

Article

Controllability Results for First Order Linear Fuzzy Differential Systems

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Abstract: In this paper, we investigate the controllability of first order linear fuzzy differential systems. We use the direct construction method to derive the controllability results for three types of first order linear fuzzy controlled systems via $(c1)$ -solution and $(c2)$ -solution, respectively. An example is presented to illustrate our theoretical results.

Keywords: linear fuzzy differential systems; controllability; $(c1)$ -solution; $(c2)$ -solution

MSC: 34A37; 93C05



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1. Introduction

The study of controllability is an important area of research and classical differential controlled systems have been discussed in many papers. In [1], controllability of impulsive differential equations was studied for the first time. The authors in [2] established some sufficient conditions for controllability by applying the measure of noncompactness and Mönch's fixed point theorem. In [3], the authors presented complete controllability of the considered impulsive control system and in [4], the author gave local and global controllability properties for nonlinear systems. In [5], the controllability and the observability of continuous linear time-varying systems with norm-bounded parameter perturbations was considered. When using classical mathematical models alone, many equations need to be solved due to the uncertainty of the measurement parameters and this can be time-consuming and can be financially burdensome to compute. The disadvantages of uncertainty can be overcome by fuzzy differential systems. The concept of controllability was introduced by Kalman [6] in 1963 and has played an important role in control theory since then. We have done some research on fuzzy differential systems [7,8], in this paper we will go further investigate the controllability of first-order linear fuzzy differential systems and it can resolve the controllability problem with uncertainty more accurately.

The main contributions are as follows:

Comparing the relevant literature above, we use the direct construction method to present the controllability of three types of first order linear fuzzy differential systems in the case of $a > 0$, $(c1)$ -solution; $a > 0$, $(c2)$ -solution; $a < 0$, $(c1)$ -solution; $a < 0$, $(c2)$ -solution, respectively.

While the method used is standard in some sense the constructed approach is technical and the control function given for our problem is a function with hyperbolic sine and cosine.

The controllability of first order fuzzy differential systems in iterative feedback tuning in fuzzy control system, fuzzy control design for non-holonomic wheeled mobile and fuzzy optimal control for complex nonlinear systems is useful and for more details we refer the reader to [9–11].

In this paper, we consider the controllability of three types of first order linear fuzzy differential systems

$$\begin{cases} y'(t) = a(t)y(t) + b(t) + \tilde{d}u(t), & t \in J = [0, T], \\ y(0) = y_0, \end{cases} \tag{1}$$

and

$$\begin{cases} y'(t) + (-a(t))y(t) = b(t) + \tilde{d}u(t), & t \in J, \\ y(0) = y_0, \end{cases} \tag{2}$$

and

$$\begin{cases} y'(t) + (-b(t) + \tilde{d}u(t)) = a(t)y(t), & t \in J, \\ y(0) = y_0, \end{cases} \tag{3}$$

where $a : J \rightarrow \mathbb{R}$, $y_0 \in \mathbb{R}_F$ and $b : J \rightarrow \mathbb{R}_F$, $u : J \rightarrow \mathbb{R}_F$, $\tilde{d} \in \mathbb{R}_+$.

In Section 2, we present notations, concepts, and lemmas needed in this paper. In Section 3, we establish some theorems concerning the controllability of first order linear fuzzy differential systems. Finally, in the last section, we give an example to illustrate our main results.

2. Preliminaries

We collect some concepts which will be used throughout the paper; for more details, see for [12,13].

Denote by $\mathbb{R}_F := \{v \mid v : \mathbb{R} \rightarrow [0, 1]\}$ the class of the fuzzy subsets of the real axis satisfying the following properties:

- (X₁) v is normal (i.e., $\exists x_0 \in \mathbb{R}$ s.t. $v(x_0) = 1$).
- (X₂) v is convex fuzzy set (i.e., $v(\xi s_0 + (1 - \xi)s_1) \geq \min\{v(s_0), v(s_1)\}$) for all $s_0, s_1 \in \mathbb{R}$ and $\xi \in [0, 1]$.
- (X₃) v is upper semicontinuous on \mathbb{R} .
- (X₄) $[v]^0 = \overline{\{x \in \mathbb{R} : v(x) > 0\}}$ is compact.

Let $\alpha \in (0, 1]$. Consider the α -level set of $v \in \mathbb{R}_F$ by $[v]^\alpha = \{s \in \mathbb{R} \mid v(s) \geq \alpha\}$, which is a nonempty compact interval for all $\alpha \in (0, 1]$. We use $[v]^\alpha = [\underline{v}_\alpha, \bar{v}_\alpha]$ to denote explicitly the α -level set of v . We call \underline{v}_α and \bar{v}_α the lower and upper branches of v , respectively. We use the notation $diam([v]^\alpha) = \bar{v}_\alpha - \underline{v}_\alpha$ to denote the length of v .

Now $\forall \alpha \in [0, 1]$, $u, v \in \mathbb{R}_F$ and $\xi \in \mathbb{R}$, we define the sum $u + v$ and the produce ξu as $[u + v]^\alpha = [u]^\alpha + [v]^\alpha = [\underline{u}_\alpha + \underline{v}_\alpha, \bar{u}_\alpha + \bar{v}_\alpha]$ and $[\xi u]^\alpha = \xi [u]^\alpha$.

Consider the Hausdorff distance $D : \mathbb{R}_F \times \mathbb{R}_F \rightarrow \mathbb{R}_+ \cup \{0\}$ where $D(u, v) = \sup_{0 \leq \alpha \leq 1} d_H([u]^\alpha, [v]^\alpha) = \sup_{0 \leq \alpha \leq 1} \max\{|\underline{u}_\alpha - \underline{v}_\alpha|, |\bar{u}_\alpha - \bar{v}_\alpha|\}$ (see [14]). Then (\mathbb{R}_F, D) is a complete metric space (see [15]) and (i) $D(u + e, v + e) = D(u, v)$, $\forall u, v, e \in \mathbb{R}_F$, (ii) $D(\zeta u, \zeta v) = |\zeta|D(u, v)$, $\forall \zeta \in \mathbb{R}$, $u, v \in \mathbb{R}_F$, (iii) $D(u + e, v + \rho) \leq D(u, v) + D(e, \rho)$, $\forall u, v, e, \rho \in \mathbb{R}_F$ are satisfied.

Definition 1 (see [13] (Definition 2.1)). Let $f : [a, b] \rightarrow \mathbb{R}_F$ be measurable and integrably bounded. The integral of f over $[a, b]$, denoted by $\int_a^b f(t)dt$, is defined levelwise by the expression

$$\begin{aligned} \left[\int_a^b f(t)dt \right]^\alpha &:= \int_a^b [f(t)]^\alpha dt \\ &= \left\{ \int_a^b \tilde{f}(t)dt \mid \tilde{f} : [a, b] \rightarrow \mathbb{R}_F \text{ is a measurable selection for } [f(\cdot)]^\alpha \right\}, \end{aligned}$$

for every $\alpha \in [0, 1]$.

Throughout this paper, we use the symbol \ominus to represent the H -difference. Note that $a_1 \ominus a_2 \neq a_1 + (-1)a_2 := a_1 - a_2$. In the sequel, we fix $J = [0, T]$, for $T > 0$.

Here we simplify the classes of strongly generalized differentiable by considering case (i) and case (ii) as in paper [16].

Definition 2 (see [13] (Definition 2.2)). Let $Q : J \rightarrow \mathbb{R}_F$ and fix $n_0 \in J$. We say Q is differentiable at n_0 , if we have an element $Q'(n_0) \in \mathbb{R}_F$ such that either

(c1) for all $p > 0$ sufficiently close to 0, the H -differences $Q(n_0 + p) \ominus Q(n_0)$, $Q(n_0) \ominus Q(n_0 - p)$ exist and the limits (in the metric D)

$$\lim_{p \rightarrow 0^+} \frac{Q(n_0 + p) \ominus Q(n_0)}{p} = \lim_{p \rightarrow 0^+} \frac{Q(n_0) \ominus Q(n_0 - p)}{p} = Q'(n_0),$$

or

(c2) for all $p > 0$ sufficiently close to 0, the H -differences $Q(n_0) \ominus Q(n_0 + p)$, $Q(n_0 - p) \ominus Q(n_0)$ exist and the limits (in the metric D)

$$\lim_{p \rightarrow 0^+} \frac{Q(n_0) \ominus Q(n_0 + p)}{-p} = \lim_{p \rightarrow 0^+} \frac{Q(n_0 - p) \ominus Q(n_0)}{-p} = Q'(n_0).$$

Definition 3 (See [13] (Definition 2.5)). Let $Q : J \rightarrow \mathbb{R}_F$. We say Q is (c1)-differentiable on J if Q is differentiable in the sense (c1) in Definition 2 and its derivative is denoted D_1Q . Similarly we can define (c2)-differentiable and denote it by D_2Q .

Theorem 1 (see [13] (Theorem 2.6)). Let $Q : J \rightarrow \mathbb{R}_F$ and put $[Q(t)]^\alpha = [p_\alpha(t), q_\alpha(t)]$ for each $\alpha \in [0, 1]$.

(i) If Q is (c1)-differentiable then p_α and q_α are differentiable functions and $[D_1Q(t)]^\alpha = [p'_\alpha(t), q'_\alpha(t)]$.

(ii) If Q is (c2)-differentiable then p_α and q_α are differentiable functions and we have $[D_2Q(t)]^\alpha = [q'_\alpha(t), p'_\alpha(t)]$.

Theorem 2 (see [17] (Theorem 2.2)). Let $K : J \rightarrow \mathbb{R}_F$ be a differentiable fuzzy number-valued mapping and we suppose that the derivative K' is integrable over J . Then for each $t \in J$, we have

(a) if K is (c1)-differentiable, then $K(t) = K(b) + \int_b^t K'(s)ds$;

(b) if K is (c2)-differentiable, then $K(t) = K(b) \ominus \int_b^t -K'(s)ds$.

Theorem 3 (see [13] (Theorem 2.7)). Let Q be (c2)-differentiable on J and assume that the derivative Q' is integrable over J . Then for each $t \in J$ we have

$$Q(t) = Q(a) \ominus \int_a^t -Q'(\tau)d\tau.$$

Theorem 4 (see [18] (Theorem 2.4)). Let $K : J \rightarrow \mathbb{R}_F$ be continuous. Define the integral $G(t) := \sigma \ominus \int_v^t -K(s)ds, t \in J$, where $\sigma \in \mathbb{R}_F$ is such that the preceding H-difference exist on J . Then $G(t)$ is (c2)-differentiable and $G'(t) = K(t)$.

Consider the following conditions (here $p : \mathbb{R} \rightarrow \mathbb{R}_F$):

Hypothesis 1. For a given $t \in J$, $p(t + h) \ominus p(t)$ and $p(t) \ominus p(t - h)$ exist for $h \rightarrow 0^+$;

Hypothesis 2. For a given $t \in J$, $p(t) \ominus p(t + h)$ and $p(t - h) \ominus p(t)$ exist for $h \rightarrow 0^+$.

3. Controllability of First Order Linear Fuzzy Differential Systems

Definition 4. Fuzzy system (1)–(3) is called controllable on $[t_0, T]$ ($T > t_0$), if for an arbitrary initial state $y_0 \in \mathbb{R}_F$ at t_0 , final state $y_1 \in \mathbb{R}_F$ at time T_1 (here $t_0 = 0$ and $T_1 = T$), there exists a control $u : J \rightarrow \mathbb{R}_F$ such that the system (1)–(3) has a solution y that satisfies $y(T_1) = y_1$ (i.e., $[y(T_1)]^\alpha = [y_1]^\alpha$).

Consider the following system (see [17]):

$$\begin{cases} y'(t) = a(t)y(t) + b(t) + \tilde{d}u(t), t \in J, \\ y(0) = y_0, \end{cases}$$

where $a : J \rightarrow \mathbb{R}, y_0 \in \mathbb{R}_F$ and $b : J \rightarrow \mathbb{R}_F, u : J \rightarrow \mathbb{R}_F, \tilde{d} \in \mathbb{R}_+$.

From [17] (Theorem 3.1, $a < 0$), the (c1)-solution of (1) can be written in the form:

$$\begin{aligned} y(t) = & \cosh\left(\int_0^t a(v)dv\right)\left(y_0 + \int_0^t \left[(b(s) + \tilde{d}u(s)) \cosh\left(\int_0^s a(v)dv\right)\right. \right. \\ & \left. \left. \ominus (b(s) + \tilde{d}u(s)) \sinh\left(\int_0^s a(v)dv\right)\right] ds\right) \\ & + \sinh\left(\int_0^t a(v)dv\right)\left(y_0 + \int_0^t \left[(b(s) + \tilde{d}u(s)) \cosh\left(\int_0^s a(v)dv\right)\right. \right. \\ & \left. \left. \ominus (b(s) + \tilde{d}u(s)) \sinh\left(\int_0^s a(v)dv\right)\right] ds\right). \end{aligned}$$

From [17] (Theorem 3.1, $a < 0$), the (c2)-solution of (1) can be written in the form:

$$y(t) = e^{\int_0^t a(v)dv} y_0 \ominus e^{\int_0^t a(v)dv} \int_0^t (-b(s)) e^{-\int_0^s a(v)dv} ds \ominus e^{\int_0^t a(v)dv} \int_0^t (-\tilde{d}u(s)) e^{-\int_0^s a(v)dv} ds.$$

Thus, we consider two cases to study the controllability of (1): The (c1)-solution and the (c2)-solution.

Case 1.1 Consider $a < 0$ via (c1)-solution.

Theorem 5. In Case 1.1, system (1) is controllable, if the control function $u(t)$ is given by

$$\begin{aligned} u(t) = & \frac{1}{T_1 \tilde{d}} \left[y_1 \cosh\left(\int_0^{T_1} a(v)dv - \int_0^t a(v)dv\right) \ominus y_1 \sinh\left(\int_0^{T_1} a(v)dv - \int_0^t a(v)dv\right) \right. \\ & \left. \ominus \left(\cosh\left(\int_0^t a(v)dv\right) y_0 + \sinh\left(\int_0^t a(v)dv\right) y_0 \right) \right] \ominus \frac{1}{\tilde{d}} b(t), t \in J, \end{aligned}$$

where the H-differences exist.

Proof. Since the H -differences exist in $u(t)$, for $T_1 > 0$, we obtain

$$\begin{aligned}
 y(T_1) &= \cosh\left(\int_0^{T_1} a(v)dv\right)y_0 + \sinh\left(\int_0^{T_1} a(v)dv\right)y_0 \\
 &+ \frac{1}{T_1} \int_0^{T_1} \cosh\left(\int_0^s a(v)dv\right) \cosh\left(\int_0^s a(v)dv\right)y_1 ds \\
 &+ \frac{1}{T_1} \int_0^{T_1} \sinh\left(\int_0^s a(v)dv\right) \cosh\left(\int_0^s a(v)dv\right)y_1 ds \\
 &\ominus \frac{1}{T_1} \int_0^{T_1} \cosh\left(\int_0^s a(v)dv\right) \sinh\left(\int_0^s a(v)dv\right)y_1 ds \\
 &\ominus \frac{1}{T_1} \int_0^{T_1} \sinh\left(\int_0^s a(v)dv\right) \sinh\left(\int_0^s a(v)dv\right)y_1 ds \\
 &\ominus \frac{1}{T_1} \int_0^{T_1} \cosh\left(\int_0^{T_1} a(v)dv + \int_0^s a(v)dv\right) \cosh\left(\int_0^s a(v)dv\right)y_0 ds \\
 &\ominus \frac{1}{T_1} \int_0^{T_1} \sinh\left(\int_0^{T_1} a(v)dv + \int_0^s a(v)dv\right) \cosh\left(\int_0^s a(v)dv\right)y_0 ds \\
 &+ \frac{1}{T_1} \int_0^{T_1} \cosh\left(\int_0^{T_1} a(v)dv + \int_0^s a(v)dv\right) \sinh\left(\int_0^s a(v)dv\right)y_0 ds \\
 &+ \frac{1}{T_1} \int_0^{T_1} \sinh\left(\int_0^{T_1} a(v)dv + \int_0^s a(v)dv\right) \sinh\left(\int_0^s a(v)dv\right)y_0 ds \\
 &= \left(\cosh\left(\int_0^{T_1} a(v)dv\right)y_0 \ominus \cosh\left(\int_0^{T_1} a(v)dv\right)y_0\right) \\
 &+ \left(\sinh\left(\int_0^{T_1} a(v)dv\right)y_0 \ominus \sinh\left(\int_0^{T_1} a(v)dv\right)y_0\right) + y_1 \\
 &= y_1.
 \end{aligned}$$

Thus, the system (1) is controllable in this case. \square

Case 1.2 Consider $a < 0$ via the (c2)-solution.

Theorem 6. In Case 1.2, system (1) is controllable, if $W_0^{-1}[0, T_1]$ exists; here

$$W_0[0, T_1] = \int_0^{T_1} e^{-\int_0^s a(v)dv} \tilde{d}\tilde{d}e^{-\int_0^s a(v)dv} ds. \tag{4}$$

Proof. From $W_0[0, T_1] = \int_0^{T_1} e^{-\int_0^s a(v)dv} \tilde{d}\tilde{d}e^{-\int_0^s a(v)dv} ds$, for $T_1 > 0$, for any final state $y_1 \in \mathbb{R}_F$ we choose a control function as follows:

$$u_1(t) = -\tilde{d}e^{-\int_0^t a(v)dv} W_0^{-1}[0, T_1] \left(y_0 \ominus \int_0^{T_1} (-b(s))e^{-\int_0^s a(v)dv} ds \ominus e^{-\int_0^{T_1} a(v)dv} y_1 \right), t \in J,$$

where the H -differences exist.

Then we get

$$\begin{aligned}
 y(T_1) &= e^{\int_0^{T_1} a(v)dv} y_0 \ominus e^{\int_0^{T_1} a(v)dv} \int_0^{T_1} (-b(s)) e^{-\int_0^s a(v)dv} ds \\
 &\quad \ominus e^{\int_0^{T_1} a(v)dv} \int_0^{T_1} (-\tilde{d}u_1(s)) e^{-\int_0^s a(v)dv} ds \\
 &= e^{\int_0^{T_1} a(v)dv} y_0 \ominus e^{\int_0^{T_1} a(v)dv} \int_0^{T_1} (-b(s)) e^{\int_0^s a(v)dv} ds \\
 &\quad \ominus e^{\int_0^{T_1} a(v)dv} \int_0^{T_1} e^{-\int_0^s a(v)dv} \tilde{d}\tilde{d}e^{-\int_0^s a(v)dv} W_0^{-1}[0, T_1] \\
 &\quad \times \left(y_0 \ominus \int_0^{T_1} (-b(s)) e^{-\int_0^s a(v)dv} ds \ominus e^{-\int_0^{T_1} a(v)dv} y_1 \right) ds \\
 &= e^{\int_0^{T_1} a(v)dv} y_0 \ominus e^{\int_0^{T_1} a(v)dv} \int_0^{T_1} (-b(s)) e^{\int_0^s a(v)dv} ds \\
 &\quad \ominus e^{\int_0^{T_1} a(v)dv} y_0 + e^{\int_0^{T_1} a(v)dv} \int_0^{T_1} (-b(s)) e^{\int_0^s a(v)dv} ds \\
 &\quad + e^{\int_0^{T_1} a(v)dv} e^{-\int_0^{T_1} a(v)dv} y_1 \\
 &= y_1.
 \end{aligned}$$

That means system (1) is controllable in this case. \square

From [17] (Theorem 3.2, $a > 0$), the (c1)-solution of (1) can be written in the form:

$$y(t) = e^{\int_0^t a(v)dv} \left(y_0 + \int_0^t b(s) e^{-\int_0^s a(v)dv} ds + \int_0^t \tilde{d}u(s) e^{-\int_0^s a(v)dv} ds \right).$$

From [17] (Theorem 3.2, $a > 0$), the (c2)-solution of (1) can be written in the form:

$$\begin{aligned}
 y(t) &= \cosh \left(\int_0^t a(v)dv \right) \left(y_0 \ominus \int_0^t \left[(b(s) + \tilde{d}u(s)) \sinh \left(\int_0^s a(v)dv \right) \right. \right. \\
 &\quad \left. \left. - (b(s) + \tilde{d}u(s)) \cosh \left(\int_0^s a(v)dv \right) \right] ds \right) \\
 &\quad \ominus - \sinh \left(\int_0^t a(v)dv \right) \left(y_0 \ominus \int_0^t \left[(b(s) + \tilde{d}u(s)) \sinh \left(\int_0^s a(v)dv \right) \right. \right. \\
 &\quad \left. \left. - (b(s) + \tilde{d}u(s)) \cosh \left(\int_0^s a(v)dv \right) \right] ds \right).
 \end{aligned}$$

Thus, we consider two cases to study the controllability of (1): The (c1)-solution and the (c2)-solution.

Case 2.1 Consider $a > 0$ via the (c1)-solution.

Theorem 7. In Case 2.1, system (1) is controllable, if $V_\epsilon^{-1}[0, T_1]$ exists; here

$$V_\epsilon[0, T_1] = \int_0^{T_1} e^{-\int_0^s a(v)dv} \tilde{d}\tilde{d}e^{-\int_0^s a(v)dv} ds. \tag{5}$$

Proof. We set

$$u_0(t) = -\tilde{d}e^{-\int_0^t a(v)dv} V_\epsilon^{-1}[0, T_1] \left(y_0 \ominus \int_0^{T_1} (-b(s)) e^{-\int_0^s a(v)dv} ds \ominus e^{-\int_0^{T_1} a(v)dv} y_1 \right), \quad t \in J,$$

where the H -differences exist. The rest of proof is the same as in Case 1.2. \square

Case 2.2 Consider $a > 0$ via the (c2)-solution.

Theorem 8. In Case 2.2, system (1) is controllable, if the control function $u_2(t)$ is given by

$$\begin{aligned}
 u_2(t) &= \frac{1}{T_1 \tilde{d}} \left[\left(- \left(\cosh \left(\int_0^t a(v) dv \right) y_0 \ominus - \sinh \left(\int_0^t a(v) dv \right) y_0 \right) \right) \right. \\
 &\quad \ominus - y_1 \cosh \left(\int_0^{T_1} a(v) dv - \int_0^t a(v) dv \right) \\
 &\quad \left. \ominus y_1 \sinh \left(\int_0^{T_1} a(v) dv - \int_0^t a(v) dv \right) \right] \ominus \frac{1}{\tilde{d}} b(t), \quad t \in J,
 \end{aligned}$$

where the H -differences exist.

Proof. Since the H -differences exist in $u_2(t)$, for $T_1 > 0$, we get

$$\begin{aligned}
 &y(T_1) \\
 = &\left[\cosh \left(\int_0^{T_1} a(v) dv \right) y_0 \ominus \left(\frac{1}{T_1} \int_0^{T_1} \cosh \left(\int_0^{T_1} a(v) dv + \int_0^s a(v) dv \right) \cosh \left(\int_0^s a(v) dv \right) y_0 ds \right. \right. \\
 &\quad \left. \ominus \frac{1}{T_1} \int_0^{T_1} \sinh \left(\int_0^{T_1} a(v) dv + \int_0^s a(v) dv \right) \sinh \left(\int_0^s a(v) dv \right) y_0 ds \right] \\
 &+ \left[- \left(\frac{1}{T_1} \int_0^{T_1} \sinh \left(\int_0^{T_1} a(v) dv + \int_0^s a(v) dv \right) \cosh \left(\int_0^s a(v) dv \right) y_0 ds \right. \right. \\
 &\quad \left. \ominus \frac{1}{T_1} \int_0^{T_1} \cosh \left(\int_0^{T_1} a(v) dv + \int_0^s a(v) dv \right) \sinh \left(\int_0^s a(v) dv \right) y_0 ds \right. \\
 &\quad \left. \ominus - \sinh \left(\int_0^{T_1} a(v) dv \right) y_0 \right] + \frac{1}{T_1} \int_0^{T_1} \cosh \left(\int_0^s a(v) dv \right) \cosh \left(\int_0^s a(v) dv \right) y_1 ds \\
 &\quad \ominus \frac{1}{T_1} \int_0^{T_1} \sinh \left(\int_0^s a(v) dv \right) \sinh \left(\int_0^s a(v) dv \right) y_1 ds \\
 &\quad \ominus - \left(\frac{1}{T_1} \int_0^{T_1} \sinh \left(\int_0^s a(v) dv \right) \cosh \left(\int_0^s a(v) dv \right) y_1 ds \right. \\
 &\quad \left. \ominus \frac{1}{T_1} \int_0^{T_1} \cosh \left(\int_0^s a(v) dv \right) \sinh \left(\int_0^s a(v) dv \right) y_1 ds \right) \\
 = &y_1.
 \end{aligned}$$

Thus, system (1) is controllable in this case. \square

Consider system (2) (see [17]):

$$\begin{cases}
 y'(t) + (-a(t))y(t) = b(t) + \tilde{d}u(t), \quad t \in J, \\
 y(0) = y_0,
 \end{cases}$$

where $a : J \rightarrow \mathbb{R}$, $y_0 \in \mathbb{R}_F$ and $b : J \rightarrow \mathbb{R}_F$, $u : J \rightarrow \mathbb{R}_F$, $\tilde{d} \in \mathbb{R}_+$.

From [17] (Theorem 3.3, $a < 0$), the $(c1)$ -solution of (2) can be written in the form:

$$y(t) = e^{\int_0^t a(v) dv} \left(y_0 + \int_0^t (b(s) + \tilde{d}u(s)) e^{-\int_0^s a(v) dv} ds \right).$$

From [17] (Theorem 3.3, $a < 0$), the (c2)-solution of (2) can be written in the form:

$$\begin{aligned}
 y(t) = & \cosh \left(\int_0^t a(v)dv \right) \left(y_0 \ominus (-1) \int_0^t \left[(b(s) + \tilde{d}u(s)) \cosh \left(\int_0^s a(v)dv \right) \right. \right. \\
 & \left. \left. \ominus (b(s) + \tilde{d}u(s)) \sinh \left(\int_0^s a(v)dv \right) \right] ds \right) \\
 & + \sinh \left(\int_0^t a(v)dv \right) \left(y_0 \ominus (-1) \int_0^t \left[(b(s) + \tilde{d}u(s)) \cosh \left(\int_0^s a(v)dv \right) \right. \right. \\
 & \left. \left. \ominus (b(s) + \tilde{d}u(s)) \sinh \left(\int_0^s a(v)dv \right) \right] ds \right).
 \end{aligned}$$

Thus, we consider two cases to study the controllability of (1): The (c1)-solution and the (c2)-solution.

Case 3.1 Consider $a < 0$ via the (c1)-solution.

Theorem 9. In Case 3.1, system (2) is controllable, if $W_1^{-1}[0, T_1]$ exists; here

$$W_1[0, T_1] = \int_0^{T_1} e^{-\int_0^s a(v)dv} \tilde{d} \tilde{d} e^{-\int_0^s a(v)dv} ds. \tag{6}$$

Proof. Let

$$u_3(t) = \tilde{d} e^{-\int_0^t a(v)dv} W_1^{-1}[0, T_1] \left(e^{-\int_0^{T_1} a(v)dv} y_1 \ominus y_0 \ominus \int_0^{T_1} (-b(s)) e^{-\int_0^s a(v)dv} ds \right), \quad t \in J,$$

where the H -differences exist. The method of proof is the same as in Case 1.2, so we omit it here. \square

Case 3.2 Consider $a < 0$ via the (c2)-solution.

Theorem 10. In Case 3.2, system (2) is controllable, if the control function $u_4(t)$ is given by

$$\begin{aligned}
 u_4(t) = & \frac{1}{T_1 \tilde{d}} \left[\left(-\cosh \left(\int_0^t a(v)dv \right) y_0 - \sinh \left(\int_0^t a(v)dv \right) y_0 \right) \right. \\
 & \left. \ominus -y_1 \cosh \left(\int_0^{T_1} a(v)dv - \int_0^t a(v)dv \right) \right. \\
 & \left. -y_1 \sinh \left(\int_0^{T_1} a(v)dv - \int_0^t a(v)dv \right) \right] \ominus \frac{1}{\tilde{d}} b(t), \quad t \in J,
 \end{aligned}$$

where the H -differences exist.

Proof. The method of proof is the same as in Case 1.1, so we omit it here. \square

From [17] (Theorem 3.5, $a > 0$), the (c1)-solution of (2) can be written in the form:

$$\begin{aligned}
 y(t) = & \cosh \left(\int_0^t a(v)dv \right) \left(y_0 + \int_0^t \left[(b(s) + \tilde{d}u(s)) \sinh \left(\int_0^s a(v)dv \right) \right. \right. \\
 & \left. \left. - (b(s) + \tilde{d}u(s)) \cosh \left(\int_0^s a(v)dv \right) \right] ds \right) \\
 & \ominus -\sinh \left(\int_0^t a(v)dv \right) \left(y_0 + \int_0^t \left[(b(s) + \tilde{d}u(s)) \sinh \left(\int_0^s a(v)dv \right) \right. \right. \\
 & \left. \left. - (b(s) + \tilde{d}u(s)) \cosh \left(\int_0^s a(v)dv \right) \right] ds \right).
 \end{aligned}$$

From [17] (Theorem 3.5, $a > 0$), the (c2)-solution of (2) can be written in the form:

$$y(t) = e^{\int_0^t a(v)dv} \left(y_0 \ominus \int_0^t (-b(s) - \tilde{d}u(s))e^{-\int_0^s a(v)dv} ds \right).$$

Thus, we consider two cases to study the controllability of (1): The (c1)-solution and the (c2)-solution.

Case 4.1 Consider $a > 0$ via the (c1)-solution.

Theorem 11. In Case 4.1, system (2) is controllable, if the control function $u_5(t)$ is given by

$$u_5(t) = \frac{1}{T_1 \tilde{d}} \left[-y_1 \cosh \left(\int_0^{T_1} a(v)dv - \int_0^t a(v)dv \right) + y_1 \sinh \left(\int_0^{T_1} a(v)dv - \int_0^t a(v)dv \right) \right. \\ \left. \ominus - \left(\cosh \left(\int_0^t a(v)dv \right) y_0 \ominus - \sinh \left(\int_0^t a(v)dv \right) y_0 \right) \right] \ominus \frac{1}{\tilde{d}} b(t), t \in J,$$

where the H -differences exist.

Proof. The method of proof is the same as in Case 1.1, so we omit it here. \square

Case 4.2 Consider $a > 0$ via the (c2)-solution.

Theorem 12. In Case 4.2, system (2) is controllable, if $W_2^{-1}[0, T_1]$ exists; here

$$W_2[0, T_1] = \int_0^{T_1} e^{-\int_0^s a(v)dv} \tilde{d} \tilde{d} e^{-\int_0^s a(v)dv} ds. \tag{7}$$

Proof. Let

$$u_6(t) = -\tilde{d} e^{-\int_0^t a(v)dv} W_2^{-1}[0, T_1] \left(y_0 \ominus \int_0^{T_1} (-b(s)) e^{-\int_0^s a(v)dv} ds \ominus e^{-\int_0^{T_1} a(v)dv} y_1 \right), t \in J,$$

where the H -differences exist. The method of proof is the same as in Case 1.2, so we omit it here. \square

Next, we consider the following system (3):

$$\begin{cases} y'(t) + (-b(t) + \tilde{d}u(t)) = a(t)y(t), t \in J, \\ y(0) = y_0, \end{cases}$$

where $a : J \rightarrow \mathbb{R}, y_0 \in \mathbb{R}_F$ and $b : J \rightarrow \mathbb{R}_F, u : J \rightarrow \mathbb{R}_F, \tilde{d} \in \mathbb{R}_+$.

The method of proof is same as in systems (1) and (2), so we omit it here.

4. An Example

In this section we give an example to prove our theorems.

Example 1. Consider linear fuzzy differential equations:

$$\begin{cases} y'(t) = -y(t) + \tilde{d}u(t), t \in [0, 0.3], \\ y(0) = \gamma, \end{cases} \tag{8}$$

where $a = -1, [\gamma]^\alpha = [\alpha - 1, 1 - \alpha], y_1 = 0.6\gamma, \tilde{d} = 3$.

According to Theorem 6,

$$W_0[0, T_1] \triangleq \int_0^{0.3} 9(e^{\int_0^s dv})^2 ds = 9 \int_0^{0.3} e^{2s} ds = \frac{9}{2}(e^{0.6} - 1) = 3.6995,$$

then $W_0^{-1}[0, 0.3] = 0.1827$. Note

$$\begin{aligned} u_1(t) &= -\tilde{d}e^{-\int_0^t a(v)dv}W_0^{-1}[0, 0.3]\left(y_0 \ominus \int_0^{0.3} (-b(s))e^{-\int_0^s a(v)dv}ds \ominus e^{-\int_0^{0.3} a(v)dv}y_1\right) \\ &= -0.4439e^{-t}\gamma. \end{aligned}$$

To sum up, $W_0^{-1}[0, T_1]$ exists, so the system is controllable.

5. Conclusions

In this paper we mainly study the controllability of first order linear fuzzy differential equations with the direct construction method. In future work, we shall study the controllability of first order linear and non-linear impulsive fuzzy differential equations.

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