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Common Attractive Point Results for Two Generalized Nonexpansive Mappings in Uniformly Convex Banach Spaces

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Abstract: In this work, we study some basic properties of the set of common attractive points and prove strong convergence results for common attractive points of two generalized nonexpansive mappings in a uniformly convex Banach space. As a consequence, we obtain a common fixed point result of such mappings and apply it to solving the convex minimization problem. Finally, numerical experiments are given to support our results.

Keywords: common attractive point; generalized nonexpansive mapping; iterative method; uniformly convex Banach space; convergence; approximation; convex minimization

MSC: 47H09; 47H10; 65J15

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1. Introduction

Throughout this paper, let $\mathbb{N}$ and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively. When C is a subset of a Banach space X, a mapping $T : C \to X$ is called nonexpansive if $\|Tx - Tu\| \le \|x - u\|$ for every $x, u \in C$. The set of fixed points of T is denoted by $F(T)$; that is, $F(T) = \{z \in C : Tz = z\}$. A mapping $T : C \to X$ is said to be quasi-nonexpansive [1] if $F(T) \neq \emptyset$ and $\|Tx - z\| \le \|x - z\|$ for every $x \in C$ and for every $z \in F(T)$. It is easy to see that nonexpansive mappings with fixed points are included in the class of quasi-nonexpansive mappings.

Fixed point theory can solve not only problems in sciences and economics but also real-world problems (see [2–7] for examples). Specifically, the theory of nonexpansive mappings plays a crucial role, because it can be applied to plenty of problems, such as convex minimization problems, optimization problems, equilibrium problems and problems of proving the existence of solutions to integral and differential equations. Consequently, many generalized nonexpansive mappings have been studied in a variety of directions. In this work, we focus on a generalized mapping defined by Hardy and Rogers [8] as: a mapping $T : C \to X$ is called generalized nonexpansive if

$$\begin{aligned} \|Tx - Tu\| \le a_1\|x - u\| + a_2\|Tx - x\| + a_3\|Tu - u\| \\ + a_4\|Tu - x\| + a_5\|Tx - u\|, \end{aligned} \tag{1}$$

for every $x, u \in C$, where a_1, a_2, a_3, a_4 and a_5 are nonnegative constants such that $a_1 + a_2 + a_3 + a_4 + a_5 \le 1$. It was also mentioned that condition (1) is equivalent with

$$\begin{aligned} \|Tx - Tu\| \le a\|x - u\| + b(\|Tx - x\| + \|Tu - u\|) \\ + c(\|Tu - x\| + \|Tx - u\|), \end{aligned} \tag{2}$$

for every $x, u \in C$, where a, b and c are nonnegative constants such that $a + 2b + 2c \leq 1$. By letting $a = 1$ and $b = c = 0$, we can see that every nonexpansive mapping is a generalized nonexpansive mapping. It is well-known from [9] that every generalized nonexpansive mapping with a fixed point is a quasi-nonexpansive mapping.

In 1953, Mann [10] introduced the following iterative scheme to approximate fixed points of a nonexpansive mapping T:

$$\begin{cases} x_1 \in C \\ x_{n+1} = (1 - a_n)x_n + a_n T x_n, \quad n \in \mathbb{N}, \end{cases} \tag{3}$$

where $\{a_n\}$ is a sequence in $(0, 1)$.

It is also known that, in general, Mann iteration does not necessarily converge to fixed points of nonexpansive mappings. Thus, in 1974, Ishikawa iteration [11] was introduced to approximate fixed points of such mappings as:

$$\begin{cases} x_1 \in C \\ x_{n+1} = (1 - a_n)x_n + a_n T y_n \\ y_n = (1 - b_n)x_n + b_n T x_n, \quad n \in \mathbb{N}, \end{cases} \tag{4}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $(0, 1)$.

In 1998, Xu [12] introduced the following iterative scheme, called Mann iteration with error terms, for nonexpansive mappings:

$$\begin{cases} x_1 \in C \\ x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, \quad n \in \mathbb{N}, \end{cases} \tag{5}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $(0, 1)$ such that $a_n + b_n + c_n = 1$ and $\{u_n\}$ is a bounded sequence in C. This scheme reduces to Mann iteration if $c_n = 0$.

When studying two mappings $T_1, T_2 : C \to X$, we recall that $x \in C$ is a common fixed point of T_1 and T_2 if $T_1 x = x = T_2 x$. The set of all common fixed points of T_1 and T_2 is denoted by $F(T_1, T_2)$. Moreover, $F(T_1, T_2) = F(T_1) \cap F(T_2)$.

To study convergence theorems for common fixed points of two mappings, Das and Debata [13] and Takahashi and Tamura [14] constructed the following iterative scheme:

$$\begin{cases} x_1 \in C \\ x_{n+1} = (1 - a_n)x_n + a_n T_1 y_n \\ y_n = (1 - b_n)x_n + b_n T_2 x_n, \quad n \in \mathbb{N}, \end{cases} \tag{6}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $(0, 1)$. If $T_1 = T_2$, then the scheme reduces to an Ishikawa iterative scheme.

Lui et al. [15] introduced the following iterative process to prove weak and strong convergence theorems of common fixed points for a pair of nonexpansive and asymptytotically nonexpansive mappings:

$$\begin{cases} x_1 \in C \\ x_{n+1} = (1 - a_n)T_1 x_n + a_n T_2 y_n \\ y_n = (1 - b_n)T_1 x_n + b_n T_2 x_n, \quad n \in \mathbb{N}, \end{cases} \tag{7}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $(0, 1)$.

Recently, Ali and Ali [16] proved a convergence theorem for common fixed points of the Mann iteration for two generalized nonexpansive mappings in uniformly convex Banach spaces. They defined a sequence $\{x_n\}$ as follows:

$$\begin{cases} x_1 \in C \\ x_{n+1} = a_n x_n + b_n T_1 x_n + c_n T_2 x_n, \quad n \in \mathbb{N}, \end{cases} \tag{8}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $(0,1)$ such that $a_n + b_n + c_n = 1$.

Another concept that relates to fixed points is the concept of attractive points, which was first introduced in Hilbert spaces by Takahashi and Takeuchi [17]. Let H be a Hilbert space, and let $T : C \to H$ be a mapping, where C is a nonempty subset of H. The set of all attractive points of T is denoted by $A(T)$; that is,

$$A(T) = \{z \in H : \|z - Ty\| \le \|z - y\|, \ \forall y \in C\}. \tag{9}$$

The authors proved an ergodic convergence theorem to find an attractive point of some nonlinear mappings without assuming convexity of its domain. Moreover, the authors [17] proved that $F(T)$ relates to $A(T)$ as follows.

Lemma 1. *Let C be a nonempty closed convex subset of H, and let T be a mapping from C into itself. If $A(T) \neq \emptyset$, then $F(T) \neq \emptyset$.*

Lemma 2. *Let C be a nonempty subset of H, and let T be a mapping from C into H. Then, $A(T)$ is a closed convex subset of H.*

Furthermore, we also know the following lemma from Takahashi et al. [18] for quasi-nonexpansive mappings.

Lemma 3. *Let C be a nonempty subset of H, and let T be a quasi-nonexpansive mapping from C into H. Then, $A(T) \cap C = F(T)$.*

In 2018, Khan [19] extended the concept of attractive points to the case of two mappings in Hilbert spaces. Let $T_1, T_2 : C \to H$, where C is a nonempty subset of H. The set of all common attractive points for T_1 and T_2 is denoted by $A(T_1, T_2)$; that is,

$$A(T_1, T_2) = \{z \in H : \max(\|T_1 x - z\|, \|T_2 x - z\|) \le \|x - z\|, \ \forall x \in C\}. \tag{10}$$

Moreover, $A(T_1, T_2) = A(T_1) \cap A(T_2)$. The next properties were proven by Khan [19].

Lemma 4. *Let C be a nonempty closed convex subset of H, and let $T_1, T_2 : C \to C$ be two mappings. If $A(T_1, T_2) \neq \emptyset$, then $F(T_1, T_2) \neq \emptyset$.*

Lemma 5. *Let C be a nonempty subset of H, and let $T_1, T_2 : C \to H$ be two mappings. Then, $A(T_1, T_2)$ is a closed convex subset of H.*

Lemma 6. *Let C be a nonempty subset of H, and let $T_1, T_2 : C \to H$ be two quasi-nonexpansive mappings. Then, $A(T_1, T_2) \cap C = F(T_1, T_2)$.*

Furthermore, there are many results of common attractive point theorems in a Hilbert space (see [19–21], and references therein).

In 2013, Lin and Takahashi [22] introduced the concept of attractive points of a nonlinear mapping in the Banach spaces setting. In 2015, Zheng [23] proved convergence theorems for attractive points, defined in (9), of some generalized nonexpansive mappings in uniformly convex Banach spaces. Moreover, there are more studies about attractive points for nonlinear mappings in Banach spaces (see [22–26]).

In this paper, motivated by all results mentioned above, we study some basic properties of the set of common attractive points, defined in (10), for two nonlinear mappings in the setting of uniformly convex Banach spaces. Furthermore, we prove strong convergence theorems of common attractive points of the Mann iteration (8) for two generalized nonexpansive mappings in a uniformly convex Banach space without closedness of the domain of such mappings. Using this result, we obtain strong convergence theorems of common fixed points in a uniformly convex Banach space and solve a convex minimization problem in Hilbert spaces. Finally, to support our results, a numerical example is given.

2. Preliminaries

Let $\{x_n\}$ be a sequence in a Banach space X. We denote the strong convergence of $\{x_n\}$ to $x \in X$ by $x_n \to x$. A Banach space X is called uniformly convex if for each $\epsilon \in [0,2]$, there is $\delta_\epsilon > 0$ such that

$$\|x\| = \|u\| = 1 \quad \text{and} \quad \|x-u\| \geq \epsilon \quad \text{implies} \quad \frac{\|x+u\|}{2} < 1 - \delta_\epsilon.$$

Remark 1. *Hilbert spaces are uniformly convex (see [27]).*

Definition 1 ([28])**.** *A subset C of a normed linear space X is said to be an existence subset of X if and only if for every element $x \in X$ there is an element $u \in C$ such that*

$$\|x-u\| = d(x,C) = \inf\{\|x-z\| : z \in C\},$$

where u is called the best approximation of x denoted by $\pi(x,C)$. It is well-known that if C is a closed and convex subset of a reflexive Banach space X, then the best approximation element $\pi(x,C)$ exists and is unique for every $x \in X$.

Every uniformly convex Banach space is reflexive (see [29]). We can see that if C is a closed and convex subset of a uniformly convex Banach space X, then for every $x \in X$ there exists a unique best approximation element $\pi(x,C)$.

Definition 2 ([28])**.** *Let C be a nonempty closed and convex subset of a normed space X. If for every $x \in X$ there exists a unique $\pi(x,C) \in C$, then the mapping $\pi(x,C)$ is called a metric projection onto C; that is,*

$$\|x - \pi(x,C)\| = d(x,C), \quad \forall x \in X.$$

It is well-known that if C is a closed and convex subset of a uniformly convex Banach space X, then there exists a metric projection from X onto C.

The following result is useful for our main theorem.

Lemma 7 ([30])**.** *Suppose X is a uniformly convex Banach space and $0 < s \leq r_n \leq t < 1$ for all $n \geq 1$. If $\{x_n\}$ and $\{u_n\}$ are two sequences in X such that $\limsup_{n\to\infty} \|x_n\| \leq d$, $\limsup_{n\to\infty} \|u_n\| \leq d$ and $\limsup_{n\to\infty} \|r_n x_n + (1-r_n)u_n\| = d$ hold for some $d \geq 0$. Then, $\lim_{n\to\infty} \|x_n - u_n\| = 0$.*

3. Main Results

In this section, we begin with studying some basic properties of common attractive points for two mappings in the framework of uniformly convex Banach spaces as follows:

Lemma 8. *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space X, and let $T_1, T_2 : C \to C$ be two mappings. If $A(T_1,T_2) \neq \emptyset$, then $F(T_1,T_2) \neq \emptyset$. In particular, if $u \in A(T_1,T_2)$, then $\pi(u,C) \in F(T_1,T_2)$.*

Proof. Let $u \in A(T_1, T_2)$; then, $u \in A(T_1)$ and $u \in A(T_2)$; that is,

$$\|T_1 y - u\| \leq \|y - u\| \quad \text{and} \quad \|T_2 y - u\| \leq \|y - u\|, \;\; \forall y \in C.$$

Since C is closed and convex, a unique element $x = \pi(u, C)$ exists in C and

$$\|x - u\| \leq \|y - u\|, \;\; \forall y \in C.$$

We know that $T_1 x \in C$, so $\|x - u\| \leq \|T_1 x - u\|$. Since $u \in A(T_1)$, we get

$$\|T_1 x - u\| \leq \|x - u\| \leq \|T_1 x - u\|.$$

Combing with the uniqueness of x, it follows that $T_1 x = x$. Thus $x \in F(T_1)$. Similarly, we also obtain $x \in F(T_2)$. Therefore, $x = \pi(u, C) \in F(T_1) \cap F(T_2) = F(T_1, T_2)$. □

Lemma 9. *Let C be a nonempty subset of a Banach space X, and let $T_1, T_2 : C \to X$ be two mappings. Then, $A(T_1, T_2)$ is a closed subset of X.*

Proof. Let $\{u_n\} \subset A(T_1, T_2)$ be a sequence converging strongly to some $u \in X$. We shall prove that $u \in A(T_1, T_2)$. Indeed, for all $x \in C$ and $u_n \in A(T_1, T_2)$, the following results:

$$\|T_1 x - u_n\| \leq \|x - u_n\| \quad \text{and} \quad \|T_2 x - u_n\| \leq \|x - u_n\|, \;\; \forall n \in \mathbb{N}.$$

By letting $n \to \infty$, we obtain

$$\|T_1 x - u\| \leq \|x - u\| \quad \text{and} \quad \|T_2 x - u\| \leq \|x - u\|, \;\; \forall x \in C.$$

Thus, $u \in A(T_1, T_2)$, and hence $A(T_1, T_2)$ is closed. □

Lemma 10. *Let C be a nonempty subset of a Banach space X, and let $T_1, T_2 : C \to X$ be two quasi-nonexpansive mappings. Then, $A(T_1, T_2) \cap C = F(T_1, T_2)$.*

Proof. Let $u \in A(T_1, T_2) \cap C$. Then, $u \in A(T_1)$ and $u \in A(T_2)$. That is

$$\|T_1 y - u\| \leq \|y - u\| \quad \text{and} \quad \|T_2 y - u\| \leq \|y - u\|, \;\; \forall y \in C.$$

In particular, by choosing $y = u \in C$, we get

$$\|T_1 u - u\| \leq \|u - u\| = 0 \quad \text{and} \quad \|T_2 u - u\| \leq \|u - u\| = 0.$$

It follows that $T_1 u = u$ and $T_2 u = u$. That is $u \in F(T_1) \cap F(T_2) = F(T_1, T_2)$.

Conversely, let $u \in F(T_1, T_2)$. Since T_1 and T_2 are quasi-nonexpansive mappings, we have

$$\|T_1 y - u\| \leq \|y - u\| \quad \text{and} \quad \|T_2 y - u\| \leq \|y - u\|, \;\; \forall y \in C.$$

Then,

$$\max(\|T_1 y - u\|, \|T_2 y - u\|) \leq \|y - u\|, \;\; \forall y \in C.$$

It follows that $u \in A(T_1, T_2)$. Since $u \in C$, we get $u \in A(T_1, T_2) \cap C$.

Thus, $A(T_1, T_2) \cap C = F(T_1, T_2)$. □

Next, we prove our main results, using the iterative scheme (8) for two generalized nonexpansive mappings in uniformly convex Banach spaces. Before proving the results, we need the following important tools.

Lemma 11. *Let C be a nonempty and convex subset of a Banach space X and $T_1, T_2 : C \to C$ be two generalized nonexpansive mappings with $A(T_1, T_2) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by (8):*

$$\begin{cases} x_1 \in C \\ x_{n+1} = a_n x_n + b_n T_1 x_n + c_n T_2 x_n, \ \ \forall n \in \mathbb{N}, \end{cases}$$

with $a_n, b_n, c_n \in (0,1)$ such that $a_n + b_n + c_n = 1$. Then, $\lim_{n\to\infty} \|x_n - u\|$ exists for any $u \in A(T_1, T_2)$.

Proof. Let $\{x_n\}$ be a sequence generated by (8) and $u \in A(T_1, T_2)$. Then,

$$\max(\|T_1 x - u\|, \|T_2 x - u\|) \leq \|x - u\|, \quad \forall x \in C.$$

It follows that

$$\|T_1 x - u\| \leq \|x - u\| \quad \text{and} \quad \|T_2 x - u\| \leq \|x - u\|, \ \ \forall x \in C.$$

Consider

$$\begin{aligned} \|x_{n+1} - u\| &= \|a_n x_n + b_n T_1 x_n + c_n T_2 x_n - u\| \\ &= \|a_n(x_n - u) + b_n(T_1 x_n - u) + c_n(T_2 x_n - u)\| \\ &\leq a_n\|x_n - u\| + b_n\|T_1 x_n - u\| + c_n\|T_2 x_n - u\| \\ &\leq a_n\|x_n - u\| + b_n\|x_n - u\| + c_n\|x_n - u\| \\ &= (a_n + b_n + c_n)\|x_n - u\| \\ &= \|x_n - u\|, \qquad \forall n \in \mathbb{N}. \end{aligned}$$

This shows that the sequence $\{\|x_n - u\|\}$ is nonincreasing and bounded below for all $u \in A(T_1, T_2)$. Therefore, $\lim_{n\to\infty} \|x_n - u\|$ exists. □

Lemma 12. *Let C be a nonempty and convex subset of a uniformly convex Banach space X and $T_1, T_2 : C \to C$ be two generalized nonexpansive mappings with $A(T_1, T_2) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by (8):*

$$\begin{cases} x_1 \in C \\ x_{n+1} = a_n x_n + b_n T_1 x_n + c_n T_2 x_n, \ \ \forall n \in \mathbb{N}, \end{cases}$$

with $a_n, b_n, c_n \in (0,1)$ such that $a_n + b_n + c_n = 1$ and $0 < s \leq a_n \leq t < 1$. Then, $\{x_n\}$ is bounded, $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0$ and $\lim_{n\to\infty} \|x_n - T_2 x_n\| = 0$.

Proof. Suppose that $A(T_1, T_2) \neq \emptyset$ and $u \in A(T_1, T_2)$. By using Lemma 11, we have that $\lim_{n\to\infty} \|x_n - u\|$ exists for all $u \in A(T_1, T_2)$. Therefore, $\{\|x_n - u\|\}$ is bounded, and so is $\{x_n\}$. Let

$$\lim_{n\to\infty} \|x_n - u\| = \alpha. \tag{11}$$

We show that $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0$ and $\lim_{n\to\infty} \|x_n - T_2 x_n\| = 0$.

Since $u \in A(T_1, T_2)$, we obtain that

$$\|T_1 x_n - u\| \leq \|x_n - u\| \quad \text{and} \quad \|T_2 x_n - u\| \leq \|x_n - u\|, \quad \forall n \in \mathbb{N}.$$

Therefore

$$\limsup_{n\to\infty} \|T_1 x_n - u\| \leq \limsup_{n\to\infty} \|x_n - u\| = \alpha \tag{12}$$

and

$$\limsup_{n\to\infty} \|T_2 x_n - u\| \leq \limsup_{n\to\infty} \|x_n - u\| = \alpha. \tag{13}$$

According to (11), we derive

$$\begin{aligned}
\alpha &= \lim_{n\to\infty} \|x_{n+1} - u\| \\
&= \lim_{n\to\infty} \|a_n x_n + b_n T_1 x_n + c_n T_2 x_n - u\| \\
&= \lim_{n\to\infty} \|a_n(x_n - u) + b_n(T_1 x_n - u) + c_n(T_2 x_n - u)\| \\
&= \lim_{n\to\infty} \left\|(1-c_n)\Big(\frac{a_n}{1-c_n}(x_n - u) + \frac{b_n}{1-c_n}(T_1 x_n - u)\Big) + c_n(T_2 x_n - u)\right\|. \qquad (14)
\end{aligned}$$

By using (12), we have

$$\begin{aligned}
\limsup_{n\to\infty} \left\|\frac{a_n}{1-c_n}(x_n - u) + \frac{b_n}{1-c_n}(T_1 x_n - u)\right\| &\le \limsup_{n\to\infty} \frac{a_n + b_n}{1-c_n}\left\|x_n - u\right\| \\
&= \limsup_{n\to\infty} \|x_n - u\| \\
&= \alpha. \qquad (15)
\end{aligned}$$

To apply Lemma 7, we choose

$$\mathbf{x_n} = T_2 x_n - u, \quad \mathbf{u_n} = \frac{a_n}{1-c_n}(x_n - u) + \frac{b_n}{1-c_n}(T_1 x_n - u), \quad \text{and} \quad \mathbf{r_n} = c_n, \quad \forall n \in \mathbb{N}.$$

Since $0 < s \le a_n \le t < 1$, we get $0 < s \le 1 - (b_n + c_n) \le t < 1$. Hence, there exist $v, w \in \mathbb{R}$ such that $0 < v \le c_n \le w < 1$ for all $n \in \mathbb{N}$. We have from (13) and (15) that

$$\limsup_{n\to\infty} \|\mathbf{x_n}\| = \limsup_{n\to\infty} \|T_2 x_n - u\| \le \alpha$$

and

$$\limsup_{n\to\infty} \|\mathbf{u_n}\| = \limsup_{n\to\infty} \left\|\frac{a_n}{1-c_n}(x_n - u) + \frac{b_n}{1-c_n}(T_1 x_n - u)\right\| \le \alpha.$$

Furthermore, from (14), we have

$$\begin{aligned}
&\limsup_{n\to\infty} \|\mathbf{r_n}\mathbf{x_n} + (1-\mathbf{r_n})\mathbf{u_n}\| \\
&\qquad = \lim_{n\to\infty} \left\|(1-c_n)\Big(\frac{a_n}{1-c_n}(x_n - u) + \frac{b_n}{1-c_n}(T_1 x_n - u)\Big) + c_n(T_2 x_n - u)\right\| \\
&\qquad = \alpha.
\end{aligned}$$

Therefore, the sequences $\{\mathbf{x_n}\}$, $\{\mathbf{u_n}\}$, and $\{\mathbf{r_n}\}$ satisfy the assumptions of Lemma 7, which imply that

$$\lim_{n\to\infty} \left\|\frac{a_n}{1-c_n}(x_n - u) + \frac{b_n}{1-c_n}(T_1 x_n - u) - (T_2 x_n - u)\right\| = \lim_{n\to\infty} \|\mathbf{u_n} - \mathbf{x_n}\| = 0. \qquad (16)$$

Consider

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{1-c_n}\left\|x_{n+1} - T_2 x_n\right\| \\
&\qquad = \lim_{n\to\infty} \frac{1}{1-c_n}\left\|a_n x_n + b_n T_1 x_n + c_n T_2 x_n - T_2 x_n + u - (a_n + b_n + c_n)u\right\| \\
&\qquad = \lim_{n\to\infty} \frac{1}{1-c_n}\left\|a_n(x_n - u) + b_n(T_1 x_n - u) - (1-c_n)(T_2 x_n - u)\right\| \\
&\qquad = \lim_{n\to\infty} \left\|\frac{a_n}{1-c_n}(x_n - u) + \frac{b_n}{1-c_n}(T_1 x_n - u) - (T_2 x_n - u)\right\|.
\end{aligned}$$

We can conclude that

$$\lim_{n\to\infty} \frac{1}{1-c_n} \left\| x_{n+1} - T_2 x_n \right\| = 0.$$

Since $0 < v \leq c_n \leq w < 1$ for all $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \|x_{n+1} - T_2 x_n\| = 0. \tag{17}$$

Next, we will show that $\lim_{n\to\infty} \|x_{n+1} - T_1 x_n\| = 0$. According to (11), we have

$$\begin{aligned} \alpha &= \lim_{n\to\infty} \|x_{n+1} - u\| \\ &= \lim_{n\to\infty} \|a_n x_n + b_n T_1 x_n + c_n T_2 x_n - u\| \\ &= \lim_{n\to\infty} \|a_n(x_n - u) + b_n(T_1 x_n - u) + c_n(T_2 x_n - u)\| \\ &= \lim_{n\to\infty} \left\|(1-b_n)\left(\frac{a_n}{1-b_n}(x_n - u) + \frac{c_n}{1-b_n}(T_2 x_n - u)\right) + b_n(T_1 x_n - u)\right\|. \end{aligned} \tag{18}$$

By using (13), we have

$$\begin{aligned} \limsup_{n\to\infty} \left\|\frac{a_n}{1-b_n}(x_n - u) + \frac{c_n}{1-b_n}(T_2 x_n - u)\right\| &\leq \limsup_{n\to\infty} \frac{a_n + c_n}{1-b_n} \left\| x_n - u \right\| \\ &= \limsup_{n\to\infty} \|x_n - u\| \\ &= \alpha. \end{aligned} \tag{19}$$

Applying Lemma 7 again, we choose

$$\mathbf{x_n} = T_1 x_n - u, \quad \mathbf{u_n} = \frac{a_n}{1-b_n}(x_n - u) + \frac{c_n}{1-b_n}(T_2 x_n - u), \quad \text{and} \quad \mathbf{r_n} = b_n, \quad \forall n \in \mathbb{N}.$$

Since $0 < s \leq a_n \leq t < 1$, we get $0 < s \leq 1 - (b_n + c_n) \leq t < 1$. Hence, there exist $v', w' \in \mathbb{R}$ such that $0 < v' \leq b_n \leq w' < 1$ for all $n \in \mathbb{N}$. We have from (12) and (19) that

$$\limsup_{n\to\infty} \|\mathbf{x_n}\| = \limsup_{n\to\infty} \|T_1 x_n - u\| \leq \alpha$$

and

$$\limsup_{n\to\infty} \|\mathbf{u_n}\| = \limsup_{n\to\infty} \left\|\frac{a_n}{1-b_n}(x_n - u) + \frac{c_n}{1-b_n}(T_2 x_n - u)\right\| \leq \alpha.$$

Furthermore, from (18), we have

$$\begin{aligned} &\limsup_{n\to\infty} \|\mathbf{r_n x_n} + (1-\mathbf{r_n})\mathbf{u_n}\| \\ &\qquad = \lim_{n\to\infty} \left\|(1-b_n)\left(\frac{a_n}{1-b_n}(x_n - u) + \frac{c_n}{1-b_n}(T_2 x_n - u)\right) + b_n(T_1 x_n - u)\right\| \\ &\qquad = \alpha. \end{aligned}$$

Therefore, we can apply Lemma 7 to get

$$\lim_{n\to\infty} \left\|\frac{a_n}{1-b_n}(x_n - u) + \frac{c_n}{1-b_n}(T_2 x_n - u) - (T_1 x_n - u)\right\| = \lim_{n\to\infty} \|\mathbf{u_n} - \mathbf{x_n}\| = 0. \tag{20}$$

Consider

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{1-b_n}\Big\|x_{n+1}-T_1x_n\Big\| \\
&\quad=\lim_{n\to\infty}\frac{1}{1-b_n}\Big\|a_nx_n+b_nT_1x_n+c_nT_2x_n-T_1x_n+u-(a_n+b_n+c_n)u\Big\| \\
&\quad=\lim_{n\to\infty}\frac{1}{1-b_n}\Big\|a_n(x_n-u)+c_n(T_2x_n-u)-(1-b_n)(T_1x_n-u)\Big\| \\
&\quad=\lim_{n\to\infty}\Big\|\frac{a_n}{1-b_n}(x_n-u)+\frac{c_n}{1-b_n}(T_2x_n-u)-(T_1x_n-u)\Big\|,
\end{aligned}$$

we can conclude that

$$\lim_{n\to\infty}\frac{1}{1-b_n}\Big\|x_{n+1}-T_1x_n\Big\|=0.$$

Since $0<v'\leq b_n\leq w'<1$ for all $n\in\mathbb{N}$, we get

$$\lim_{n\to\infty}\|x_{n+1}-T_1x_n\|=0. \tag{21}$$

Finally, we will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. According to (11), we can derive

$$\begin{aligned}
\alpha&=\lim_{n\to\infty}\|x_{n+1}-u\| \\
&=\lim_{n\to\infty}\|a_nx_n+b_nT_1x_n+c_nT_2x_n-u\| \\
&=\lim_{n\to\infty}\|a_n(x_n-u)+b_n(T_1x_n-u)+c_n(T_2x_n-u)\| \\
&=\lim_{n\to\infty}\Big\|a_n(x_n-u)+(1-a_n)\Big(\frac{b_n}{1-a_n}(T_1x_n-u)+\frac{c_n}{1-a_n}(T_2x_n-u)\Big)\Big\|.
\end{aligned}\tag{22}$$

By using (12) and (13), we have

$$\begin{aligned}
\limsup_{n\to\infty}\Big\|\frac{b_n}{1-a_n}(T_1x_n-u)+\frac{c_n}{1-a_n}(T_2x_n-u)\Big\|&\leq\limsup_{n\to\infty}\frac{b_n+c_n}{1-a_n}\Big\|x_n-u\Big\| \\
&=\limsup_{n\to\infty}\|x_n-u\| \\
&=\alpha.
\end{aligned}\tag{23}$$

To apply Lemma 7, we choose

$$\mathbf{x_n}=x_n-u,\quad \mathbf{u_n}=\frac{b_n}{1-a_n}(T_1x_n-u)+\frac{c_n}{1-a_n}(T_2x_n-u),\quad\text{and}\quad \mathbf{r_n}=a_n,\quad\forall n\in\mathbb{N}.$$

We have from (11) and (23) that

$$\limsup_{n\to\infty}\|\mathbf{x_n}\|=\limsup_{n\to\infty}\|x_n-u\|\leq\alpha$$

and

$$\limsup_{n\to\infty}\|\mathbf{u_n}\|=\limsup_{n\to\infty}\Big\|\frac{b_n}{1-a_n}(T_1x_n-u)+\frac{c_n}{1-a_n}(T_2x_n-u)\Big\|\leq\alpha.$$

Furthermore, from (22), we have

$$\begin{aligned}
&\limsup_{n\to\infty}\|\mathbf{r_n}\mathbf{x_n}+(1-\mathbf{r_n})\mathbf{u_n}\| \\
&\qquad=\lim_{n\to\infty}\Big\|(1-a_n)\Big(\frac{b_n}{1-a_n}(T_1x_n-u)+\frac{c_n}{1-a_n}(T_2x_n-u)\Big)+a_n(x_n-u)\Big\| \\
&\qquad=\alpha.
\end{aligned}$$

Therefore, we can apply Lemma 7 to get

$$\lim_{n\to\infty}\left\|\frac{b_n}{1-a_n}(T_1x_n-u)+\frac{c_n}{1-a_n}(T_2x_n-u)-(x_n-u)\right\| = \lim_{n\to\infty}\|\mathbf{u_n}-\mathbf{x_n}\|=0. \tag{24}$$

Consider

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{1-a_n}\left\|x_{n+1}-x_n\right\| \\
&\quad=\lim_{n\to\infty}\frac{1}{1-a_n}\left\|a_nx_n+b_nT_1x_n+c_nT_2x_n-x_n+u-(a_n+b_n+c_n)u\right\| \\
&\quad=\lim_{n\to\infty}\frac{1}{1-a_n}\left\|b_n(T_1x_n-u)+c_n(T_2x_n-u)-(1-a_n)(x_n-u)\right\| \\
&\quad=\lim_{n\to\infty}\left\|\frac{b_n}{1-a_n}(T_1x_n-u)+\frac{c_n}{1-a_n}(T_2x_n-u)-(x_n-u)\right\|.
\end{aligned}$$

We can conclude that

$$\lim_{n\to\infty}\frac{1}{1-a_n}\left\|x_{n+1}-x_n\right\|=0.$$

Since $0<s\le a_n\le t<1$, we get

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{25}$$

Note that

$$\|x_n-T_1x_n\|\le\|x_n-x_{n+1}\|+\|x_{n+1}-T_1x_n\|,\quad \forall n\in\mathbb{N}, \tag{26}$$

and

$$\|x_n-T_2x_n\|\le\|x_n-x_{n+1}\|+\|x_{n+1}-T_2x_n\|,\quad \forall n\in\mathbb{N}. \tag{27}$$

By (17), (21), (25) and $n\to\infty$ in (26) and (27), we get

$$\lim_{n\to\infty}\|x_n-T_1x_n\|=0 \quad\text{and}\quad \lim_{n\to\infty}\|x_n-T_2x_n\|=0$$

as desired. □

Next, we establish a strong convergence theorem of common attractive points for two generalized nonexpansive mappings, in the sense of Hardy and Roger [8] defined in (2), in a uniformly convex Banach space without assuming the closedness of the domain of such mappings.

Theorem 1. *Let C be a nonempty and convex subset of a uniformly convex Banach space X and $T_1, T_2 : C\to C$ be two generalized nonexpansive mappings with $A(T_1,T_2)\neq\emptyset$. Let $\{x_n\}$ be a sequence generated by (8):*

$$\begin{cases} x_1\in C \\ x_{n+1}=a_nx_n+b_nT_1x_n+c_nT_2x_n, \quad \forall n\in\mathbb{N}, \end{cases}$$

with $a_n, b_n, c_n\in(0,1)$ such that $a_n+b_n+c_n=1$ and $0<s\le a_n\le t<1$. Then, $\{x_n\}$ converges strongly to $z\in A(T_1,T_2)$ if and only if $\liminf_{n\to\infty} d(x_n, A(T_1,T_2))=0$ or $\limsup_{n\to\infty} d(x_n, A(T_1,T_2))=0$.

Proof. Suppose that $x_n\to u\in A(T_1,T_2)$. Then, for each $\epsilon>0$, there exists $m_0\in\mathbb{N}$ such that

$$\|x_n-u\|<\epsilon$$

for all $n\ge m_0$. Therefore, we obtain

$$d(x_n, A(T_1,T_2))=\inf\{\|x_n-u\| : u\in A(T_1,T_2)\}\le\|x_n-u\|<\epsilon,\quad \forall n\ge m_0.$$

It follows that $\lim_{n\to\infty} d(x_n, A(T_1, T_2)) = 0$, and hence

$$\liminf_{n\to\infty} d(x_n, A(T_1, T_2)) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(x_n, A(T_1, T_2)) = 0.$$

Conversely, if $\limsup_{n\to\infty} d(x_n, A(T_1, T_2)) = 0$, then $\lim_{n\to\infty} d(x_n, A(T_1, T_2)) = 0$.

Assume that $\liminf_{n\to\infty} d(x_n, A(T_1, T_2)) = 0$. This means that $\{d(x_n, A(T_1, T_2))\}$ contains a subsequence $\{d(x_{n_k}, A(T_1, T_2))\}$ such that

$$\lim_{k\to\infty} d(x_{n_k}, A(T_1, T_2)) = 0.$$

By Lemma 11, we have that $\lim_{n\to\infty} \|x_n - u\|$ exists for all $u \in A(T_1, T_2)$. Then,

$$\lim_{n\to\infty} \|x_n - u\| = \lim_{k\to\infty} \|x_{n_k} - u\|.$$

It follows that

$$\lim_{n\to\infty} (\inf\{\|x_n - u\| : u \in A(T_1, T_2)\}) = \lim_{k\to\infty} (\inf\{\|x_{n_k} - u\| : u \in A(T_1, T_2)\}) = 0.$$

That is,

$$\lim_{n\to\infty} d(x_n, A(T_1, T_2)) = 0.$$

Next, we shall show that $\{x_n\}$ is a Cauchy sequence. By Lemma 11, we have $\|x_{n+1} - u\| \leq \|x_n - u\|$ for all $u \in A(T_1, T_2)$. In fact, for any $n, m \in \mathbb{N}$, without of generality, we may set $m > n$. Then,

$$\|x_m - u\| \leq \|x_n - u\|, \quad \forall u \in A(T_1, T_2).$$

Consider

$$\|x_n - x_m\| \leq \|x_n - u\| + \|u - x_m\| \leq 2\|x_n - u\|.$$

Since u is arbitrary, we may take infimum all over $u \in A(T_1, T_2)$ on both sides to get

$$\|x_n - x_m\| \leq 2\inf\{\|x_n - u\| : u \in A(T_1, T_2)\} = 2d(x_n, A(T_1, T_2)).$$

From $\lim_{n\to\infty} d(x_n, A(T_1, T_2)) = 0$, we get $\lim_{n\to\infty} \|x_n - x_m\| = 0$. This means that $\{x_n\}$ is a Cauchy sequence in a uniformly convex Banach space X. Thus, there exists $z \in X$ such that

$$\lim_{n\to\infty} \|x_n - z\| = 0.$$

By Lemma 12, we have

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|x_n - T_2 x_n\| = 0.$$

Thus,

$$\lim_{n\to\infty} \|z - T_1 x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|z - T_2 x_n\| = 0.$$

Next, we show that $z \in A(T_1, T_2)$. Since T_1 is generalized nonexpansive mapping, we can use (1) to get that there exist nonnegative constants a, b, c with $a + 2b + 2c \leq 1$ such that

$$\begin{aligned} \|T_1 x - T_1 x_n\| &\leq a\|x - x_n\| + b(\|x - T_1 x\| + \|x_n - T_1 x_n\|) \\ &\quad + c(\|x - T_1 x_n\| + \|x_n - T_1 x\|), \quad \forall x \in C, \ \forall n \in \mathbb{N}. \end{aligned}$$

Thus

$$\begin{aligned} \|T_1 x - T_1 x_n\| &\leq a\|x - x_n\| + b(\|x - x_n\| + \|x_n - T_1 x\| + \|x_n - T_1 x_n\|) \\ &\quad + c(\|x - T_1 x_n\| + \|x_n - T_1 x\|), \quad \forall x \in C, \ \forall n \in \mathbb{N}. \end{aligned}$$

Taking limit on both sides, we get

$$\begin{aligned}\|T_1x - z\| &\le a\|x - z\| + b(\|x - z\| + \|z - T_1x\| + \|z - z\|) + c(\|x - z\| + \|z - T_1x\|)\\ &= (a + b + c)\|x - z\| + (b + c)\|z - T_1x\|.\end{aligned}$$

Hence, $(1 - b - c)\|T_1x - z\| \le (a + b + c)\|x - z\|, \quad \forall x \in C$.

Since $a + 2b + 2c \le 1$, we have

$$\frac{a + b + c}{1 - b - c} \le \frac{1 - b - c}{1 - b - c} = 1.$$

Thus, $\|T_1x - z\| \le \|x - z\|$ for all $x \in C$. This means that $z \in A(T_1)$.

Similarly, we have $z \in A(T_2)$. Therefore, $z \in A(T_1, T_2)$. □

By applying Theorem 1, we also obtain the following result.

Corollary 1. *Let C be a nonempty and convex subset of a uniformly convex Banach space X and $T_1, T_2 : C \to C$ be two generalized nonexpansive mappings with $F(T_1, T_2) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by (8) with $a_n, b_n, c_n \in (0, 1)$ such that $a_n + b_n + c_n = 1$ and $0 < s \le a_n \le t < 1$ for all $n \in \mathbb{N}$.*

(1) Suppose that $\liminf_{n\to\infty} d(x_n, F(T_1, T_2)) = 0$ or $\limsup_{n\to\infty} d(x_n, F(T_1, T_2)) = 0$. Then, $\{x_n\}$ converges strongly to $z \in A(T_1, T_2)$. If C is closed, then $\{x_n\}$ converges strongly to $z \in F(T_1, T_2)$.

(2) Suppose that $\{x_n\}$ converges strongly to the common attractive point of T_1 and T_2; then, $\liminf_{n\to\infty} d(x_n, A(T_1, T_2)) = 0$ or $\limsup_{n\to\infty} d(x_n, A(T_1, T_2)) = 0$. If C is closed, then $\liminf_{n\to\infty} d(x_n, F(T_1, T_2)) = 0$ or $\limsup_{n\to\infty} d(x_n, F(T_1, T_2)) = 0$.

Proof. Since $F(T_1, T_2) \neq \emptyset$, we have T_1, T_2 are quasi-nonexpansive mappings. By Lemma 10, we have $A(T_1, T_2) \cap C = F(T_1, T_2)$, which implies that, $A(T_1, T_2) \neq \emptyset$.

(1) Suppose that

$$\liminf_{n\to\infty} d(x_n, F(T_1, T_2)) = 0 \quad \text{or} \quad \limsup_{n\to\infty} d(x_n, F(T_1, T_2)) = 0.$$

Since $F(T_1, T_2) \subset A(T_1, T_2)$, we have

$$d(x_n, A(T_1, T_2)) \le d(x_n, F(T_1, T_2)), \ \forall n \in \mathbb{N}.$$

Then,

$$\liminf_{n\to\infty} d(x_n, A(T_1, T_2)) \le \liminf_{n\to\infty} d(x_n, F(T_1, T_2)) = 0,$$

or

$$\limsup_{n\to\infty} d(x_n, A(T_1, T_2)) \le \limsup_{n\to\infty} d(x_n, F(T_1, T_2)) = 0.$$

By Theorem 1, we get $x_n \to z \in A(T_1, T_2)$. If C is closed, then $z \in C$. It follows that $x_n \to z \in F(T_1, T_2)$.

(2) Assume that $x_n \to z \in A(T_1, T_2)$. By Theorem 1, we get

$$\liminf_{n\to\infty} d(x_n, A(T_1, T_2)) = 0 \quad \text{or} \quad \limsup_{n\to\infty} d(x_n, A(T_1, T_2)) = 0.$$

If C is closed, then $z \in C$. Since $A(T_1, T_2) \cap C = F(T_1, T_2)$, we have $x_n \to z \in F(T_1, T_2)$. It follows that $\liminf_{n\to\infty} d(x_n, F(T_1, T_2)) = 0$ or $\limsup_{n\to\infty} d(x_n, F(T_1, T_2)) = 0$. □

Next, we aim to apply Corollary 1 to solve a convex minimization problem in Hilbert spaces. The general formulation of the convex minimization problem is defined by

$$\min_x \{f(x) + g(x)\}. \tag{28}$$

In a Hilbert space H, the solution of problem (28) is usually considered under the following assumptions:

(i) g is a lower semicontinuous function and properly convex from H into $\mathbb{R} \cup \{+\infty\}$;
(ii) f is a convex differentiable function from H into $\mathbb{R}$, with ∇f being ℓ-Lipschitz constant for some $\ell > 0$—that is, $\|\nabla f(x) - \nabla f(y)\| \leq \ell \|x - y\|$ for all $x, y \in \mathbb{R}^n$.

We denote the set of all solutions of (28) by $\operatorname{argmin}(f+g)$, and it is well-known that finding a solution of problem (28) is equivalent to finding a zero-solution $x^\star$ such that:

$$0 \in \partial g(x^\star) + \nabla f(x^\star), \tag{29}$$

where ∇f is the gradient operator of function f and ∂g is the subdifferential of function g; see [31] for more details. Dealing with fixed point theory, Parikh and Boyd [32] solved the problem (29) by using the proximal gradient technique; that is, if $x^\star$ solves (29), then $x^\star$ is a fixed point of the proximal operator:

$$x^\star = \operatorname{prox}_{\kappa g}(I - \kappa \nabla f)(x^\star),$$

where κ is a positive parameter, $\operatorname{prox}_{\kappa g} = (I + \kappa \partial g)^{-1}$ and I is the identity operator.

If we set $T_1 := \operatorname{prox}_{\kappa_1 g}(I - \kappa_1 \nabla f)$ and $T_2 := \operatorname{prox}_{\kappa_2 g}(I - \kappa_2 \nabla f)$ where $\kappa_1, \kappa_2 \in \left(0, \frac{2}{\ell}\right)$, then T_1 and T_2 are nonexpansive mappings; see [33–35] for more details.

We denote $S^\star = \{x^\star : x^\star \in \operatorname{argmin}(f+g)\}$. The following result is a consequence of Corollary 1.

Corollary 2. *Let g be a lower semicontinuous function and proper convex from a real Hilbert space H into $\mathbb{R} \cup \{+\infty\}$, and let f be a convex differentiable function from H into $\mathbb{R}$ with ∇f being ℓ-Lipschitz constant for some $\ell > 0$. Let $\{x_n\}$ be a sequence generated by (8) under the same conditions of parameters as in Corollary 1 with $S^\star \neq \emptyset$. If $\liminf_{n\to\infty} d(x_n, S^\star) = 0$ or $\limsup_{n\to\infty} d(x_n, S^\star) = 0$, then $\{x_n\}$ converges strongly to an element in argmin$(f+g)$.*

Proof. Let T_1 and T_2 be the forward–backward operators of f and g with respect to κ_1 and κ_2, respectively, where $\kappa_1, \kappa_2 \in \left(0, \frac{2}{\ell}\right)$. Then, $T_1 :=\operatorname{prox}_{\kappa_1 g}(I - \kappa_1 \nabla f)$ and $T_2 :=\operatorname{prox}_{\kappa_2 g}(I - \kappa_2 \nabla f)$ are nonexpansive mappings which imply generalized nonexpansiveness. By Proposition 26.1 in [31], $F(T_1, T_2) = F(T_1) \cap F(T_2) = \operatorname{argmin}(f+g)$. Using (1) of Corollary 1, we obtain that $\{x_n\}$ converges strongly to an element in $\operatorname{argmin}(f+g)$. $\square$

Remark 2. *A convex minimization problem (28) can be applied to solving many real world problems, such as image/signal processing, regression and data classification; see [36–38]. Hence, the convergence result of Corollary 2 can be applied for solving those important problems.*

Another condition for strong convergence results was introduced by Senter and Dotson [39]. A mapping $T : C \to C$ satisfies condition (A), if there exists a nondecreasing function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ and $h(\gamma) > 0$ for every $\gamma > 0$ such that

$$h(d(u, A(T))) \leq \|u - Tu\|, \quad \forall u \in C.$$

Chidume and Ali extended above condition to two mappings [40] as follows: Two mappings $T_1, T_2 : C \to C$ are said to satisfy condition (A') if there exists a nondecreasing function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ and $h(\gamma) > 0$ for every $\gamma > 0$ such that

$$h(d(u, A(T_1, T_2))) \leq \|u - T_1 u\| \quad \text{or} \quad h(d(u, A(T_1, T_2))) \leq \|u - T_2 u\|, \quad \forall u \in C.$$

In the following theorem, we obtain a strong convergence theorem for common attractive points of two generalized nonexpansive mappings that satisfy condition (A').

Theorem 2. *Let C be a nonempty and convex subset of a uniformly convex Banach space X and $T_1, T_2 : C \to C$ be two generalized nonexpansive mappings. Let $\{x_n\}$ be a sequence generated by (8):*

$$\begin{cases} x_1 \in C \\ x_{n+1} = a_n x_n + b_n T_1 x_n + c_n T_2 x_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

with $a_n, b_n, c_n \in (0,1)$ such that $a_n + b_n + c_n = 1$ and $0 < s \leq a_n \leq t < 1$. Suppose that $A(T_1, T_2) \neq \emptyset$ and T_1, T_2 satisfy condition (A'); then, $\{x_n\}$ converges strongly to a common attractive point of T_1 and T_2.

Proof. By Lemma 12, we have

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0 = \lim_{n\to\infty} \|x_n - T_2 x_n\|.$$

Since T_1 and T_2 satisfy condition (A'), there exists a nondecreasing function $h : [0,\infty) \to [0,\infty)$ with $h(0) = 0$ and $h(\gamma) > 0$ for all $\gamma > 0$ such that

$$h(d(x_n, A(T_1, T_2))) \leq \|x_n - T_1 x_n\| \quad \text{or} \quad h(d(x_n, A(T_1, T_2))) \leq \|x_n - T_2 x_n\|.$$

It follows that

$$0 \leq \lim_{n\to\infty} h(d(x_n, A(T_1, T_2))) \leq \lim_{n\to\infty} \|x_n - T_1 x_n\| = 0,$$

or

$$0 \leq \lim_{n\to\infty} h(d(x_n, A(T_1, T_2))) \leq \lim_{n\to\infty} \|x_n - T_2 x_n\| = 0.$$

In both cases, we get

$$\lim_{n\to\infty} h(d(x_n, A(T_1, T_2))) = 0.$$

Since $h : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $h(0) = 0$ and $h(\gamma) > 0$ for all $\gamma > 0$, we get

$$\lim_{n\to\infty} d(x_n, A(T_1, T_2)) = 0.$$

It follows that

$$\liminf_{n\to\infty} d(x_n, A(T_1, T_2)) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(x_n, A(T_1, T_2)) = 0.$$

By Theorem 1, we conclude that $\{x_n\}$ converges strongly to a common attractive point of T_1 and T_2. □

Corollary 3. *Let C be a nonempty and convex subset of a uniformly convex Banach space X and $T_1, T_2 : C \to C$ be two generalized nonexpansive mappings. Let $\{x_n\}$ be a sequence generated by (8) with $a_n, b_n, c_n \in (0,1)$ such that $a_n + b_n + c_n = 1$ and $0 < s \leq a_n \leq t < 1$ for all $n \in \mathbb{N}$. Suppose that $F(T_1, T_2) \neq \emptyset$ and T_1, T_2 satisfy condition (A'); then, $\{x_n\}$ converges strongly to a common attractive point of T_1 and T_2. If C is closed, then $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.*

Proof. Since $F(T_1, T_2) \neq \emptyset$, we have T_1, T_2 are quasi-nonexpansive mappings. By Lemma 10, we have $A(T_1, T_2) \cap C = F(T_1, T_2)$. It follows that $A(T_1, T_2) \neq \emptyset$. By Theorem 2, we have $x_n \to z \in A(T_1, T_2)$. If C is closed, then $z \in F(T_1, T_2)$. □

We end this section by providing some numerical experiments to illustrate the performance of iteration (8) for supporting our main results.

Example 1. *Let $X = \mathbb{R}$ with the usual norm and $C = (0,\infty)$. Suppose $T_1, T_2 : C \to C$ are defined by*

$$T_1 x = \begin{cases} \frac{x}{4}, & x \in (0,1) \\ \frac{x}{16}, & x \in [1,\infty), \end{cases}$$

and

$$T_2 x = \begin{cases} \frac{x}{5}, & x \in (0,1) \\ \frac{x}{25}, & x \in [1,\infty). \end{cases}$$

Then, T_1 and T_2 are generalized nonexpansive mappings such that $0 \in A(T_1, T_2)$ but $F(T_1, T_2) = \emptyset$. We chose the parameters $a_n = \frac{n}{50n-1}, b_n = \frac{n}{50n-1}, c_n = \frac{48n-1}{50n-1}$ and initial point $x_1 = \frac{1}{2}$. Table 1 shows the values of x_n, $|x_n - 0|$, $|x_n - T_1 x_n|$ and $|x_n - T_2 x_n|$ of iteration $n = 1, 2, 3, \ldots, 10$.

Table 1. Numerical experiment of the iteration process (8).

Iteration No.	x_n	$\lvert x_n - 0\rvert$	$\lvert x_n - T_1 x_n\rvert$	$\lvert x_n - T_2 x_n\rvert$
1	0.5000	0.5000	0.4688	0.4800
2	0.0300	0.0300	0.0281	0.0288
3	0.0018	0.0018	0.0017	0.0017
4	1.0743×10^{-4}	1.0743×10^{-4}	1.0071×10^{-4}	1.0313×10^{-4}
5	6.4186×10^{-6}	6.4186×10^{-6}	6.0175×10^{-6}	6.1619×10^{-6}
6	3.8338×10^{-7}	3.8338×10^{-7}	3.5942×10^{-7}	3.6804×10^{-7}
7	2.2894×10^{-8}	2.2894×10^{-8}	2.1463×10^{-8}	2.1978×10^{-8}
8	1.3669×10^{-9}	1.3669×10^{-9}	1.2815×10^{-9}	1.3122×10^{-9}
9	8.1603×10^{-11}	8.1603×10^{-11}	7.6502×10^{-11}	7.8339×10^{-11}
10	4.8712×10^{-12}	4.8712×10^{-12}	4.5667×10^{-12}	4.6763×10^{-12}

It is evident from Table 1 that $x_n \to 0 \in A(T_1, T_2)$, with the errors $|x_n - 0| \to 0$, $|x_n - T_1 x_n| \to 0$ and $|x_n - T_2 x_n| \to 0$. Moreover, Figure 1 shows the convergence behavior of the iterative process (8).

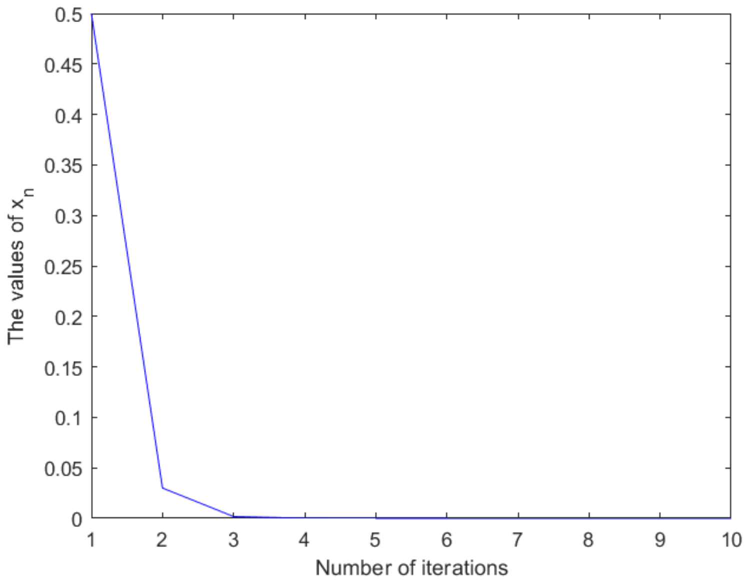

Figure 1. The graph of the convergence behavior of the iteration process (8).

Next, under control conditions from Theorem 1, we compared the rates of convergence for the sequences $\{x_n\}$ generated by (6)–(8), as shown in Table 2.

Table 2. The values of $\{x_n\}$ for different iteration processes.

Iteration No.	x_n **of Iteration** (6)	x_n **of Iteration** (7)	x_n **of Iteration** (8)
1	0.5000	0.5000	0.5000
2	0.4904	0.0306	0.0300
3	0.4811	0.0019	0.0018
4	0.4720	1.1509×10^{-4}	1.0743×10^{-4}
5	0.4631	7.0541×10^{-6}	6.4186×10^{-6}
6	0.4544	4.3238×10^{-7}	3.8338×10^{-7}
7	0.4458	2.6503×10^{-8}	2.2894×10^{-8}
8	0.4374	1.6245×10^{-9}	1.3669×10^{-9}
9	0.4292	9.9578×10^{-11}	8.1603×10^{-11}
10	0.4211	6.1039×10^{-12}	4.8712×10^{-12}

From Table 2, we can see that iteration process (8) performs with a better rate of convergence than iteration processes (6) and (7).

4. Conclusions

In this paper, we studied some basic properties of the set of common attractive points, defined in (10), of two nonlinear mappings in the setting of uniformly convex Banach spaces. Furthermore, using the Mann iteration (8), we proved strong convergence theorems of common attractive points for two generalized nonexpansive mappings, in the sense of Hardy and Rogers [8], in a uniformly convex Banach space without the closedness assumption of the domain of such mappings. Using this result, we obtained strong convergence theorems of common fixed points in a uniformly convex Banach space and solved some convex minimize problems. Finally, we constructed a numerical example to support our main result.

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