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Integrable Solutions for Gripenberg-Type Equations with m-Product of Fractional Operators and Applications to Initial Value Problems

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Abstract: In this paper, we deal with the existence of integrable solutions of Gripenberg-type equations with m-product of fractional operators on a half-line $\mathbb{R}^+ = [0, \infty)$. We prove the existence of solutions in some weighted spaces of integrable functions, i.e., the so-called L_1^N-solutions. Because such a space is not a Banach algebra with respect to the pointwise product, we cannot follow the idea of the proof for continuous solutions, and we prefer a fixed point approach concerning the measure of noncompactness to obtain our results. Appropriate measures for this space and some of its subspaces are introduced. We also study the problem of uniqueness of solutions. To achieve our goal, we utilize a generalized Hölder inequality on the noted spaces. Finally, to validate our results, we study the solvability problem for some particularly interesting cases and initial value problems.

Keywords: weighted Lebesgue spaces; measure of noncompactness; fractional calculus; Gripenberg-type equations; initial value problem; generalized Hölder inequality

MSC: 45G10; 47H30; 47H08; 47N20

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1. Introduction

In [1], Gripenberg investigated the problem of existence of continuous solutions for the following quadratic integral equation:

$$x(t) = k\left(g_1(t) + \int_0^t a_1(t-s)x(s)\,ds\right)\left(g_2(t) + \int_0^t a_2(t-s)x(s)\,ds\right)$$

for $t \geq 0$. The above problem is studied for its use in epidemic models and has various applications in mathematical biology, such as in models of the spread of diseases that do not induce permanent immunity (for SI models, cf. [1,2]). Some generalizations of this equation, in particular for convolution kernels yielding fractional-order Riemann–Liuoville integral operators, have been studied in the space of continuous functions [3–5], in various Banach algebras [6–8], and in classical Lebesgue L_p-spaces [9–12]. It is important that most of these results are related to problems on compact intervals only (see [10] for a result directly related to the above equation).

Our generalization of these studies is twofold. First, we examine the case of σ-finite measure space by examining the existence of a solution in the corresponding weighted space, which results in the need to study the properties of operators and measures of noncompactness in such spaces. Some new solution spaces are also proposed. Second, we will consider equations not only for linear integral operators of convolution type, but also for a general class of fractional operators composed with the Nemytskii superposition operators, and we are not limited to quadratic integral equations.

Different types of the infectious disease samples are based on possibly discontinuous data functions, so we are encouraged to check the discontinuous solutions of these problems. This remains in close connection with the purely mathematical motivation for studying integral equations with discontinuous solutions. Equations on unbounded intervals are worth investigating in certain weighted Lebesgue spaces, and results are known for the Hammerstein and Urysohn integral equations examined in weighted L_p-spaces in [13,14]. Note that the study of solutions in $L_1(\mathbb{R})$ requires significant restrictions on the growth of functions (cf. [15], for instance). Here, we omit this problem.

Quadratic integral equations, however, require some special tools. In [16], the existence and the uniqueness of $L_1^N(\mathbb{R}^+)$-solutions to the functional-integral equation

$$x(t) = h(t, x(\varphi_3(t))) + \Big(g_1(t) + g_3(t) \cdot (Gx)(\varphi_1(t))\Big)\left(g_2(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(s, x(\varphi_2(s)))}{(t-s)^{1-\alpha}} ds\right)$$

has been verified, where $0 < \alpha < 1$, by compactness arguments, using the Schauder fixed point theorem. Here, without the context of functional parts, we study the more general problem and for the product of more than two operators, and we also consider some subspaces of L_1^N, and the right-hand side considered here is more general (i.e., superposed with some nonlinear function). Some assumptions on operators are also relaxed.

At this point it is worth noting one more extension considered here, specifically, about considering the product of not just two operators, but any finite number of them. It was proposed by Brestovanská [17] in the context of the Gripenberg equation and by Brestovanská and Medved' [6] for fractional integral operators. As the recent studies of epidemic models are based rather on SIR or SEIR models than the SI model as in [1], it seems to be worthwhile to study the product of more than two operators (cf. [18,19]), so we propose to create a mathematical basis for this theory, especially for fractional-type operators ([5,10], for instance). Let us note that in the previously investigated cases, all operators were considered as acting on the same space, so it should be a Banach algebra, which is not necessary in our approach. So far, it has been usual to consider operators that allow the application of the Banach fixed point theorem or proofs that are based on a weakly Picard operator technique, unfortunately, in both cases in Banach algebras (cf. [6,8] for a discussion of prior proof methods and their limitations). By constructing a suitable measure of weak noncompactness in the solution space, we are able to apply the Darbo theorem and even the Schauder theorem here. However, for m-tuples of appropriate spaces, we must carefully construct the contraction constant for measures of weak noncompactness, so that it is indeed less than 1.

In this paper, we apply a Hölder inequality modified in the case of the weighted space $L_1^N(\mathbb{R}^+),\ N > 0$ and use it to study solutions of the equation

$$x(t) = \int_0^t K(t,\varphi) h\left(\varphi,\ \prod_{i=1}^m \left(g_i(\varphi) + \frac{1}{\Gamma(\alpha_i)} \int_0^\varphi \frac{f_i(s, x(s))}{(\varphi - s)^{1-\alpha_i}} ds\right)\right) d\varphi, \tag{1}$$

where $0 < \alpha_i < 1,\ i = 1, \cdots, m$. In particular, we will study the case of $K(t,\varphi) = \frac{(t-\varphi)^{-\gamma}}{\Gamma(1-\gamma)}$. As applications of our outcomes, we will examine some special cases and, in addition, the initial value problem of the form

$$\begin{cases} \frac{dz(\varphi)}{d\varphi} = h\left(\varphi,\ \prod_{i=1}^m \left(g_i(\varphi) + \frac{1}{\Gamma(\alpha_i)} \int_0^\varphi \frac{f_i(s,z(s))}{(\varphi-s)^{1-\alpha_i}}\, ds\right)\right) \\ z(0) = 0,\ \ \varphi \in \mathbb{R}^+,\ \ 0 < \gamma, \alpha_i < 1,\ i = 1, \cdots m, \end{cases} \tag{2}$$

which is a differential form of (1) with some special kernels K.

Here we waive the monotonicity assumptions presented in [20–25] and use a general set of assumptions to study the generalized Gripenberg equation in the case of m-product of fractional operators (1) in $L_1^N(\mathbb{R}^+)$, which, however, is not a Banach algebra with respect to the pointwise product. Certain relationships between sets of values of the operators under consideration will replace the requirement to operate in Banach algebras. We also examine the uniqueness problem for solutions of the problem under investigation.

Finally, we check the solvability of some special cases and initial value problems to validate the results. Defining appropriate new measures of weak noncompactness in certain subspaces of L_1^N establishes the foundation for considering solutions in such cases as well. We get our outcomes by utilizing the fixed point approach, fractional calculus, and some measures of noncompactness. Finally, let us note that some comments about the possibility of some numerical treatment of this problem can be found in Remark 2 (cf. also [26]).

2. Notation and Auxiliary Facts

Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^+ = [0, \infty)$, $J = [0, T] \subset \mathbb{R}^+$. Denote by $L_p^N = L_p^N(\mathbb{R}^+)$, $1 \leq p < \infty$ the weighted Lebesgue spaces of exponential type, i.e., the spaces of measurable functions x endowed with the norm

$$\|x\|_{L_p^N} = \left(\int_0^\infty e^{-Ns} |x(s)|^p \, ds \right)^{\frac{1}{p}} < \infty, \quad N > 0.$$

In the particular case $N = 0$. we get the classical L_p-spaces for $1 \leq p < \infty$ with their norm. Such a space, although very simple, seems to be sufficient for our research.

Let $C(D)$ denote the space of continuous functions on a bounded and closed subset $\emptyset \neq D \subset \mathbb{R}$. Let $T > 0$ be arbitrary and let X be a nonempty and bounded subset of $C(D)$.

For $x \in X$ and $\varepsilon > 0$, we assign by $\omega^T(x, \varepsilon)$ the (minimal) modulus of continuity of the function x, on the interval $[0, T]$, i.e.,

$$\omega^T(x, \varepsilon) = \sup\{|x(t_2) - x(t_1)| : \ t_1, t_2 \in [0, T], \ |t_1 - t_2| \leq \varepsilon\}.$$

We need to discuss some definitions and properties of fractional differential and integral operators that will be useful in sequel (cf. [11,16,27–29]). Let $\Gamma(\alpha) = \int_0^\infty e^{-\varphi} \varphi^{\alpha-1} \, d\varphi$.

Definition 1. *Let $y \in L_1^{loc}$ and $\alpha \in \mathbb{R}^+$. The fractional Riemann–Liuouville integral of order α of the function y is defined as*

$$I^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{y(s)}{(t-s)^{1-\alpha}} \, ds, \quad \alpha > 0, \quad \varphi > 0,$$

provided that the integral is almost everywhere pointwise defined (i.e., it is convergent).

Here is a good place to justify the choice of our function space. Namely, the operator I^α is well-defined on the set of a locally integrable function on $[0, \infty)$. Moreover, it defines a bounded transformation of any of the Banach spaces of locally integrable functions with a weight of exponential type for any choice of N, which will be clarified in the next proposition. First, for completeness, let us recall a definition of the Caputo fractional derivative.

Definition 2. *Let y be an absolutely continuous function (AC) on the finite interval J. The Caputo fractional derivative of order $\alpha \in (0, 1)$ of y, for $t \in J$ is defined by*

$$D^\alpha y(t) = I^{1-\alpha} D y(t), \quad D = \frac{d}{dt}.$$

Proposition 1.

(a) Let $0 < \beta, \alpha < 1$ and suppose that $f, Df \in L_1$. Then the Riemann–Liouville operator

1. *Has a semigroup property* $I^\beta I^\alpha f(t) = I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t)$,
2. *Is the inverse operator for the Caputo differential operator* $DI^\alpha f(t) = I^\alpha Df(t)$, *whenever* $f(0) = 0$,
3. $D^\alpha I^\beta f(t) = I^{\beta-\alpha} f(t)$, $\alpha < \beta$,
4. $D^\alpha I^\beta f(t) = D^{\alpha-\beta} f(t)$, $\alpha > \beta$, *if moreover* $f(0) = 0$.

(b) *(cf. [29], Theorem 5.7) The fractional operator* I^α, $\alpha > 0$ *when acting on* L_p^N $(N > 0,$ $1 \le p \le \infty)$ *has the following properties:*

(1) *The operator* I^α *maps* L_p^N *into* L_p^N *and is continuous,*

(2) $\|I^\alpha f\|_{L_p^N} \le (\frac{p}{N})^\alpha \|f\|_{L_p^N}$ *for* $1 \le p < \infty$,

(3) $\|I^\alpha f\|_{L_p^N} \le (\frac{1}{N})^\alpha \|f\|_{L_p^N}$ *for* $p = \infty$.

We now need to recall a basic nonlinear operator and some of its properties on considered weighted Lebesgue spaces.

Definition 3 ([30])**.** *Assume that a function* $h(t, x) : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *satisfies the Carathéodory conditions, i.e., it is measurable in* t *for any* $x \in \mathbb{R}$ *and continuous in* x *for almost all* $t \in \mathbb{R}^+$. *For each measurable function* x, *we denote by* F_h *the Nemytskii (superposition) operator defined by*

$$F_h(x)(t) = h(t, x(t)) \,,\; t \in \mathbb{R}^+.$$

Lemma 1 ([16])**.** *Suppose that the function* h *satisfies the Carathéodory conditions and the following growth condition: there exist a function* $a \in L_q^N$ *and a constant* $b \ge 0$ *such that*

$$|h(t, x)| \le a(t) + b \cdot |x|^{\frac{p}{q}} \tag{3}$$

for all $t \in \mathbb{R}^+$ *and* $x \in \mathbb{R}$. *Then* F_h *maps* L_p^N *into* L_q^N $(p, q \ge 1)$ *and is continuous.*

We will need also the following version of the Scorza–Dragoni theorem:

Theorem 1. *Let* $h : J \times \mathbb{R} \to \mathbb{R}$ *be a function fulfilling Carathéodory conditions. Then there exists a closed set* $D_\varepsilon \subset J$, $\varepsilon > 0$ *such that* $h|_{D_\varepsilon \times \mathbb{R}}$ *is continuous with* $meas(D_\varepsilon^c) \le \varepsilon$, *where* $D_\varepsilon^c = J \setminus D_\varepsilon$.

Assume that $(E, \|\cdot\|)$ is a Banach space with the zero element denoted by θ and let $B_r = \{u \in L_1^N : \|u\|_{L_1^N} \le r\}$, $r > 0$. Let $\emptyset \ne \mathcal{M}_E$ and $\mathcal{N}_E^W$ be the family of all nonempty, bounded subsets and the subfamily containing all relatively weakly compact sets of E, respectively. The symbols Conv Y and $\overline{Y}^W$ denote the convex hull and the weak closure of a set Y, respectively.

Definition 4 ([31])**.** *The function* $\mu : \mathcal{M}_E \to [0, \infty)$ *is called a regular measure of weak noncompactness in E if it fulfills:*

(i) $\mu(Z) = 0 \iff Z \in \mathcal{N}_E^W$,

(ii) $X \subset Z \Rightarrow \mu(X) \le \mu(Z)$,

(iii) $\mu(\overline{Z}^W) = \mu(Conv\, Z) = \mu(Z)$,

(iv) $\mu(\lambda Z) = |\lambda| \cdot \mu(Z)$, *for* $\lambda \in \mathbb{R}$,

(v) $\mu(X + Z) \le \mu(X) + \mu(Z)$,

(vi) $\mu(X \cup Z) = \max\{\mu(X), \mu(Z)\}$,

(vii) *If* $\emptyset \ne Z_n \subset E$ *be a sequence of bounded and closed subsets,* $Z_n = \overline{Z}_n^W$ *such that* $Z_{n+1} \subset Z_n$, $n = 1, 2, \cdots$ *with* $\lim_{n\to\infty} \mu(Z_n) = 0$, *then* $Z_\infty = \bigcap_{n=1}^{\infty} Z_n \ne \emptyset$.

De Blasi presented the following general definition of a measure of weak noncompactness β (cf. [21], for instance):

$$\beta(Z) = \inf\{r > 0 : \text{ there exists a weakly compact subset } X \text{ of } E \text{ such that } Z \subset X + B_r\}.$$

However, it is important to find an analytical formula for a measure of weak noncompactness in L_p^N $(p \geq 1)$ in the sense of Definition 4. The space of integrable functions over a space with σ-finite measure, i.e., here $L_1(\mathbb{R}^+)$, has rather complicated weak compactness criteria. Nevertheless, although the space we are studying contains functions defined on the half-line, the use of the weight $w(t) = e^{-Nt}$ not only extends the set of functions, but also allows us to use the results for a finite measure.

Note that our weighted space can be treated as $L_1(\mu)$, where $d\mu = e^{-Nt}dt$ and μ is a finite measure. This allows us to apply the well-known criterion of weak compactness in spaces of integrable functions with finite measure (cf. [32], p. 888). Namely, we have:

Proposition 2. *Let X be a nonempty and bounded subset X of the space L_1^N, $\varepsilon > 0$ and let*

$$c^T(X) = \lim_{\varepsilon \to 0} \sup_{x \in X} \left\{ \sup \left\{ \int_D e^{-Nt} |x(t)|\, dt : D \subset [0,T],\ meas(D) \leq \varepsilon \right\} \right\},$$

$$c(X) = \lim_{T \to \infty} c^T(X), \tag{4}$$

$$d(X) = \lim_{T \to \infty} \sup \left\{ \int_T^\infty e^{-Nt} |x(t)|\, dt : x \in X \right\}. \tag{5}$$

Then the following quantity

$$\gamma(X) = c(X) + d(X) \tag{6}$$

becomes a measure of weak noncompactness on the space L_1^N, which is equivalent to the De Blasi measure of weak noncompactness.

Proof. The proof is straightforward. It suffices to observe that our assumptions reformulated in terms of the measure μ with the Radon–Nikodym derivative $w(t) = \frac{d\mu}{dt}$ are exactly those in the original result. Thus our proposition follows from [21], Theorem 4. Moreover, its equivalence with the De Blasi measure of noncompactness β

$$\beta(A) \leq \gamma(A) \leq 2\beta(A) \quad , \quad A \in \mathcal{M}_{L_1^N}$$

follows from [21], Theorem 5. □

Some comments about subspaces of L_1^N like L_p^N $(p > 1)$ or Orlicz spaces and measures of weak noncompactness therein will be presented at the end of the paper.

Although we will construct a set on which our solving operator will be a contraction with respect to the measure of noncompactness, we will ultimately use the classical Schauder theorem:

Theorem 2 ([33])**.** *Let $C \neq \emptyset$ be a nonempty, closed, convex, and bounded subset of a Banach space E. Let $V : C \to C$ be a completely continuous mapping. Then V has at least one fixed point in C.*

3. Main Results

Equation (1) can be written in the following operator form, which allows us to directly exploit the properties of the operators on the selected weighted Lebesgue space:

$$x(t) = H(x)(t) = \int_0^t K(t,\varphi) h(\varphi, (Ax)(\varphi))\, d\varphi = K_0 F_h A(x)(t),$$

where $K_0x(t) = \int_0^t K(t,\varphi)x(\varphi)\,d\varphi$ is a linear integral operator with the kernel K and

$$F_h(A(x)) = h(\varphi, A(x)) \quad , \quad A(x) = \prod_{i=1}^{m} A_i(x) = \prod_{i=1}^{m} \Big(g_i + I^{\alpha_i} F_{f_i}(x)\Big),$$

where F_{f_i} are the superposition operators as in Definition 3, and I^{α_i} are fractional Riemann–Liouville integral operators.

Let $\sum_{i=1}^{m} \frac{1}{p_i} = 1,\ p_i \geq 1,\ p > 1$ and consider the following set of assumptions:

(i) $g_i : \mathbb{R}^+ \to \mathbb{R}$, where $g_i \in L^N_{p_i}$ for $i = 1, \cdots, m$,

(ii) Suppose that $h, f_i : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R},\ i = 1, \cdots, m$, satisfy Carathéodory conditions,

(iii) There exist $b, b_i \geq 0$ $(i = 1, \ldots, m)$ and positive functions $a \in L^N_p,\ a_i \in L^N_{p_i}$ such that

$$|h(t,x)| \leq a(t) + b|x|^{\frac{1}{p}}, \quad |f_i(t,x)| \leq a_i(t) + b_i|x|^{\frac{1}{p_i}},\ i = 1, \cdots, m \tag{7}$$

for all $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$,

(iv) Let $K(t,s) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be measurable such that $K_0x(t) = \int_0^t K(t,\varphi)x(\varphi)\,d\varphi$ maps L^N_p into L^N_1 continuously with

$$\|K_0\|_{L^N_1} = \Big\| \|K(t,\cdot)\|_{L_q^{N(1-q)}} \Big\|_{L^N_1}, \quad \text{where } q = \frac{p}{p-1}.$$

(v) Suppose that for the functions described in (i)–(iv) there additionally exists a constant $r > 0$ satisfying the following inequality:

$$\|K_0\|_{L^N_1}\left[\|a\|_{L^N_p} + b\prod_{i=1}^{m}\left(\|g_i\|_{L^N_{p_i}} + \left(\frac{p_i}{N}\right)^{\alpha_i}\left(\|a_i\|_{L^N_{p_i}} + b_i \cdot r^{\frac{1}{p_i}}\right)\right)^{\frac{1}{p}}\right] \leq r.$$

Proposition 3. *Under assumption (iv):*

(a) The operator $K_0x(t) = \int_0^t K(t,s)x(s)\,ds$ *maps the space* L^N_p *into* L^N_1 *and is continuous satisfying*

$$\|K_0(x)\|_{L^N_1} = \Big\| \|K(t,\cdot)\|_{L_q^{N(1-q)}} \Big\|_{L^N_1} \|x\|_{L^N_p}, \quad q = \frac{p}{p-1}.$$

(b) Assume that $1 = \sum_{i=1}^{m} \frac{1}{p_i},\ p_i \geq 1$ *and let* $g_i \in L^N_{p_i}, i = 1, \cdots, m$, *we have*

$$\Big\| \prod_{i=1}^{m} g_i \Big\|_{L^N_1} \leq \prod_{i=1}^{m} \|g_i\|_{L^N_{p_i}}.$$

Proof. (a) Indeed, by assumption (iv) and the Hölder inequality, ([29], Inequality 1.38), we get

$$\begin{aligned}
|K_0(x)(t)| &= \left| \int_0^t K(t,\varphi)x(\varphi)\,d\varphi \right| \\
&\leq \left(\int_0^\infty \left| e^{\frac{N\varphi}{p}} K(t,\varphi) \right|^q d\varphi \right)^{\frac{1}{q}} \left(\int_0^\infty \left| e^{\frac{-N\varphi}{p}} x(\varphi) \right|^p d\varphi \right)^{\frac{1}{p}} \\
&= \left(\int_0^\infty e^{-N\varphi(1-q)} \Big| K(t,\varphi) \Big|^q d\varphi \right)^{\frac{1}{q}} \|x\|_{L^N_p}, \\
&= \|K(t,\cdot)\|_{L_q^{N(1-q)}} \|x\|_{L^N_p}, \qquad q = \frac{p}{p-1}.
\end{aligned}$$

Consequently,

$$\|K_0(x)\|_{L^N_1} \leq \|K_0\|_{L^N_1} \|x\|_{L^N_p}.$$

(b) Now, directly applying the definition of the norm in the space under consideration together with the Hölder inequality, we obtain

$$\begin{aligned}
\left\| \prod_{i=1}^{m} g_i \right\|_{L_1^N} &= \int_0^\infty e^{-N\varphi} \left| \prod_{i=1}^{m} g_i(\varphi) \right| d\varphi \\
&= \int_0^\infty \left| e^{-N\varphi \sum_{i=1}^{m} \frac{1}{p_i}} \prod_{i=1}^{m} g_i(\varphi) \right| d\varphi \\
&= \left\| e^{\frac{-N\varphi}{p_1}} g_1 \cdots e^{\frac{-N\varphi}{p_m}} g_m \right\|_{L_1} \\
&\leq \left\| e^{\frac{-N\varphi}{p_1}} g_1 \right\|_{L_{p_1}} \cdots \left\| e^{\frac{-N\varphi}{p_m}} g_m \right\|_{L_{p_m}} \\
&\leq \left(\int_0^\infty e^{-N\varphi} \left| g_1(\varphi) \right|^{p_1} d\varphi \right)^{\frac{1}{p_1}} \cdots \left(\int_0^\infty e^{-N\varphi} \left| g_1(\varphi) \right|^{p_m} d\varphi \right)^{\frac{1}{p_m}} \\
&= \|g_1\|_{L_{p_1}^N} \cdots \|g_m\|_{L_{p_m}^N} = \prod_{i=1}^{m} \|g_i\|_{L_{p_i}^N}.
\end{aligned}$$

□

We are now ready to study the solvability problem for (1) in the weighted Lebesgue space L_1^N (e.g., on $\mathbb{R}^+$).

Theorem 3. *Let the assumptions (i)–(v) be satisfied. Fix an arbitrary $N > 0$. If, in addition, the constant connecting our assumptions*

$$W := \left(b\|K_0\|_{L_1^N} \prod_{i=1}^{m} b_i^{\frac{1}{p}} \left(\frac{p_i}{N} \right)^{\frac{\alpha_i}{p}} \right) < 1,$$

then Equation (1) has at least one solution $x \in L_1^N$ on $\mathbb{R}^+$.

Proof. Although the proof of our theorem requires a precise study of the relations of many spaces and operators, to make it easier to follow, we will adopt its main steps from [34] taking into account, however, the peculiarities of our solution space.

Step 1. Using assumptions (ii), (iii), and Lemma 1, we see that F_{f_i} maps L_1^N into $L_{p_i}^N$ $(i = 1, \ldots, m)$ and is continuous. Because the operators $I^{\alpha_i} : L_{p_i}^N \to L_{p_i}^N$ are continuous, it follows from assumption (i) that the operators $A_i : L_1^N \to L_{p_i}^N$ are continuous. Then, using Proposition $3_{(b)}$, we find that $A : L_1^N \to L_1^N$ is continuous. Finally, the operator F_h is a continuous mapping from L_1^N into L_p^N by virtue of assumption (iv) and then by Proposition $3_{(a)}$, we obtain that $H(x) = K_0 F_h(A(x)) : L_1^N \to L_1^N$ is continuous.

Step 2. Using Proposition $3_{(a)}$ and our assumptions, we have an estimation

$$\begin{aligned}
\|H(x)\|_{L_1^N} &= \|K_0 F_h(A(x))\|_{L_1^N} \\
&\leq \|K_0\|_{L_1^N} \|F_h(A(x))\|_{L_p^N} \\
&\leq \|K_0\|_{L_1^N} \|a + b|A(x)|^{\frac{1}{p}}\|_{L_p^N}.
\end{aligned}$$

By Proposition $3_{(b)}$, we obtain $\|(A(x))^{\frac{1}{p}}\|_{L_p^N} = \|A(x)\|_{L_1^N}^{\frac{1}{p}} \leq \prod_{i=1}^{m} \|A_i(x)\|_{L_{p_i}^N}^{\frac{1}{p}}$, and then

$$\begin{aligned}
\|H(x)\|_{L_1^N} &\le \|K_0\|_{L_1^N}\left[\|a\|_{L_p^N} + b\prod_{i=1}^{m}\left\|g_i + I^{\alpha_i}F_{f_i}(x)\right\|_{L_{p_i}^N}^{\frac{1}{p}}\right] \\
&\le \|K_0\|_{L_1^N}\left[\|a\|_{L_p^N} + b\prod_{i=1}^{m}\left(\|g_i\|_{L_{p_i}^N} + \|I^{\alpha_i}F_{f_i}(x)\|_{L_{p_i}^N}\right)^{\frac{1}{p}}\right] \\
&\le \|K_0\|_{L_1^N}\left[\|a\|_{L_p^N} + b\prod_{i=1}^{m}\left(\|g_i\|_{L_{p_i}^N} + \left(\frac{p_i}{N}\right)^{\alpha_i}\left\|f_i(\varphi,x)\right\|_{L_{p_i}^N}\right)^{\frac{1}{p}}\right] \\
&\le \|K_0\|_{L_1^N}\left[\|a\|_{L_p^N} + b\prod_{i=1}^{m}\left(\|g_i\|_{L_{p_i}^N} + \left(\frac{p_i}{N}\right)^{\alpha_i}\left\|a_i + b_i|x|^{\frac{1}{p_i}}\right\|_{L_{p_i}^N}\right)^{\frac{1}{p}}\right] \\
&\le \|K_0\|_{L_1^N}\left[\|a\|_{L_p^N} + b\prod_{i=1}^{m}\left(\|g_i\|_{L_{p_i}^N} + \left(\frac{p_i}{N}\right)^{\alpha_i}\left(\|a_i\|_{L_{p_i}^N} + b_i\|x^{\frac{1}{p_i}}\|_{L_{p_i}^N}\right)\right)^{\frac{1}{p}}\right] \\
&\le \|K_0\|_{L_1^N}\left[\|a\|_{L_p^N} + b\prod_{i=1}^{m}\left(\|g_i\|_{L_{p_i}^N} + \left(\frac{p_i}{N}\right)^{\alpha_i}\left(\|a_i\|_{L_{p_i}^N} + b_i\|x\|_{L_1^N}^{\frac{1}{p_i}}\right)\right)^{\frac{1}{p}}\right].
\end{aligned}$$

Then the operator $H : L_1^N \to L_1^N$. For $x \in B_r$, where r is given by assumption (v) and $B_r = \{u \in L_1^N : \|u\|_{L_1^N} \le r\}$, we have

$$\|H(x)\|_{L_1^N} \le \|K_0\|_{L_1^N}\left[\|a\|_{L_p^N} + b\prod_{i=1}^{m}\left(\|g_i\|_{L_{p_i}^N} + \left(\frac{p_i}{N}\right)^{\alpha_i}\left(\|a_i\|_{L_{p_i}^N} + b_i\cdot r^{\frac{1}{p_i}}\right)\right)^{\frac{1}{p}}\right] \le r.$$

We then get an invariant ball for H and the operator $H : B_r \to B_r$ is continuous.

Step 3. We need to prove that H is a contraction with respect to the measure of weak noncompactness γ on the ball B_r. Let $\emptyset \ne X \subset B_r$ and $x \in X$. For arbitrary $\varepsilon > 0$ and $T > 0$ such that for any $D \subset [0,T]$ with $meas(D) \le \varepsilon$, by Proposition 3, we obtain

$$\begin{aligned}
&\int_D e^{-N\varphi}|(H(x))(\varphi)|d\varphi \le \|K_0F_h(A(x))\|_{L_1^N(D)} \\
\le\ & \|K_0\|_{L_1^N(D)}\|F_h(A(x))\|_{L_p^N(D)} \\
\le\ & \|K_0\|_{L_1^N(D)}\|a + b|A(x)|^{\frac{1}{p}}\|_{L_p^N(D)} \\
\le\ & \|K_0\|_{L_1^N(D)}\left[\|a\|_{L_p^N(D)} + b\|A(x)\|_{L_1^N(D)}^{\frac{1}{p}}\right] \\
\le\ & \|K_0\|_{L_1^N(D)}\left[\|a\|_{L_p^N(D)} + b\left\|\prod_{i=1}^{m}A_i(x)\right\|_{L_1^N(D)}^{\frac{1}{p}}\right] \\
\le\ & \|K_0\|_{L_1^N(D)}\left[\|a\|_{L_p^N(D)} + b\prod_{i=1}^{m}\left(\|g_i\|_{L_{p_i}^N(D)} + \left(\frac{p_i}{N}\right)^{\alpha_i}\left(\|a_i\|_{L_{p_i}^N(D)} + b_i\|x^{\frac{1}{p_i}}\|_{L_{p_i}^N(D)}\right)\right)^{\frac{1}{p}}\right] \\
\le\ & \|K_0\|_{L_1^N(D)}\left[\|a\|_{L_p^N(D)} + b\prod_{i=1}^{m}\left(\|g_i\|_{L_{p_i}^N(D)}\right.\right. \\
& \left.\left. + \left(\frac{p_i}{N}\right)^{\alpha_i}\left(\|a_i\|_{L_{p_i}^N(D)} + b_i\left(\int_D e^{-N\varphi}|x(\varphi)|d\varphi\right)^{\frac{1}{p_i}}\right)\right)^{\frac{1}{p}}\right].
\end{aligned}$$

Since $g_i, a_i \in L_{p_i}^N$ and $a \in L_p^N$, we have

$$\lim_{\varepsilon\to 0}\left\{\sup_{x\in X}\left\{\sup\left(\ \|a\|_{L_p^N(D)} :\ D \subset \mathbb{R}^+,\ meas(D) \le\ \varepsilon\right)\right\}\right\} = 0$$

so

$$\lim_{\varepsilon\to 0}\left\{\sup_{x\in X}\left\{\sup\left(\|g_i\|_{L^N_{p_i}(D)}+\left(\frac{p_i}{N}\right)^{\alpha_i}\|a_i\|_{L^N_{p_i}(D)}:D\subset\mathbb{R}^+, meas(D)\leq\ \varepsilon\right)\right\}\right\}=0.$$

From Equation (4) and as $\left(\int_D e^{-N\varphi}|x(\varphi)|d\varphi\right)^{\frac{1}{p}}\leq\int_D e^{-N\varphi}|x(\varphi)|d\varphi$, we have

$$c(H(X))\leq\left(b\|K_0\|_{L^N_1(D)}\prod_{i=1}^{m}b_i^{\frac{1}{p}}\left(\frac{p_i}{N}\right)^{\frac{\alpha_i}{p}}\right)\cdot c(X)=W\cdot c(X). \tag{8}$$

Thus for any $T>0$ and $x\in X$, we get

$$\int_T^\infty e^{-N\varphi}|(H(x))(\varphi)|d\varphi\leq\|K_0\|_{L^N_1(T)}\Bigg[\|a\|_{L^N_p(T)}$$
$$+\quad b\prod_{i=1}^{m}\left(\|g_i\|_{L^N_{p_i}(T)}+\left(\frac{p_i}{N}\right)^{\alpha_i}\left(\|a_i\|_{L^N_{p_i}(T)}+b_i\left(\int_T^\infty e^{-N\varphi}|x(\varphi)|d\varphi\right)^{\frac{1}{p_i}}\right)\right)^{\frac{1}{p}}\Bigg].$$

We then pass with T to the limit $T\to\infty$ and by the definition of $d(X)$ in (5), we get

$$d(H(X))\leq W\cdot d(X). \tag{9}$$

Adding Equations (8) and (9) on both sides and by the definition of γ, i.e., formula (6), we have

$$\gamma(H(X))\leq W\cdot\gamma(X). \tag{10}$$

Step 4. Define $B_r^1=Conv(H(B_r))$, $B_r^2=Conv(H(B_r^1))$ and so on, where B_r is as in Step 2. We have a decreasing sequence $\{B_r^n\}$, i.e., $B_r^{n+1}\subset B_r^n$ for $n\in\mathbb{N}$. Obviously, all the sets in this sequence are closed and convex, and hence weakly closed. From Step 3, we have $\gamma(H(X))\leq W\cdot\gamma(X)$ for all bounded subsets X of B_r, so

$$\gamma(B_r^n)\leq W^n\gamma(B_r),$$

which implies that $\lim_{n\to\infty}\gamma(B_r^n)=0$. Then, from property (vii) of Definition 4, it follows that the set $Q=\cap_{n=1}^\infty B_r^n\neq\emptyset$ is closed, convex, and weakly compact (as $\gamma(Q)=0$). Moreover, $H(Q)$ is a subset of Q. We must prove that this set contains a more regular invariant subset that is not only weakly relatively compact in L_1^N, but also strongly relatively compact.

Step 5. Since $\gamma(Q)=0$, by virtue of the criterion of the weak compactness (cf. (6)), and then for arbitrary sequence $\{x_n\}\subset Q$, there exists $T>0$ such that for all n the following inequality is satisfied:

$$\int_T^\infty e^{-Ns}|x_n(s)|\,ds\leq\frac{\varepsilon}{4}. \tag{11}$$

Considering the functions $h(\varphi,x), f_i(\varphi,x),\ i=1,\cdots,m$ on the set $[0,T]\times\mathbb{R}$, and $K(t,\varphi)$ on the set $[0,T]\times[0,T]$, then it follows from Theorem 1 that there exists a closed set $D_\varepsilon\subset[0,T]$ such that $meas(D_\varepsilon^c)\leq\varepsilon$, $g_i\big|_{D_\varepsilon}$, $i=1,\cdots,m$ are continuous and $h\big|_{D_\varepsilon\times\mathbb{R}}, K\big|_{D_\varepsilon\times[0,T]}, f_i\big|_{D_\varepsilon\times\mathbb{R}},\ i=1,\cdots,m$ are continuous. Hence we conclude that $K\big|_{D_\varepsilon\times[0,T]}$ is uniformly continuous.

Denote by $H_n(t) = K_0 F_h(A(x_n))(t)$, where $A(x_n) = \prod_{i=1}^{m} A_i(x_n),\ n \in \mathbb{N}$. We need to show that the operator (H_n) is equibounded and equicontinuous in the space $C(D_\varepsilon)$. Then

$$
\begin{aligned}
|H_n(t)| &\le |K_0 F_h(A(x_n))(t)| \le \left| \int_0^t K(t,\varphi) h(t,(A(x_n))(\varphi)) d\varphi \right| \\
&\le \overline{K} \int_0^t \left(|a(\varphi)| + b|A(x_n)(\varphi)|^{\frac{1}{p}} \right) d\varphi \\
&\le \overline{K} \int_0^t \left[d + b \left(\prod_{i=1}^{m} |A_i(x_n)(\varphi)| \right)^{\frac{1}{p}} \right] d\varphi \\
&= \overline{K} \int_0^t \left[d + b \prod_{i=1}^{m} \left(|g_i(\varphi)| + \frac{1}{\Gamma(\alpha_i)} \int_0^{\varphi} \frac{|f_i(s,x_n(s))|}{(\varphi - s)^{1-\alpha_i}} ds \right)^{\frac{1}{p}} \right] d\varphi \\
&\le \overline{K} \int_0^t \left[d + b \prod_{i=1}^{m} \left(d_i + \frac{1}{\Gamma(\alpha_i)} \int_0^{\varphi} \frac{|a_i(s)| + b_i |x_n(s)|^{\frac{1}{p_i}}}{(\varphi - s)^{1-\alpha_i}} ds \right)^{\frac{1}{p}} \right] d\varphi \\
&\le \overline{K} \int_0^t \left[d + b \prod_{i=1}^{m} \left(d_i + \frac{c_i + b_i (r^*)^{\frac{1}{p_i}}}{\Gamma(\alpha_i)} \int_0^{\varphi} \frac{ds}{(\varphi - s)^{1-\alpha_i}} ds \right)^{\frac{1}{p}} \right] d\varphi \\
&\le \overline{K} \int_0^t \left[d + b \prod_{i=1}^{m} \left(d_i + \frac{c_i + b_i (r^*)^{\frac{1}{p_i}}}{\Gamma(\alpha_i + 1)} T^{\alpha_i} \right)^{\frac{1}{p}} \right] d\varphi \\
&\le \overline{K} \cdot T \left[d + b \prod_{i=1}^{m} \left(d_i + \frac{c_i + b_i (r^*)^{\frac{1}{p_i}}}{\Gamma(\alpha_i + 1)} T^{\alpha_i} \right)^{\frac{1}{p}} \right],
\end{aligned}
$$

where $\overline{K} = \max\{K(t,s) : (t,s) \in D_\varepsilon \times [0,T]\}$, and $|a(\varphi)| \le d,\ |g_i(\varphi)| \le d_i,\ |a_i(\varphi)| \le c_i,\ |x_n(\varphi))| \le r^*$ for $\varphi \in D_\varepsilon$. It follows from the above estimate that $\{H_n\}$ is equibounded in $C(D_\varepsilon)$. Then, for any $\varphi_1, \varphi_2 \in D_\varepsilon$, where $\varphi_1 < \varphi_2$, we obtain

$$
\begin{aligned}
& |A_i x_n(\varphi_2) - A_i x_n(\varphi_1)| \\
=\ & \left| g_i(\varphi_2) + \frac{1}{\Gamma(\alpha_i)} \int_0^{\varphi_2} \frac{f_i(s,x_n(s))}{(\varphi_2 - s)^{1-\alpha_i}} ds - g_i(\varphi_1) - \frac{1}{\Gamma(\alpha_i)} \int_0^{\varphi_1} \frac{f_i(s,x_n(s))}{(\varphi_1 - s)^{1-\alpha_i}} ds \right| \\
\le\ & |g_i(\varphi_2) - |g_i(\varphi_1)| + \left| \frac{1}{\Gamma(\alpha_i)} \int_0^{\varphi_2} \frac{f_i(s,x_n(s))}{(\varphi_2 - s)^{1-\alpha_i}} ds - \frac{1}{\Gamma(\alpha_i)} \int_0^{\varphi_1} \frac{f_i(s,x_n(s))}{(\varphi_2 - s)^{1-\alpha_i}} ds \right| \\
& + \left| \frac{1}{\Gamma(\alpha_i)} \int_0^{\varphi_1} \frac{f_i(s,x_n(s))}{(\varphi_2 - s)^{1-\alpha_i}} ds - \frac{1}{\Gamma(\alpha_i)} \int_0^{\varphi_1} \frac{f_i(s,x_n(s))}{(\varphi_1 - s)^{1-\alpha}} ds \right| \\
\le\ & \omega^T(g_i, |\varphi_2 - \varphi_1|) + \frac{1}{\Gamma(\alpha_i)} \int_{\varphi_1}^{\varphi_2} \frac{|f_i(s,x_n(s))|}{(\varphi_2 - s)^{1-\alpha_i}} ds \\
& + \frac{1}{\Gamma(\alpha_i)} \int_0^{\varphi_1} \frac{|f_i(s,x_n(s))|}{(\varphi_2 - s)^{1-\alpha_i} - (\varphi_1 - s)^{1-\alpha_i}} ds \\
\le\ & \omega^T(g_i, |\varphi_2 - \varphi_1|) + \frac{c_i + b_i \cdot (r^*)^{\frac{1}{p_i}}}{\Gamma(\alpha_i + 1)} \Big((\varphi_2 - \varphi_1)^{\alpha_i} + (\varphi_2^{\alpha_i} - \varphi_1^{\alpha_i}) - (\varphi_2 - \varphi_1)^{\alpha_i} \Big),
\end{aligned}
$$

where $\omega^T(g_i, \cdot)$ refers to the (minimal) modulus of continuity of the functions $g_i, i = 1, \cdots, m$ on the set D_ε. Recall that they are continuous on this set. According to the mean value theorem, there exists y such that $\varphi_1 < y < \varphi_2$ such that $\varphi_2^\alpha - \varphi_1^\alpha = \alpha y^{\alpha-1}(\varphi_2 - \varphi_1) \le (\varphi_2 - \varphi_1)$, then we have

$$
|A_i(x_n)(\varphi_2) - A_i(x_n)(\varphi_1)| \le \omega^T(g_i, |\varphi_2 - \varphi_1|) + \frac{c_i + b_i \cdot (r^*)^{\frac{1}{p_i}}}{\Gamma(\alpha_i + 1)} \Big((\varphi_2 - \varphi_1) \Big). \qquad (12)
$$

Moreover, for any $t_1, t_2 \in D_\varepsilon$ such that $t_1 < t_2$ we obtain

$$\begin{aligned}
|H_n(t_2) - H_n(t_1)| & \\
\leq & \left| \int_0^{t_2} K(t_2,\varphi)h(\varphi,(A(x_n))(\varphi))d\varphi - \int_0^{t_2} K(t_1,\varphi)h(\varphi,(A(x_n))(\varphi))d\varphi \right| \\
& + \left| \int_0^{t_2} K(t_1,\varphi)h(\varphi,(A(x_n))(\varphi))d\varphi - \int_0^{t_1} K(t_1,\varphi)h(\varphi,(A(x_n))(\varphi))d\varphi \right| \\
\leq & \int_0^{t_2} |K(t_2,\varphi) - K(t_1,\varphi)||h(\varphi,(A(x_n))(\varphi))|d\varphi \\
& + \int_{t_1}^{t_2} |K(t_1,\varphi)||h(\varphi,(A(x_n))(\varphi))|d\varphi \\
\leq & \omega^T(K,\cdot)\big(d + b\cdot|A(x_n)|^{\frac{1}{p}}\big)T + \overline{K}\big(d + b\cdot|A(x_n)|^{\frac{1}{p}}\big)(t_2 - t_1) \\
\leq & \omega^T(K,\cdot)\left(d + b\cdot\prod_{i=1}^m \left(d_i + \frac{c_i + b_i(r^*)^{\frac{1}{p_i}}}{\Gamma(\alpha_i+1)}T^{\alpha_i}\right)^{\frac{1}{p}}\right)T \\
& + \overline{K}\left(d + b\cdot\prod_{i=1}^m \left(d_i + \frac{c_i + b_i(r^*)^{\frac{1}{p_i}}}{\Gamma(\alpha_i+1)}T^{\alpha_i}\right)^{\frac{1}{p}}\right)(t_2 - t_1),
\end{aligned} \tag{13}$$

where $\omega^T(K,\cdot)$ refers to the modulus of continuity of the function K on the sets D_ε. The inequalities (12) and (13) are obtained because $Q \subset B_r$.

Since $\{x_n\}) \subset Q$, we obtain $\gamma(\{x_n\}) \leq \gamma(Q) = 0$. If, in inequalities (12) and (13), we pass to the limit with $(\varphi_2 - \varphi_1) \to 0$ and $(t_2 - t_1) \to 0$, respectively, we see that the obtained inequalities allow us to estimate moduli of continuity independently of x_n. Set

$$Y = \sup\{\, \|H_n|_{D_\varepsilon}\|_\infty : n \in \mathbb{N}\}.$$

As we just proved that $\{H_n\}$ is equibounded in $C(D_\varepsilon)$, the number Y is finite. Recall that $D_\varepsilon \subset [0,T]$ is closed, so as a closed subset of the compact set $[0,T]$ is also compact. Clearly, continuous functions on a compact domain are uniformly continuous, the sets $D_\varepsilon \times [-Y,Y]$ are compact, and we can conclude that the functions $h\big|_{D_\varepsilon\times[-Y,Y]}, f_i\big|_{D_\varepsilon\times[-Y,Y]}$, are continuous. Thus, we conclude that $\{H_n\} = \{K_0F_h(A(x_n)) : n \in \mathbb{N}\}$ is equicontinuous and equibounded in $C(D_\varepsilon)$. It follows from the Ascoli–Arzéla theorem ([33], for instance) that $\{H_n|_{D_\varepsilon}\}$ is relatively compact in the space $C(D_\varepsilon)$.

We note that the above reasoning about the set D_ε does not depend on the choice of the value of ε. We can then construct a sequence of closed sets $D_{\delta_n} \subset [0,T]$ such that $meas(D^c_{\delta_n}) \to 0$ as $n \to \infty$ with the property that $\{H_n|_{D_{\delta_n}}\}$ is relatively compact in every space $C(D_{\delta_n})$. It means that passing to subsequence if necessary, the sequence (H_n) is a Cauchy sequence in each space $C(D_{\delta_n})$, $n = 1,2,\ldots$.

Now, we need to control the behavior of the set $\{H_n|_{D^c_{\delta_n}}\}$. Because the set $H(Q)$ is weakly compact in L^N_1, we get its uniform integrability (cf. Proposition 2), so we can choose a number $\delta > 0$ such that for every closed set $D_\delta \subset [0,T]$, we get

$$\int_{D^c_\delta} e^{-Ns}|(H(x))(s)|\, ds \leq \frac{\varepsilon}{4}, \tag{14}$$

for any $x \in Q$. Keeping in mind that the sequence $(H(x_n))$ is a Cauchy sequence in the spaces $C(D_l)$, $l = 1,2,\cdots$, we can choose a number l_0 such that $meas(D^c_{l_0}) \leq \delta$ (with $meas(D_{l_0}) > 0$), and for arbitrary natural numbers $n, m \geq l_0$ we have

$$|(H(x_n))(s) - (H(x_m))(s)| \leq \frac{\varepsilon \cdot e^{N\cdot meas(D_{l_0})}}{4\cdot meas(D_{l_0})} \tag{15}$$

for any $s \in D_{l_0}$. Using (14) and (15), we get

$$\begin{aligned}
\int_0^T e^{-Ns}|(H(x_n))(s) - (H(x_m))(s)|\, ds &= \int_{D_{l_0}} e^{-Ns}|(H(x_n))(s) - (H(x_m))(s)|\, ds \\
&+ \int_{D_{l_0}^c} e^{-Ns}|(H(x_n))(s) - (H(x_m))(s)|\, ds \\
&\leq \int_{D_{l_0}} e^{-Ns} \cdot \frac{\varepsilon \cdot e^{N \cdot meas(D_{l_0})}}{4 \cdot meas(D_{l_0})}\, ds + \int_{D_{l_0}^c} e^{-Ns}|(H(x_n))(s)|\, ds + \int_{D_{l_0}^c} e^{-Ns}|(H(x_m))(s)|\, ds \\
&\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{3\varepsilon}{4}
\end{aligned}$$

and then by (11)

$$\begin{aligned}
\int_0^\infty e^{-Ns}|(H(x_n))(s) - (H(x_m))(s)|\, ds &= \int_0^T e^{-Ns}|(H(x_n))(s) - (H(x_m))(s)|\, ds \\
&+ \int_T^\infty e^{-Ns}|(H(x_n))(s) - (H(x_m))(s)|\, ds \leq \varepsilon,
\end{aligned}$$

which implies that $\|(H(x_n)) - H(x_m))\|_{L_1^N} \leq \varepsilon$, so $(H(x_n))$ is a Cauchy sequence in L_1^N. As this space is complete, the sequence $(H(x_n))$ is convergent in this space, and because we can choose a norm convergent subsequence from any sequence from $H(Q)$, this set is a relatively compact norm in L_1^N.

Step 6. Let $Q_0 = Conv(H(Q))$ and using the Mazur theorem, we obtain that Q_0 is again compact in L_1^N. Moreover, by definition $Q_0 \subset H(Q) \subset Q$, so $H(Q_0) \subset H(Q) \subset Conv(H(Q)) = Q_0$ and then $H : Q_0 \to Q_0$ and is continuous. We can then apply Theorem 2 for $H|_{Q_0}$, which completes the proof. □

Remark 1. *The outcomes referenced in Theorem 3 can also be applied to some subspaces of L_1^N, such as the spaces $L_p^N(\mathbb{R}^+)$ $(p > 1)$ or more generally to weighted Orlicz spaces L_φ^w (cf. [35]) under a proper set of hypotheses (cf. [36]). In the second case, it is interesting to observe that weakly compact sets in L_1^N can be characterized in terms of Orlicz spaces (due to de la Vallée Poussin criterion of uniform integrability). However, the case of σ-finite measure require some effort (cf. [37]).*

Interestingly, there are only a few known analytic formulas for measures of weak noncompactness in function spaces that can be adapted for these cases. Namely, we can define the following the measures: if $X \subset L_p^N$ is bounded, then we can put

$$\begin{aligned}
\mu(X) &= \lim_{\varepsilon \to 0}\left\{\sup_{x \in X}\left\{\sup\left[\int_D e^{-Nt} \cdot |x(t)|^p\, dt,\ D \subset I,\ measD \leq \varepsilon\right]\right\}\right\} \\
&\quad + \lim_{T \to \infty} \sup\left\{\int_T^\infty e^{-Nt}|x(t)|^p\, dt : x \in X\right\}.
\end{aligned}$$

The above formula generalize from [38] (formulated for $N = 0$ only), which is based on a different compactness criterion.

Moreover, for $X \subset L_\phi^N$ we can define a measure of weak $\sigma(\phi, \phi^*)$ measure of noncompactness (cf. [39], Example 1):

$$\mu_w(X) = \lim_{\varepsilon \to 0} \sup_{x \in X} \frac{1}{\varepsilon} \int_{[0,\infty)} \phi\Big(\varepsilon \cdot e^{-Nt} \cdot x(t)\Big) dt$$

for some class of Orlicz functions ϕ satisfying the Δ_2 condition.

Although the study of integral equations in subspaces of integrable functions seems a perfectly natural approach, it is novel for the problem we are studying. The starting point is the measures of weak noncompactness we define above, and the subsequent results are expected.

Theorem 4. *Suppose that the assumptions of Theorem 3 are satisfied, but instead of assumption (iii), consider the following hypotheses:*

(vi) *There exist $a, a_i \in L_1^N$ $(i = 1, \ldots, m)$ such that $|h(s,0)| \leq a(s)$, $|f_i(s,0)| \leq a_i(s)$, $s \geq 0$ and*

$$|h(s,x) - h(s,y)| \leq b|x-y|^{\frac{1}{p}} \ , \ |f_i(s,x) - f_i(s,y)| \leq b_i|x-y|^{\frac{1}{p_i}}, \qquad i = 1, \cdots, m,$$

where $x, y \in Q$ and Q is defined as in the proof of Theorem 3,

(vii) *If for some constant $\mathbb{W} \geq 0$, the following inequality is satisfied:*

$$\begin{aligned}
\mathbb{W} \quad \leq \quad & b\|K_0\|_{L_1^N}\Bigg[\left(\frac{p_1}{N}\right)^{\alpha_1}\prod_{i=2}^{m}\left(\|g_i\|_{L_{p_i}^N} + \left(\frac{p_i}{N}\right)^{\alpha_i}\left[\|a_i\|_{L_{p_i}^N} + b_i \cdot r^{\frac{1}{p_i}}\right]\right)b_1 \cdot \mathbb{W}^{\frac{1}{p_1}} \\
+ \quad & \left(\|g_1\|_{L_{p_1}^N} + \left(\frac{p_1}{N}\right)^{\alpha_1}\left[\|a_1\|_{L_{p_1}^N} + b_1 \cdot r^{\frac{1}{p_1}}\right]\right)\left(\frac{p_2}{N}\right)^{\alpha_2} \\
\times \quad & \prod_{i=3}^{m}\left(\|g_i\|_{L_{p_i}^N} + \left(\frac{p_i}{N}\right)^{\alpha_i}\left[\|a_i\|_{L_{p_i}^N} + b_i \cdot r^{\frac{1}{p_i}}\right]\right)b_2 \cdot \mathbb{W}^{\frac{1}{p_2}} + \ \cdots \ + \\
+ \quad & \prod_{i=1}^{m-1}\left(\|g_i\|_{L_{p_i}^N} + \left(\frac{p_i}{N}\right)^{\alpha_i}\left[\|a_i\|_{L_{p_i}^N} + b_i \cdot r^{\frac{1}{p_i}}\right]\right)\left(\frac{p_m}{N}\right)^{\alpha_m} b_m \cdot \mathbb{W}^{\frac{1}{p_m}}\Bigg]^{\frac{1}{p}},
\end{aligned}$$

then $\mathbb{W} = 0$, where r is defined in assumption (v).

With this set of assumptions, Equation (1) has a unique solution $x \in L_1^N$ (in the set Q).

Proof. Using assumption (vi), we obtain

$$\begin{aligned}
\Big||h(s,x)| - |h(s,0)|\Big| \quad & \leq \quad |h(s,x) - h(s,0)| \leq b|x|^{\frac{1}{p}} \\
\Rightarrow |h(s,x)| \quad & \leq \quad |u(s,0)| + b|x|^{\frac{1}{p}} \leq a(s) + b|x|^{\frac{1}{p}}.
\end{aligned}$$

Similarly, we can prove that $|f_i(\varphi,x)| \leq a_i(\varphi) + b_i|x|^{\frac{1}{p_i}}$, $i = 1, \cdots, m$. Then from Theorem 3 we conclude that (1) has at least one solution $x \in L_1^N$ on $\mathbb{R}^+$.

Next, let x and y be any two solutions of Equation (1). We then obtain

$$\begin{aligned}
& \|x - y\|_{L_1^N} = \Big\|K_0F_h(A(x)) - K_0F_h(A(y))\Big\|_{L_1^N} \\
\leq \quad & \|K_0\|_{L_1^N}\|F_h(A(x)) - F_h(Ay)\|_{L_p^N} \\
\leq \quad & \|K_0\|_{L_1^N}\|b|A(x) - A(y)|^{\frac{1}{p}}\|_{L_p^N} \\
= \quad & b\|K_0\|_{L_1^N}\|A(x) - A(y)\|_{L_1^N}^{\frac{1}{p}} \\
= \quad & b\|K_0\|_{L_1^N}\left\|\prod_{i=1}^{m}A_i(x) - \prod_{i=1}^{m}A_i(y)\right\|_{L_1^N}^{\frac{1}{p}} \\
= \quad & b\|K_0\|_{L_1^N}\Big\|A_1(x)A_2(x)\cdots A_m(x) - A_1(y)A_2(y)\cdots A_m(y)\Big\|_{L_1^N}^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&\le b\|K_0\|_{L_1^N}\bigg[\Big\|A_1(x)A_2(x)\cdots A_m(x)-A_1(y)A_2(x)\cdots A_m(x)\Big\|_{L_1^N}\\
&\quad+\Big\|A_1(y)A_2(x)\cdots A_m(x)-A_1(y)A_2(y)A_2(x)\cdots A_m(x)\Big\|_{L_1^N}\\
&\quad+\cdots+\Big\|A_1(y)A_2(y)\cdots A_{m-1}(y)A_m(x)-A_1(y)A_2(y)\cdots A_{m-1}(y)A_m(y)\Big\|_{L_1^N}\bigg]^{\frac{1}{p}}\\
&= b\|K_0\|_{L_1^N}\bigg[\Big\||A_1(x)-A_1(y)|A_2(x)\cdots A_m(x)\Big\|_{L_1^N}\\
&\quad+\Big\|A_1(y)|A_2(x)-A_2(y)|A_3(x)\cdots A_m(x)\Big\|_{L_1^N}\\
&\quad+\cdots+\Big\|A_1(y)A_2(y)\cdots A_{m-1}(y)|A_m(x)-A_m(y)|\Big\|_{L_1^N}\bigg]^{\frac{1}{p}}\\
&\le b\|K_0\|_{L_1^N}\bigg[\Big\|A_1(x)-A_1(y)\Big\|_{L_{p_1}^N}\prod_{i=2}^{m}\|A_i(x)\|_{L_{p_i}^N}\\
&\quad+\|A_1(y)\|_{L_{p_1}^N}\Big\|A_2(x)-A_2(y)\Big\|_{L_{p_2}^N}\prod_{i=3}^{m}\|A_i(x)\|_{L_{p_i}^N}\\
&\quad+\cdots+\prod_{i=1}^{m-1}\|A_i(y)\|_{L_{p_i}^N}\Big\|A_m(x)-A_m(y)\Big\|_{L_{p_m}^N}\bigg]^{\frac{1}{p}}\\
&\le b\|K_0\|_{L_1^N}\bigg[\Big\|I^{\alpha_1}|f_1(\varphi,x)-f_1(\varphi,y)|\Big\|_{L_{p_1}^N}\prod_{i=2}^{m}\left(\|g_i\|_{L_{p_i}^N}+\left(\frac{p_i}{N}\right)^{\alpha_i}\left[\|a_i\|_{L_{p_i}^N}+b_i\cdot r^{\frac{1}{p_i}}\right]\right)\\
&\quad+\left(\|g_1\|_{L_{p_1}^N}+\left(\frac{p_1}{N}\right)^{\alpha_1}\left[\|a_1\|_{L_{p_1}^N}+b_1\cdot r^{\frac{1}{p_1}}\right]\right)\Big\|I^{\alpha_2}|f_2(\varphi,x)-f_2(\varphi,y)|\Big\|_{L_{p_2}^N}\\
&\quad\times\prod_{i=3}^{m}\left(\|g_i\|_{L_{p_i}^N}+\left(\frac{p_i}{N}\right)^{\alpha_i}\left[\|a_i\|_{L_{p_i}^N}+b_i\cdot r^{\frac{1}{p_i}}\right]\right)\\
&\quad+\cdots+\prod_{i=1}^{m-1}\left(\|g_i\|_{L_{p_i}^N}+\left(\frac{p_i}{N}\right)^{\alpha_i}\left[\|a_i\|_{L_{p_i}^N}+b_i\cdot r^{\frac{1}{p_i}}\right]\right)\Big\|I^{\alpha_m}|f_m(\varphi,x)-f_m(\varphi,y)|\Big\|_{L_{p_m}^N}\bigg]^{\frac{1}{p}}\\
&\le b\|K_0\|_{L_1^N}\bigg[\left(\frac{p_1}{N}\right)^{\alpha_1}\Big\|f_1(\varphi,x)-f_1(\varphi,y)\Big\|_{L_{p_1}^N}\prod_{i=2}^{m}\left(\|g_i\|_{L_{p_i}^N}+\left(\frac{p_i}{N}\right)^{\alpha_i}\left[\|a_i\|_{L_{p_i}^N}+b_i\cdot r^{\frac{1}{p_i}}\right]\right)\\
&\quad+\left(\|g_1\|_{L_{p_1}^N}+\left(\frac{p_1}{N}\right)^{\alpha_1}\left[\|a_1\|_{L_{p_1}^N}+b_1\cdot r^{\frac{1}{p_1}}\right]\right)\left(\frac{p_2}{N}\right)^{\alpha_2}\Big\|f_2(\varphi,x)-f_2(\varphi,y)\Big\|_{L_{p_2}^N}\\
&\quad\times\prod_{i=3}^{m}\left(\|g_i\|_{L_{p_i}^N}+\left(\frac{p_i}{N}\right)^{\alpha_i}\left[\|a_i\|_{L_{p_i}^N}+b_i\cdot r^{\frac{1}{p_i}}\right]\right)\\
&\quad+\cdots+\prod_{i=1}^{m-1}\left(\|g_i\|_{L_{p_i}^N}+\left(\frac{p_i}{N}\right)^{\alpha_i}\left[\|a_i\|_{L_{p_i}^N}+b_i\cdot r^{\frac{1}{p_i}}\right]\right)\left(\frac{p_m}{N}\right)^{\alpha_m}\Big\|f_m(\varphi,x)-f_m(\varphi,y)\Big\|_{L_{p_m}^N}\bigg]^{\frac{1}{p}}\\
&\le b\|K_0\|_{L_1^N}\bigg[\left(\frac{p_1}{N}\right)^{\alpha_1}\Big\|b_1|x-y|^{\frac{1}{p_1}}\Big\|_{L_{p_1}^N}\prod_{i=2}^{m}\left(\|g_i\|_{L_{p_i}^N}+\left(\frac{p_i}{N}\right)^{\alpha_i}\left[\|a_i\|_{L_{p_i}^N}+b_i\cdot r^{\frac{1}{p_i}}\right]\right)\\
&\quad+\left(\|g_1\|_{L_{p_1}^N}+\left(\frac{p_1}{N}\right)^{\alpha_1}\left[\|a_1\|_{L_{p_1}^N}+b_1\cdot r^{\frac{1}{p_1}}\right]\right)\left(\frac{p_2}{N}\right)^{\alpha_2}\Big\|b_2|x-y|^{\frac{1}{p_2}}\Big\|_{L_{p_2}^N}\\
&\quad\times\prod_{i=3}^{m}\left(\|g_i\|_{L_{p_i}^N}+\left(\frac{p_i}{N}\right)^{\alpha_i}\left[\|a_i\|_{L_{p_i}^N}+b_i\cdot r^{\frac{1}{p_i}}\right]\right)\\
&\quad+\cdots+\prod_{i=1}^{m-1}\left(\|g_i\|_{L_{p_i}^N}+\left(\frac{p_i}{N}\right)^{\alpha_i}\left[\|a_i\|_{L_{p_i}^N}+b_i\cdot r^{\frac{1}{p_i}}\right]\right)\left(\frac{p_m}{N}\right)^{\alpha_m}\Big\|b_m|x-y|^{\frac{1}{p_m}}\Big\|_{L_{p_m}^N}\bigg]^{\frac{1}{p}}
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & b\|K_0\|_{L_1^N}\Bigg[\left(\frac{p_1}{N}\right)^{\alpha_1}\prod_{i=2}^{m}\left(\|g_i\|_{L_{p_i}^N}+\left(\frac{p_i}{N}\right)^{\alpha_i}\left[\|a_i\|_{L_{p_i}^N}+b_i\cdot r^{\frac{1}{p_i}}\right]\right)b_1\|x-y\|_{L_1^N}^{\frac{1}{p_1}} \\
&+\left(\|g_1\|_{L_{p_1}^N}+\left(\frac{p_1}{N}\right)^{\alpha_1}\left[\|a_1\|_{L_{p_1}^N}+b_1\cdot r^{\frac{1}{p_1}}\right]\right)\left(\frac{p_2}{N}\right)^{\alpha_2} \\
&\times\prod_{i=3}^{m}\left(\|g_i\|_{L_{p_i}^N}+\left(\frac{p_i}{N}\right)^{\alpha_i}\left[\|a_i\|_{L_{p_i}^N}+b_i\cdot r^{\frac{1}{p_i}}\right]\right)b_2\|x-y\|_{L_1^N}^{\frac{1}{p_2}} \\
&+\cdots+\prod_{i=1}^{m-1}\left(\|g_i\|_{L_{p_i}^N}+\left(\frac{p_i}{N}\right)^{\alpha_i}\left[\|a_i\|_{L_{p_i}^N}+b_i\cdot r^{\frac{1}{p_i}}\right]\right)\left(\frac{p_m}{N}\right)^{\alpha_m}b_m\|x-y\|_{L_1^N}^{\frac{1}{p_m}}\Bigg]^{\frac{1}{p}}.
\end{aligned}
$$

Using assumption (vii) together with the above inequality, we obtain $x=y$ almost everywhere, and this concludes the proof. □

4. Applications

In the last part of the paper, we will check the existence of solutions for the special case of Equation (1) and the solvability of the initial value problem (2) through Theorem 3.

4.1. Fractional Gripenberg Equations

Consider the following generalized Gripenberg integral equation:

$$x(t)=\int_0^t\frac{(t-\varphi)^{-\gamma}}{\Gamma(1-\gamma)}h\left(\varphi,\prod_{i=1}^{m}\left(g_i(\varphi)+\frac{1}{\Gamma(\alpha_i)}\int_0^{\varphi}\frac{f_i(s,x(s))}{(\varphi-s)^{1-\alpha_i}}ds\right)\right), \tag{16}$$

where $t\in\mathbb{R}^+,\ \gamma,\alpha_i\in(0,1)$. Equation (16) is related to

$$x(t)=I^{1-\gamma}h\left(t,\ \prod_{i=1}^{m}\left(g_i(t)+I^{\alpha_i}f_i(t,x(t))\right)\right),\ t\in\mathbb{R}^+,\gamma,\alpha_i\in(0,1).$$

Equation (16) is a special case of (1), as we can put $K(t,\varphi)=\frac{(t-\varphi)^{-\gamma}}{\Gamma(1-\gamma)},\ \gamma\in(0,1)$. Thus for Riemann–Liouville fractional operators we have

Corollary 1. *Let assumptions of Theorem 3 be satisfied with $K(t,s)=\frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)},\ \gamma\in(0,1)$. Then Equation (16) has at least one solution $x\in L_1^N$ on $\mathbb{R}^+$.*

Corollary 2. *Let assumptions of Theorem 4 be fulfilled with $K(t,s)=\frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)},\ \gamma\in(0,1)$. Then Equation (16) has a unique solution $x\in Q\subset L_1^N$.*

4.2. Initial Value Problems

We will now look at the result concerning IVP (2). As a solution to a differential problem, we require here nothing more from the function than its absolute continuity.

Definition 5. *A function z is a solution of IVP (2) if it is absolutely continuous and satisfies IVP (2).*

Theorem 5. *Let the assumptions of Theorem 3 be fulfilled with $K(t,\varphi)=\frac{(t-\varphi)^{-\gamma}}{\Gamma(1-\gamma)}$. Then the initial value problem (2) has a solution $z(t)=I^{\gamma}x(t)$, where x fulfills (16).*

Proof. Let x satisfy Equation (16). Put

$$z(t)=I^{\gamma}x(t),\ \ \gamma\in(0,1), \tag{17}$$

and then, by applying the semigroup property for fractional integrals

$$\begin{aligned} z(t) &= I^{\gamma} x(t) \\ &= I^{\gamma} I^{1-\gamma} h\Big(t, \prod_{i=1}^{m} \Big(g_i(t) + I^{\alpha_i} f_i(t, x(t))\Big)\Big) \\ &= I^{1} h\Big(t, \prod_{i=1}^{m} \Big(g_i(t) + I^{\alpha_i} f_i(t, x(t))\Big)\Big) \\ &= \int_0^t h\Big(\varphi, \prod_{i=1}^{m} \Big(g_i(\varphi) + \frac{1}{\Gamma(\alpha_i)} \int_0^{\varphi} \frac{f_i(s, x(s))}{(\varphi - s)^{1-\alpha_i}} ds\Big)\Big). \end{aligned}$$

We obtain that z is absolutely continuous, so, in particular, the derivative $D^{\gamma} z(t)$ exists and $z(0) = 0$. Further,

$$\begin{aligned} D^{\gamma} z(t) &= I^{1-\gamma} \frac{d}{dt} z(t) = I^{1-\gamma} \frac{d}{dt} I^{\gamma} x(t) \\ &= I^{1-\gamma} \frac{d}{dt} I^{\gamma} \Big(I^{1-\gamma} h\Big(t, \prod_{i=1}^{m} \Big(g_i(t) + I^{\alpha_i} f_i(t, x(t))\Big)\Big)\Big) \\ &= \frac{d}{dt} \Big(I^{1-\gamma} I^{\gamma} \Big) I^{1-\gamma} h\Big(t, \prod_{i=1}^{m} \Big(g_i(t) + I^{\alpha_i} f_i(t, x(t))\Big)\Big) \\ &= \frac{d}{dt} I^{1} I^{1-\gamma} h\Big(t, \prod_{i=1}^{m} \Big(g_i(t) + I^{\alpha_i} f_i(t, x(t))\Big)\Big) \\ &= (\frac{d}{dt} I^{1}) I^{1-\gamma} h\Big(t, \prod_{i=1}^{m} \Big(g_i(t) + I^{\alpha_i} f_i(t, x(t))\Big)\Big) \\ &= I^{1-\gamma} h\Big(t, \prod_{i=1}^{m} \Big(g_i(t) + I^{\alpha_i} f_i(t, x(t))\Big)\Big) \\ &= x(t). \end{aligned}$$

Applying Equation (17), we obtain

$$\begin{aligned} \frac{d}{dt} z(t) &= \frac{d}{dt} \Big(I^{\gamma} x(t) \Big) \\ &= \frac{d}{dt} I^{\gamma} \Big(I^{1-\gamma} h\Big(t, \prod_{i=1}^{m} \Big(g_i(t) + I^{\alpha_i} f_i(t, x(t))\Big)\Big)\Big) \\ &= \frac{d}{dt} I^{1} h\Big(t, \prod_{i=1}^{m} \Big(g_i(t) + I^{\alpha_i} f_i(t, x(t))\Big)\Big) \\ &= h\Big(t, \prod_{i=1}^{m} \Big(g_i(t) + I^{\alpha_i} f_i(t, x(t))\Big)\Big), \end{aligned}$$

which means that if $x(t)$ fulfills Equation (16), then $z(t) = I^{\gamma} x(t)$ satisfies the initial value problem (2), which completes the proof. □

Remark 2. *We should complete the paper with a note on the numerical treatment of the considered problem. Because our considerations are based on properly developed Gronwall-type inequalities (cf. Step 3 of the proof of Theorem 3), presenting an interesting numerical example requires discrete equivalents of such inequalities. For the case $m = 1$ one can find such a result, e.g., in a recently published paper [40] or in [3], but quadratic problems and for the pointwise product of m operators have not yet been studied, and their inclusion here would definitely go beyond the intended scope and volume of the paper. However, the main problem in this case is the use of numerical methods for non-continuous functions. This problem with continuous data for some fractional equations is* *described in [26], for instance. Integrable solutions for some (non-quadratic) singular problems were studied in [41]. However, research in Orlicz's spaces remains an open problem.*

Basic approximation and numerical treatment for quadratic integral equations (related to the quadratic Chandrasekhar equation) can be found in [42], where a class of approximate methods for solution of this equation in a general context is presented. Interestingly, this paper also uses the same type of weight, although in the space of continuous functions. Thus, our paper provides a basis for further research.

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