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Encouraging Students' Motivation and Involvement in STEM Degrees by the Execution of Real Applications in Mathematical Subjects: The Population Migration Problem

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Abstract: This paper presents a simplified model of the population migration problem, addressed to first-year engineering students in order to show them the use of linear algebra tools. The study consists of predicting the census in the city centre and in the suburbs, determining the city population equilibrium point, and making a sociological interpretation of population flows. This practical problem is part of the seminar "Applications of Linear Algebra in Engineering", which is being held at the Universitat Politècnica de Catalunya-BarcelonaTech (UPC). This seminar consists in the learning of linear algebra by the implementation of real applications where mathematical tools are required to resolve them. This paper presents an application of linear algebra to the population migration problem and analyses students' appreciation through anonymous surveys and personal interviews. The surveys assessed students' motivation towards the subject of linear algebra and their learning of mathematical concepts. Personal interviews were conducted for students in order to let them express in detail their opinion about the seminar. The results confirm that the introduction of real applications in the learning of mathematics increases students' motivation and involvement, which implies an improvement in students' performance in the first courses of STEM degrees.

Keywords: engineering; linear algebra; mathematics; population migration; STEM; students' motivation

MSC: 97D30; 97D40; 97H60; 97M10; 97M50

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1. Introduction

For a long time, there have been discussions about the syllabus of mathematical subjects in first-year courses of engineering degrees regarding the contents needed for technological disciplines and how to teach them to make students aware of this necessity [1].

Currently, there is an increasing concern that derives from the alarming fact that European undergraduate students show less interest in STEM (sciences, technology, engineering, and mathematics) fields compared to other fields [2]. Therefore, STEM education professionals are working to identify the causes of this fact [3] and have determined that there are various interrelated factors, both extrinsic (science capital, learning opportunities, socio-economic status) and intrinsic (interest linked to self-efficacy issues, attitude towards science, perceived social expectations) [4].

In addition, high occupancy demands for STEM professions are foreseen [5], and it is expected to increase in the forthcoming years. Likewise, technological skills are becoming more specialized [6], and thus STEM professions are essential for technological development and economic growth of present societies [2].

The low interest in STEM disciplines is mainly due to the high failure rate in the first courses of STEM degrees [7,8], especially in mathematical subjects. This is considered one of the main difficulties for engineering students in first courses of their degrees [9], which entails a loss of student engagement [10] in the first year courses and in most cases results in the dropout of these degrees [11].

The influence of students' sense of belonging in the classroom should also be considered in improving students' motivation and involvement [12], which is especially relevant in STEM degrees [13]. The sense of class belonging is directly linked to engagement and can be as important to persist in STEM degrees as self-efficacy and intrinsic interest in these fields [7]. To settle this failure rate, it is necessary to revise mathematical subjects' syllabi and teaching methodologies as well as to encourage students' motivation and engagement [11,14,15], giving them awareness of the connections between mathematics and technological disciplines and with their future careers and illustrating real applications of the concepts and tools learnt in mathematical subjects.

Mathematics is fundamental to many professions, especially science, technology, and engineering, and it is a gateway to many scientific and technological fields [16]. Mathematics should be an integrated part of STEM disciplines, such that it constitutes specific practices related to those fields [9]. It is essential to teach mathematics using real problems in order to achieve the involvement of students, ensuring that it is significant for them; it is also essential to develop students' mathematical thinking through practical applications, rather than teaching a set of disconnected concepts and skills. If mathematics was taught as an applied subject, connections between mathematics and STEM disciplines would be reinforced, as it is shown in [16].

There is a wide consensus about the promotion of STEM degrees by the encouragement of students' motivation and involvement through the contextualization of mathematical concepts in the technological disciplines of the following courses of these degrees. It is unquestionable that applied problems used in mathematical subjects of STEM degrees encourage students' motivation and engagement [11,13,17,18], as well as the improvement in their learning [19], which improves their academic performance [20–22].

There are several investigations that support the idea that active learning has a profitable effect on the improvement in students' engagement and in their performance, as indicated in [23–25]. The use of problem-solving methodologies in mathematical subjects of STEM degrees is a profitable tool to increase students' involvement. With problem-solving, students are given a problem based on a real situation, which will lead them to learn the required mathematical concepts to develop the problem. In addition, a problem-solving methodology will facilitate students' critical thinking skills, since they will have the opportunity to consider different assumptions and discuss and make decisions about this problem. Problem-solving also lets students be more active, both individually and in groups: they can exchange knowledge, engage in peer learning, and work together [26]. Therefore, problem-solving helps students acquire skills and competences such as autonomy, continuous learning, critical thinking, teamwork, planning, organization, and communications [26], which are essential for their careers [27,28].

The objective of this study is that students understand the need for mathematical concepts to solve practical problems. With this work, it is expected that students' motivation towards the learning of mathematics in first year courses will increase, and as a result, their performance is expected to improve, as confirmed by previous studies [21,29]. At the same time, problem-solving will let students develop required skills for the execution of their degree and for their engineering careers [30].

This article focuses on the analysis of the population migration problem:

- How can migration flow problems be solved using linear algebra?
- How do students of technological degrees value the implementation of this mathematical application in the learning of linear algebra?

2. Materials and Methods

This study is part of the work "Applications of Mathematics in Engineering" [22], which emerged with the proposal of providing connections between mathematics and STEM education, through the execution of real practical problems to engineering students, with the intention of improving their engagement in first year courses of engineering degrees. The aim of this work is to analyse and solve demographic problems using mathematical tools in order to illustrate applications of linear algebra for engineering students, with the objective that they will realize of the significance of mathematics for the development of their future profession, which will entail an improvement of students' motivation in the first courses of these degrees and therefore an increase in their academic performance.

The seminar is being taught at the Industrial Engineering Bachelor's Degree from the Barcelona School of Industrial Engineering (ETSEIB) of the Universitat Politècnica de Catalunya-BarcelonaTech (UPC), a public university specialising in STEM degrees. The work "Applications of Mathematics in Engineering" comprises two seminars, "Applications of Mathematics in Engineering I: Linear Algebra" and "Applications of Mathematics in Engineering II: Multivariable Calculus", which began in the 2019/2020 academic year. Each seminar is held in one semester (the first and the second, respectively) and consists of weekly sessions one hour and a half in length. These sessions have also been given in the 2020/2021 academic year.

The two seminars, "Applications of Mathematics in Engineering I: Linear Algebra" and "Applications of Mathematics in Engineering II: Multivariable Calculus" consist of voluntary sessions, aimed at first course undergraduate students of engineering degrees. These sessions are organized according to the contents of the Linear Algebra and Multivariable Calculus syllabus, respectively, in order to show students that the concepts they are learning in ordinary mathematical subjects are essential for their degree and for their future career.

The sessions "Applications of Mathematics in Engineering I: Linear Algebra", consists of 10 sessions, which are indicated in Table 1.

Table 1. Applications of Linear Algebra in Engineering ([22]).

Session	Title
1	"Complex Numbers on the Study of Price Fluctuations"
2	"Complex Numbers on the Study of Alternating Current"
3	"Indeterminate Systems: Control Variables"
4	"Mesh Flushes: a Basis of Conservative Fluxes Vector Subspace"
5	"Addition and Intersection of Vector Subspaces in Discrete Dynamical Systems"
6	"Linear Applications and Associated Matrix"
7	"Basis Changes"
8	"Eigenvalues, Eigenvectors and Diagonalization in Engineering"
9	"Modal Analysis in Discrete Dynamical Systems"
10	"Difference Equations"

This article focuses on one of the applications studied in two of the sessions of the seminar "Applications of Linear Algebra in Engineering". In each of these two sessions of this seminar—Session 6, "Linear Applications and Associated Matrix", and Session 8—"Eigenvalues, Eigenvectors and Diagonalization in Engineering"—the population migration problem was analysed and solved illustrating the use of mathematical concepts related to linear algebra. In Session 6, the population migration problem was studied using concepts related to linear applications and associated matrix. In Session 8, eigenvalues, eigenvectors, and diagonalization are used to analyse and solve this demographic problem.

The problem presented here is a simplified model of the population migration for educational purposes. It does not consider sociological factors of migration that would give rise to more complex models, since this is not the aim of the study.

With the aim of assessing students' appreciation of the population migration problem, anonymous questionnaires and personal interviews were conducted at the end of each session. The results obtained have contributed to analysing the material developed in these sessions.

Questionnaires assess the influence of this implementation on the students attending these two sessions, in relation to the execution of demographic problems, the motivation towards the subject of linear algebra, and the learning of mathematical concepts. These surveys consisted of several statements, which were valued on a 5-point scale (1 = Strongly disagree, 2 = Disagree, 3 = Nor agree nor disagree, 4 = Agree, 5 = Strongly agree). In addition, students could add a comment expressing their opinion and explaining their impression about each session.

In order to extract further information from the students attending the sessions, personal interviews were conducted, where they could express in detail their impression and opinion about these sessions. With the aim of evading partiality in their comments, the person who interviewed students was a doctoral student instead of a teacher.

The responses to the surveys and to the interviews undertaken after the sessions were analysed to assess the influence of the implementation of the population migration problem on the students attending these sessions.

3. Results

3.1. Analysis of Population Migration Using Linear Algebra

The 10 sessions of the seminar "Applications of Mathematics in Engineering I: Linear Algebra" consisted of applied problems related to mathematical concepts developed in the subject of linear algebra, whose content is organized into four mains topics:

1. Algebraic structures (complex numbers, matrix and determinants, and systems of equations);
2. Vector spaces and linear applications (vector spaces, vector subspaces, and linear applications);
3. Reduction of linear applications (diagonalization, eigenvalues, and eigenvectors);
4. Resolution of linear discrete dynamical systems (difference equations, and linear discrete dynamical systems).

A major part of the population migration problem introduced in this article includes linear algebra, as detailed below.

This exercise is intended to be developed as the student progresses in the Linear Algebra course, essentially in four phases:

(a) Basic concepts of vector space and linear application, which are explained in the second topic of the subject, are used in the development of the population migration problem (Sections 3.1.1–3.1.3).
(b) The matrix of an application and its use, learnt at the end of the second topic of the linear algebra subject, are concepts are necessary to solve the problems in Sections 3.1.4–3.1.6.
(c) Eigenvalues, eigenvectors, diagonalization, and the calculation of matrix powers, which are the concepts studied in the third topic of the subject, are used in Sections 3.1.7 and 3.1.8.
(d) Extensions: some of the concepts developed in Sections 3.1.9–3.1.12 are explained in the fourth topic of the Linear Algebra subject.

3.1.1. Description of a Population Migration Problem

Considering a population exchange city centre and periphery, assuming, to simplify, that the total number of habitants is constant and equal to 1,000,000 inhabitants, and supposing a decreasing census in the centre of the city—1,000,000 inhabitants in 2010, 600,000 inhabitants in 2015, and 400,000 inhabitants in 2020 (provisional) [31]—the following points must be answered:

1. Inhabitants in the centre of the city in 2025.
2. Inhabitants in the centre of the city in 2030, 2035, etc.
3. Equilibrium point.
4. In general, inhabitants in 2010 + 5k.

5. Which definitive census in 2020 would be alarming?
6. Sociological interpretation.

3.1.2. A Linear Algebra Approach

The aim is to answer the above questions by means of elementary Linear Algebra tools. Therefore, the first step is to model the population migration as a linear map.

$$\text{Vector space } E = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{array}{l} x_1 = \text{ inhabitants in centre} \\ x_2 = \text{ inhabitants in periphery} \end{array} \right\} \subset \mathbb{R}^2 \tag{1}$$

$$\underbrace{E}_{\text{inhabitants in } t} \xrightarrow{f} \underbrace{E}_{\text{inhabitants in } t+5} \tag{2}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \xrightarrow{f} \begin{pmatrix} x_1(t+5) \\ x_2(t+5) \end{pmatrix} \tag{3}$$

If x_1, x_2 are expressed in terms of millions of inhabitants, the census data show that:

$$\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{2010} \xrightarrow{f} \underbrace{\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}}_{2015} \xrightarrow{f} \underbrace{\begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}}_{2020 \text{ (provisional)}} \longrightarrow \ldots \tag{4}$$

A key point is that f can be assumed to be linear because of two natural hypotheses: homogeneity (same behaviour in any neighbourhood of the city) and proportionality (the behaviours do not depend on the number of inhabitants).

3.1.3. Inhabitants in the Centre in 2025

The first question can be answered by means of the basic formula for a linear map f:

$$f(\alpha_1 x_1 + \alpha_2 x_2 + \ldots) = \alpha_1 f(x_1) + \alpha_2 f(x_2) + \ldots \tag{5}$$

In our case, it allows us to compute $f\begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}$:

$$\begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} \Leftrightarrow \begin{array}{c} 0.4 = \alpha + 0.6\beta \\ 0.6 = 0.4\beta \end{array} \Leftrightarrow \begin{array}{c} \alpha = -1/2 \\ \beta = 3/2 \end{array} \tag{6}$$

$$f\begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix} = \alpha f\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta f\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix} \Rightarrow \tag{7}$$

$$\Rightarrow 300{,}000 \text{ inhabitants in the centre in 2025} \tag{8}$$

3.1.4. The Matrix

Below, the matrix of the linear map f in ordinary basis is obtained in order to simplify the computation:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} \tag{9}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \alpha' \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} + \beta' \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \alpha' \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\prime \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} \Leftrightarrow \begin{array}{c} \alpha' = -3/2 \\ \beta = 5/2 \end{array} \tag{10}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \alpha' \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} + \beta' \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix} = -\frac{3}{2}\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} + \frac{5}{2}\begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.9 \end{pmatrix} \tag{11}$$

Hence, we can conclude:

$$\text{Matrix in ordinary basis} = \begin{pmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{pmatrix} \equiv A \tag{12}$$

In particular:

$$\begin{pmatrix} x_1(t+5) \\ x_2(t+5) \end{pmatrix} = \underbrace{\begin{pmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{pmatrix}}_{A} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \tag{13}$$

3.1.5. Inhabitants in the Centre in 2030, 2035, etc.

By means of matrix A, the two first questions can be answered in a very simple way:

$$\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{2010} \overset{A}{\to} \underbrace{\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}}_{2015} \overset{A}{\to} \underbrace{\begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}}_{2020} \overset{A}{\to} \underbrace{\begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix}}_{2025} \overset{A}{\to} \underbrace{\begin{pmatrix} 0.25 \\ 0.75 \end{pmatrix}}_{2030} \overset{A}{\to} \underbrace{\begin{pmatrix} 0.225 \\ 0.775 \end{pmatrix}}_{2035} \to \ldots \Rightarrow \tag{14}$$

$$\Rightarrow 250{,}000 \text{ inhabitants in the centre in } 2030, \qquad 225{,}000 \text{ in } 2035\ldots \tag{15}$$

3.1.6. Equilibrium Point

Moreover, again by using A, the equilibrium point, if it exists, can be computed:

$$\begin{pmatrix} {x_1}^e \\ {x_2}^e \end{pmatrix} \to (A) \begin{pmatrix} {x_1}^e \\ {x_2}^e \end{pmatrix} \Leftrightarrow \begin{array}{l} {x_1}^e = 0.6{x_1}^e + 0.1{x_2}^e \\ {x_2}^e = 0.4{x_1}^e + 0.9{x_2}^e \end{array} \Leftrightarrow \ldots \Leftrightarrow \begin{array}{l} {x_1}^e = 0.2 \\ {x_2}^e = 0.8 \end{array} \tag{16}$$

Therefore, the equilibrium point is 200,000 inhabitants in the city centre.

3.1.7. Inhabitants in 2010 + 5k

However, let us see that this elementary use of A is not sufficient in order to solve question 4. As the calculation is done in five-year periods:

$$\begin{cases} t \ = \ 2010 \ + \ 5k, \ k = 0, \ 1, \ 2, \ \ldots \\ \qquad x(k) \equiv x(2010 + 5k) \end{cases} \tag{17}$$

$$\underbrace{x(0)}_{2010} \overset{A}{\to} \underbrace{x(1)}_{2015} \overset{A}{\to} \underbrace{x(2)}_{2020} \to \ldots \tag{18}$$

$$\begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} = A \cdots A \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = A^k \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{pmatrix}^k \begin{pmatrix} 1 \\ 0 \end{pmatrix} = ? \tag{19}$$

3.1.8. Eigenvalues, Eigenvectors, and Diagonalization

In order to compute A^k, more sophisticated tools are needed, namely the techniques of diagonalization by means of the eigenvalues and eigenvectors. The eigenvalues are obtained by means of the characteristic polynomial:

$$Q_A(t) = \det\begin{pmatrix} 0.6 - t & 0.1 \\ 0.4 & 0.9 - t \end{pmatrix} = t^2 - \frac{3}{2}t + \frac{1}{2} = (t-1)\left(t - \frac{1}{2}\right) \Leftrightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = \frac{1}{2} \end{cases} \tag{20}$$

The eigenvectors corresponding to these eigenvalues are:

$$\begin{array}{l} v_1 = \text{Ker}(A - 1I) = \left[\begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix}\right] \\ v_2 = \text{Ker}\left(A - \frac{1}{2}I\right) = \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] \end{array} \tag{21}$$

We conclude:

$$S^{-1}AS = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}, \text{ being } S = \begin{pmatrix} 0.2 & 1 \\ 0.8 & -1 \end{pmatrix} \tag{22}$$

Thus:

$$A^k = S\begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}^k S^{-1} = S\begin{pmatrix} 1 & 0 \\ 0 & 1/2^k \end{pmatrix} S^{-1} \tag{23}$$

Finally:

$$x(k) = S\begin{pmatrix} 1 & 0 \\ 0 & 1/2^k \end{pmatrix} S^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \ldots = \begin{pmatrix} 0.2 + 0.8\frac{1}{2^k} \\ \ldots \end{pmatrix} \tag{24}$$

The solution obtained shows the millions of inhabitants in the city centre, in the quinquennium k. If we calculate it for different values of k, we see that it coincides with the results obtained before; see Table 2.

Table 2. Number of inhabitants in the city centre in the quinquennium k.

Quinquennium	Year	Inhabitants
$k = 0$	2010	1,000,000
$k = 1$	2015	600,000
$k = 2$	2020	400,000
$k = 3$	2025	300,000
$k = 4$	2030	250,000
$k = 5$	2035	225,000
...	...	...
$k \to \infty$	$\to \infty$	200,000

As shown in Table 2, if $k \to \infty$, the population in the city center tends to an equilibrium point with 200,000 inhabitants, which confirms the study done previously.

Indeed, the general theory of dynamical systems ensures the existence of an equilibrium point because $|\lambda_1|, |\lambda_2| \leq 1$.

3.1.9. Which Definitive Census in 2020 Would Be Alarming?

In addition, Question 5 can be answered: the situation would be alarming if the centre empties, that is to say, if the equilibrium point is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. This happens for a transition matrix A' such that $A'\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$:

$$A' = \begin{pmatrix} 0.6 & a \\ 0.4 & b \end{pmatrix} \text{ in equilibrium with } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{25}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.6 & a \\ 0.4 & b \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Leftrightarrow \begin{matrix} a = 0 \\ b = 1 \end{matrix} \tag{26}$$

This matrix is characterised by the census in 2020:

$$A' = \begin{pmatrix} 0.6 & 0 \\ 0.4 & 1 \end{pmatrix} \Leftrightarrow A'\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.36 \\ 0.64 \end{pmatrix} \tag{27}$$

Therefore, it would be alarming if a definitive census in 2020 showed 360,000 inhabitants instead of 400,000 inhabitants.

3.1.10. Sociological Interpretation

Finally, let us see the difference between cases A and A' from a sociological point of view:

$$\begin{aligned} A:\ & x_1(k+1) = 0.6x_1(k) + 0.1x_2(k) \\ A':\ & x_1(k+1) = 0.6x_1(k) + 0x_2(k) \end{aligned} \tag{28}$$

By comparing A and A', it is deduced that alarm does not depend on the percentage of inhabitants who move from centre to periphery but it does depend on the percentage of inhabitants who return from periphery to centre.

3.1.11. Linear Systems Determined by Consecutive Values of the States

We consider homogeneous linear systems of the form

$$x(k+1) = Ax(k),\ k = 0, 1 \ldots,\ A \in M_n(\mathbb{R}) \tag{29}$$

where $x(k) \in \mathbb{R}^n$ is the state variable.

Sometimes it is difficult to empirically calculate the entries of the transition matrix A, whereas it is easy to measure consecutive values of the states:

$$x(0),\ x(1),\ x(2) \ldots \tag{30}$$

Let us see how the matrix A can be obtained from these data.

As in the above example, let us consider the linear map

$$f:\ \mathbb{R}^n \to \mathbb{R}^n\ ,\ x(k) \xrightarrow{f} x(k+1) \tag{31}$$

Then, the consecutive values of the states can be seen as successive images of $x(0)$.

$$x(0) \xrightarrow{f} x(1) \xrightarrow{f}\ \ldots\ \xrightarrow{f} x(n-1) \xrightarrow{f} x(n) \tag{32}$$

If $x(0), \ldots, x(n-1)$ are linearly independent, it is clear that the matrix of f in this basis is:

$$\overline{A} = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_1 \\ 1 & 0 & \cdots & 0 & a_2 \\ 0 & 1 & \cdots & 0 & a_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_n \end{pmatrix} \tag{33}$$

being

$$x(n) = a_1x(0) + \ldots + a_nx(n-1) \tag{34}$$

Notice that $\overline{A}$ is a Sylvester or companion matrix, such that it is a non-derogatory matrix (that is to say, each eigenvalue has a unique eigenvector), and the coefficients of its characteristic polynomial are just the opposite of the coefficients in the last column of the Sylvester matrix (see (40)).

From $\overline{A}$, the matrix A of f in the ordinary basis can be obtained simply as

$$A = S\overline{A}S^{-1},\ S = (x(0)\ \ldots\ x(n-1)) \tag{35}$$

where the columns of S are the coordinates of the states $x(0), \ldots, x(n-1)$. Clearly the columns of the matrix $S\overline{A}$ are the coordinates of the states $x(1), \ldots, x(n)$.

Summarizing, one has:

Proposition 1. *Let*

$$x(k+1) = Ax(k),\ k = 0, 1 \ldots,\ A \in M_n(\mathbb{R}) \tag{36}$$

be an homogeneous linear system in $\mathbb{R}^n$*, where*

$$x(0),\ \ldots,\ x(n-1) \text{ are linearly independent vectors} \tag{37}$$

$$x(n) = a_1 x(0) + \ldots + a_{\mathrm{n}} x(n-1) \tag{38}$$

Then:

1. *The matrix of f in ordinary basis is:*

$$A = (x(1)\ \ldots\ x(n))\ (x(0)\ \ldots\ x(n-1))^{-1} \tag{39}$$

2. *The characteristic polynomial of A is:*

$$Q_A(t) = t^n - a_n t^{n-1} - \ldots - a_2 t - a_1 \tag{40}$$

where:

$$x(n) = a_1 x(0) + \ldots + a_n x(n-1) \tag{41}$$

In particular,

$$1 \text{ is an eigenvalue of } A \Leftrightarrow a_1 + \ldots + a_n = 1 \tag{42}$$

which is a necessary condition for the existence of an equilibrium point.

3. *A is a non-derogatory matrix, that is to say, each eigenvalue has a unique eigenvector.*

In particular, there is only a Jordan block for each eigenvalue, so that

$$A \text{ diagonalizes} \Leftrightarrow A \text{ has n distinct eigenvalues} \tag{43}$$

If in addition 1 is a dominant eigenvalue, then its eigenvector is the equilibrium point.

3.1.12. Learning Theory to Solve Problems

We have seen that concrete problems can be solved by means of elementary linear algebra. Now, let us see that the solution is easier if further linear algebra tools are used. This fact contrasts with some topics that focus the attention on the "problems", underestimating the "theory".

Indeed, by means of the above proposition, the computations in our problem can be simplified as follows

$$\overline{A} = \begin{pmatrix} 0 & -1/2 \\ 1 & 3/2 \end{pmatrix} \tag{44}$$

$$S = \begin{pmatrix} 1 & 0.6 \\ 0 & 0.4 \end{pmatrix} \tag{45}$$

$$A = \begin{pmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{pmatrix} \begin{pmatrix} 1 & -3/2 \\ 0 & 5/2 \end{pmatrix} = \begin{pmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{pmatrix} \tag{46}$$

$$Q_A(t) = t^2 - \frac{3}{2}t + \frac{1}{2} = (t-1)\left(t - \frac{1}{2}\right) \tag{47}$$

Thus, the eigenvector with an eigenvalue of 1 is the equilibrium point:

$$x^e = \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix} \tag{48}$$

3.2. Students' Surveys and Interview Results

The number of students attending the seminar "Application of Linear Algebra in Engineering" undertaken in the first semesters of the 2019/2020 and 2020/2021 academic years was 20 [22], all of them answered to the surveys and interviews, and the results achieved in both academic years did not show relevant differences.

The population migration problem has been treated in two sessions of this seminar as mentioned above: in Session 6, "Linear Applications and Associated Matrix" and in Session 8, "Eigenvalues, Eigenvectors and Diagonalization in Engineering".

The answers to some of the questions held in Sessions 6 and 8, where the migration population problem was studied, are shown.

According to the surveys answers, all the students thought that the sessions "Linear Applications and Associated Matrix" and "Eigenvalues, Eigenvectors and Diagonalization in Engineering" let them know of applications of demographic problems; see Figure 1.

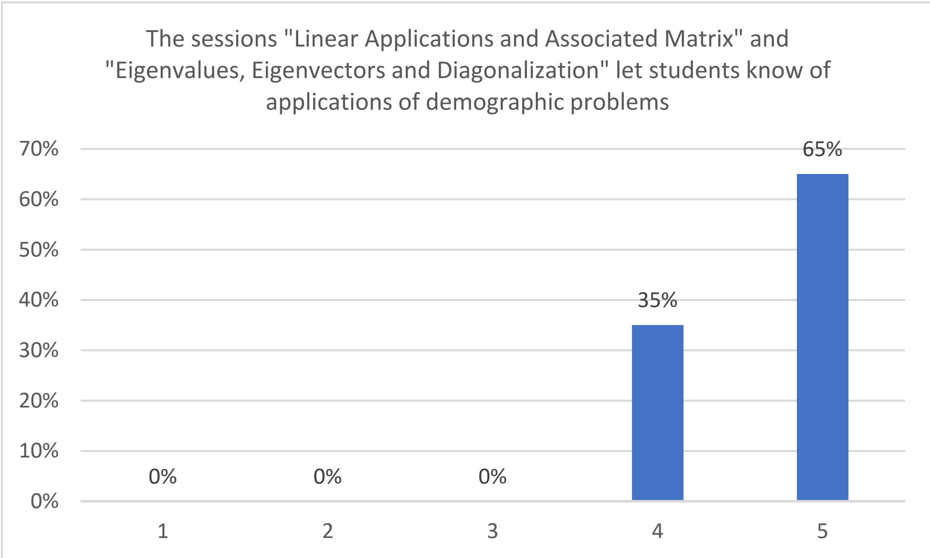

Figure 1. Answers to the question: the sessions "Linear Applications and Associated Matrix" and "Eigenvalues, eigenvectors and Diagonalization in Engineering" let students know of applications of demographic problems.

All the students agreed that the study of the migration population problem increased their motivation to understand the subject of linear algebra; see Figure 2.

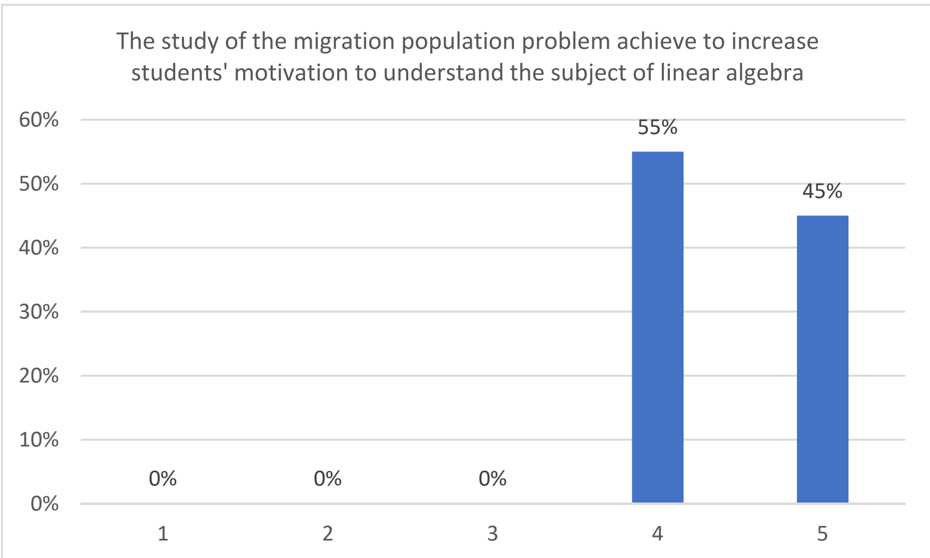

Figure 2. Answers to the question of whether the study of the migration population problem increased students' motivation to understand the subject of linear algebra.

Most of the students stated that the execution of the population migration problem improved their learning of mathematical concepts; see Figure 3.

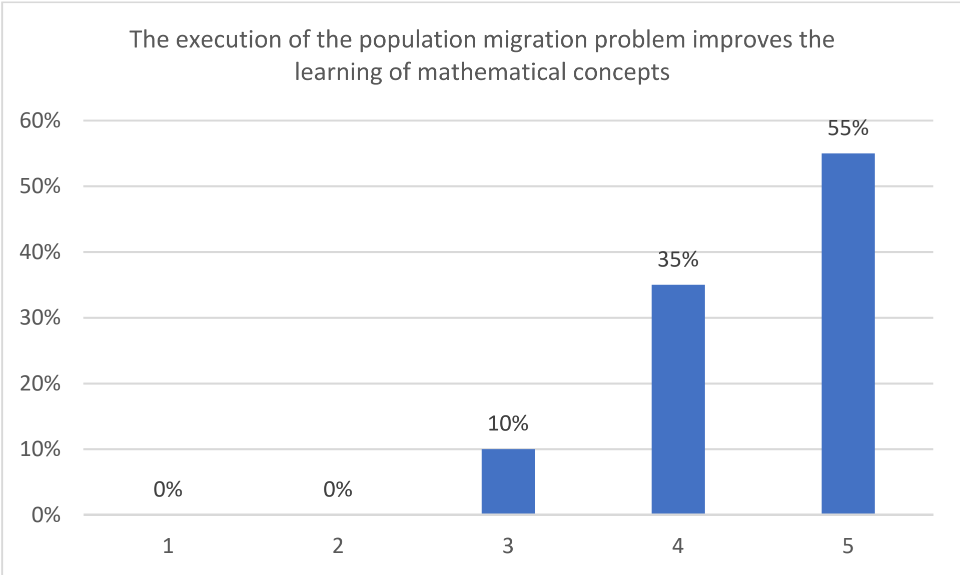

Figure 3. Answers to the question of whether the execution of the population migration problem improves the learning of mathematical concepts.

There are also noteworthy comments from students expressed in the open section of the anonymous questionnaires in the 2019/2020 and 2020/2021 academic years. The main themes were as follows:

- The population migration problem has helped students understand and learn the concepts of eigenvalues and eigenvectors.
- This application has let students realize of the importance of Linear Algebra to solve real social problems.
- Real applications, such as demographic control, motivated students to study linear algebra.

The information extracted from the students attending these two sessions was a follows:

- The sessions "Linear Applications and Associated Matrix" and "Eigenvalues, Eigenvectors and Diagonalization in Engineering" let students solve problems with real applications, such as the population migration problem.
- This application shows students what they will be capable of doing in the next courses, and it improves their motivation and involvement.
- The use of the concepts of linear algebra in the population migration problem, namely linear applications, matrices, eigenvalues, and eigenvectors, lets students know the importance of Linear Algebra for engineering degrees and for their STEM careers.
- Students classified the application of population flows and demographic control as one of the most interesting applications studied in the seminar.
- Students value positively knowing applications in sociology such as the population migration problem.

4. Discussion

Considering the students' survey and interview results, the implementation of the population migration problem in the seminar "Applications of Linear Algebra in Engineering" has been very positive, as explained below.

By analysing the results presented in Figure 2, it is shown that all the students agreed that the study of the population migration problem achieve to increase their motivation to the subject Linear Algebra, what implies the decrease of the dropout, as it is related to motivation [11], student achievement [14] and academic performance [13].

The results shown in Figure 3 confirm that 90% of the students think that the execution of the population migration problem let them learn mathematical tools through real applications. This improves their motivation in mathematical subjects, as is shown in previous studies such as [20].

Moreover, as Figure 1 shows, all the students stated that the sessions “Linear Applications and Associated Matrix” and “Eigenvalues, Eigenvectors and Diagonalization in Engineering” let students understand applications in engineering and other areas such as sociology, which motivated them to the study of mathematical subjects, as was confirmed in other works [11,17,18].

This work confirms the relevance of solving real problems in the teaching of mathematics, as shown in [26]. Problem-solving methodology improves students’ critical thinking skills and provides key skills and competences for their future careers [27,28].

The responses to the questions given to the students in the personal interviews about the sessions “Linear Applications and Associated Matrix” and “Eigenvalues, Eigenvectors and Diagonalization in Engineering” show that the population migration problem amazed students because they were unaware of linear algebra’s applications in demographic problems. It can also be emphasized that students’ motivation and engagement increased when they realized the importance of linear algebra for their STEM degree and improved their interest towards the subject.

These findings demonstrate that the execution of the population migration problem improves the understanding of linear algebra concepts, as was shown in [19], which increases students’ performance in mathematical subjects of STEM degrees, as was studied previously in [21].

5. Conclusions

This study was implemented at the Universitat Politècnica de Catalunya-BarcelonaTech (UPC). The objective of this work is to provide connections between mathematics and STEM education. The execution of the migration population problem in the seminar “Applications of Linear Algebra in Engineering” has let students know how to analyse demographic problems using concepts from linear algebra. This work confirms the possibility of illustrating in engineering degrees the concepts and basic results of linear algebra, such as the computation of the matrix by means of consecutive states, eigenvectors and eigenvalues, equilibrium points, and stability. Moreover, first-year engineering students have realized the importance of mathematics in solving real problems. Thus, students’ involvement in mathematical subjects has improved, and therefore, their performance will increase. The exercises are appropriate for first-year students since they only require basic concepts of linear algebra to be solved.

In view of the success of the implementation of the population migration problem in the seminar “Applications of Linear Algebra in Engineering”, the development of more real applications is planned. Furthermore, the project “Applications of Mathematics in Engineering”, formed by this seminar “Applications of Mathematics in Engineering I: Linear Algebra” and the seminar “Applications of Mathematics in Engineering II: Multivariable Calculus”, is programmed to be undertaken in future academic years. Likewise, surveys and interviews will be conducted among students attending the seminars, with the aim of analysing the applications that students consider more interesting and more useful for learning mathematics, as well as with the objective of collecting a greater sample of surveys results and more information about students’ experience in these sessions.

This work confirms that it is possible to increase students’ engagement through the introduction of STEM applications in the learning of mathematical subjects, which entails an enhancement of the learning of mathematics and, hence, an improvement in students’ performance and a decline in the dropout in the early stages of engineering degrees. This involves an increasing interest in STEM degrees, which are considered essential for the technological development and economic growth of present-day society.

Since the results obtained confirm that the implementation of real applications increases students' motivation and involvement, which implies a decrease in the dropout rates of engineering degrees, the future aim of this project is to introduce the developed of applications in the seminar "Applications of Mathematics in Engineering I: Linear Algebra" in the ordinary syllabus of the Linear Algebra subject within the Industrial Engineering bachelor's degree from the Barcelona School of Industrial Engineering (ETSEIB) of the UPC. The impact of the inclusion of these applications in the linear algebra program will be assessed regarding the influence on students' motivation as well as on students' academic performance. If the results obtained are positive, there is a plan to introduce the applications developed in the seminar "Applications of Mathematics in Engineering II: Multivariable Calculus" in the subject of Calculus Multivariable within the Industrial Engineering bachelor's degree from the ETSEIB of the UPC.

In the same way, other mathematical applications linked to other studies in STEM fields could be developed and introduced into the mathematical subjects of the corresponding study plans of the Universitat Politècnica de Catalunya-BarcelonaTech (UPC), a public university specializing in STEM degrees. In this way, the connection of mathematics with the rest of the STEM disciplines would be much clearer, and students would be able to see the application of mathematics in their different careers from the first year of studies, which would contribute to an improvement in the quality of education.

These implementations will require the implication of the governing bodies of the UPC, who are acquainted with these seminaries and give support to proposals whose aims are the improvement of students' motivation, the increase in academic performance and the decrease in the dropout in the early stages of STEM degrees.

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