}^{j_1}_{i}g)$; consequently, $Im(T_4({}^{j_1}_{i_1}h) - T_4({}^{j_1}_{i_1}f)) \subset Im(T_4({}^{i}_{i_1}f))$. Thus, the homomorphism $\theta^1({}^{j_2}_{i_2}\mathbf{f})$ is the same for all ${}^j_i\mathbf{f}$ satisfying Condition (1).

Similarly for ${}^{j_2}_{i_2}\mathbf{f}$ and ${}^j_i\tilde{\mathbf{f}}$ satisfying the condition similar to (1)

$${}^{j_2}_{j}\mathbf{f} \circ {}^{j}_{i}\tilde{\mathbf{f}} = {}^{j_2}_{i_2}\tilde{\mathbf{f}} \circ {}^{i_2}_{i}\mathbf{f}, \tag{7}$$

where ${}^n_k\tilde{\mathbf{f}} \in Hom_{l,w}({}^kM, {}^nM)$ for the corresponding n, k, we deduce that, for each ${}^j_i\tilde{\mathbf{h}} \in Hom_{l,w}({}^iM, {}^jM)$ such that

$${}^{j_2}_{j}\mathbf{f} \circ {}^{j}_{i}\tilde{\mathbf{h}} = {}^{j_2}_{i_2}\tilde{\mathbf{f}} \circ {}^{i_2}_{i}\mathbf{f}, \tag{8}$$

there exists ${}^{j_1}_{i}\tilde{\mathbf{g}} \in Hom_{l,w}({}^iM, {}^{j_1}M)$ such that ${}^j_i\tilde{\mathbf{h}} = {}^j_i\mathbf{f} + {}^{j}_{j_1}\mathbf{f} \circ {}^{j_1}_{i}\tilde{\mathbf{g}}$. From the proof above, it follows that the homomorphism $\theta_1({}^{j_2}_{i_2}\tilde{\mathbf{f}})$ exists, and it is the same for all ${}^j_i\mathbf{f}$ satisfying Condition (7). From ${}^{j}_{j_1}f \circ ({}^{j_1}_{i}g + {}^{j_1}_{i}\tilde{g}) = {}^{j}_{j_1}f \circ {}^{j_1}_{i}g + {}^{j}_{j_1}f \circ {}^{j_1}_{i}\tilde{g}$ and $({}^{j_1}_{i}g + {}^{j_1}_{i}\tilde{g}) \circ {}^{i}_{i_1}f = {}^{j_1}_{i}g \circ {}^{i}_{i_1}f + {}^{j_1}_{i}\tilde{g} \circ {}^{i}_{i_1}f$ Formulas (2) and (3) follow.

Formulas (5) and (6) are obtained by the iteration of the proof above for ${}^{j_3}_{j_2}\mathbf{f}$ and ${}^{j_3}_{j_2}\mathbf{f} \circ {}^{j_2}_{i_2}\mathbf{f}$. □

Remark 3. *Take now the following diagram for the category* ${}^{s,\tau}_{\mu}\check{\mathcal{M}}$ *with* $s \in \{\mathsf{eg}, \mathsf{e}\}$

$$\begin{array}{c} {}^{j_1'}M_0 \to {}^{j_1}M \xrightarrow[{}^{j}_{j_1}\mathbf{f}]{} {}^{j}M \xrightarrow[{}^{j_2}_{j}\mathbf{f}]{} {}^{j_2}M \to {}^{j_2'}M_0 \\ {}^{i_1}_{j_1}\mathbf{f}\downarrow \\ {}^{i_1'}M_0 \to {}^{i_1}M \xrightarrow[{}^{i}_{i_1}\mathbf{f}]{} {}^{i}M \xrightarrow[{}^{i_2}_{i}\mathbf{f}]{} {}^{i_2}M \to {}^{i_2'}M_0 \end{array} \tag{9}$$

with exact rows in the subcategories ${}_{\mu_1}\check{\mathcal{M}}_1$ *for the upper row and* ${}_{\mu_2}\check{\mathcal{M}}_1$ *for the lower row with* $\mu_1 \cup \mu_2 \subset \mu$ *and an injective G-graded left B-module* ${}^{i}X$*, where* ${}^{i_m}_{j_m}f' = {}^{i}_{j}f'$ *for each m. Therefore, a homomorphism* ${}^{i}_{j}\mathbf{f} \in Hom_{l,w}({}^{j}M, {}^{i}M)$ *exists such that*

$$ {}^{i}_{j}\mathbf{f} \circ {}^{j}_{j_1}\mathbf{f} = {}^{i}_{i_1}\mathbf{f} \circ {}^{i_1}_{j_1}\mathbf{f}. \tag{10}$$

This induces a homomorphism ${}^{i_2}_{j_2}\mathbf{f} \in Hom_{l,w}({}^{j}M, {}^{i}M)$ *such that*

$$ {}^{i_2}_{j_2}\mathbf{f} \circ {}^{j_2}_{j}\mathbf{f} = {}^{j_2}_{i}\mathbf{f} \circ {}^{i}_{j}\mathbf{f}. \tag{11}$$

For a covariant functor T on ${}^{s,\tau}_{\mu}\check{\mathcal{M}}$ *the diagram*

$$\begin{array}{ccc} T({}^{j}M) & \xrightarrow[T({}^{j_2}_{j}\mathbf{f})]{} & T({}^{j_2}M) \\ T({}^{i}_{j}\mathbf{f})\downarrow & & T({}^{i_2}_{j_2}\mathbf{f})\downarrow \\ T({}^{i}M) & \xrightarrow[T({}^{i_2}_{i}\mathbf{f})]{} & T({}^{i_2}M) \end{array} \tag{12}$$

is commutative and implies an existence of a homomorphism

$$\theta^1({}^{i_1}_{j_1}f) : Coker(T_4({}^{j_2}_{j}f)) \to Coker(T_4({}^{i_2}_{i}f)) \text{ and } (\theta^1({}^{i_1}_{j_1}\mathbf{f}))' = \theta^1({}^{i_1}_{j_1}f'), \tag{13}$$

Since $T_3({}^{j_2}_{j}f') = id_{T_3({}^{j}B)}$, $T_3({}^{i_2}_{i}f') = id_{T_3({}^{i}B)}$, $T_3({}^{i}_{j}f') = T_3({}^{i_2}_{j_2}f')$.

For a contravariant functor T, directions of all arrows in the latter diagram are inverse, and it induces a homomorphism

$$\theta_1({}^{i_1}_{j_1}f) : Ker(T_4({}^{i_2}_{i}f)) \to Ker(T_4({}^{j_2}_{j}f)) \text{ and } (\theta_1({}^{i_1}_{j_1}\mathbf{f}))' = \theta_1({}^{i_1}_{j_1}f'). \tag{14}$$

Symmetrically to Proposition 8, one gets the following:

Proposition 9. *Let the conditions of Remark 3 be satisfied. Then, the homomorphisms* $\theta^1({}^{i_1}_{j_1}\mathbf{f})$ *and* $\theta_1({}^{i_1}_{j_1}\mathbf{f})$ *for the category* ${}^{s,\tau}_{\mu}\check{\mathcal{M}}$ *are independent of a choice of* ${}^{i}_{j}\mathbf{f}$ *satisfying Conditions* (10) *and* (11) *in Remark 3 such that* θ^1 *and* θ_1 *are additive:*

$$\theta^1({}^{i_1}_{j_1}f + {}^{i_1}_{j_1}\tilde{f}) = \theta^1({}^{i_1}_{j_1}f) + \theta^1({}^{i_1}_{j_1}\tilde{f}), \tag{15}$$

$$\theta_1({}^{i_1}_{j_1}f + {}^{i_1}_{j_1}\tilde{f}) = \theta_1({}^{i_1}_{j_1}f) + \theta_1({}^{i_1}_{j_1}\tilde{f}) \tag{16}$$

for G-graded left B-modules ${}^{k}X$, ${}^{n}X$ *for each* ${}^{n}_{k}\mathbf{f}$ *and* ${}^{n}_{k}\tilde{\mathbf{f}}$ *in* $Hom_{l,w}({}^{k}M, {}^{n}M)$ *for the corresponding n, k. Moreover, for the following diagram:*

$$\begin{array}{l}
{}^{p'_1}M_0 \to {}^{p_1}M \xrightarrow[{}^{p}_{p_1}\mathbf{f}]{} {}^{p}M \xrightarrow[{}^{p_2}_{p}\mathbf{f}]{} {}^{p_2}M \to {}^{p'_2}M_0 \\
\qquad\quad {}^{j_1}_{p_1}\mathbf{f}\downarrow \\
{}^{j'_1}M_0 \to {}^{j_1}M \xrightarrow[{}^{j}_{j_1}\mathbf{f}]{} {}^{j}M \xrightarrow[{}^{j_2}_{j}\mathbf{f}]{} {}^{j_2}M \to {}^{j'_2}M_0 \\
\qquad\quad {}^{i_1}_{j_1}\mathbf{f}\downarrow \\
{}^{i'_1}M_0 \to {}^{i_1}M \xrightarrow[{}^{i}_{i_1}\mathbf{f}]{} {}^{i}M \xrightarrow[{}^{i_2}_{i}\mathbf{f}]{} {}^{i_2}M \to {}^{i'_2}M_0
\end{array} \tag{17}$$

with exact rows in the subcategories ${}_{\mu_k}\check{\mathcal{M}}_1$ *with* $\mu_1 \cup \mu_2 \cup \mu_3 \subset \mu$, $k \in \{1,2,3\}$ *correspondingly, with* ${}^{j_m}_{p_m}f' = {}^{j}_{p}f'$ *and* ${}^{i_m}_{j_m}f' = {}^{i}_{j}f'$ *for each m and injective G-graded left B-modules* ${}^{j}X$ *and* ${}^{i}X$, θ^1 *and* θ_1 *are transitive:*

$$\theta^1({}^{i_1}_{j_1}\mathbf{f} \circ {}^{j_1}_{p_1}\mathbf{f}) = \theta^1({}^{i_1}_{j_1}\mathbf{f}) \circ \theta^1({}^{j_1}_{p_1}\mathbf{f}), \tag{18}$$

$$\theta_1({}^{i_1}_{j_1}\mathbf{f} \circ {}^{j_1}_{p_1}\mathbf{f}) = \theta_1({}^{j_1}_{p_1}\mathbf{f}) \circ \theta_1({}^{i_1}_{j_1}\mathbf{f}). \tag{19}$$

Definition 6. *Let*

$${}^{i'_1}M_0 \to {}^{i_1}M \xrightarrow[{}^{i}_{i_1}\mathbf{f}]{} {}^{i}M \xrightarrow[{}^{i_2}_{i}\mathbf{f}]{} {}^{i_2}M \to {}^{i'_2}M_0 \text{ and} \tag{20}$$

$${}^{j'_1}M_0 \to {}^{j_1}M \xrightarrow[{}^{j}_{j_1}\mathbf{f}]{} {}^{j}M \xrightarrow[{}^{j_2}_{j}\mathbf{f}]{} {}^{j_2}M \to {}^{j'_2}M_0 \tag{21}$$

be two exact sequences in the subcategory ${}_{\mu}\check{\mathcal{M}}_1$*, where* ${}^{i}X$ *is the projective G-graded left B-module, and* ${}^{j}X$ *is the injective G-graded left B-module (see also Remarks 2 and 3). For a covariant additive functor T, let*

$$S_1T({}^{i_2}M) = (T_1(G), T_3(B), Ker(T_4({}^{i}_{i_1}f))) \text{ and} \tag{22}$$

$$S^1T({}^{j_1}M) = (T_1(G), T_3(B), \ Coker(T_4({}^{j_2}_{j}f))) \text{ with the ring } T_2(\mathcal{T}). \tag{23}$$

Lemma 1. *If there are exact sequences* (20) *and* (21) *as in Definition 6 and*

$${}^{l'_1}M_0 \to {}^{l_1}M \xrightarrow[{}^{l}_{l_1}\mathbf{f}]{} {}^{l}M \xrightarrow[{}^{i_2}_{l}\mathbf{f}]{} {}^{i_2}M \to {}^{i'_2}M_0 \text{ and} \tag{24}$$

$${}^{j'_1}M_0 \to {}^{j_1}M \xrightarrow[{}^{n}_{j_1}\mathbf{f}]{} {}^{n}M \xrightarrow[{}^{n_2}_{n}\mathbf{f}]{} {}^{n_2}M \to {}^{n'_2}M_0 \tag{25}$$

in the category ${}_{\mu}\check{\mathcal{M}}_1$*, where* ${}^{l}X$ *is the projective G-graded left B-module,* ${}^{n}M$ *is the injective G-graded left B-module, then* $Ker(T_4({}^{i}_{i_1}f))$ *and* $Ker(T_4({}^{l}_{l_1}f))$ *are isomorphic, also* $Coker(T_4({}^{j_2}_{j}f))$ *and* $Coker(T_4({}^{n_2}_{n}f))$ *are isomorphic.*

Proof. Definition 6 implies that there are exact sequences

$$S_1T({}^{i'_2}M_0) \to S_1T({}^{i_2}M) \to T({}^{i_1}M) \to T({}^{i}M) \text{ and} \tag{26}$$

$$T({}^{j}M) \to T({}^{j_2}M) \to S^1T({}^{j_1}M) \to S^1T({}^{j'_1}M_0) \text{ and} \tag{27}$$

$$\bar{S}_1T({}^{i'_2}M_0) \to \bar{S}_1T({}^{i_2}M) \to T({}^{l_1}M) \to T({}^{l}M) \text{ and} \tag{28}$$

$$T({}^{n}M) \to T({}^{n_2}M) \to \bar{S}^1T({}^{j_1}M) \to \bar{S}^1T({}^{j'_1}M_0), \tag{29}$$

where $\bar{S}_1T({}^{i_2}M) = (T_1(G), T_3(B), T_4({}^{l}_{l_1}f)))$, $\bar{S}^1T({}^{j_1}M) = (T_1(G), T_3(B), Coker(T_4({}^{n_2}_{n}f)))$ with the ring $T_2(\mathcal{T})$. Therefore, (26)–(29) induce homomorphisms $\theta_1(id_{{}^{i_2}M}) : S_1T({}^{i_2}M) \to \bar{S}_1T({}^{i_2}M)$ and $\theta_1(id_{{}^{i_2}M}) : \bar{S}_1T({}^{i_2}M) \to S_1T({}^{i_2}M)$, also $\theta^1(id_{{}^{j_1}M}) : S^1T({}^{j_1}M) \to \bar{S}^1T({}^{j_1}M)$ and $\theta^1(id_{{}^{j_1}M}) : \bar{S}^1T({}^{j_1}M) \to S^1T({}^{j_1}M)$. In view of Propositions 8 and 9, the G-graded left B-modules $Ker(T_4({}^{i}_{i_1}f))$ and $Ker(T_4({}^{l}_{l_1}f))$ are isomorphic; also, $Coker(T_4({}^{j_2}_{j}f))$ and $Coker(T_4({}^{n_2}_{n}f))$ are isomorphic. □

Definition 7. *Let ${}^{i_2}M$ and ${}^{l_2}M$ belong to the category ${}^{s,\tau}_{\mu}\check{\mathcal{M}}$ with $s \in \{\mathsf{eg}, \mathsf{e}\}$, ${}^{l_2}_{i_2}\mathbf{f} \in Hom_{l,w}({}^{i_2}M, {}^{l_2}M)$ and let T be a covariant additive functor, where $s \in \{\mathsf{eg}, \mathsf{e}\}$. The homomorphisms $\theta_1({}^{l_2}_{i_2}\mathbf{f})$ and $\theta^1({}^{l_2}_{i_2}\mathbf{f})$ define homomorphisms*

$$S_1T({}^{l_2}_{i_2}\mathbf{f}) : S_1T({}^{i_2}M) \to S_1T({}^{l_2}M) \text{ and} \tag{30}$$

$$S^1T({}^{l_2}_{i_2}\mathbf{f}) : S^1T({}^{i_2}M) \to S^1T({}^{l_2}M). \tag{31}$$

The functor S_1T (or S^1T) is called a left (right correspondingly) satellite of the functor T. Then, by induction, the satellites of higher order are defined:

$$S_{n+1}T = S_1(S_nT),\ S_0T = T, \tag{32}$$

$$S^{n+1}T = S^1(S^nT), S^0T = T. \tag{33}$$

It is put that $S_nT = S^{-n}T$ for each $n \in \mathbf{Z}$.

Remark 4. *In view of Propositions 7 and 8, the left and right satellites S_nT and S^nT are covariant additive functors on the category ${}^{s,\tau}_{\mu}\check{\mathcal{M}}$. For the contravariant additive functor T, we get that*

$$S_1T({}^{i_2}M) = (T_1(G), T_3(B), Ker(T_4({}^{l_2}_{l}f))), \tag{34}$$

$$S^1T({}^{i_2}M) = (T_1(G), T_3(B), Coker(T_4({}^{i}_{i_1}f))), \tag{35}$$

$$S_1T({}^{i'_2}M_0) \to S_1T({}^{i_2}M) \to T({}^{l_2}M) \to T({}^{l}M), \tag{36}$$

$$T({}^{i}M) \to T({}^{i_1}M) \to S^1T({}^{i_2}M) \to S^1T({}^{i'_2}M_0), \tag{37}$$

$$S_1T({}^{j_2}_{i_2}\mathbf{f}) : S_1T({}^{j_2}M) \to S_1T({}^{i_2}M), \tag{38}$$

$$S^1T({}^{j_2}_{i_2}\mathbf{f}) : S^1T({}^{j_2}M) \to S^1T({}^{i_2}M) \tag{39}$$

analogous to Remarks 2 and 3 and Lemma 1 with the ring $T_2(\mathcal{T})$. Therefore, S_nT and S^nT also are contravariant additive functors.

Corollary 2. *If the additive functor T is exact on the right, then $S^nT_4 = 0$ for each $n > 0$. If the additive functor T is exact on the left, then $S^nT_4 = 0$ for each $n < 0$. If the additive functor T is exact, then $S^nT_4 = 0$ for each $n \neq 0$.*

Proposition 10. *Assume that the functor T is additive and covariant (or contravariant). If the G-graded left B-module ${}^{i_2}X$ is projective (or injective correspondingly), then $S^nT_4({}^{i_2}X) = 0$ for each $n < 0$. If ${}^{i_2}X$ is injective (or projective correspondingly), then $S^nT_4({}^{i_2}X) = 0$ for each $n > 0$.*

Proof. If the G-graded left B-module ${}^{i_2}X$ is projective, then we put ${}^{i}M = {}^{i_2}M$ and ${}^{i_1}X = 0$ in the exact sequence

$${}^{i'_1}M_0 \to {}^{i_1}M_0 \to {}^{i}M \to {}^{i_2}M \to {}^{i'_2}M_0$$

in the subcategory ${}_{\mu_1}\check{\mathcal{M}}_1$ with $\mu_1 \subset \mu$ and $i_2 \in {}_{\mu_1}\Omega$. If the G-graded left B-module ${}^{i_2}X$ is injective, then we put ${}^{j}M = {}^{i_2}M$ and ${}^{j_2}X = 0$ in the exact sequence

$${}^{i'_2}M_0 \to {}^{i_2}M \to {}^{j}M \to {}^{j_2}M_0 \to {}^{j'_2}M_0$$

in the subcategory ${}_{\mu_1}\check{\mathcal{M}}_1$ with $\mu_1 \subset \mu$.

Then, the assertions of this proposition follow from Proposition 9 and Lemma 1. □

Proposition 11. *Let*

$$ {}^{i_1'}M_0 \to {}^{i_1}M \to {}^{i}M \to {}^{i_2}M \to {}^{i_2'}M_0 \text{ and} $$
$$ {}^{i_2'}M_0 \to {}^{i_2}M \to {}^{j}M \to {}^{j_2}M \to {}^{j_2'}M_0 $$

be exact sequences in the category ${}_{\mu}\check{\mathcal{M}}_1$ *with a projective G-graded left B-module* ${}^{i}X$ *and an injective left B-module* ${}^{j}X$*. If the functor T is covariant (or contravariant), then*

$$ S_{n+1}T({}^{i_2}M) = S_nT({}^{i_1}M), $$
$$ S^{n+1}T({}^{i_2}M) = S^nT({}^{j_2}M), $$
$$ (\text{or } S_{n+1}T({}^{i_2}M) = S_nT({}^{j_2}M), $$
$$ S^{n+1}T({}^{i_2}M) = S^nT({}^{i_1}M) \text{ correspondingly}). $$

Proof. This follows from Proposition 10 using exact sequences (36) and (37) in Remark 4, (26) and (27) in Lemma 1. □

Theorem 1. *Assume that a diagram*

$$ \begin{array}{ccccccccc} {}^{i_1'}M_0 \to & {}^{i_1}M & \xrightarrow[{}^{i}_{i_1}\mathbf{f}]{} & {}^{i}M & \xrightarrow[{}^{i_2}_{i}\mathbf{f}]{} & {}^{i_2}M & \to {}^{i_2'}M_0 \\ & {}^{j_1}_{i_1}\mathbf{f}\downarrow & & {}^{j}_{i}\mathbf{f}\downarrow & & {}^{j_2}_{i_2}\mathbf{f}\downarrow & \\ {}^{j_1'}M_0 \to & {}^{j_1}M & \xrightarrow[{}^{j}_{j_1}\mathbf{f}]{} & {}^{j}M & \xrightarrow[{}^{j_2}_{j}\mathbf{f}]{} & {}^{j_2}M & \to {}^{j_2'}M_0 \end{array} \tag{40} $$

is commutative with exact rows in the category ${}^{s,\tau}_{\mu}\check{\mathcal{M}}$*, where* $s \in \{\mathsf{eg}, \mathsf{e}\}$*. If a functor T is additive and covariant (or contravariant), then there exists a commutative diagram*

$$ \begin{array}{ccccccccccc} \to S^{n-1}T({}^{i_2}M) & \longrightarrow & S^nT({}^{i_1}M) & \longrightarrow & S^nT({}^{i}M) & \longrightarrow & S^nT({}^{i_2}M) & \longrightarrow & S^{n+1}T({}^{i_1}M) \to \\ \downarrow S^{n-1}T({}^{j_2}_{i_2}\mathbf{f}) & & \downarrow S^nT({}^{j_1}_{i_1}\mathbf{f}) & & \downarrow S^nT({}^{j}_{i}\mathbf{f}) & & \downarrow S^nT({}^{j_2}_{i_2}\mathbf{f}) & & \downarrow S^{n+1}T({}^{j_1}_{i_1}\mathbf{f}) \\ \to S^{n-1}T({}^{j_2}M) & \longrightarrow & S^nT({}^{j_1}M) & \longrightarrow & S^nT({}^{j}M) & \longrightarrow & S^nT({}^{j_2}M) & \longrightarrow & S^{n+1}T({}^{j_1}M) \to \end{array} \tag{41} $$

(or

$$ \begin{array}{ccccccccccc} \leftarrow S_{n-1}T({}^{i_2}M) & \longleftarrow & S_nT({}^{i_1}M) & \longleftarrow & S_nT({}^{i}M) & \longleftarrow & S_nT({}^{i_2}M) & \longleftarrow & S_{n+1}T({}^{i_1}M) \leftarrow \\ \uparrow S_{n-1}T({}^{j_2}_{i_2}\mathbf{f}) & & \uparrow S_nT({}^{j_1}_{i_1}\mathbf{f}) & & \uparrow S_nT({}^{j}_{i}\mathbf{f}) & & \uparrow S_nT({}^{j_2}_{i_2}\mathbf{f}) & & \uparrow S_{n+1}T({}^{j_1}_{i_1}\mathbf{f}) \\ \leftarrow S_{n-1}T({}^{j_2}M) & \longleftarrow & S_nT({}^{j_1}M) & \longleftarrow & S_nT({}^{j}M) & \longleftarrow & S_nT({}^{j_2}M) & \longleftarrow & S_{n+1}T({}^{j_1}M) \leftarrow \end{array} \tag{42} $$

correspondingly).

Proof. We consider an exact sequence

$$ {}^{l_1'}M_0 \to {}^{l_1}M \to {}^{l}M \to {}^{i_2}M \to {}^{i_2'}M_0 \tag{43} $$

with a projective G-graded left B-module ${}^{l}X$ in the subcategory ${}_{\mu_1}\check{\mathcal{M}}_1$ with $\mu_1 \subset \mu$ and $i_2 \in \mu_1$. Therefore, there exists a diagram

$$ \begin{array}{ccccccccc} {}^{l_1'}M_0 \to & {}^{l_1}M & \xrightarrow[{}^{l}_{l_1}\mathbf{f}]{} & {}^{l}M & \xrightarrow[{}^{i_2}_{l}\mathbf{f}]{} & {}^{i_2}M & \to {}^{i_2'}M_0 \\ & & & & & {}^{i_2}_{i_2}\mathbf{f}\downarrow & \\ {}^{i_1'}M_0 \to & {}^{i_1}M & \xrightarrow[{}^{i}_{i_1}\mathbf{f}]{} & {}^{i}M & \xrightarrow[{}^{i_2}_{i}\mathbf{f}]{} & {}^{i_2}M & \to {}^{i_2'}M_0, \end{array} \tag{44} $$

where ${}^{i_2}_{i_2}\mathbf{f} = id_{{}^{i_2}M}$. Note that, if ${}^{j}_{i}\mathbf{f} \in Hom_{l,w_{i,j}}({}^{i}M, {}^{j}M)$, then $Ker({}^{j}_{i}f) = \{x \in {}^{i}X : {}^{j}_{i}f(x) = 0\}$; consequently, ${}^{j}_{i}f(x+y) = {}^{j}_{i}f(x) + {}^{j}_{i}f(y) = 0$ and ${}^{j}_{i}f(bx) = {}^{j}_{i}f'(b){}^{j}_{i}f(x) = 0$ for each x and y in $Ker({}^{j}_{i}f)$, $b \in {}^{i}B$, hence $Ker({}^{j}_{i}f)$ is a ${}^{i}G$-graded left ${}^{i}B$ submodule in ${}^{i}X$. On the other hand, Definitions 6, 7, and Lemma 1 imply that $S^nT_3({}^{j}_{i}f') = T_3({}^{j}_{i}f')$ and $S_nT_3({}^{j}_{i}f') = T_3({}^{j}_{i}f')$ for each $n \in \mathbf{Z}$, where $S^nT_3({}^{j}_{i}f') = (S^nT({}^{j}_{i}\mathbf{f}))'$. In view of Proposition 8, the homomorphism exists as follows:

$$\theta_1({}^{i_2}_{i_2}f) : Ker(T_4({}^{l}_{l_1}f)) \to Ker(T_4({}^{i}_{i_1}f)).$$

The latter homomorphism induces a homomorphism ${}^{i_1}_{i_2}\mathbf{p}_1 : S_1T({}^{i_2}M) \to T({}^{i_1}M)$ such that $T_4({}^{i}_{i_1}f) \circ {}^{i_1}_{i_2}p_1 = 0$, where ${}^{i_1}_{i_2}\mathbf{p}_1 = ({}^{i_1}_{i_2}p'_1, {}^{i_1}_{i_2}p_1)$, ${}^{i_1}_{i_2}p'_1 = T({}^{i_1}_{i_2}f')$, ${}^{i_1}_{i_2}p'_1 : T_3({}^{i_2}B) \to T_3({}^{i_1}B)$, ${}^{i_1}_{i_2}p_1 : {}^{i_2}P_1 \to {}^{i_1}Y$, $S_1T({}^{i_2}M) = (T_1({}^{i_2}G), T_3({}^{i_2}B), {}^{i_2}P_1)$; ${}^{i_1}Y = T_4({}^{i_1}X)$ for the ring $T_2(\mathcal{T})$. Then, we consider an exact sequence

$$0 \to {}^{i_1}X \xrightarrow[{}^{k}_{i_1}f]{} {}^{k}X \xrightarrow[{}^{k_2}_{k}f]{} {}^{k_2}X \to 0, \tag{45}$$

where ${}^{k}X$ and ${}^{k_2}X$ are ${}^{i_1}G$-graded left ${}^{i_1}B$-modules, where ${}^{k}X$ is injective. Therefore, $\theta^1({}^{i_1}_{i_1}\mathbf{f})$ with ${}^{i_1}_{i_1}\mathbf{f} = id_{{}^{i_1}M}$ induces a homomorphism

${}^{i_1}_{i_2}\mathbf{p}^1 : T({}^{i_2}M) \to S^1T({}^{i_1}M)$ such that ${}^{i_1}_{i_2}p^1 \circ T_4({}^{i_2}_{i}f) = 0$, where ${}^{i_1}_{i_2}\mathbf{p}^1 = ({}^{i_1}_{i_2}p'^1, {}^{i_1}_{i_2}p^1)$, ${}^{i_1}_{i_2}p^1 : {}^{i_2}Y \to {}^{i_1}Q$, ${}^{i_1}_{i_2}p'^1 : T_3({}^{i_2}B) \to T_3({}^{i_1}B)$, where $S^1T({}^{i_1}M) = (T_1({}^{i_1}G), T_3({}^{i_1}B), {}^{i_1}Q)$, ${}^{i_2}Y = T_4({}^{i_2}X)$. By virtue of Propositions 8 and 9, the homomorphisms $\mathbf{p}^1$ and $\mathbf{p}_1$ are independent of choices of auxiliary sequences (43), (45) satisfying the conditions imposed above.

Iterating this procedure in n, we infer that there exists an infinite exact sequence

$$\cdots \to S^{n-1}T({}^{i_2}M) \xrightarrow[{}^{i_1}_{i_2}\mathbf{p}^n]{} S^nT({}^{i_1}M) \xrightarrow[S^nT({}^{i}_{i_1}\mathbf{f})]{} S^nT({}^{i}M)$$
$$\xrightarrow[S^nT({}^{i_2}_{i}\mathbf{f})]{} S^nT({}^{i_2}M) \xrightarrow[{}^{i_1}_{i_2}\mathbf{p}^{n+1}]{} S^{n+1}T({}^{i_1}M) \to \ldots,$$

where $n \in \mathbf{Z}$. It remains to prove that diagram (41) is commutative in squares containing $\mathbf{p}^n$. Take any exact sequence

$$^{p'_1}M_0 \to {}^{p_1}M \to {}^{p}M \to {}^{j_2}M \to {}^{j'_2}M_0 \tag{46}$$

with a projective ${}^{j_2}G$-graded left ${}^{j_2}B$-module ${}^{p}X$, in the subcategory ${}_{\mu_2}\check{\mathcal{M}}_1$ with $\mu_2 \subset \mu$ and $j_2 \in \mu_2$, where ${}^{k}G = {}^{j_2}G$ and ${}^{k}B = {}^{j_2}B$ for each $k \in \{p'_1, p_1, p, j'_2\}$. Using (40), (43), we choose a diagram

$$\begin{array}{c} {}^{l'_1}M_0 \to {}^{l_1}M \xrightarrow[{}^{l}_{l_1}\mathbf{f}]{} {}^{l}M \xrightarrow[{}^{i_2}_{l}\mathbf{f}]{} {}^{i_2}M \to {}^{i'_2}M_0 \\ \qquad\qquad\qquad\qquad {}^{j_2}_{i_2}\mathbf{f}\downarrow \\ {}^{j'_1}M_0 \to {}^{j_1}M \xrightarrow[{}^{j}_{j_1}\mathbf{f}]{} {}^{j}M \xrightarrow[{}^{j_2}_{j}\mathbf{f}]{} {}^{j_2}M \to {}^{j'_2}M_0, \end{array} \tag{47}$$

where ${}^{k}G = {}^{i_2}G$ and ${}^{k}B = {}^{i_2}B$ for each $k \in \{l'_1, l_1, l, i'_2\}$. To diagram (47), there corresponds a homomorphism from $S_1T({}^{i_2}M)$ to $T({}^{j_1}M)$ such that it is the composition of homomorphisms from $S_1T({}^{i_2}M)$ into $T({}^{i_1}M)$ and from $T({}^{i_1}M)$ into $T({}^{j_1}M)$. From (40), (43), and (46), it follows that there exists a diagram

$$\begin{array}{c} {}^{l'_1}M_0 \to {}^{l_1}M \xrightarrow[{}^{l}_{l_1}\mathbf{f}]{} {}^{l}M \xrightarrow[{}^{i_2}_{l}\mathbf{f}]{} {}^{i_2}M \to {}^{i'_2}M_0 \\ \qquad\qquad\qquad\qquad {}^{j_1}_{i_2}\mathbf{f}\downarrow \end{array} \tag{48}$$

$$\begin{array}{c}
{}^{p'_1}M_0 \to {}^{p_1}M \xrightarrow[{}^{p}_{p_1}\mathbf{f}]{} {}^{p}M \xrightarrow[{}^{j_1}_{p}\mathbf{f}]{} {}^{j_1}M \to {}^{j'_1}M_0 \\
\qquad\qquad\qquad {}^{j_2}_{j_1}\mathbf{f}\downarrow \\
{}^{j'_1}M_0 \to {}^{j_1}M \xrightarrow[{}^{j}_{j_1}\mathbf{f}]{} {}^{j}M \xrightarrow[{}^{j_2}_{j}\mathbf{f}]{} {}^{j_2}M \to {}^{j'_2}M_0.
\end{array}$$

Applying Proposition 8, we infer that the homomorphism from $S_1T({}^{i_2}M)$ to $T({}^{j_1}M)$ is the composition of homomorphisms from $S_1T({}^{i_2}M)$ into $S_1T({}^{j_2}M)$ and from $S_1T({}^{j_2}M)$ into $T({}^{j_1}M)$.

For the contravariant functor, we deduce that homomorphisms ${}^{i_2}_{i_1}\mathbf{q}_1 : S_1T({}^{i_1}M) \to T({}^{i_2}M)$ and ${}^{i_2}_{i_1}\mathbf{q}^1 : T({}^{i_1}M) \to S^1T({}^{i_2}M)$ exist. The rest of the proof is similar. □

Theorem 2. *Assume that a sequence*

$$ {}^{i_1}M_0 \to {}^{i_1}M \xrightarrow[{}^{i}_{i_1}\mathbf{f}]{} {}^{i}M \xrightarrow[{}^{i_2}_{i}\mathbf{f}]{} {}^{i_2}M \to {}^{i_2}M_0 \tag{49}$$

is exact in the category ${}^{s,\tau}_{\mu}\check{\mathcal{M}}$*, where* $s \in \{\mathsf{eg}, \mathsf{e}\}$*. Assume also that the additive functor T is covariant. Then, there exists an infinite sequence*

$$\begin{array}{c}
\cdots \to S^{n-1}T({}^{i_2}M) \xrightarrow[{}^{i_1}_{i_2}\mathbf{p}^n]{} S^nT({}^{i_1}M) \xrightarrow[S^nT({}^{i}_{i_1}\mathbf{f})]{} S^nT({}^{i}M) \\
\to S^nT({}^{i_2}M) \to S^{n+1}T({}^{i_1}M) \to \dots,
\end{array} \tag{50}$$

where $n \in \mathbf{Z}$*, such that* $S^nT_4({}^{i}_{i_1}f) \circ {}^{i_1}_{i_2}p^n = 0$ *and* $S^nT_4({}^{i_2}_{i}f) \circ S^nT_4({}^{i}_{i_1}f) = 0$ *and* ${}^{i_1}_{i_2}p^{n+1} \circ S^nT_4({}^{i_2}_{i}f) = 0$*.*

Proof. For the sequence

$$ {}^{i_1}X \xrightarrow[{}^{i}_{i_1}f]{} {}^{i}X \xrightarrow[{}^{i_2}_{i}f]{} {}^{i_2}X,$$

the equation ${}^{i_2}_{i}f \circ {}^{i}_{i_1}f = 0$ is satisfied; consequently, for the sequence

$$S^nT({}^{i_1}M) \to S^nT({}^{i}M) \to S^nT({}^{i_2}M),$$

one gets that $S^nT_4({}^{i_2}_{i}f) \circ S^nT_4({}^{i}_{i_1}f) = 0$. For the sequence

$$S^nT({}^{i}M) \xrightarrow[S^nT({}^{i_2}_{i}\mathbf{f})]{} S^nT({}^{i_2}M) \xrightarrow[{}^{i_1}_{i_2}\mathbf{p}^{n+1}]{} S^{n+1}T({}^{i_1}M),$$

we get that ${}^{i_1}_{i_2}p^{n+1} \circ S^nT_4({}^{i_2}_{i}f) = 0$ for each $n \geq 0$, where ${}^{i_1}_{i_2}\mathbf{p}^{n+1} = ({}^{i_1}_{i_2}p'^{n+1}, {}^{i_1}_{i_2}p^{n+1})$. Consider now the case $n < 0$. This variant using iterations with S_1, S_n can be reduced to

$$S_1T({}^{i}M) \xrightarrow[S_1T({}^{i_2}_{i}\mathbf{f})]{} S_1T({}^{i_2}M) \xrightarrow[{}^{i_1}_{i_2}\mathbf{p}_1]{} T({}^{i_1}M).$$

The homomorphism ${}^{i_1}_{i_2}\mathbf{p}_1 \circ S_1T({}^{i_2}_{i}\mathbf{f})$ is induced from the diagram

$$\begin{array}{c}
{}^{l_2}M_0 \to {}^{l_2}M \xrightarrow[{}^{l_1}_{l_2}\mathbf{f}]{} {}^{l_1}M \xrightarrow[{}^{i}_{l_1}\mathbf{f}]{} {}^{i}M \to {}^{i}M_0 \\
\qquad\qquad\qquad {}^{i_2}_{i}\mathbf{f}\downarrow \\
{}^{j_2}M_0 \to {}^{j_2}M \xrightarrow[{}^{j_1}_{j_2}\mathbf{f}]{} {}^{j_1}M \xrightarrow[{}^{i_2}_{j_1}\mathbf{f}]{} {}^{i_2}M \to {}^{i_2}M_0 \\
\qquad\qquad\qquad {}^{i_2}_{i_2}\mathbf{f}\downarrow \\
{}^{i_1}M_0 \to {}^{i_1}M \xrightarrow[{}^{i}_{i_1}\mathbf{f}]{} {}^{i}M \xrightarrow[{}^{i_2}_{i}\mathbf{f}]{} {}^{i_2}M \to {}^{i_2}M_0
\end{array}$$

with the exact upper and middle rows (horizontal lines) in the subcategories ${}_{\mu_k}\check{\mathcal{M}}_1$ such that $\mu_1 \cup \mu_2 \subset \mu$, $k \in \{1,2\}$ correspondingly, with a projective ${}^iG$-graded left ${}^iB$-module ${}^{l_1}X$ and a projective ${}^{i_2}G$-graded left ${}^{i_2}B$-module ${}^{j_1}X$, where ${}^{l_2}X$ is a ${}^iG$-graded left ${}^iB$-module, ${}^{j_2}X$ is a ${}^{i_2}G$-graded left ${}^{i_2}B$-module. Therefore, for the homomorphism $\theta_1({}^{i_2}_i\mathbf{f})$ corresponding to the diagram

$$\begin{array}{ccccccc} {}^{l_2}M_0 \to {}^{l_2}M & \xrightarrow[{}^{l_1}_{l_2}\mathbf{f}]{} & {}^{l_1}M & \xrightarrow[{}^{i}_{l_1}\mathbf{f}]{} & {}^iM \to {}^iM_0 \\ & & {}^{i}_{l_1}\mathbf{f}\downarrow & & {}^{i_2}_{i}\mathbf{f}\downarrow & \\ {}^{i_1}M_0 \to {}^{i_1}M & \xrightarrow[{}^{i}_{i_1}\mathbf{f}]{} & {}^iM & \xrightarrow[{}^{i_2}_{i}\mathbf{f}]{} & {}^{i_2}M \to {}^{i_2}M_0 \end{array}$$

we infer that $(\theta_1({}^{i_2}_i\mathbf{f}))_4 = 0$, since the induced homomorphism ${}^{i_1}_{l_2}\mathbf{f} : {}^{l_2}M \to {}^{i_1}M$ is such that ${}^{i_1}_{l_2}f = 0$. Similarly for the sequence

$$S^nT({}^{i_2}M) \xrightarrow[{}^{i_1}_{i_2}\mathbf{p}^{n+1}]{} S^{n+1}T({}^{i_1}M) \xrightarrow[S^{n+1}T({}^{i}_{i_1}\mathbf{f})]{} S^{n+1}T({}^iM),$$

the equality is satisfied $S^{n+1}T_4({}^{i}_{i_1}f) \circ {}^{i_1}_{i_2}p^{n+1} = 0$. □

Theorem 3. *Assume that there exists an exact sequence*

$${}^{i_1}M_0 \to {}^{i_1}M \xrightarrow[{}^{i}_{i_1}\mathbf{f}]{} {}^iM \xrightarrow[{}^{i_2}_{i}\mathbf{f}]{} {}^{i_2}M \to {}^{i_2}M_0 \tag{51}$$

in the category ${}^{s,\tau}_{\mu}\check{\mathcal{M}}$ *with* $s \in \{\mathtt{eg}, \mathtt{e}\}$. *If* T *is an additive covariant (or contravariant) half-exact functor, then there exists an exact sequence*

$$\cdots \to S^{n-1}T({}^{i_2}M) \xrightarrow[{}^{i_1}_{i_2}\mathbf{p}^n]{} S^nT({}^{i_1}M) \xrightarrow[S^nT({}^{i}_{i_1}\mathbf{f})]{} S^nT({}^iM) \tag{52}$$

$$\to S^nT({}^{i_2}M) \to S^{n+1}T({}^{i_1}M) \to \dots, \ (or$$

$$\cdots \to S^{n-1}T({}^{i_1}M) \xrightarrow[{}^{i_1}_{i_2}\mathbf{p}^n]{} S^nT({}^{i_2}M) \xrightarrow[S^nT({}^{i_2}_{i}\mathbf{f})]{} S^nT({}^iM) \tag{53}$$

$$\to S^nT({}^{i_1}M) \to S^{n+1}T({}^{i_2}M) \to \dots$$

correspondingly).

Proof. In view of Proposition A2 in [34] and the conditions of this theorem, we infer that there exists a commutative diagram

$$\begin{array}{ccccc} T({}^{i_1}M) & \xrightarrow[T({}^{i}_{i_1}\mathbf{f})]{} & T({}^iM) & \xrightarrow[T({}^{i_2}_{i}\mathbf{f})]{} & T({}^{i_2}M) \\ T({}^{l_1}_{i_1}\mathbf{f})\downarrow & & T({}^{l}_{i}\mathbf{f})\downarrow & & T({}^{l_2}_{i_2}\mathbf{f})\downarrow \\ (T({}^{l_1}M))_0 \to T({}^{l_1}M) & \xrightarrow[T({}^{l}_{l_1}\mathbf{f})]{} & T({}^lM) & \xrightarrow[T({}^{l_2}_{l}\mathbf{f})]{} & T({}^{l_2}M) \end{array}$$

with exact rows, since the functor T is half-exact. By virtue of Lemma A1 in [34], the sequence

$$Ker(T_4({}^{l_1}_{i_1}f)) \to Ker(T_4({}^{l}_{i}f)) \to Ker(T_4({}^{l_2}_{i_2}f))$$

is exact. That is, the sequence

$$S_1T({}^{i_1}M) \to S_1T({}^iM) \to S_1T({}^{i_2}M)$$

is exact. Consider now an exact sequence

$${}^{l_1}M_0 \to {}^{l_1}M \to {}^lM \to {}^iM \to {}^iM_0$$

in the subcategory ${}_{\mu_1}\check{\mathcal{M}}_1$ with $\mu_1 \subset \mu$ and $i \in \mu_1$, so that ${}^{k}G = {}^{i}G$ and ${}^{k}B = {}^{i}B$ for each $k \in \{l, l_1\}$, where the ${}^{i}G$-graded left ${}^{i}B$-module ${}^{l}X$ is projective. Put $Y = Ker({}^{i_2}_{i}f \circ {}^{i}_{l}f)$; hence, there exist exact sequences

$$\begin{array}{l} 0 \to Y \to {}^{l}X \to {}^{i_2}X \to 0 \textit{ and} \\ 0 \to {}^{i}X \to Y \to {}^{i_1}X \to 0 \end{array}$$

and the commutative diagram

$$\begin{array}{ccccccc} & & T({}^{i}M) & \xrightarrow[{}^{l}_{i}\mathbf{h}]{} & T(K) & \xrightarrow[{}^{i_1}_{l}\mathbf{g}]{} & T({}^{i_1}M) \\ & & T({}^{l}_{i}\mathbf{f}) \downarrow & & {}^{l}_{l}\mathbf{g} \downarrow & & \downarrow \\ (T({}^{l}M))_0 & \to & T({}^{l}M) & \xrightarrow[T({}^{l}_{l}\mathbf{f})]{} & T({}^{l}M) & \xrightarrow[(T_3({}^{i_1}_{l}f'),0)]{} & (T({}^{i_1}M))_0 \end{array}$$

with $K = ({}^{i}G, {}^{i}B, Y)$ and exact rows, ${}^{i}K = K$, ${}^{i}Y = Y$. Therefore, there exists the exact sequence

$$Ker(T_4({}^{l}_{i}f)) \to Ker({}^{l}_{l}g) \to T_4({}^{i_1}X)$$

by Lemma A1 in [34]. On the other hand, the homomorphism from $Ker({}^{l}_{l}g)$ into $T_4({}^{i_1}X)$ coincides with the homomorphism from $S_1T_4({}^{i_2}X)$ into $T_4({}^{i_1}X)$. Therefore, the sequence

$$S_1T({}^{i}M) \to S_1T({}^{i_2}M) \to T({}^{i_1}M)$$

is exact.

We consider an exact sequence

$${}^{l_1}M_0 \to {}^{l_1}M \to {}^{l}M \to {}^{i_2}M \to {}^{i_2}M_0, \tag{54}$$

where ${}^{l}X$ is the projective ${}^{i}G$-graded left ${}^{i}B$-module. Consider a ${}^{i}G$-graded left ${}^{i}B$-submodule Y of ${}^{i}X \oplus {}^{l}X$ such that, for each $(x_i, x_l) \in Y$ with $x_i \in {}^{i}X$ and $x_l \in {}^{l}X$, the equality ${}^{i_2}_{l}f(x_l) = {}^{i_2}_{i}f(x_i)$ is satisfied. Certainly, there are homomorphisms ${}_{i}\pi : Y \to {}^{i}X$ and ${}_{l}\pi : Y \to {}^{l}X$ induced by the maps $(x_i, x_l) \mapsto x_i$ and $(x_i, x_l) \mapsto x_l$. Note that there are else homomorphisms: ${}_{i_1}\rho : {}^{i_1}X \to Y$ and ${}_{l_1}\rho : {}^{l_1}X \to Y$ induced by the maps $x_{i_1} \mapsto ({}^{i}_{i_1}f(x_{i_1}), 0)$ and $x_{l_1} \mapsto (0, {}^{l}_{l_1}f(x_{l_1}))$ for each $x_{i_1} \in {}^{i_1}X$ and $x_{l_1} \in {}^{l_1}X$. Therefore, there exists a commutative diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & 0 & \longrightarrow & {}^{l_1}X & \longrightarrow & {}^{l_1}X & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & {}^{i_1}X & \longrightarrow & {}^{i}Y & \longrightarrow & {}^{l}X & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & {}^{i_1}X & \longrightarrow & {}^{i}X & \longrightarrow & {}^{i_2}X & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

with exact rows and columns, and with ${}^{i}K = ({}^{i}G, {}^{i}B, {}^{i}Y)$, where the sequence

$$0 \to {}^{i_1}X \to Y \to {}^{l}X \to 0$$

splits, since the ${}^{i}G$-graded left ${}^{i}B$-module ${}^{l}X$ is projective. Therefore, there exists a commutative diagram

$$\begin{array}{ccccccc} & & (T({}^{l_1}M))_0 & \to & T({}^{l_1}M) & \xrightarrow[T({}^{l_1}_{l_1}\mathbf{f})]{} & T({}^{l_1}M) \to (T({}^{l_1}M))_0 \\ & & \downarrow & & \downarrow & & \downarrow \\ (T({}^{i_1}M))_0 & \to & T({}^{i_1}M) & \longrightarrow & T({}^{i}K) & \longrightarrow & T({}^{l}M) \end{array}$$

with exact rows. From Proposition A2 in [34], it follows that the sequence

$$Ker(T_4({}^{l}_{l_1}f)) \xrightarrow[\xi]{} T_4({}^{i_1}X) \xrightarrow[\eta]{} Coker(T_4({}_{l_1}\rho))$$

is exact. Then, we infer that $Ker(\eta) = Ker(T_4({}^{i}_{i_1}f))$, since the sequence $T_4({}^{l_1}X) \to T_4(Y) \to T_4({}^{i}X)$ is exact. Thus, the sequence

$$Ker(T_4({}^{l}_{l_1}f)) \xrightarrow[\xi]{} T_4({}^{i_1}X) \to T_4({}^{i}X)$$

is exact. Then, we prove that $\xi = {}^{i_1}_{i_2}q$. Assume that sequences (51) and (53) are identical. In this case, there exists an embedding homomorphism

$$\tilde{\xi} : Ker(T_4({}^{l}_{l_1}f)) \to T_4({}^{l_1}X).$$

In general, we consider the following commutative diagram:

$$\begin{array}{ccccccccc} {}^{l_1}M_0 & \to & {}^{l_1}M & \xrightarrow[{}^{l}_{l_1}\mathbf{f}]{} & {}^{l}M & \xrightarrow[{}^{i_2}_{l}\mathbf{f}]{} & {}^{i_2}M & \to & {}^{i_2}M_0 \\ & & {}^{i_1}_{l_1}\mathbf{f}\downarrow & & {}^{i}_{l}\mathbf{f}\downarrow & & {}^{i_2}_{i_2}\mathbf{f}\downarrow & & \\ {}^{i_1}M_0 & \to & {}^{i_1}M & \xrightarrow[{}^{i}_{i_1}\mathbf{f}]{} & {}^{i}M & \xrightarrow[{}^{i_2}_{i}\mathbf{f}]{} & {}^{i_2}M & \to & {}^{i_2}M_0 \end{array}$$

with ${}^{i_2}_{i_2}\mathbf{f} = id_{{}^{i_2}M}$. This induces the commutative diagram:

$$\begin{array}{ccc} Ker(T_4({}^{l}_{l_1}f)) & \xrightarrow[\tilde{\xi}]{} & T_4({}^{l_1}X) \\ id\downarrow & & T_4({}^{i_1}_{l_1}f)\downarrow \\ Ker(T_4({}^{l_1}_{l_1}f)) & \xrightarrow[\xi]{} & T({}^{i_1}M). \end{array}$$

This implies that $\xi = T_4({}^{i_1}_{l_1}f) \circ \tilde{\xi} = {}^{i_1}_{i_2}q_1$, hence the sequence

$$S_1T({}^{i_2}M) \to T({}^{i_1}M) \to T({}^{i}M)$$

is exact, where, as usual, $\circ$ denotes the composition of maps. This implies that the sequence

$$S_1T({}^{i_1}M) \to S_1T({}^{i}M) \to S_1T({}^{i_2}M) \to T({}^{i_1}M) \to T({}^{i}M) \to T({}^{i_2}M)$$

is exact. By the dual proof to the above, one gets that the sequence

$$T({}^{i_1}M) \to T({}^{i}M) \to T({}^{i_2}M) \to S^1T({}^{i_1}M) \to S^1T({}^{i}M) \to S^1T({}^{i_2}M)$$

is also exact. Thus, the functors S_1T and S^1T are half-exact. Proceeding this proof by induction, one gets that the sequence (52) is exact. □

4. Conclusions

In this article, new satellites of functors for non-associative algebras and modules over non-associative algebras with metagroup relations are investigated.

The results presented above can be used for subsequent studies of structure of non-associative algebras, modules and homological complexes over them and their homomorphisms. In particular, it also is possible to continue investigations of non-associative generalized Cayley–Dickson algebras, related with their cohomologies, noncommutative geometry, algebraic geometry, operator theory, spectral theory, PDEs, their applications in the sciences, etc. [6,9,12,14,18,21,23,31,33,36,37]. It can also be applied in information technologies for an antinoise coding in radio communication systems and classification of flows of information [38–40].

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

Appendix A

For the convenience of readers, necessary definitions from the book [31] and previous articles [15,16,35] are recalled. However, a reader familiar with them may skip these definitions.

Definition A1. *Let G be a set with a single-valued binary operation (multiplication)* $G^2 \ni (a,b) \mapsto ab \in G$ *defined on G satisfying the conditions:*

$$for\ each\ a\ and\ b\ in\ G, there\ is\ a\ unique\ x \in G\ with\ ax = b\ and \tag{A1}$$

$$\begin{gathered} a\ unique\ y \in G\ exists\ satisfying\ ya = b,\ which\ are\ denoted\ by \\ x = a \setminus b = Div_l(a,b)\ and\ y = b/a = Div_r(a,b); \end{gathered} \tag{A2}$$

correspondingly,

$$there\ exists\ a\ neutral\ (i.e.,\ unit)\ element\ e_G = e \in G: \tag{A3}$$

$$eg = ge = g\ for\ each\ g \in G.$$

The set of all elements $h \in G$ *commuting and associating with G:*

$$Com(G) := \{a \in G :\ \forall\, b \in G,\ ab = ba\}, \tag{A4}$$

$$N_l(G) := \{a \in G :\ \forall\, b \in G, \forall\, c \in G,\ (ab)c = a(bc)\}, \tag{A5}$$

$$N_m(G) := \{a \in G :\ \forall\, b \in G, \forall\, c \in G,\ (ba)c = b(ac)\}, \tag{A6}$$

$$N_r(G) := \{a \in G :\ \forall b \in G, \forall c \in G,\ (bc)a = b(ca)\}, \tag{A7}$$

$$N(G) := N_l(G) \cap N_m(G) \cap N_r(G); \tag{A8}$$

$$\mathcal{C}(G) := Com(G) \cap N(G)$$

is called the center $\mathcal{C}(G)$ *of G.*

We call G a metagroup if a set G possesses a single-valued binary operation and satisfies Conditions $(A1)$–$(A3)$ *and*

$$(ab)c = \mathrm{t}_3(a,b,c)a(bc) \tag{A9}$$

for each a, b and c in G, where $\mathrm{t}_3(a,b,c) \in \mathbf{\Psi}$*,* $\mathbf{\Psi} \subset \mathcal{C}(G)$*; where* t_3 *shortens a notation* $\mathrm{t}_{3,G}$*, where* $\mathbf{\Psi}$ *denotes a (proper or improper) subgroup of* $\mathcal{C}(G)$*.*

In view of the nonassociativity of G in general, a product of several elements of G is specified as usual by opening "(" and closing ")" parentheses. For elements $a_1, \dots, a_n$ *in G, we shall denote shortly by* $\{a_1, \dots, a_n\}_{q(n)}$ *the product, where a vector* $q(n)$ *indicates an order of pairwise multiplications of elements in the row* $a_1, \dots, a_n$ *in braces in the following manner. Enumerate positions: before* a_1 *by 1, between* a_1 *and* a_2 *by 2,..., by n between* a_{n-1} *and* a_n*, by* $n+1$ *after* a_n*. Then, put* $q_j(n) = (k,m)$ *if there are k opening "(" and m closing ")" parentheses in the ordered product at the j-th position of the type* $) \dots)(\dots ($*, where k and m are nonnegative integers,* $q(n) = (q_1(n), \dots, q_{n+1}(n))$ *with* $q_1(n) = (k,0)$ *and* $q_{n+1}(n) = (0,m)$*.*

Definition A2. *Let A be an algebra over an associative unital ring* $\mathcal{T}$ *such that A has a natural structure of a* $(\mathcal{T},\mathcal{T})$*-bimodule with a multiplication map* $A \times A \to A$*, which is right and left distributive* $a(b+c) = ab + ac$*,* $(b+c)a = ba + ca$*, also satisfying the following identities*

$r(ab) = (ra)b$, $(ar)b = a(rb)$, $(ab)r = a(br)$, $s(ra) = (sr)a$ and $(ar)s = a(rs)$ for any a, b and c in A, r and s in $\mathcal{T}$. Let G be a metagroup and $\mathcal{T}$ be an associative unital ring.

Henceforth, the ring $\mathcal{T}$ will be supposed commutative, if something else will not be specified.

Then, by , $\mathcal{T}[G]$ is denoted a metagroup algebra over $\mathcal{T}$ of all formal sums $s_1a_1 + \ldots + s_na_n$ satisfying Conditions $(A10)$–$(A12)$ below, where n is a positive integer, $s_1,\ldots,s_n$ are in $\mathcal{T}$ and $a_1,\ldots,a_n$ belong to G:

$$sa = as \; for \; each \; s \; in \; \mathcal{T} \; and \; a \; in \; G, \tag{A10}$$

$$s(ra) = (sr)a \; for \; each \; s \; and \; r \; in \; \mathcal{T}, \; and \; a \in G, \tag{A11}$$

$$r(ab) = (ra)b, \; (ar)b = a(rb), \; (ab)r = a(br) \tag{A12}$$

for each a and b in G, $r \in \mathcal{T}$.

Definition A3. *Let $\mathcal{R}$ be a ring, which may be non-associative relative to the multiplication. If there exists a mapping $\mathcal{R} \times M \to M$, $\mathcal{R} \times M \ni (a, m) \mapsto am \in M$ such that $a(m + k) = am + ak$ and $(a + b)m = am + bm$ for each a and b in $\mathcal{R}$, m and k in M, then M will be called a generalized left $\mathcal{R}$-module or shortly: left $\mathcal{R}$-module or left module over $\mathcal{R}$.*

If $\mathcal{R}$ is a unital ring and $1m = m$ for each $m \in M$, then, M is called a left unital module over $\mathcal{R}$, where 1 denotes the unit element in the ring $\mathcal{R}$. A right $\mathcal{R}$-module is defined symmetrically.

If M is a left and right $\mathcal{R}$-module, then it is called a two-sided $\mathcal{R}$-module or a $(\mathcal{R}, \mathcal{R})$-bimodule. If M is a left $\mathcal{R}$-module and a right $\mathcal{S}$-module, then it is called a $(\mathcal{R}, \mathcal{S})$-bimodule.

Let G be a metagroup. Take a metagroup algebra $A = \mathcal{T}[G]$ and a two-sided A-module M, where $\mathcal{T}$ is an associative unital ring (see Definition A2). Let M_g be a two-sided $\mathcal{T}$-module for each $g \in G$, where G is the metagroup. Let M have the decomposition $M = \sum_{g \in G} M_g$ as a two-sided $\mathcal{T}$-module. Let M also satisfy the following conditions:

$$hM_g = M_{hg} \; and \; M_gh = M_{gh}, \tag{A13}$$

$$(bh)x_g = b(hx_g) \; and \; x_g(bh) = (x_gh)b \; and \; bx_g = x_gb, \tag{A14}$$

$$(hs)x_g = \mathrm{t}_3(h, s, g)h(sx_g) \; and \; (hx_g)s = \mathrm{t}_3(h, g, s)h(x_gs) \tag{A15}$$

$$and \; (x_gh)s = \mathrm{t}_3(g, h, s)x_g(hs)$$

for every h, g, s in G and $b \in \mathcal{T}$ and $x_g \in M_g$. Then, a two-sided A-module M satisfying Conditions $(A13)$–$(A15)$ will be called smashly G-graded. For short, it also will be called "G-graded" instead of "smashly G-graded". In particular, if the module M is G-graded and splits into a direct sum $M = \bigoplus_{g \in G} M_g$ of two-sided $\mathcal{T}$-submodules M_g, then we will say that M is directly G-graded. For a nontrivial (nonzero) G-graded module X with the nontrivial metagroup G, it will be supposed that there exists $g \in G$ such that $X_g \neq X_e$, if something else will not be outlined.

G-graded left and right A-modules are similarly defined. Henceforward, speaking about A-modules (left, right or two-sided), it will be supposed that they are G-graded, and it will be written for short "an A-module" instead of "a G-graded A-module", unless otherwise specified.

If P and N are left A-modules and a homomorphism $\gamma : P \to N$ is such that $\gamma(ax) = a\gamma(x)$ for each $a \in A$ and $x \in P$, then γ is called a left A-homomorphism. Right A-homomorphisms for right A-modules are analogously defined. For two-sided A modules, a left and right A-homomorphism is called an A-homomorphism.

For left $\mathcal{T}$-modules M and N by $Hom_{\mathcal{T}}(M, N)$, a family of all left $\mathcal{T}$-homomorphisms from M into N is denoted. A similar notation is used for a a family of all $\mathcal{T}$-homomorphisms (or right $\mathcal{T}$-homomorphisms) of two-sided $\mathcal{T}$-modules (or right $\mathcal{T}$-modules correspondingly). If an algebra A is specified, a homomorphism may be written shortly instead of an A-homomorphism.

Definition A4. *Let M and P and N be two-sided A-modules, where A is a non-associative metagroup algebra over a commutative associative unital ring $\mathcal{T}$. An A-homomorphism (isomorphism) $f : M \to P$ is called a right (operator) A-homomorphism (isomorphism) if it is such for M and N as right A-modules, which is $f(x + y) = f(x) + f(y)$ and $f(xa) = f(x)a$ for each x and y in M*

and $a \in A$ (see also Definition A3). If an algebra A is specified, a homomorphism (isomorphism) may be written shortly instead of an A-homomorphism (an A-isomorphism respectively).

Definition A5. *Assuming that G is a metagroup, $A = \mathcal{T}[G]$ is a metagroup algebra $A = \mathcal{T}[G]$ and X is a two-sided A-module, where $\mathcal{T}$ is an associative unital ring. We denote by G^n the n-fold direct product of G with itself such that G^n is a metagroup, where $n \geq 2$ is a natural number. We consider a two-sided $\mathcal{T}$-module $X_{\{g_1,...,g_n\}_{q(n)}}$ for each $g_1,\ldots,g_n$ in G and a vector $q(n)$ indicating an order of pairwise multiplications in the braces $\{g_1, \ldots, g_n\}$ (see Definition 1 in [15]). Suppose that X has the following decomposition:*

$$X = \sum_{g_1 \in G,...,g_n \in G} X_{\{g_1,...,g_n\}_{l(n)}} \tag{A16}$$

as the two-sided $\mathcal{T}$-module, where $\{g_1\}_{l(1)} = g_1$, $\{g_1, g_2\}_{l(2)} = g_1 g_2$ and by induction $\{g_1, \ldots, g_n, g_{n+1}\}_{l(n+1)} = \{g_1, \ldots, g_n\}_{l(n)} g_{n+1}$ for each $n \geq 2$. Assume also that X satisfies the following conditions:

$$\text{there exists a } \mathcal{T}-\text{linear isomorphism}$$

$$\theta(g_1, \ldots, g_n; q(n), v(n)) : X_{\{g_1,...,g_n\}_{v(n)}} \to X_{\{g_1,...,g_n\}_{q(n)}} \tag{A17}$$

$$\text{such that } \theta(g_1, \ldots, g_n; q(n), v(n))(x_{\{g_1,...,g_n\}_{v(n)}}) = t_n(g_1, \ldots, g_n; q(n), v(n)) x_{\{g_1,...,g_n\}_{v(n)}}$$

for each $x_{\{g_1,...,g_n\}_{v(n)}} \in X_{\{g_1,...,g_n\}_{v(n)}}$, where $t_n(g_1, \ldots, g_n; q(n), v(n)) \in \mathbf{\Psi}$ is such that
$\{g_1, \ldots, g_n\}_{q(n)} = t_n(g_1, \ldots, g_n; q(n), v(n))\{g_1, \ldots, g_n\}_{v(n)}$,
$t_n(g_1, \ldots, g_n; q(n), v(n)) = t_n(g_1, \ldots, g_n; q(n), v(n)|id)$
(see also Lemma 1 and Example 2 in [15]);

$$\text{there exist } \mathcal{T}-\text{linear isomorphisms}$$

$$\theta_l(g_0, g_1, \ldots, g_n; l(n), l(n)) : g_0 X_{\{g_1,...,g_n\}_{l(n)}} \to X_{\{(g_0 g_1),...,g_n\}_{l(n)}} \tag{A18}$$

$$\text{and } \theta_r(g_1, \ldots, g_n, g_{n+1}; l(n), l(n)) : X_{\{g_1,...,g_n\}_{l(n)}} g_{n+1} \to X_{\{g_1,...,(g_n g_{n+1})\}_{l(n)}}$$

$$(bg_0) x_{\{g_1,...,g_n\}_{l(n)}} = b(g_0 x_{\{g_1,...,g_n\}_{l(n)}}) \tag{A19}$$

$$\text{and } x_{\{g_1,...,g_n\}_{l(n)}}(bg_{n+1}) = (x_{\{g_1,...,g_n\}_{l(n)}} g_{n+1}) b$$

$$\text{and } b x_{\{g_1,...,g_n\}_{l(n)}} = x_{\{g_1,...,g_n\}_{l(n)}} b,$$

$$(g_0 g_{n+1}) x_{\{g_1,...,g_n\}_{l(n)}} = t_3(g_0, g_{n+1}, g) g_0 (g_{n+1} x_{\{g_1,...,g_n\}_{l(n)}}) \tag{A20}$$

$$\text{and } (g_0 x_{\{g_1,...,g_n\}_{l(n)}}) g_{n+1} = t_3(g_0, g, g_{n+1}) g_0 (x_{\{g_1,...,g_n\}_{l(n)}} g_{n+1})$$

$$\text{and } (x_{\{g_1,...,g_n\}_{l(n)}} g_0) g_{n+1} = t_3(g, g_0, g_{n+1}) x_{\{g_1,...,g_n\}_{l(n)}} (g_0 g_{n+1})$$

for every $b \in \mathcal{T}$, $x_{\{g_1,...,g_n\}_{l(n)}} \in X_{\{g_1,...,g_n\}_{l(n)}}$, elements $g_0, g_1,\ldots,g_n, g_{n+1}$ in the metagroup G, vectors $q(n)$ and $v(n)$ indicating orders of pairwise multiplications, where $g = \{g_1, \ldots, g_n\}_{l(n)}$. Then, a two-sided A-module X satisfying Conditions $(A16)$–$(A20)$ will be called smashly G^n-graded. For short, it also will be called "G^n-graded" instead of "smashly G^n-graded". In particular, if the module X is G^n-graded and splits into a direct sum

$$X = \bigoplus_{g_1 \in G,...,g_n \in G} X_{\{g_1,...,g_n\}_{l(n)}} \tag{A21}$$

of two-sided $\mathcal{T}$-submodules $X_{\{g_1,...,g_n\}_{l(n)}}$, then we will say that that X is directly G^n-graded.

Similarly, G^n-graded left and right A-modules are defined.

References

1. Bourbaki, N. Espaces vectoriels topologiques. In *Algèbre*; Springer: Berlin, Germany, 2007; Chapter 1–3.
2. Georgantas, G.T. Derivations in central separable algebras. *Glasgow Math. J.* **1978**, *19*, 75–77. [CrossRef]
3. Van Oystaeyen, F. Separable algebras. In *Handbook of Algebra*; Hazewinkel, M., Ed.; Elsevier: North Holland, The Netherlands, 2000; Volume 2, pp. 463–505.
4. Pierce, R.S. *Associative Algebras*; Springer: New York, NY, USA, 1982.
5. Rumynin, D.A. Cohomomorphisms of separable algebras. *Algebra Log.* **1994**, *33*, 233–237. [CrossRef]
6. Cartan, H.; Eilenberg, S. *Homological Algebra*; Princeton University Press: Princeton, NJ, USA, 1956.
7. Hochschild, G. On the cohomology theory for associative algebras. *Ann. Mathem.* **1946**, *47*, 568–579. [CrossRef]
8. Bredon, G.E. *Sheaf Theory*; McGarw-Hill: New York, NY, USA, 2012.
9. Pommaret, J.F. *Systems of Partial Differential Equations and Lie Pseudogroups*; Gordon and Breach Science Publishers: New York, NY, USA, 1978.
10. Baez, J.C. The octonions. *Bull. Am. Math. Soc.* **2002**, *39*, 145–205. [CrossRef]
11. Dickson, L.E. *The Collected Mathematical Papers*; Chelsea Publishing Co.: New York, NY, USA, 1975; Volume 1–5.
12. Gürsey, F.; Tze, C.-H. *On the Role of Division, Jordan and Related Algebras in Particle Physics*; World Scientific Publishing Co.: Singapore, 1996.
13. Kantor, I.L.; Solodovnikov, A.S. *Hypercomplex Numbers*; Springer: Berlin, Germany, 1989.
14. Ludkovsky, S.V. Wrap groups of connected fiber bundles: Their structure and cohomologies. *Int. J. Math. Game Theory Algebra* **2009**, *19*, 53–128.
15. Ludkowski, S.V. Cohomology theory of non-associative algebras with metagroup relations. *Axioms* **2019**, *8*, 78. [CrossRef]
16. Ludkowski, S.V. Separability of non-associative algebras with metagroup relations. *Axioms* **2019**, *8*, 139. [CrossRef]
17. Ludkowski, S.V. Smashed and twisted wreath products of metagroups. *Axioms* **2019**, *8*, 127. [CrossRef]
18. Ludkowski, S.V. Integration of vector Sobolev type PDE over octonions. *Complex Variab. Elliptic Equat.* **2016**, *61*, 1014–1035.
19. Ludkovsky, S.V. Normal families of functions and groups of pseudoconformal diffeomorphisms of quaternion and octonion variables. *J. Mathem. Sci. N. Y.* **2008**, *150*, 2224–2287. [CrossRef]
20. Ludkovsky, S.V.; Sprössig, W. Ordered representations of normal and super-differential operators in quaternion and octonion Hilbert spaces. *Adv. Appl. Clifford Alg.* **2010**, *20*, 321–342.
21. Ludkovsky, S.V.; Sprössig, W. Spectral theory of super-differential operators of quaternion and octonion variables. *Adv. Appl. Clifford Alg.* **2011**, *21*, 165–191.
22. Nichita, F.F. Unification theories: New results and examples. *Axioms* **2021**, *8*, 60. [CrossRef]
23. Schafer, R.D. *An Introduction to Nonassociative Algebras*; Academic Press: New York, NY, USA, 1966.
24. Allcock, D. Reflection groups and octave hyperbolic plane. *J. Algebra* **1998**, *213*, 467–498. [CrossRef]
25. Belov, A.; Bokut, L.; Rowen, L.; Yu, J.-T. The Jacobian conjecture, together with Specht and Burnside-type problems. *Automorphisms Birational Affin. Geom.* **2014**, *79*, 249–285.
26. Markl, M.; Shnider, S.; Stasheff, J. *Operads in Algebra, Topology and Physics*; Mathematical Surveys and Monographs; American Mathematical Society: Providence, RI, USA, 2002; Volume 96.
27. Zhevlakov, K.A.; Slin'ko, A.M.; Shestakov, I.P.; Shirshov, A.I. *Rings That Are Nearly Associative*; Academic Press: New York, NY, USA, 1982.
28. Dirac, P.A.M. *Die Prinzipen der Quantenmechanik*; Hirzel: Leipzig, Germany, 1930.
29. Ludkowski, S.V. Decompositions of PDE over Cayley–Dickson algebras. *Rendic. dell'Ist. di Math. dell'Università di Trieste. Nuova Ser.* **2014**, *46*, 1–23.
30. Ludkovsky, S.V. Line integration of Dirac operators over octonions and Cayley–Dickson algebras. *Computat. Meth. Funct. Theory* **2012**, *12*, 279–306. [CrossRef]
31. Bruck, R.H. *A Survey of Binary Systems*; Springer: Berlin, Germany, 1971.
32. Pickert, G. *Projektive Ebenen*; Springer: Berlin, Geramany, 1955.
33. Pickert, G. Doppelebenen und loops. *J. Geom.* **1991**, *41*, 133–144. [CrossRef]
34. Ludkowski, S.V. Torsion for homological complexes of non-associative algebras with metagroup relations. *Axioms* **2021**, *10*, 319. [CrossRef]
35. Ludkowski, S.V. Homotopism of homological complexes over non-associative algebras with metagroup relations. *Mathematics* **2021**, *9*, 734. [CrossRef]
36. Lee, D.-W. On the digital cohomology modules. *Mathematics* **2020**, *8*, 1451. [CrossRef]
37. Bourbaki, N. Algèbre Homologique; In *Algèbre*; Springer: Berlin, Germany, 2007; Chapter 10.
38. Blahut, R.E. *Algebraic Codes for Data Transmission*; Cambridge University Press: Cambridge, UK, 2003.
39. Shang, Y. Lie algebraic discussion for affinity based information diffusion in social networks. *Open Phys.* **2017**, *15*, 705–711. [CrossRef]
40. Paramonov, A.A.; Zung, H.V. Joint use of frequency-time division and antinoise coding in radio communication systems with FHS. *Russ. Technol. J.* **2021**, *9*, 77–84. [CrossRef]