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Bertrand and Mannheim Curves of Spherical Framed Curves in a Three-Dimensional Sphere

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Abstract: We investigated differential geometries of Bertrand curves and Mannheim curves in a three-dimensional sphere. We clarify the conditions for regular spherical curves to become Bertrand and Mannheim curves. Then, we concentrate on Bertrand and Mannheim curves of singular spherical curves. As singular spherical curves, we considered spherical framed curves. We define Bertrand and Mannheim curves of spherical framed curves. We give conditions for spherical framed curves to become Bertrand and Mannheim curves.

Keywords: Bertrand curves; Mannheim curves; spherical regular curves; spherical framed curves; singularity

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1. Introduction

In differential geometries, Bertrand and Mannheim curves are classical objects, which have been deeply studied in the Euclidean space [1–5]. Given a curve γ, a Bertrand curve is a curve $\overline{\gamma}$ such that the principal normal vector field of γ coincides with the principal normal vector field of $\overline{\gamma}$. Another type of associated curve is the Mannheim curve such that the bi-normal vector field of γ coincides with the principal normal vector field of $\overline{\gamma}$. Bertrand and Mannheim curves have an important role and a wide range of applications, which are used in computer-aided geometric design, computer-aided manufacturing, and physical sciences [6–8].

Recently, mathematicians have paid attention to Bertrand and Mannheim curves in other spaces, such as in a three-dimensional sphere and in non-flat space form [9–14]. In the three-dimensional sphere, a Bertrand curve is a spherical curve whose principal normal geodesic is the same as the principal normal geodesic of another spherical curve. A Mannheim curve is a spherical curve whose principal normal geodesic is the same as the bi-normal geodesic of another spherical curve. In order to define the principal normal geodesic vector, a non-degenerate condition is required. However, for regular Bertrand and Mannheim curves, the existence condition is not sufficient in general. In [15], the non-degenerate condition for Bertrand or Mannheim curves of regular curves in the three-dimensional Euclidean space was added. Moreover, the existence the conditions of the Bertrand and Mannheim curves of framed curves were discussed.

In this paper, we would like to treat Bertrand and Mannheim curves in the three-dimensional sphere. We investigate not only Bertrand and Mannheim curves of spherical regular curves, but also Bertrand and Mannheim curves of spherical singular curves. In Section 2, we clarify the conditions for spherical regular curves to become Bertrand and Mannheim curves, respectively (Theorems 2 and 3). As an application of our results, we clarify the relations between Bertrand curves (respectively, Mannheim curves) and general helices. Then, we consider singular spherical curves. As singular spherical curves, we considered spherical framed curves. A spherical framed curve is a smooth curve endowed with a moving frame. It is a generalization of a Legendre curve in the unit spherical bundle over the unit sphere (cf. [16]) and of a framed curve in the Euclidean space (cf. [17]). In Section 3, we define Bertrand and Mannheim curves of spherical framed curves. Then, we give conditions for spherical framed curves to become Bertrand and Mannheim curves, respectively (Theorems 6 and 7). Moreover, we give some examples to illustrate our results.

All maps and manifolds considered in this paper are differentiable of class C^∞.

2. Regular Spherical Curves

Let $\mathbb{R}^4$ be the four-dimensional Euclidean space equipped with the inner product $\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4$, where $\boldsymbol{a} = (a_1, a_2, a_3, a_4)$ and $\boldsymbol{b} = (b_1, b_2, b_3, b_4) \in \mathbb{R}^4$. The norm of $\boldsymbol{a}$ is given by $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$. Let $\boldsymbol{a}_i \in \mathbb{R}^4$ be vectors $\boldsymbol{a}_i = (a_{i1}, a_{i2}, a_{i3}, a_{i4})$ for $i = 1, 2, 3$. The vector product is given by:

$$\boldsymbol{a}_1 \times \boldsymbol{a}_2 \times \boldsymbol{a}_3 = \sum_{i=1}^{4} \det(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{e}_i)\boldsymbol{e}_i,$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ is the canonical basis on $\mathbb{R}^4$. Then, we have $(\boldsymbol{a}_1 \times \boldsymbol{a}_2 \times \boldsymbol{a}_3) \cdot \boldsymbol{a}_i = 0$ for $i = 1, 2, 3$. Let $S^3 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\}$ be the unit sphere. We define the following two sets $\Delta = \{(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) \in S^3 \times S^3 \times S^3 \mid \boldsymbol{a}_1 \cdot \boldsymbol{a}_2 = \boldsymbol{a}_1 \cdot \boldsymbol{a}_3 = \boldsymbol{a}_2 \cdot \boldsymbol{a}_3 = 0\}$ and $\Delta_2 = \{(\boldsymbol{a}_1, \boldsymbol{a}_2) \in S^3 \times S^3 \mid \boldsymbol{a}_1 \cdot \boldsymbol{a}_2 = 0\}$. Then, Δ and Δ_2 are six- and five-dimensional smooth manifolds.

Note that for $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in \Delta$, if we denote $\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{d}$, then:

$$\boldsymbol{d} \times \boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{c},\ \boldsymbol{c} \times \boldsymbol{d} \times \boldsymbol{a} = \boldsymbol{b},\ \boldsymbol{b} \times \boldsymbol{c} \times \boldsymbol{d} = -\boldsymbol{a}.$$

Let I be an interval of $\mathbb{R}$, and let $\gamma : I \to S^3$ be a regular spherical curve, that is $\dot{\gamma}(t) \neq 0$ for all $t \in I$, where $\dot{\gamma}(t) = (d\gamma/dt)(t)$.

Definition 1. *We say that γ is non-degenerate or γ satisfies the non-degenerate condition if $\gamma(t) \times \dot{\gamma}(t) \times \ddot{\gamma}(t) \neq 0$ for all $t \in I$.*

Let s be the arc-length parameter of γ, that is $|\gamma'(s)| = 1$ for all s. If $|\gamma''(s)| \neq 1$ for all s, then the tangent vector, the principal normal geodesic vector, and the bi-normal geodesic vector are given by:

$$\boldsymbol{t}(s) = \gamma'(s),\ \boldsymbol{n}(s) = \frac{\gamma''(s) + \gamma(s)}{|\gamma''(s) + \gamma(s)|},\ \boldsymbol{b}(s) = \gamma(s) \times \boldsymbol{t}(s) \times \boldsymbol{n}(s),$$

respectively. In fact,

$$|\gamma''(s) + \gamma(s)|^2 = |\gamma''(s)|^2 + 2\gamma''(s) \cdot \gamma(s) + |\gamma(s)|^2 = |\gamma''(s)|^2 - 2\gamma'(s) \cdot \gamma'(s) + 1 = |\gamma''(s)|^2 - 1,$$

we have $|\gamma''(s)| \neq 1$ if and only if $|\gamma''(s) + \gamma(s)| \neq 0$. Then, $\{\gamma(s), \boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)\}$ is a moving frame of γ, and we have the Frenet–Serret formula:

$$\begin{pmatrix} \gamma'(s) \\ \boldsymbol{t}'(s) \\ \boldsymbol{n}'(s) \\ \boldsymbol{b}'(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \kappa(s) & 0 \\ 0 & -\kappa(s) & 0 & \tau(s) \\ 0 & 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \gamma(s) \\ \boldsymbol{t}(s) \\ \boldsymbol{n}(s) \\ \boldsymbol{b}(s) \end{pmatrix}, \tag{1}$$

where $\kappa(s)$ and $\tau(s)$ are the curvature and the torsion of $\gamma(s)$, respectively. Moreover,

$$\begin{aligned} \kappa(s) &= |\gamma''(s) + \gamma(s)| = |(\gamma''(s) + \gamma(s)) \cdot \boldsymbol{n}(s)| = |\gamma''(s) \cdot \boldsymbol{n}(s)| \\ &= |\boldsymbol{t}'(s) \cdot (-(\boldsymbol{b}(s) \times \gamma(s) \times \boldsymbol{t}(s)))| = |\boldsymbol{b}(s) \cdot (\gamma(s) \times \boldsymbol{t}(s) \times \boldsymbol{t}'(s))| \\ &= |\gamma(s) \times \boldsymbol{t}(s) \times \boldsymbol{t}'(s)| = |\gamma(s) \times \gamma'(s) \times \gamma''(s)| \end{aligned}$$

and

$$\begin{aligned}\tau(s) &= -\boldsymbol{t}'(s)\cdot\boldsymbol{n}(s) = -(\gamma(s)\times\boldsymbol{t}(s)\times\boldsymbol{n}'(s))\cdot\boldsymbol{n}(s)\\ &= \det(\gamma(s),\boldsymbol{t}(s),\boldsymbol{n}(s),\boldsymbol{n}'(s)) = \frac{\det(\gamma(s),\gamma'(s),\gamma''(s),\gamma'''(s))}{|\gamma''(s)+\gamma(s)|^2}\\ &= \frac{\det(\gamma(s),\gamma'(s),\gamma''(s),\gamma'''(s))}{\kappa^2(s)}.\end{aligned}$$

Since $\kappa^2(s) = |\gamma(s)\times\gamma'(s)\times\gamma''(s)|^2 = |\gamma''(s)+\gamma(s)|^2 = |\gamma''(s)|^2 - 1$, we have that $|\gamma''(s)| \neq 1$ if and only if the curvature $\kappa(s)$ does not vanish, that is γ is non-degenerate.

If $\gamma(t)\times\dot\gamma(t)\times\ddot\gamma(t) \neq 0$ for all $t \in I$, then the tangent vector, the principal normal geodesic vector, and the bi-normal geodesic vector are given by:

$$\boldsymbol{t}(t) = \frac{\dot\gamma(t)}{|\dot\gamma(t)|},\ \boldsymbol{n}(t) = -\boldsymbol{b}(t)\times\gamma(t)\times\boldsymbol{t}(t),\ \boldsymbol{b}(t) = \frac{\gamma(t)\times\dot\gamma(t)\times\ddot\gamma(t)}{|\gamma(t)\times\dot\gamma(t)\times\ddot\gamma(t)|}.$$

Then, $\{\gamma(t),\boldsymbol{t}(t),\boldsymbol{n}(t),\boldsymbol{b}(t)\}$ is a moving frame of γ, and we have the Frenet–Serret formula:

$$\begin{pmatrix}\dot\gamma(t)\\ \dot{\boldsymbol{t}}(t)\\ \dot{\boldsymbol{n}}(t)\\ \dot{\boldsymbol{b}}(t)\end{pmatrix} = \begin{pmatrix} 0 & |\dot\gamma(t)| & 0 & 0\\ -|\dot\gamma(t)| & 0 & |\dot\gamma(t)|\kappa(t) & 0\\ 0 & -|\dot\gamma(t)|\kappa(t) & 0 & |\dot\gamma(t)|\tau(t)\\ 0 & 0 & -|\dot\gamma(t)|\tau(t) & 0\end{pmatrix}\begin{pmatrix}\gamma(t)\\ \boldsymbol{t}(t)\\ \boldsymbol{n}(t)\\ \boldsymbol{b}(t)\end{pmatrix}, \qquad (2)$$

where

$$\kappa(t) = \frac{|\gamma(t)\times\dot\gamma(t)\times\ddot\gamma(t)|}{|\dot\gamma(t)|^3},\ \tau(t) = \frac{\det(\gamma(t),\dot\gamma(t),\ddot\gamma(t),\dddot\gamma(t))}{|\gamma(t)\times\dot\gamma(t)\times\ddot\gamma(t)|^2}.$$

Note that in order to define $\boldsymbol{t}(t),\boldsymbol{n}(t),\boldsymbol{b}(t),\kappa(t)$ and $\tau(t)$, we assumed that γ is non-degenerate.

As a well-known result, we recall the fundamental theorem of regular curves (cf. [18]).

Theorem 1. *Let $\kappa,\tau : I \to \mathbb{R}$ be smooth functions and $\kappa(s) > 0$ for all $s \in I$. Then, there exists a regular spherical curve $\gamma : I \to S^3$ whose associated curvature and torsion are $\kappa(s)$ and $\tau(s)$. Moreover, s is the arc-length parameter of γ.*

2.1. Bertrand Curves of Regular Spherical Curves

Let γ and $\overline{\gamma} : I \to S^3$ be non-degenerate curves with $\overline{\gamma} \not\equiv \pm\gamma$.

Definition 2. *We say that γ and $\overline{\gamma}$ are Bertrand mates if the principal normal geodesics of γ and $\overline{\gamma}$ are parallel at the corresponding points. We also say that γ is a Bertrand curve if there exists a non-degenerate curve $\overline{\gamma}$ such that γ and $\overline{\gamma}$ are Bertrand mates.*

Assume that γ and $\overline{\gamma}$ are Bertrand mates, then there exists a smooth function $\varphi : I \to \mathbb{R}$ such that $\overline{\gamma}(t) = \cos\varphi(t)\gamma(t) - \sin\varphi(t)\boldsymbol{n}(t)$ and $\overline{\boldsymbol{n}}(t) = \sin\varphi(t)\gamma(t) + \cos\varphi(t)\boldsymbol{n}(t)$ for all $t \in I$.

Remark 1. *If $\overline{\gamma} = -\gamma$, we have that the principal normal geodesics of γ and $\overline{\gamma}$ are parallel at corresponding points, then γ and $\overline{\gamma}$ are always Bertrand mates. This is why we assumed $\overline{\gamma} \not\equiv -\gamma$.*

We take the arc-length parameter s of γ.

Lemma 1. *Let $\gamma : I \to S^3$ be a non-degenerate curve parameterized by the arc-length. If γ and $\overline{\gamma}$ are Bertrand mates with $\overline{\gamma}(s) = \cos\varphi(s)\gamma(s) - \sin\varphi(s)\boldsymbol{n}(s)$, then φ is a constant with $\sin\varphi \neq 0$.*

Proof. By differentiating $\overline{\gamma}(s) = \cos\varphi(s)\gamma(s) - \sin\varphi(s)\boldsymbol{n}(s)$, we have

$$|\dot{\overline{\gamma}}(s)|\overline{\boldsymbol{t}}(s) = -\varphi'(s)\sin\varphi(s)\gamma(s) + (\cos\varphi(s) + \kappa(s)\sin\varphi(s))\boldsymbol{t}(s) + \\ -\varphi'(s)\cos\varphi(s)\boldsymbol{n}(s) - \tau(s)\sin\varphi(s)\boldsymbol{b}(s).$$

Since $\overline{\boldsymbol{n}}(s) = \sin\varphi(s)\gamma(s) + \cos\varphi(s)\boldsymbol{n}(s)$, we have $\varphi'(s) = 0$ for all $s \in I$. Therefore, φ is a constant. If $\sin\varphi = 0$, then $\overline{\gamma}(s) = \pm\gamma(s)$ for all $s \in I$. Hence, φ is a constant with $\sin\varphi \neq 0$. □

By $\overline{\gamma}(s) = \cos\varphi(s)\gamma(s) - \sin\varphi(s)\boldsymbol{n}(s)$ and $\overline{\boldsymbol{n}}(s) = \sin\varphi(s)\gamma(s) + \cos\varphi(s)\boldsymbol{n}(s)$, there exists a smooth function $\theta : I \to \mathbb{R}$ such that:

$$\begin{pmatrix} \overline{\boldsymbol{t}}(s) \\ \overline{\boldsymbol{b}}(s) \end{pmatrix} = \begin{pmatrix} \cos\theta(s) & -\sin\theta(s) \\ \sin\theta(s) & \cos\theta(s) \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(s) \\ \boldsymbol{b}(s) \end{pmatrix}.$$

Lemma 2. *Let $\gamma : I \to S^3$ be a non-degenerate curve parameterized by the arc-length. Suppose that φ is a constant with $\sin\varphi \neq 0$. If γ and $\overline{\gamma}$ are Bertrand mates with $\overline{\gamma}(s) = \cos\varphi\gamma(s) - \sin\varphi\boldsymbol{n}(s)$ and $\overline{\boldsymbol{t}}(s) = \cos\theta(s)\boldsymbol{t}(s) - \sin\theta(s)\boldsymbol{b}(s)$, then θ is a constant.*

Proof. By differentiating $\overline{\gamma}(s) = \cos\varphi\gamma(s) - \sin\varphi\boldsymbol{n}(s)$, we have

$$|\dot{\overline{\gamma}}(s)|\overline{\boldsymbol{t}}(s) = (\cos\varphi + \kappa(s)\sin\varphi)\boldsymbol{t}(s) - \tau(s)\sin\varphi\boldsymbol{b}(s).$$

Thus, by differentiating $\boldsymbol{t}(s) \cdot \overline{\boldsymbol{t}}(s)$, we have

$$\begin{aligned} \frac{d}{ds}\cos\theta(s) &= \frac{d}{ds}(\boldsymbol{t}(s) \cdot \overline{\boldsymbol{t}}(s)) \\ &= (-\gamma(s) + \kappa(s)\boldsymbol{n}(s)) \cdot \overline{\boldsymbol{t}}(s) + \boldsymbol{t}(s) \cdot (-|\dot{\overline{\gamma}}(s)|\overline{\gamma}(s) + |\dot{\overline{\gamma}}(s)|\overline{\kappa}(s)\overline{\boldsymbol{n}}(s)) = 0. \end{aligned}$$

Hence, θ is a constant. □

Theorem 2. *Let $\gamma : I \to S^3$ be a non-degenerate curve parameterized by the arc-length. Suppose that $\tau(s) \neq 0$ for all $s \in I$ and φ is a constant with $\sin\varphi \neq 0$. Then, γ and $\overline{\gamma}$ are Bertrand mates with $\overline{\gamma}(s) = \cos\varphi\gamma(s) - \sin\varphi\boldsymbol{n}(s)$ if and only if there exists a constant θ with $\sin\theta \neq 0$ such that*

$$-\kappa(s)\sin\theta + \tau(s)\cos\theta = \cot\varphi\sin\theta \tag{3}$$

and

$$\frac{\sin\varphi}{\sin\theta}\tau(s) > 0, \ (-\sin\varphi + \kappa(s)\cos\varphi)\cos\theta + \tau(s)\cos\varphi\sin\theta > 0 \tag{4}$$

for all $s \in I$.

Proof. Suppose that γ and $\overline{\gamma}$ are Bertrand mates and $\overline{\gamma}(s) = \cos\varphi\gamma(s) - \sin\varphi\boldsymbol{n}(s)$, $\overline{\boldsymbol{n}}(s) = \sin\varphi\gamma(s) + \cos\varphi\boldsymbol{n}(s)$ for all $s \in I$. Note that s is not the arc-length parameter of $\overline{\gamma}$. By differentiating $\overline{\gamma}(s) = \cos\varphi\gamma(s) - \sin\varphi\boldsymbol{n}(s)$, we have

$$|\dot{\overline{\gamma}}(s)|\overline{\boldsymbol{t}}(s) = (\cos\varphi + \kappa(s)\sin\varphi)\boldsymbol{t}(s) - \tau(s)\sin\varphi\boldsymbol{b}(s).$$

Since $\overline{\boldsymbol{t}}(s) = \cos\theta\boldsymbol{t}(s) - \sin\theta\boldsymbol{b}(s)$, we have $|\dot{\overline{\gamma}}(s)|\cos\theta = \cos\varphi + \kappa(s)\sin\varphi$ and $|\dot{\overline{\gamma}}(s)|\sin\theta = \tau(s)\sin\varphi$. It follows that

$$(\cos\varphi + \kappa(s)\sin\varphi)\sin\theta - \tau(s)\sin\varphi\cos\theta = 0.$$

As $\sin\varphi \neq 0$, we have

$$-\kappa(s)\sin\theta + \tau(s)\cos\theta = \cot\varphi\sin\theta.$$

As $|\dot{\overline{\gamma}}(s)|\sin\theta = \tau(s)\sin\varphi$ and $\tau(s) \neq 0$ for all $s \in I$, we have $\sin\theta \neq 0$ and $\dfrac{\sin\varphi}{\sin\theta}\tau(s) > 0$ for all $s \in I$. Moreover, by differentiating $\overline{\boldsymbol{t}}(s) = \cos\theta\boldsymbol{t}(s) - \sin\theta\boldsymbol{b}(s)$, we have

$$\begin{aligned} -|\dot{\overline{\gamma}}(s)|\overline{\gamma}(s) + |\dot{\overline{\gamma}}(s)|\overline{\kappa}(s)\overline{\boldsymbol{n}}(s) &= (-\gamma(s) + \kappa(s)\boldsymbol{n}(s))\cos\theta + \tau(s)\sin\theta\boldsymbol{n}(s) \\ &= -\cos\theta\gamma(s) + (\kappa(s)\cos\theta + \tau(s)\sin\theta)\boldsymbol{n}(s). \end{aligned}$$

It follows that $|\dot{\overline{\gamma}}(s)|\overline{\kappa}(s) = (-\sin\varphi + \kappa(s)\cos\varphi)\cos\theta + \tau(s)\cos\varphi\sin\theta$. Since $\overline{\kappa}(s) > 0$ for all $s \in I$, we have

$$(-\sin\varphi + \kappa(s)\cos\varphi)\cos\theta + \tau(s)\cos\varphi\sin\theta > 0$$

for all $s \in I$.

Conversely, suppose that there exists a constant θ with $\sin\theta \neq 0$ such that $-\kappa(s)\sin\theta + \tau(s)\cos\theta = \cot\varphi\sin\theta$, $\dfrac{\sin\varphi}{\sin\theta}\tau(s) > 0$, $(-\sin\varphi + \kappa(s)\cos\varphi)\cos\theta + \tau(s)\cos\varphi\sin\theta > 0$, and $\overline{\gamma}(s) = \cos\varphi\gamma(s) - \sin\varphi\boldsymbol{n}(s)$. By differentiating $\overline{\gamma}$, we have

$$\begin{aligned} \dot{\overline{\gamma}}(s) &= (\cos\varphi + \kappa(s)\sin\varphi)\boldsymbol{t}(s) - \tau(s)\sin\varphi\boldsymbol{b}(s) = \frac{\sin\varphi}{\sin\theta}\tau(s)(\cos\theta\boldsymbol{t}(s) - \sin\theta\boldsymbol{b}(s)), \\ \ddot{\overline{\gamma}}(s) &= \frac{\sin\varphi}{\sin\theta}\big(-\tau(s)\cos\theta\gamma(s) + \tau'(s)\cos\theta\boldsymbol{t}(s) \\ &\quad + \tau(s)(\kappa(s)\cos\theta + \tau(s)\sin\theta)\boldsymbol{n}(s) - \tau'(s)\sin\theta\boldsymbol{b}(s)\big). \end{aligned}$$

By a direct calculation, we have

$$\begin{aligned} &\overline{\gamma}(s) \times \dot{\overline{\gamma}}(s) \times \ddot{\overline{\gamma}}(s) \\ &= \frac{\sin^2\varphi}{\sin^2\theta}\tau^2(s)((-\sin\varphi + \kappa(s)\cos\varphi)\cos\theta + \tau(s)\cos\varphi\sin\theta)(\sin\theta\boldsymbol{t}(s) + \cos\theta\boldsymbol{b}(s)). \end{aligned}$$

As assumption, we have

$$|\dot{\overline{\gamma}}(s)| = \frac{\sin\varphi}{\sin\theta}\tau(s) > 0,\ \overline{\gamma}(s) \times \dot{\overline{\gamma}}(s) \times \ddot{\overline{\gamma}}(s) \neq 0$$

for all $s \in I$.

Thus, $\overline{\gamma}$ is regular and non-degenerate. Moreover, we have

$$\overline{\boldsymbol{t}}(s) = \frac{\dot{\overline{\gamma}}(s)}{|\dot{\overline{\gamma}}(s)|} = \cos\theta\boldsymbol{t}(s) - \sin\theta\boldsymbol{b}(s),\ \overline{\boldsymbol{b}}(s) = \frac{\overline{\gamma}(s) \times \dot{\overline{\gamma}}(s) \times \ddot{\overline{\gamma}}(s)}{|\overline{\gamma}(s) \times \dot{\overline{\gamma}}(s) \times \ddot{\overline{\gamma}}(s)|} = \sin\theta\boldsymbol{t}(s) + \cos\theta\boldsymbol{b}(s).$$

It follows that

$$\overline{\boldsymbol{n}}(s) = -\overline{\boldsymbol{b}}(s) \times \overline{\gamma}(s) \times \overline{\boldsymbol{t}}(s) = \sin\varphi\gamma(s) + \cos\varphi\boldsymbol{n}(s).$$

Therefore, γ and $\overline{\gamma}$ are Bertrand mates. □

Remark 2. *With the same assumption as in Theorem 2, suppose that γ and $\overline{\gamma}$ are Bertrand mates with $\overline{\gamma}(s) = \cos\varphi\gamma(s) - \sin\varphi\boldsymbol{n}(s)$. Then, the following results hold:*

(1) *Both the curvature κ and torsion τ of the Bertrand curve can be constants (Example 1).*

(2) *If $\cos\varphi = 0$, then $\kappa(s) = \dfrac{\tau(s)}{\sin\theta}\cos\theta$, $\dfrac{\tau(s)}{\sin\theta}\sin\varphi > 0$ and $\sin\varphi\cos\theta < 0$ by Equations (3) and (4). It follows that $\kappa(s) < 0$. Hence, $\cos\varphi \neq 0$.*

(3) *If $\cos\theta = 0$, then $\kappa(s) = -\cot\varphi$, $\dfrac{\tau(s)}{\sin\theta}\sin\varphi > 0$ and $\tau(s)\cos\varphi\sin\theta > 0$ by Equations (3) and (4). It follows that $\kappa(s) < 0$. Hence, $\cos\theta \neq 0$.*

Proposition 1. *With the same assumption as in Theorem 2, suppose that γ and $\overline{\gamma}$ are Bertrand mates with $\overline{\gamma}(s) = \cos\varphi\gamma(s) - \sin\varphi\boldsymbol{n}(s)$. Then, the curvature $\overline{\kappa}$ and the torsion $\overline{\tau}$ of $\overline{\gamma}$ are given by*

$$\overline{\kappa}(s) = \frac{\sin\theta}{\tau(s)\sin\varphi}((-\sin\varphi + \kappa(s)\cos\varphi)\cos\theta + \tau(s)\cos\varphi\sin\theta), \quad (5)$$

$$\overline{\tau}(s) = \frac{\sin^2\theta}{\tau(s)\sin^2\varphi}. \quad (6)$$

Proof. Since $\overline{\gamma}(s) = \cos\varphi\gamma(s) - \sin\varphi\boldsymbol{n}(s)$, we have

$$\dot{\overline{\gamma}}(s) = (\cos\varphi + \kappa(s)\sin\varphi)\boldsymbol{t}(s) - \tau(s)\sin\varphi\boldsymbol{b}(s) = \frac{\sin\varphi}{\sin\theta}\tau(s)(\cos\theta\boldsymbol{t}(s) - \sin\theta\boldsymbol{b}(s)).$$

Therefore,

$$\begin{aligned}
\ddot{\overline{\gamma}}(s) &= \frac{\sin\varphi}{\sin\theta}\big(- \tau(s)\cos\theta\gamma(s) + \tau'(s)\cos\theta\boldsymbol{t}(s) \\
&\quad + \tau(s)(\kappa(s)\cos\theta + \tau(s)\sin\theta)\boldsymbol{n}(s) - \tau'(s)\sin\theta\boldsymbol{b}(s)\big), \\
\dddot{\overline{\gamma}}(s) &= \frac{\sin\varphi}{\sin\theta}\big(- 2\tau'(s)\cos\theta\gamma(s) + ((-\tau(s) + \tau''(s))\cos\theta - \kappa(s)\tau(s)(\kappa(s)\cos\theta + \tau(s)\sin\theta))\boldsymbol{t}(s) \\
&\quad + (2\tau'(s)(\kappa(s)\cos\theta + \tau(s)\sin\theta) + \tau(s)(\kappa'(s)\cos\theta + \tau'(s)\sin\theta))\boldsymbol{n}(s) \\
&\quad + (-\tau''(s)\sin\theta + \tau^2(s)(\kappa(s)\cos\theta + \tau(s)\sin\theta))\boldsymbol{b}(s)\big).
\end{aligned}$$

Since

$$\begin{aligned}
|\dot{\overline{\gamma}}(s)| &= \frac{\sin\varphi}{\sin\theta}\tau(s), \\
|\overline{\gamma}(s) \times \dot{\overline{\gamma}}(s) \times \ddot{\overline{\gamma}}(s)| &= \frac{\sin^2\varphi}{\sin^2\theta}\tau^2(s)((-\sin\varphi + \kappa(s)\cos\varphi)\cos\theta + \tau(s)\cos\varphi\sin\theta), \\
\det(\overline{\gamma}(s), \dot{\overline{\gamma}}(s), \ddot{\overline{\gamma}}(s), \dddot{\overline{\gamma}}(s)) &= \frac{\sin^2\varphi}{\sin^2\theta}\tau^3(s)((-\sin\varphi + \kappa(s)\cos\varphi)\cos\theta + \tau(s)\cos\varphi\sin\theta)^2,
\end{aligned}$$

we have

$$\begin{aligned}
\overline{\kappa}(s) &= \frac{|\overline{\gamma}(s) \times \dot{\overline{\gamma}}(s) \times \ddot{\overline{\gamma}}(s)|}{|\dot{\overline{\gamma}}(s)|^3} \\
&= \frac{\sin\theta}{\tau(s)\sin\varphi}((-\sin\varphi + \kappa(s)\cos\varphi)\cos\theta + \tau(s)\cos\varphi\sin\theta), \\
\overline{\tau}(s) &= \frac{\det(\overline{\gamma}(s), \dot{\overline{\gamma}}(s), \ddot{\overline{\gamma}}(s), \dddot{\overline{\gamma}}(s))}{|\overline{\gamma}(s) \times \dot{\overline{\gamma}}(s) \times \ddot{\overline{\gamma}}(s)|^2} \\
&= \frac{\sin^2\theta}{\tau(s)\sin^2\varphi}.
\end{aligned}$$

□

Remark 3. *Suppose that γ and $\overline{\gamma}$ are Bertrand mates with $\overline{\gamma}(s) = \cos\varphi\gamma(s) - \sin\varphi\boldsymbol{n}(s)$. By Equation (6), we have $\overline{\tau}(s) \neq 0$ for all $s \in I$.*

Proposition 2. *With the same assumption as in Theorem 2, suppose that γ and $\overline{\gamma}$ are Bertrand mates with $\overline{\gamma}(s) = \cos\varphi\gamma(s) - \sin\varphi\boldsymbol{n}(s)$. Then, there exists a constant θ with $\sin\theta \neq 0$ such that the following formulas hold:*

(1) $\tau(s)\sin\varphi = |\dot{\overline{\gamma}}(s)|\sin\theta$, $|\dot{\overline{\gamma}}(s)|\overline{\tau}(s)\sin\varphi = \sin\theta$.

(2) $|\dot{\overline{\gamma}}(s)|\overline{\tau}(s)\cos\varphi = -\kappa(s)\sin\theta + \tau(s)\cos\theta$, $\tau(s)\cos\varphi = |\dot{\overline{\gamma}}(s)|(\overline{\kappa}(s)\sin\theta + \overline{\tau}(s)\cos\theta)$.

(3) $\cos\varphi + \kappa(s)\sin\varphi = |\dot{\overline{\gamma}}(s)|\cos\theta$, $|\dot{\overline{\gamma}}(s)|(\cos\varphi - \overline{\kappa}(s)\sin\varphi) = \cos\theta$.

(4) $|\dot{\overline{\gamma}}(s)|(\sin\varphi+\overline{\kappa}(s)\cos\varphi)=\kappa(s)\cos\theta+\tau(s)\sin\theta$, $\sin\varphi-\kappa(s)\cos\varphi=|\dot{\overline{\gamma}}(s)|(-\overline{\kappa}(s)\cos\theta+\overline{\tau}(s)\sin\theta)$.

Proof. By Definition 2 and the proof of Theorem 2, we have

$$\begin{aligned}\overline{\gamma}(s) &= \cos\varphi\gamma(s)-\sin\varphi\boldsymbol{n}(s),\\ \overline{\boldsymbol{n}}(s) &= \sin\varphi\gamma(s)+\cos\varphi\boldsymbol{n}(s),\\ \overline{\boldsymbol{t}}(s) &= \cos\theta\boldsymbol{t}(s)-\sin\theta\boldsymbol{b}(s),\\ \overline{\boldsymbol{b}}(s) &= \sin\theta\boldsymbol{t}(s)+\cos\theta\boldsymbol{b}(s).\end{aligned}$$

We write the moving frame of γ in terms of the moving frame of $\overline{\gamma}$:

$$\begin{aligned}\gamma(s) &= \cos\varphi\overline{\gamma}(s)+\sin\varphi\overline{\boldsymbol{n}}(s),\\ \boldsymbol{n}(s) &= -\sin\varphi\overline{\gamma}(s)+\cos\varphi\overline{\boldsymbol{n}}(s),\\ \boldsymbol{t}(s) &= \cos\theta\overline{\boldsymbol{t}}(s)+\sin\theta\overline{\boldsymbol{b}}(s),\\ \boldsymbol{b}(s) &= -\sin\theta\overline{\boldsymbol{t}}(s)+\cos\theta\overline{\boldsymbol{b}}(s).\end{aligned}$$

By differentiating $\overline{\gamma}(s),\overline{\boldsymbol{n}}(s),\overline{\boldsymbol{t}}(s),\overline{\boldsymbol{b}}(s)$ and $\gamma(s),\boldsymbol{n}(s),\boldsymbol{t}(s),\boldsymbol{b}(s)$, we obtain the formulas. □

Proposition 2 leads to the following result.

Corollary 1. *With the same assumption as in Theorem 2, suppose that γ and $\overline{\gamma}$ are Bertrand mates with $\overline{\gamma}(s)=\cos\varphi\gamma(s)-\sin\varphi\boldsymbol{n}(s)$. Then, the following relations hold:*

(1) $\tau(s)\overline{\tau}(s)\sin^2\varphi=\sin^2\theta$.
(2) $\tau(s)\overline{\tau}(s)\cos^2\varphi=(-\kappa(s)\sin\theta+\tau(s)\cos\theta)(\overline{\kappa}(s)\sin\theta+\overline{\tau}(s)\cos\theta)$.
(3) $(\cos\varphi+\kappa(s)\sin\varphi)(\cos\varphi-\overline{\kappa}(s)\sin\varphi)=\cos^2\theta$.
(4) $(\sin\varphi-\kappa(s)\cos\varphi)(\sin\varphi+\overline{\kappa}(s)\cos\varphi)=(\kappa(s)\cos\theta+\tau(s)\sin\theta)(-\overline{\kappa}(s)\cos\theta+\overline{\tau}(s)\sin\theta)$.

A twisted curve γ in S^3 (i.e., a curve with torsion $\tau\not\equiv 0$) is said to be a *helix* if its curvature and torsion are non-zero constants. More generally, a twisted curve γ in S^3 is a *general helix* if and only if there exists a constant a such that $\tau(s)=a\kappa(s)\pm 1$ (for details, see [19]). Then, we clarify the relations between Bertrand curves and general helices in S^3.

Proposition 3. *With the same assumption as in Theorem 2, suppose that γ and $\overline{\gamma}$ are Bertrand mates with $\overline{\gamma}(s)=\cos\varphi\gamma(s)-\sin\varphi\boldsymbol{n}(s)$. If $\varphi=\pm\theta$, then γ and $\overline{\gamma}$ are general helices.*

Proof. Since γ and $\overline{\gamma}$ are Bertrand mates, we have $-\kappa(s)\sin\theta+\tau(s)\cos\theta=\cot\varphi\sin\theta$ by Theorem 2. If $\varphi=\theta$, we have $\kappa(s)\sin\theta=(\tau(s)-1)\cos\theta$. As $\cos\theta\neq 0$ (cf. Remark 2 (3)), we have $\tau(s)=\kappa(s)\tan\theta+1$. By Proposition 1, we have $\overline{\tau}(s)=\dfrac{\sin^2\theta}{\tau(s)\sin^2\varphi}=\dfrac{1}{\tau(s)}$. Moreover,

$$\begin{aligned}\overline{\kappa}(s) &= \frac{\kappa(s)\cos^2\theta+(\tau(s)-1)\cos\theta\sin\theta}{\tau(s)}=\frac{\dfrac{(\tau(s)-1)\cos^3\theta}{\sin\theta}+(\tau(s)-1)\cos\theta\sin\theta}{\tau(s)}\\ &= \left(1-\frac{1}{\tau(s)}\right)\frac{\cos^3\theta+\cos\theta\sin^2\theta}{\sin\theta}=(1-\overline{\tau}(s))\cot\theta.\end{aligned}$$

It follows that $\overline{\tau}(s)=-\overline{\kappa}(s)\tan\theta+1$. If $\varphi=-\theta$, by a calculation similar to the case of $\varphi=\theta$, we have $\tau(s)=\kappa(s)\tan\theta-1$ and $\overline{\tau}(s)=-\overline{\kappa}(s)\tan\theta-1$. Thus, γ and $\overline{\gamma}$ are general helices. □

Proposition 4. *Let $\gamma : I \to S^3$ be a general helix with $\tau(s) = a\kappa(s) \pm 1$ and $a \neq 0$. Suppose that $\kappa(s)$ is not a constant and $\tau(s) \neq 0$ for all $s \in I$. If $\tau(s) = a\kappa(s) + 1 > 0$ for all $s \in I$ or $\tau(s) = a\kappa(s) - 1 < 0$ for all $s \in I$, then γ is a Bertrand curve.*

Proof. If $\tau(s) = a\kappa(s) + 1 > 0$ for all $s \in I$, we take $\varphi = \theta$, $\sin\theta = a/\sqrt{1+a^2}$ and $\cos\theta = 1/\sqrt{1+a^2}$. If $\tau(s) = a\kappa(s) - 1 < 0$ for all $s \in I$, we take $\varphi = -\theta$, $\sin\theta = a/\sqrt{1+a^2}$ and $\cos\theta = 1/\sqrt{1+a^2}$. Then, κ and τ satisfy Equations (3) and (4) in Theorem 2. Hence, γ is a Bertrand curve. □

Remark 4. *With the same assumption as in Proposition 4, if $\tau(s) = a\kappa(s) + 1 < 0$ for all $s \in I$ or $\tau(s) = a\kappa(s) - 1 > 0$ for all $s \in I$, we conclude there are not any constants angles φ and θ, such that κ and τ satisfy Equations (3) and (4) in Theorem 2. Hence, γ is not a Bertrand curve.*

Example 1. *Let $\gamma : I \to S^3$ be a helix with the curvature $\kappa = a$ and torsion $\tau = b$, where $a,\ b$ are constants and $a > 0,\ b \neq 0$. We consider the following four cases:*

$$(i)\ b > 0, b \neq 1,\ (ii)\ b = 1,\ (iii)\ b < 0, b \neq -1,\ (iv)\ b = -1.$$

In the case (i), we take $\varphi = \theta$, $\sin\theta = (b-1)/\sqrt{a^2+(b-1)^2}$ and $\cos\theta = a/\sqrt{a^2+(b-1)^2}$. In the case (ii), we take $\varphi = \pi/4$, $\sin\theta = 1/\sqrt{1+(a+1)^2}$ and $\cos\theta = (a+1)/\sqrt{1+(a+1)^2}$. In the case (iii), we take $\varphi = -\theta$, $\sin\theta = (b+1)/\sqrt{a^2+(b+1)^2}$ and $\cos\theta = a/\sqrt{a^2+(b+1)^2}$. In the case (iv), we take $\varphi = \pi/4$, $\sin\theta = -1/\sqrt{1+(a+1)^2}$ and $\cos\theta = (a+1)/\sqrt{1+(a+1)^2}$. Then, κ and τ satisfy Equations (3) and (4) in Theorem 2. Hence, γ is a Bertrand curve.

2.2. Mannheim Curves of Regular Spherical Curves

Let γ and $\overline{\gamma} : I \to S^3$ be non-degenerate curves with $\overline{\gamma} \not\equiv \gamma$.

Definition 3. *We say that γ and $\overline{\gamma}$ are Mannheim mates if the principal normal geodesic of γ and the bi-normal geodesic of $\overline{\gamma}$ are parallel at the corresponding points. We also say that γ is a Mannheim curve if there exists a non-degenerate curve $\overline{\gamma}$ such that γ and $\overline{\gamma}$ are Mannheim mates.*

Assume that γ and $\overline{\gamma}$ are Mannheim mates, then there exists a smooth function $\varphi : I \to \mathbb{R}$ such that $\overline{\gamma}(t) = \cos\varphi(t)\gamma(t) - \sin\varphi(t)\boldsymbol{n}(t)$ and $\overline{\boldsymbol{b}}(t) = \sin\varphi(t)\gamma(t) + \cos\varphi(t)\boldsymbol{n}(t)$ for all $t \in I$.

Remark 5. *If $\overline{\gamma} = -\gamma$, then γ and $\overline{\gamma}$ are not Mannheim mates.*

We take the arc-length parameter s of γ.

Lemma 3. *Let $\gamma : I \to S^3$ be a non-degenerate curve parameterized by the arc-length. If γ and $\overline{\gamma}$ are Mannheim mates with $\overline{\gamma}(s) = \cos\varphi(s)\gamma(s) - \sin\varphi(s)\boldsymbol{n}(s)$, then φ is a constant with $\sin\varphi \neq 0$.*

Proof. By differentiating $\overline{\gamma}(s) = \cos\varphi(s)\gamma(s) - \sin\varphi(s)\boldsymbol{n}(s)$, we have

$$\begin{aligned}|\dot{\overline{\gamma}}(s)|\overline{\boldsymbol{t}}(s) = &- \varphi'(s)\sin\varphi(s)\gamma(s) + (\cos\varphi(s) + \kappa(s)\sin\varphi(s))\boldsymbol{t}(s)\\ &- \varphi'(s)\cos\varphi(s)\boldsymbol{n}(s) - \tau(s)\sin\varphi(s)\boldsymbol{b}(s).\end{aligned}$$

Since $\overline{\boldsymbol{b}}(s) = \sin\varphi(s)\gamma(s) + \cos\varphi(s)\boldsymbol{n}(s)$, we have $\varphi'(s) = 0$ for all $s \in I$. Therefore, φ is a constant. If $\sin\varphi = 0$, then $\overline{\gamma}(s) = \pm\gamma(s)$ for all $s \in I$. Hence, φ is a constant with $\sin\varphi \neq 0$. □

Theorem 3. *Let $\gamma : I \to S^3$ be a non-degenerate curve parameterized by the arc-length. Suppose that φ is a constant with $\sin\varphi \neq 0$. Then, γ and $\overline{\gamma}$ are Mannheim mates with $\overline{\gamma}(s) = \cos\varphi\gamma(s) - \sin\varphi\boldsymbol{n}(s)$ if and only if*

$$(\kappa^2(s) + \tau^2(s) - 1)\sin\varphi\cos\varphi = \kappa(s)(\sin^2\varphi - \cos^2\varphi) \tag{7}$$

and

$$\tau'(s)\sin\varphi\cos\varphi - (\kappa'(s)\tau(s) - \kappa(s)\tau'(s))\sin^2\varphi > 0 \tag{8}$$

for all $s \in I$.

Proof. Suppose that γ and $\overline{\gamma}$ are Mannheim mates and $\overline{\gamma}(s) = \cos\varphi\gamma(s) - \sin\varphi\boldsymbol{n}(s)$, $\overline{\boldsymbol{b}}(s) = \sin\varphi\gamma(s) + \cos\varphi\boldsymbol{n}(s)$ for all $s \in I$. Note that s is not the arc-length parameter of $\overline{\gamma}$. By differentiating $\overline{\gamma}(s) = \cos\varphi\gamma(s) - \sin\varphi\boldsymbol{n}(s)$, we have $|\dot{\overline{\gamma}}(s)|\overline{\boldsymbol{t}}(s) = (\cos\varphi + \kappa(s)\sin\varphi)\boldsymbol{t}(s) - \tau(s)\sin\varphi\boldsymbol{b}(s)$. Since $\overline{\gamma}$ is regular, we have $|\dot{\overline{\gamma}}(s)| = \sqrt{(\cos\varphi + \kappa(s)\sin\varphi)^2 + (\tau(s)\sin\varphi)^2} \neq 0$ for all $s \in I$. Since $\overline{\boldsymbol{b}}(s) = \sin\varphi\gamma(s) + \cos\varphi\boldsymbol{n}(s)$, there exists a smooth function $\theta : I \to \mathbb{R}$ such that

$$\begin{pmatrix} \overline{\boldsymbol{t}}(s) \\ \overline{\boldsymbol{n}}(s) \end{pmatrix} = \begin{pmatrix} \cos\theta(s) & -\sin\theta(s) \\ \sin\theta(s) & \cos\theta(s) \end{pmatrix} \begin{pmatrix} \boldsymbol{b}(s) \\ \boldsymbol{t}(s) \end{pmatrix}.$$

Then, $-|\dot{\overline{\gamma}}(s)|\sin\theta(s) = \cos\varphi + \kappa(s)\sin\varphi$ and $|\dot{\overline{\gamma}}(s)|\cos\theta(s) = -\tau(s)\sin\varphi$. It follows that

$$(\cos\varphi + \kappa(s)\sin\varphi)\cos\theta(s) - \tau(s)\sin\varphi\sin\theta(s) = 0. \tag{9}$$

By differentiating $\overline{\boldsymbol{t}}(s) = \cos\theta(s)\boldsymbol{b}(s) - \sin\theta(s)\boldsymbol{t}(s)$, we have

$$\begin{aligned} -|\dot{\overline{\gamma}}(s)|\overline{\gamma}(s) + |\dot{\overline{\gamma}}(s)|\overline{\kappa}(s)\overline{\boldsymbol{n}}(s) &= \sin\theta(s)\gamma(s) - \theta'(s)\cos\theta(s)\boldsymbol{t}(s) \\ &\quad + (-\kappa(s)\sin\theta(s) - \tau(s)\cos\theta(s))\boldsymbol{n}(s) - \theta'(s)\sin\theta(s)\boldsymbol{b}(s). \end{aligned}$$

Since $\overline{\boldsymbol{b}}(s) = \sin\varphi\gamma(s) + \cos\varphi\boldsymbol{n}(s)$, we have

$$(\sin\varphi - \kappa(s)\cos\varphi)\sin\theta(s) - \tau(s)\cos\varphi\cos\theta(s) = 0. \tag{10}$$

By Equations (9) and (10), we have

$$\begin{pmatrix} -\tau(s)\sin\varphi & \cos\varphi + \kappa(s)\sin\varphi \\ \sin\varphi - \kappa(s)\cos\varphi & -\tau(s)\cos\varphi \end{pmatrix} \begin{pmatrix} \sin\theta(s) \\ \cos\theta(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus,

$$\det\begin{pmatrix} -\tau(s)\sin\varphi & \cos\varphi + \kappa(s)\sin\varphi \\ \sin\varphi - \kappa(s)\cos\varphi & -\tau(s)\cos\varphi \end{pmatrix} = 0.$$

It follows that

$$(\kappa^2(s) + \tau^2(s) - 1)\sin\varphi\cos\varphi = \kappa(s)(\sin^2\varphi - \cos^2\varphi). \tag{11}$$

By differentiating $\overline{\boldsymbol{n}}(s) = \sin\theta(s)\boldsymbol{b}(s) + \cos\theta(s)\boldsymbol{t}(s)$, we have

$$\begin{aligned} &|\dot{\overline{\gamma}}(s)|\overline{\tau}(s)\sin\varphi\gamma(s) + |\dot{\overline{\gamma}}(s)|\overline{\kappa}(s)\sin\theta(s)\boldsymbol{t}(s) + |\dot{\overline{\gamma}}(s)|\overline{\tau}(s)\cos\varphi\boldsymbol{n}(s) - |\dot{\overline{\gamma}}(s)|\overline{\kappa}(s)\cos\theta(s)\boldsymbol{b}(s) \\ &= -\cos\theta(s)\gamma(s) - \theta'(s)\sin\theta(s)\boldsymbol{t}(s) + (\kappa(s)\cos\theta(s) - \tau(s)\sin\theta(s))\boldsymbol{n}(s) + \theta'(s)\cos\theta(s)\boldsymbol{b}(s). \end{aligned}$$

Thus, $|\dot{\overline{\gamma}}(s)|\overline{\kappa}(s) = -\theta'(s)$. Since $\overline{\kappa}(s) > 0$ for all $s \in I$, we have $\theta'(s) < 0$ for all $s \in I$. By differentiating (9), we have

$$(\kappa'(s) - \theta'(s)\tau(s))\sin\varphi\cos\theta(s) - (\theta'(s)(\cos\varphi + \kappa(s)\sin\varphi) + \tau'(s)\sin\varphi)\sin\theta(s) = 0. \tag{12}$$

By Equations (9) and (12), we have

$$\begin{pmatrix} \tau(s)\sin\varphi & \cos\varphi+\kappa(s)\sin\varphi \\ \theta'(s)(\cos\varphi+\kappa(s)\sin\varphi)+\tau'(s)\sin\varphi & (\kappa'(s)-\theta'(s)\tau(s))\sin\varphi \end{pmatrix}\begin{pmatrix} \sin\theta(s) \\ \cos\theta(s) \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus,

$$\theta'(s)=\frac{-(\tau'(s)\sin\varphi\cos\varphi-(\kappa'(s)\tau(s)-\kappa(s)\tau'(s))\sin^2\varphi)}{\tau^2(s)\sin^2\varphi+(\cos\varphi+\kappa(s)\sin\varphi)^2}.$$

It follows that $\tau'(s)\sin\varphi\cos\varphi-(\kappa'(s)\tau(s)-\kappa(s)\tau'(s))\sin^2\varphi>0$ for all $s\in I$.

Conversely, suppose that $(\kappa^2(s)+\tau^2(s)-1)\sin\varphi\cos\varphi=\kappa(s)(\sin^2\varphi-\cos^2\varphi)$, $\tau'(s)\sin\varphi\cos\varphi-(\kappa'(s)\tau(s)-\kappa(s)\tau'(s))\sin^2\varphi>0$ and $\overline{\gamma}(s)=\cos\varphi\gamma(s)-\sin\varphi\boldsymbol{n}(s)$ for all $s\in I$.

By differentiating $\overline{\gamma}(s)=\cos\varphi\gamma(s)-\sin\varphi\boldsymbol{n}(s)$, we have

$$\begin{aligned}\dot{\overline{\gamma}}(s) &= (\cos\varphi+\kappa(s)\sin\varphi)\boldsymbol{t}(s)-\tau(s)\sin\varphi\boldsymbol{b}(s),\\ \ddot{\overline{\gamma}}(s) &= -(\cos\varphi+\kappa(s)\sin\varphi)\gamma(s)+\kappa'(s)\sin\varphi\boldsymbol{t}(s)\\ &\quad+(\kappa(s)\cos\varphi+(\kappa^2(s)+\tau^2(s))\sin\varphi)\boldsymbol{n}(s)-\tau'(s)\sin\varphi\boldsymbol{b}(s).\end{aligned}$$

By a direct calculation, we have

$$\overline{\gamma}(s)\times\dot{\overline{\gamma}}(s)\times\ddot{\overline{\gamma}}(s)=(\tau'(s)\sin\varphi\cos\varphi-(\kappa'(s)\tau(s)-\kappa(s)\tau'(s))\sin^2\varphi)(\sin\varphi\gamma(s)+\cos\varphi\boldsymbol{n}(s)).$$

As the assumption,

$$\begin{aligned}&\tau'(s)\sin\varphi\cos\varphi-(\kappa'(s)\tau(s)-\kappa(s)\tau'(s))\sin^2\varphi\\ &=\sin\varphi(\tau'(s)(\cos\varphi+\kappa(s)\sin\varphi)-\kappa'(s)\tau(s)\sin\varphi)>0\end{aligned}$$

for all $s\in I$, we have

$$|\dot{\overline{\gamma}}(s)|=\sqrt{(\cos\varphi+\kappa(s)\sin\varphi)^2+(\tau(s)\sin\varphi)^2}\neq 0,\ \overline{\gamma}(s)\times\dot{\overline{\gamma}}(s)\times\ddot{\overline{\gamma}}(s)\neq 0$$

for all $s\in I$. Thus, $\overline{\gamma}(s)$ is regular and non-degenerate. Moreover, we have

$$\overline{\boldsymbol{b}}(s)=\frac{\overline{\gamma}(s)\times\dot{\overline{\gamma}}(s)\times\ddot{\overline{\gamma}}(s)}{|\overline{\gamma}(s)\times\dot{\overline{\gamma}}(s)\times\ddot{\overline{\gamma}}(s)|}=\sin\varphi\gamma(s)+\cos\varphi\boldsymbol{n}(s).$$

Therefore, γ and $\overline{\gamma}$ are Mannheim mates. □

Remark 6. *With the same assumption as in Theorem 3, suppose that γ and $\overline{\gamma}$ are Mannheim mates with $\overline{\gamma}(s)=\cos\varphi\gamma(s)-\sin\varphi\boldsymbol{n}(s)$. Then, the following results hold:*

(1) *Both the curvature κ and torsion τ of γ can not be constants.*
(2) *If θ is a constant, then $|\dot{\overline{\gamma}}(s)|\overline{\kappa}(s)=0$. Hence, θ is not a constant.*
(3) *If $\cos\varphi=0$, then $\kappa(s)=0$ by Equation (7). Hence, $\cos\varphi\neq 0$.*

Proposition 5. *With the same assumption as in Theorem 3, suppose that γ and $\overline{\gamma}$ are Mannheim mates with $\overline{\gamma}(s)=\cos\varphi\gamma(s)-\sin\varphi\boldsymbol{n}(s)$. Then, the curvature $\overline{\kappa}$ and the torsion $\overline{\tau}$ of $\overline{\gamma}$ are given by*

$$\begin{aligned}\overline{\kappa}(s) &= \frac{\tau'(s)\sin\varphi\cos\varphi-(\kappa'(s)\tau(s)-\kappa(s)\tau'(s))\sin^2\varphi}{((\cos\varphi+\kappa(s)\sin\varphi)^2+\tau^2(s)\sin^2\varphi)^{\frac{3}{2}}},\\ \overline{\tau}(s) &= -\frac{\kappa'(s)}{2(\tau'(s)\sin\varphi\cos\varphi-(\kappa'(s)\tau(s)-\kappa(s)\tau'(s))\sin^2\varphi)}.\end{aligned}$$

Proof. Since $\overline{\gamma}(s) = \cos\varphi\gamma(s) - \sin\varphi\boldsymbol{n}(s)$, we have

$$\dot{\overline{\gamma}}(s) = (\cos\varphi + \kappa(s)\sin\varphi)\boldsymbol{t}(s) - \tau(s)\sin\varphi\boldsymbol{b}(s).$$

Therefore,

$$\begin{aligned}\ddot{\overline{\gamma}}(s) &= -(\cos\varphi + \kappa(s)\sin\varphi)\gamma(s) + \kappa'(s)\sin\varphi\boldsymbol{t}(s)\\ &\quad + (\kappa(s)\cos\varphi + (\kappa^2(s)+\tau^2(s))\sin\varphi)\boldsymbol{n}(s) - \tau'(s)\sin\varphi\boldsymbol{b}(s),\\ \dddot{\overline{\gamma}}(s) &= -2\kappa'(s)\sin\varphi\gamma(s) + \big(\kappa''(s)\sin\varphi - \kappa(s)(\kappa(s)\cos\varphi + (\kappa^2(s)+\tau^2(s))\sin\varphi)\\ &\quad -\cos\varphi - \kappa(s)\sin\varphi\big)\boldsymbol{t}(s) + \big(\kappa'(s)\cos\varphi + 3(\kappa(s)\kappa'(s)+\tau(s)\tau'(s))\sin\varphi\big)\boldsymbol{n}(s)\\ &\quad + \big(\tau(s)(\kappa(s)\cos\varphi + (\kappa^2(s)+\tau^2(s))\sin\varphi) - \tau''(s)\sin\varphi\big)\boldsymbol{b}(s).\end{aligned}$$

By differentiating

$$(\kappa^2(s)+\tau^2(s)-1)\sin\varphi\cos\varphi = \kappa(s)(\sin^2\varphi - \cos^2\varphi),$$

we have

$$2(\kappa(s)\kappa'(s)+\tau(s)\tau'(s))\sin\varphi\cos\varphi = \kappa'(s)(\sin^2\varphi - \cos^2\varphi).$$

It follows that

$$\det(\overline{\gamma}(s), \dot{\overline{\gamma}}(s), \ddot{\overline{\gamma}}(s), \dddot{\overline{\gamma}}(s)) = -\frac{1}{2}\kappa'(s)\big(\tau'(s)\sin\varphi\cos\varphi - (\kappa'(s)\tau(s) - \kappa(s)\tau'(s))\sin^2\varphi\big).$$

Since

$$\begin{aligned}|\dot{\overline{\gamma}}(s)| &= \big((\cos\varphi + \kappa(s)\sin\varphi)^2 + \tau^2(s)\sin^2\varphi\big)^{\frac{1}{2}},\\ |\overline{\gamma}(s)\times\dot{\overline{\gamma}}(s)\times\ddot{\overline{\gamma}}(s)| &= \tau'(s)\sin\varphi\cos\varphi - (\kappa'(s)\tau(s) - \kappa(s)\tau'(s))\sin^2\varphi,\end{aligned}$$

we have

$$\begin{aligned}\overline{\kappa}(s) &= \frac{|\overline{\gamma}(s)\times\dot{\overline{\gamma}}(s)\times\ddot{\overline{\gamma}}(s)|}{|\dot{\overline{\gamma}}(s)|^3}\\ &= \frac{\tau'(s)\sin\varphi\cos\varphi - (\kappa'(s)\tau(s) - \kappa(s)\tau'(s))\sin^2\varphi}{\big((\cos\varphi + \kappa(s)\sin\varphi)^2 + \tau^2(s)\sin^2\varphi\big)^{\frac{3}{2}}},\\ \overline{\tau}(s) &= \frac{\det(\overline{\gamma}(s), \dot{\overline{\gamma}}(s), \ddot{\overline{\gamma}}(s), \dddot{\overline{\gamma}}(s))}{|\overline{\gamma}(s)\times\dot{\overline{\gamma}}(s)\times\ddot{\overline{\gamma}}(s)|^2}\\ &= -\frac{\kappa'(s)}{2\big(\tau'(s)\sin\varphi\cos\varphi - (\kappa'(s)\tau(s) - \kappa(s)\tau'(s))\sin^2\varphi\big)}.\end{aligned}$$

□

Proposition 6. *With the same assumption as in Theorem 3, suppose that γ and $\overline{\gamma}$ are Mannheim mates with $\overline{\gamma}(s) = \cos\varphi\gamma(s) - \sin\varphi\boldsymbol{n}(s)$. Then, there exists a smooth function $\theta : I \to \mathbb{R}$ such that the following formulas hold:*

(1) $|\dot{\overline{\gamma}}(s)|\cos\varphi = -\sin\theta(s)$, $\kappa(s)\sin\varphi + \cos\varphi = -|\dot{\overline{\gamma}}(s)|\sin\theta(s)$.
(2) $|\dot{\overline{\gamma}}(s)|\overline{\tau}(s)\sin\varphi = -\cos\theta(s)$, $\tau(s)\sin\varphi = -|\dot{\overline{\gamma}}(s)|\cos\theta(s)$.
(3) $|\dot{\overline{\gamma}}(s)|\sin\varphi = -\kappa(s)\sin\theta(s) - \tau(s)\cos\theta(s)$, $\tau(s)\cos\varphi = -|\dot{\overline{\gamma}}(s)|\overline{\tau}(s)\sin\theta(s)$.
(4) $|\dot{\overline{\gamma}}(s)|\overline{\tau}(s)\cos\varphi = \kappa(s)\cos\theta(s) - \tau(s)\sin\theta(s)$, $-\sin\varphi + \kappa(s)\cos\varphi = |\dot{\overline{\gamma}}(s)|\overline{\tau}(s)\cos\theta(s)$.

Proof. By Definition 3 and the proof of Theorem 3, we have

$$\begin{aligned}\overline{\gamma}(s) &= \cos\varphi\gamma(s) - \sin\varphi\boldsymbol{n}(s),\\ \overline{\boldsymbol{b}}(s) &= \sin\varphi\gamma(s) + \cos\varphi\boldsymbol{n}(s),\\ \overline{\boldsymbol{t}}(s) &= \cos\theta(s)\boldsymbol{b}(s) - \sin\theta(s)\boldsymbol{t}(s),\\ \overline{\boldsymbol{n}}(s) &= \sin\theta(s)\boldsymbol{b}(s) + \cos\theta(s)\boldsymbol{t}(s).\end{aligned}$$

We write the moving frame of γ in terms of the moving frame of $\overline{\gamma}$:

$$\begin{aligned}\gamma(s) &= \cos\varphi\overline{\gamma}(s) + \sin\varphi\overline{\boldsymbol{b}}(s),\\ \boldsymbol{n}(s) &= -\sin\varphi\overline{\gamma}(s) + \cos\varphi\overline{\boldsymbol{b}}(s),\\ \boldsymbol{t}(s) &= \cos\theta(s)\overline{\boldsymbol{t}}(s) + \sin\theta(s)\overline{\boldsymbol{n}}(s),\\ \boldsymbol{b}(s) &= -\sin\theta(s)\overline{\boldsymbol{t}}(s) + \cos\theta(s)\overline{\boldsymbol{n}}(s).\end{aligned}$$

By differentiating $\overline{\gamma}(s), \overline{\boldsymbol{n}}(s), \overline{\boldsymbol{t}}(s), \overline{\boldsymbol{b}}(s)$ and $\gamma(s), \boldsymbol{n}(s), \boldsymbol{t}(s), \boldsymbol{b}(s)$, we obtain the formulas. □

Proposition 6 leads to the following result.

Corollary 2. *With the same assumption as in Theorem 3, suppose that γ and $\overline{\gamma}$ are Mannheim mates with $\overline{\gamma}(s) = \cos\varphi\gamma(s) - \sin\varphi\boldsymbol{n}(s)$. Then, the following relations hold:*

(1) $\cos\varphi(\kappa(s)\sin\varphi + \cos\varphi) = \sin^2\theta(s)$.
(2) $\tau(s)\overline{\tau}(s)\sin^2\varphi = \cos^2\theta(s)$.
(3) $\tau(s)\sin\varphi\cos\varphi = \overline{\tau}(s)\sin\theta(s)(\kappa(s)\sin\theta(s) + \tau(s)\cos\theta(s))$.
(4) $\cos\varphi(-\sin\varphi + \kappa(s)\cos\varphi) = \cos\theta(s)(\kappa(s)\cos\theta(s) - \tau(s)\sin\theta(s))$.

We obtain the relation between Mannheim curves and general helices in S^3 in next proposition.

Proposition 7. *Let $\gamma : I \to S^3$ be a non-degenerate twisted curve parameterized by arc-length. If γ is a general helix, then γ is not a Mannheim curve.*

Proof. Since γ is a general helix, we have that the curvature κ and torsion τ of γ satisfy $\tau(s) = a\kappa(s) \pm 1$, where a is a constant. If γ is a Mannheim curve, by Equation (7) in Theorem 3, we have

$$\big((a^2+1)\kappa(s) \pm 2a\big)\sin\varphi\cos\varphi = \sin^2\varphi - \cos^2\varphi,$$

then κ is a constant by $\sin\varphi\cos\varphi \neq 0$ (cf. Remark 6 (3)). It follows that τ is a constant. This is a contradiction (cf. Remark 6 (1)). Hence, γ is not a Mannheim curve. □

Example 2. *We consider the smooth functions $\kappa, \tau : (-\pi/2, \pi/2) \to \mathbb{R}$ as $\kappa(s) = \cos s$ and $\tau(s) = \sin s$. By Theorem 1, there exists a spherical regular curve $\gamma : (-\pi/2, \pi/2) \to S^3$ whose associated curvature and torsion are κ, τ and s is the arc-length parameter of γ. If we take $\varphi = \pi/4$, then κ, τ satisfy Equations (7) and (8) in Theorem 3. Hence, γ is a Mannheim curve.*

3. Spherical Framed Curves

Definition 4. *We say that $(\gamma, \nu_1, \nu_2) : I \to \Delta$ is a spherical framed curve if $\dot{\gamma}(t) \cdot \nu_1(t) = 0$ and $\dot{\gamma}(t) \cdot \nu_2(t) = 0$ for all $t \in I$. We say that $\gamma : I \to S^3$ is a spherical framed base curve if there exists $(\nu_1, \nu_2) : I \to \Delta_2$ such that (γ, ν_1, ν_2) is a spherical framed curve.*

We denote $\mu(t) = \gamma(t) \times \nu_1(t) \times \nu_2(t)$. Then, $\{\gamma(t), \nu_1(t), \nu_2(t), \mu(t)\}$ is a moving frame along the spherical framed base curve $\gamma(t)$ in S^3, and we have the Frenet–Serret-type formula:

$$\begin{pmatrix} \dot{\gamma}(t) \\ \dot{\nu}_1(t) \\ \dot{\nu}_2(t) \\ \dot{\mu}(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \alpha(t) \\ 0 & 0 & \ell(t) & m(t) \\ 0 & -\ell(t) & 0 & n(t) \\ -\alpha(t) & -m(t) & -n(t) & 0 \end{pmatrix} \begin{pmatrix} \gamma(t) \\ \nu_1(t) \\ \nu_2(t) \\ \mu(t) \end{pmatrix},$$

where $\alpha(t) = \dot{\gamma}(t) \cdot \mu(t)$, $\ell(t) = \dot{\nu}_1(t) \cdot \nu_2(t)$, $m(t) = \dot{\nu}_1(t) \cdot \mu(t)$ and $n(t) = \dot{\nu}_2(t) \cdot \mu(t)$. We call the mapping (α, ℓ, m, n) *the curvature of the spherical framed curve* (γ, ν_1, ν_2). Note that t_0 is a singular point of γ if and only if $\alpha(t_0) = 0$.

Definition 5. *Let (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2) : I \to \Delta$ be spherical framed curves. We say that (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ are congruent as spherical framed curves if there exists $A \in SO(4)$ such that $\widetilde{\gamma}(t) = A(\gamma(t))$, $\widetilde{\nu}_1(t) = A(\nu_1(t))$ and $\widetilde{\nu}_2(t) = A(\nu_2(t))$ for all $t \in I$.*

We have the existence and uniqueness theorems for spherical framed curves in terms of the curvatures. The proofs are similar to the cases of Legendre curves in the unit tangent bundle ([16]) and framed curves in the Euclidean space ([17]), so we omit them.

Theorem 4 (Existence theorem for spherical framed curves)**.** *Let $(\alpha, \ell, m, n) : I \to \mathbb{R}^4$ be a smooth mapping. Then, there exists a spherical framed curve $(\gamma, \nu_1, \nu_2) : I \to \Delta$ whose curvature is given by (α, ℓ, m, n).*

Theorem 5 (Uniqueness theorem for spherical framed curves)**.** *Let (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2) : I \to \Delta$ be spherical framed curves with curvatures (α, ℓ, m, n) and $(\widetilde{\alpha}, \widetilde{\ell}, \widetilde{m}, \widetilde{n})$, respectively. Then, (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ are congruent as spherical framed curves if and only if the curvatures (α, ℓ, m, n) and $(\widetilde{\alpha}, \widetilde{\ell}, \widetilde{m}, \widetilde{n})$ coincide.*

Let $(\gamma, \nu_1, \nu_2) : I \to \Delta$ be a spherical framed curve with curvature (α, ℓ, m, n). For the normal plane spanned by $\nu_1(t)$ and $\nu_2(t)$, there are other frames by rotations (cf. [20]). We define $(\widetilde{\nu}_1(t), \widetilde{\nu}_2(t)) \in \Delta_2$ by

$$\begin{pmatrix} \widetilde{\nu}_1(t) \\ \widetilde{\nu}_2(t) \end{pmatrix} = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} \begin{pmatrix} \nu_1(t) \\ \nu_2(t) \end{pmatrix},$$

where $\theta(t)$ is a smooth function. Then, $(\gamma, \widetilde{\nu}_1, \widetilde{\nu}_2) : I \to \Delta$ is also a spherical framed curve and $\widetilde{\mu}(t) = \mu(t)$. By a direct calculation, we have

$$\begin{aligned} \dot{\widetilde{\nu}}_1(t) &= (\ell(t) - \dot{\theta}(t)) \sin\theta(t) \nu_1(t) + (\ell(t) - \dot{\theta}(t)) \cos\theta(t) \nu_2(t) + (m(t)\cos\theta(t) - n(t)\sin\theta(t))\mu(t), \\ \dot{\widetilde{\nu}}_2(t) &= (\ell(t) - \dot{\theta}(t)) \cos\theta(t) \nu_1(t) + (\ell(t) - \dot{\theta}(t)) \sin\theta(t) \nu_2(t) + (m(t)\sin\theta(t) + n(t)\cos\theta(t))\mu(t). \end{aligned}$$

If we take a smooth function $\theta : I \to \mathbb{R}$ that satisfies $\dot{\theta}(t) = \ell(t)$, then we call the frame $\{\widetilde{\nu}_1(t), \widetilde{\nu}_2(t), \mu(t)\}$ an adapted frame along $\gamma(t)$. It follows that the Frenet–Serret-type formula is given by

$$\begin{pmatrix} \dot{\gamma}(t) \\ \dot{\widetilde{\nu}}_1(t) \\ \dot{\widetilde{\nu}}_2(t) \\ \dot{\mu}(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \alpha(t) \\ 0 & 0 & 0 & \widetilde{m}(t) \\ 0 & 0 & 0 & \widetilde{n}(t) \\ -\alpha(t) & -\widetilde{m}(t) & -\widetilde{n}(t) & 0 \end{pmatrix} \begin{pmatrix} \gamma(t) \\ \nu_1(t) \\ \nu_2(t) \\ \mu(t) \end{pmatrix},$$

where $\widetilde{m}(t)$ and $\widetilde{n}(t)$ are given by

$$\begin{pmatrix} \widetilde{m}(t) \\ \widetilde{n}(t) \end{pmatrix} = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} \begin{pmatrix} m(t) \\ n(t) \end{pmatrix}.$$

3.1. Bertrand Curves of Spherical Framed Curves

Let (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : I \to \Delta$ be spherical framed curves with curvatures (α, ℓ, m, n) and $(\overline{\alpha}, \overline{\ell}, \overline{m}, \overline{n})$, respectively. Suppose that $\overline{\gamma} \not\equiv \pm\gamma$.

Definition 6. *We say that (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are Bertrand mates if there exists a smooth function $\varphi : I \to \mathbb{R}$ such that $\overline{\gamma}(t) = \cos\varphi(t)\gamma(t) - \sin\varphi(t)\nu_1(t)$ and $\overline{\nu}_1(t) = \sin\varphi(t)\gamma(t) + \cos\varphi(t)\nu_1(t)$ for all $t \in I$. We also say that $(\gamma, \nu_1, \nu_2) : I \to \Delta$ is a Bertrand curve if there exists a spherical framed curve $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : I \to \Delta$ such that (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are Bertrand mates.*

Lemma 4. *Under the notations of Definition 6, if (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are Bertrand mates, then φ is a constant with $\sin\varphi \neq 0$.*

Proof. By differentiating $\overline{\gamma}(t) = \cos\varphi(t)\gamma(t) - \sin\varphi(t)\nu_1(t)$, we have

$$\begin{aligned}\overline{\alpha}(t)\overline{\mu}(t) = &- \dot{\varphi}(t)\sin\varphi(t)\gamma(t) - \dot{\varphi}(t)\cos\varphi(t)\nu_1(t) - \ell(t)\sin\varphi(t)\nu_2(t)\\ &+ (\alpha(t)\cos\varphi(t) - m(t)\sin\varphi(t))\mu(t).\end{aligned}$$

Since $\overline{\nu}_1(t) = \sin\varphi(t)\gamma(t) + \cos\varphi(t)\nu_1(t)$, we have $\dot{\varphi}(t) = 0$ for all $t \in I$. Therefore, φ is a constant. If $\sin\varphi = 0$, then $\overline{\gamma}(t) = \pm\gamma(t)$ for all $t \in I$. Hence, φ is a constant with $\sin\varphi \neq 0$. □

Theorem 6. *Let $(\gamma, \nu_1, \nu_2) : I \to \Delta$ be a spherical framed curve with curvature (α, ℓ, m, n). Then, (γ, ν_1, ν_2) is a Bertrand curve if and only if there exist a constant φ with $\sin\varphi \neq 0$ and a smooth function $\theta : I \to \mathbb{R}$ such that*

$$\ell(t)\sin\varphi\cos\theta(t) + (\alpha(t)\cos\varphi - m(t)\sin\varphi)\sin\theta(t) = 0 \tag{13}$$

for all $t \in I$.

Proof. Suppose that (γ, ν_1, ν_2) is a Bertrand curve. By Lemma 4, there exist a spherical framed curve $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ and a constant φ with $\sin\varphi \neq 0$ such that $\overline{\gamma}(t) = \cos\varphi\gamma(t) - \sin\varphi\nu_1(t)$ and $\overline{\nu}_1(t) = \sin\varphi\gamma(t) + \cos\varphi\nu_1(t)$ for all $t \in I$. By differentiating $\overline{\gamma}(t) = \cos\varphi\gamma(t) - \sin\varphi\nu_1(t)$, we have $\overline{\alpha}(t)\overline{\mu}(t) = -\ell(t)\sin\varphi\nu_2(t) + (\alpha(t)\cos\varphi - m(t)\sin\varphi)\mu(t)$. Since $\overline{\nu}_1(t) = \sin\varphi\gamma(t) + \cos\varphi\nu_1(t)$, there exists a smooth function $\theta : I \to \mathbb{R}$ such that

$$\begin{pmatrix} \overline{\nu}_2(t) \\ \overline{\mu}(t) \end{pmatrix} = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix} \begin{pmatrix} \nu_2(t) \\ \mu(t) \end{pmatrix}. \tag{14}$$

Then, we have

$$\overline{\alpha}(t)\sin\theta(t) = -\ell(t)\sin\varphi,\ \overline{\alpha}(t)\cos\theta(t) = \alpha(t)\cos\varphi - m(t)\sin\varphi.$$

It follows that $\ell(t)\sin\varphi\cos\theta(t) + (\alpha(t)\cos\varphi - m(t)\sin\varphi)\sin\theta(t) = 0$ for all $t \in I$.

Conversely, suppose that there exists a smooth function $\theta : I \to \mathbb{R}$ such that $\ell(t)\sin\varphi \cos\theta(t) + (\alpha(t)\cos\varphi - m(t)\sin\varphi)\sin\theta(t) = 0$ for all $t \in I$. We define a mapping $(\gamma, \nu_1, \nu_2) : I \to \Delta$ by $\overline{\gamma}(t) = \cos\varphi\gamma(t) - \sin\varphi\nu_1(t)$, $\overline{\nu}_1(t) = \sin\varphi\gamma(t) + \cos\varphi\nu_1(t)$ and $\overline{\nu}_2(t) = \cos\theta(t)\nu_2(t) - \sin\theta(t)\mu(t)$. Then, $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ is a spherical framed curve. Therefore, (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are Bertrand mates. □

Proposition 8. *Suppose that (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : I \to \Delta$ are Bertrand mates, where $\overline{\gamma}(t) = \cos\varphi\gamma(t) - \sin\varphi\nu_1(t)$ and $\ell(t)\sin\varphi\cos\theta(t) + (\alpha(t)\cos\varphi - m(t)\sin\varphi)\sin\theta(t) = 0$ for all $t \in I$. Then, the curvature $(\overline{\alpha}, \overline{\ell}, \overline{m}, \overline{n})$ of $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ is given by*

$$\begin{aligned}
\overline{\alpha}(t) &= -\ell(t)\sin\varphi\sin\theta(t) + (\alpha(t)\cos\varphi - m(t)\sin\varphi)\cos\theta(t),\\
\overline{\ell}(t) &= \ell(t)\cos\varphi\cos\theta(t) - (\alpha(t)\sin\varphi + m(t)\cos\varphi)\sin\theta(t),\\
\overline{m}(t) &= \ell(t)\cos\varphi\sin\theta(t) + (\alpha(t)\sin\varphi + m(t)\cos\varphi)\cos\theta(t),\\
\overline{n}(t) &= n(t) - \dot{\theta}(t).
\end{aligned}$$

Proof. By Equation (14), we have $\overline{\nu}_2(t) = \cos\theta(t)\nu_2(t) - \sin\theta(t)\mu(t)$. By differentiating, we have

$$\begin{aligned}
-\overline{\ell}(t)\overline{\nu}_1(t) + \overline{n}(t)\overline{\mu}(t) =& \alpha(t)\sin\theta(t)\gamma(t) + (-\ell(t)\cos\theta(t) + m(t)\sin\theta(t))\nu_1(t)\\
&+ (n(t) - \dot{\theta}(t))\sin\theta(t)\nu_2(t) + (n(t) - \dot{\theta}(t))\cos\theta(t)\mu(t).
\end{aligned}$$

Since $\overline{\nu}_1(t) = \sin\varphi\gamma(t) + \cos\varphi\nu_1(t)$ and $\overline{\mu}(t) = \sin\theta(t)\nu_2(t) + \cos\theta(t)\mu(t)$, we have

$$\overline{\ell}(t)\sin\varphi = -\alpha(t)\sin\theta(t),\ \overline{\ell}(t)\cos\varphi = \ell(t)\cos\theta(t) - m(t)\sin\theta(t),\ \overline{n}(t) = n(t) - \dot{\theta}(t).$$

It follows that

$$\overline{\ell}(t) = \ell(t)\cos\varphi\cos\theta(t) - (\alpha(t)\sin\varphi + m(t)\cos\varphi)\sin\theta(t).$$

Moreover, by differentiating $\overline{\mu}(t) = \sin\theta(t)\nu_2(t) + \cos\theta(t)\mu(t)$, we have

$$\begin{aligned}
-\overline{\alpha}(t)\overline{\gamma}(t) - \overline{m}(t)\overline{\nu}_1(t) - \overline{n}(t)\overline{\nu}_2(t) =& -\alpha(t)\cos\theta(t)\gamma(t) - (\ell(t)\sin\theta(t) + m(t)\cos\theta(t))\nu_1(t)\\
&+ (\dot{\theta}(t) - n(t))\cos\theta(t)\nu_2(t) + (n(t) - \dot{\theta}(t))\sin\theta(t)\mu(t).
\end{aligned}$$

Since $\overline{\gamma}(t) = \cos\varphi\gamma(t) - \sin\varphi\nu_1(t)$, $\overline{\nu}_1(t) = \sin\varphi\gamma(t) + \cos\varphi\nu_1(t)$ and $\overline{\nu}_2(t) = \cos\theta(t)\nu_2(t) - \sin\theta(t)\mu(t)$, we have

$$\begin{aligned}
\overline{\alpha}(t)\cos\varphi + \overline{m}(t)\sin\varphi &= \alpha(t)\cos\theta(t),\\
\overline{\alpha}(t)\sin\varphi - \overline{m}(t)\cos\varphi &= -\ell(t)\sin\theta(t) - m(t)\cos\theta(t).
\end{aligned}$$

It follows that

$$\overline{\alpha}(t) = -\ell(t)\sin\varphi\sin\theta(t) + (\alpha(t)\cos\varphi - m(t)\sin\varphi)\cos\theta(t)$$

and

$$\overline{m}(t) = \ell(t)\cos\varphi\sin\theta(t) + (\alpha(t)\sin\varphi + m(t)\cos\varphi)\cos\theta(t).$$

□

Corollary 3. *Let $(\gamma, \nu_1, \nu_2) : I \to \Delta$ be a spherical framed curve with curvature (α, ℓ, m, n). If $\ell(t) = 0$ for all $t \in I$, then (γ, ν_1, ν_2) is a Bertrand curve.*

Proof. If we take $\theta(t) = 0$, then Equation (13) is satisfied. □

Let $(\gamma, \nu_1, \nu_2) : I \to \Delta$ be a spherical framed curve with curvature (α, ℓ, m, n). If we take an adapted frame $\{\widetilde{\nu}_1(t), \widetilde{\nu}_2(t), \mu(t)\}$, then the curvature is given by $(\alpha, 0, \widetilde{m}, \widetilde{n})$. By Corollary 3, we have the following.

Corollary 4. *For an adapted frame, $(\gamma, \widetilde{\nu}_1, \widetilde{\nu}_2)$ is always a Bertrand curve.*

Proposition 9. *Suppose that (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : I \to \Delta$ are Bertrand mates with curvatures (α, ℓ, m, n) and $(\overline{\alpha}, \overline{\ell}, \overline{m}, \overline{n})$, respectively. Then, there exist a constant φ with $\sin\varphi \neq 0$ and a smooth function $\theta : I \to \mathbb{R}$ such that the following formulas hold:*

(1) $\ell(t)\sin\varphi = -\overline{\alpha}(t)\sin\theta(t)$, $\overline{\ell}(t)\sin\varphi = -\alpha(t)\sin\theta(t)$.
(2) $\alpha(t)\cos\varphi - m(t)\sin\varphi = \overline{\alpha}(t)\cos\theta(t)$, $\overline{\alpha}(t)\cos\varphi + \overline{m}(t)\sin\varphi = \alpha(t)\cos\theta(t)$.

(3) $\ell(t)\cos\varphi = \overline{\ell}(t)\cos\theta(t) + \overline{m}(t)\sin\theta(t)$, $\overline{\ell}(t)\cos\varphi = \ell(t)\cos\theta(t) - m(t)\sin\theta(t)$.

(4) $\alpha(t)\sin\varphi + m(t)\cos\varphi = -\overline{\ell}(t)\sin\theta(t) + \overline{m}(t)\cos\theta(t)$,
$-\overline{\alpha}(t)\sin\varphi + \overline{m}(t)\cos\varphi = \ell(t)\sin\theta(t) + m(t)\cos\theta(t)$.

Proof. By Definition 6 and the proof of Theorem 6, we have

$$\begin{aligned}
\overline{\gamma}(t) &= \cos\varphi\gamma(t) - \sin\varphi\nu_1(t),\\
\overline{\nu}_1(t) &= \sin\varphi\gamma(t) + \cos\varphi\nu_1(t),\\
\overline{\nu}_2(t) &= \cos\theta(t)\nu_2(t) - \sin\theta(t)\mu(t),\\
\overline{\mu}(t) &= \sin\theta(t)\nu_2(t) + \cos\theta(t)\mu(t).
\end{aligned}$$

We write the moving frame of γ in terms of the moving frame of $\overline{\gamma}$:

$$\begin{aligned}
\gamma(t) &= \cos\varphi\overline{\gamma}(t) + \sin\varphi\overline{\nu}_1(t),\\
\nu_1(t) &= -\sin\varphi\overline{\gamma}(t) + \cos\varphi\overline{\nu}_1(t),\\
\nu_2(t) &= \cos\theta(t)\overline{\nu}_2(t) + \sin\theta(t)\overline{\mu}(t),\\
\mu(t) &= -\sin\theta(t)\overline{\nu}_2(t) + \cos\theta(t)\overline{\mu}(t).
\end{aligned}$$

By differentiating $\overline{\gamma}(t), \overline{\nu}_1(t), \overline{\nu}_2(t), \overline{\mu}(t)$ and $\gamma(t), \nu_1(t), \nu_2(t), \mu(t)$, we obtain the formulas. □

By Proposition 9, we have the following relations.

Corollary 5. *With the same assumption as in Proposition 9, suppose that (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are Bertrand mates, then the following relations hold:*

(1) $\ell(t)\overline{\ell}(t)\sin^2\varphi = \alpha(t)\overline{\alpha}(t)\sin^2\theta(t)$.

(2) $(\alpha(t)\cos\varphi - m(t)\sin\varphi)(\overline{\alpha}(t)\cos\varphi + \overline{m}(t)\sin\varphi) = \alpha(t)\overline{\alpha}(t)\cos^2\theta(t)$.

(3) $\ell(t)\overline{\ell}(t)\cos^2\varphi = (\ell(t)\cos\theta(t) - m(t)\sin\theta(t))(\overline{\ell}(t)\cos\theta(t) + \overline{m}(t)\sin\theta(t))$.

(4) $(\alpha(t)\sin\varphi + m(t)\cos\varphi)(\overline{\alpha}(t)\sin\varphi - \overline{m}(t)\cos\varphi)$
$= (\ell(t)\sin\theta(t) + m(t)\cos\theta(t))(\overline{\ell}(t)\sin\theta(t) - \overline{m}(t)\cos\theta(t))$.

3.2. Mannheim Curves of Spherical Framed Curves

Let (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : I \to \Delta$ be spherical framed curves with curvatures (α, ℓ, m, n) and $(\overline{\alpha}, \overline{\ell}, \overline{m}, \overline{n})$, respectively. Suppose that $\overline{\gamma} \not\equiv \pm\gamma$.

Definition 7. *We say that (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are Mannheim mates if there exists a smooth function $\varphi : I \to \mathbb{R}$ such that $\overline{\gamma}(t) = \cos\varphi(t)\gamma(t) - \sin\varphi(t)\nu_1(t)$ and $\overline{\nu}_2(t) = \sin\varphi(t)\gamma(t) + \cos\varphi(t)\nu_1(t)$ for all $t \in I$. We also say that $(\gamma, \nu_1, \nu_2) : I \to \Delta$ is a Mannheim curve if there exists a spherical framed curve $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : I \to \Delta$ such that (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are Mannheim mates.*

Lemma 5. *Under the notations of Definition 7, if (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are Mannheim mates, then φ is a constant with $\sin\varphi \neq 0$.*

Proof. By differentiating $\overline{\gamma}(t) = \cos\varphi(t)\gamma(t) - \sin\varphi(t)\nu_1(t)$, we have

$$\begin{aligned}
\overline{\alpha}(t)\overline{\mu}(t) = &-\dot{\varphi}(t)\sin\varphi(t)\gamma(t) - \dot{\varphi}(t)\cos\varphi(t)\nu_1(t) - \ell(t)\sin\varphi(t)\nu_2(t)\\
&+ (\alpha(t)\cos\varphi(t) - m(t)\sin\varphi(t))\mu(t).
\end{aligned}$$

Since $\overline{\nu}_2(t) = \sin\varphi(t)\gamma(t) + \cos\varphi(t)\nu_1(t)$, we have $\dot{\varphi}(t) = 0$ for all $t \in I$. Therefore, φ is a constant. If $\sin\varphi = 0$, then $\overline{\gamma}(t) = \pm\gamma(t)$ for all $t \in I$. Hence, φ is a constant with $\sin\varphi \neq 0$. □

Theorem 7. *Let $(\gamma, \nu_1, \nu_2) : I \to \Delta$ be a spherical framed curve with curvature (α, ℓ, m, n). Then, (γ, ν_1, ν_2) is a Mannheim curve if and only if there exist a constant φ with $\sin\varphi \neq 0$ and a smooth function $\phi : I \to \mathbb{R}$ such that*

$$-\ell(t)\sin\varphi\sin\phi(t) + (\alpha(t)\cos\varphi - m(t)\sin\varphi)\cos\phi(t) = 0 \tag{15}$$

for all $t \in I$.

Proof. Suppose that (γ, ν_1, ν_2) is a Mannheim curve. By Lemma 5, there exist a spherical framed curve $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ and a constant φ with $\sin\varphi \neq 0$ such that $\overline{\gamma}(t) = \cos\varphi\gamma(t) - \sin\varphi\nu_1(t)$ and $\overline{\nu}_2(t) = \sin\varphi\gamma(t) + \cos\varphi\nu_1(t)$ for all $t \in I$. By differentiating $\overline{\gamma}(t) = \cos\varphi\gamma(t) - \sin\varphi\nu_1(t)$, we have $\overline{\alpha}(t)\overline{\mu}(t) = (\alpha(t)\cos\varphi - m(t)\sin\varphi)\mu(t) - \ell(t)\sin\varphi\nu_2(t)$. Since $\overline{\nu}_2(t) = \sin\varphi\gamma(t) + \cos\varphi\nu_1(t)$, there exists a smooth function $\phi : I \to \mathbb{R}$ such that

$$\begin{pmatrix} \overline{\mu}(t) \\ \overline{\nu}_1(t) \end{pmatrix} = \begin{pmatrix} \cos\phi(t) & -\sin\phi(t) \\ \sin\phi(t) & \cos\phi(t) \end{pmatrix} \begin{pmatrix} \nu_2(t) \\ \mu(t) \end{pmatrix}. \tag{16}$$

Then, we have

$$\overline{\alpha}(t)\cos\phi(t) = -\ell(t)\sin\varphi, \quad -\overline{\alpha}\sin\phi(t) = \alpha(t)\cos\varphi - m(t)\sin\varphi.$$

It follows that $-\ell(t)\sin\varphi\sin\phi(t) + (\alpha(t)\cos\varphi - m(t)\sin\varphi)\cos\phi(t) = 0$ for all $t \in I$.

Conversely, suppose that $-\ell(t)\sin\varphi\sin\phi(t) + (\alpha(t)\cos\varphi - m(t)\sin\varphi)\cos\phi(t) = 0$ for all $t \in I$. We define a mapping $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : I \to \Delta$ by $\overline{\gamma}(t) = \cos\varphi\gamma(t) - \sin\varphi\nu_1(t)$, $\overline{\nu}_1(t) = \sin\phi(t)\nu_2(t) + \cos\phi(t)\mu(t)$ and $\overline{\nu}_2(t) = \sin\varphi\gamma(t) + \cos\varphi\nu_1(t)$. Then, $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ is a spherical framed curve. Therefore, (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ are Mannheim mates. □

Proposition 10. *Suppose that (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : I \to \Delta$ are Mannheim mates, where $\overline{\gamma}(t) = \cos\varphi\gamma(t) - \sin\varphi\nu_1(t)$ and $-\ell(t)\sin\varphi\sin\phi(t) + (\alpha(t)\cos\varphi - m(t)\sin\varphi)\cos\phi(t) = 0$ for all $t \in I$. Then, the curvature $(\overline{\alpha}, \overline{\ell}, \overline{m}, \overline{n})$ of $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ is given by*

$$\begin{aligned}
\overline{\alpha}(t) &= -\ell(t)\sin\varphi\cos\phi(t) + (-\alpha(t)\cos\varphi + m(t)\sin\varphi)\sin\phi(t), \\
\overline{\ell}(t) &= \ell(t)\cos\varphi\sin\phi(t) - (\alpha(t)\sin\varphi + m(t)\cos\varphi)\cos\phi(t), \\
\overline{m}(t) &= \dot{\phi}(t) - n(t), \\
\overline{n}(t) &= -\ell(t)\cos\varphi\cos\phi(t) - (\alpha(t)\sin\varphi + m(t)\cos\varphi)\sin\phi(t).
\end{aligned}$$

Proof. By Equation (16), we have $\overline{\mu}(t) = \cos\phi(t)\nu_2(t) - \sin\phi(t)\mu(t)$. By differentiating, we have

$$\begin{aligned}
-\overline{\alpha}(t)\overline{\gamma}(t) - \overline{m}(t)\overline{\nu}_1(t) - \overline{n}(t)\overline{\nu}_2(t) &= \alpha(t)\sin\phi(t)\gamma + (-\ell(t)\cos\phi(t) + m(t)\sin\phi(t))\nu_1(t) \\
&\quad + (n(t) - \dot{\phi}(t))\sin\phi(t)\nu_2(t) + (n(t) - \dot{\phi}(t))\cos\phi(t)\mu(t).
\end{aligned}$$

Since $\overline{\gamma}(t) = \cos\varphi\gamma(t) - \sin\varphi\nu_1(t)$, $\overline{\nu}_1(t) = \sin\phi(t)\nu_2(t) + \cos\phi(t)\mu(t)$ and $\overline{\nu}_2(t) = \sin\varphi\gamma(t) + \cos\varphi\nu_1(t)$, we have

$$\begin{aligned}
\overline{m}(t) &= \dot{\phi}(t) - n(t), \\
\overline{\alpha}(t)\cos\varphi + \overline{n}(t)\sin\varphi &= -\alpha(t)\sin\phi(t), \\
\overline{\alpha}(t)\sin\varphi - \overline{n}(t)\cos\varphi &= -\ell(t)\cos\phi(t) + m(t)\sin\phi(t).
\end{aligned}$$

It follows that

$$\overline{\alpha}(t) = -\ell(t)\sin\varphi\cos\phi(t) + (-\alpha(t)\cos\varphi + m(t)\sin\varphi)\sin\phi(t)$$

and

$$\overline{n}(t) = \ell(t)\cos\varphi\cos\phi(t) - (\alpha(t)\sin\varphi + m(t)\cos\varphi)\sin\phi(t).$$

Moreover, by differentiating $\overline{\nu}_1(t) = \sin\phi(t)\nu_2(t) + \cos\phi(t)\mu(t)$, we have

$$\begin{aligned}\overline{\ell}(t)\overline{\nu}_2(t) + \overline{m}(t)\overline{\mu}(t) =& -\alpha(t)\cos\phi(t)\gamma(t) - (\ell(t)\sin\phi(t) + m(t)\cos\phi(t))\nu_1(t)\\ &+ (\dot{\phi}(t) - n(t))\cos\phi(t)\nu_2(t) + (n(t) - \dot{\phi}(t))\sin\phi(t)\mu(t).\end{aligned}$$

Since $\overline{\nu}_2(t) = \sin\varphi\gamma(t) + \cos\varphi\nu_1(t)$, we have

$$\begin{aligned}\overline{\ell}(t)\sin\varphi &= -\alpha(t)\cos\phi(t),\\ \overline{\ell}(t)\cos\varphi &= -(\ell(t)\sin\phi(t) + m(t)\cos\phi(t)).\end{aligned}$$

It follows that

$$\overline{\ell}(t) = -\ell(t)\cos\varphi\sin\phi(t) - (\alpha(t)\sin\varphi + m(t)\cos\varphi)\cos\phi(t).$$

□

Corollary 6. *Let $(\gamma, \nu_1, \nu_2) : I \to \Delta$ be a spherical framed curve with curvature (α, ℓ, m, n). If $\ell(t) = 0$ for all $t \in I$, then (γ, ν_1, ν_2) is a Mannheim curve.*

Proof. If we take $\phi(t) = \pi/2$, then Equation (15) is satisfied. □

Let $(\gamma, \nu_1, \nu_2) : I \to \Delta$ be a spherical framed curve with curvature (α, ℓ, m, n). If we take an adapted frame $\{\widetilde{\nu}_1(t), \widetilde{\nu}_2(t), \mu(t)\}$, then the curvature of $(\gamma, \widetilde{\nu}_1, \widetilde{\nu}_2)$ is given by $(\alpha, 0, \widetilde{m}, \widetilde{n})$. By Corollary 6, we have the following.

Corollary 7. *For an adapted frame, $(\gamma, \widetilde{\nu}_1, \widetilde{\nu}_2)$ is always a Mannheim curve.*

Proposition 11. *Suppose that (γ, ν_1, ν_2) and $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : I \to \Delta$ are Mannheim mates with curvatures (α, ℓ, m, n) and $(\overline{\alpha}, \overline{\ell}, \overline{m}, \overline{n})$, respectively. Then, there exist a constant φ with $\sin\varphi \neq 0$ and a smooth function $\phi : I \to \mathbb{R}$ such that the following formulas hold:*

(1) $\alpha(t)\cos\varphi - m(t)\sin\varphi = -\overline{\alpha}(t)\sin\phi(t)$, $\overline{\alpha}(t)\cos\varphi + \overline{n}(t)\sin\varphi = -\alpha(t)\sin\phi(t)$.
(2) $\ell(t)\sin\varphi = -\overline{\alpha}(t)\cos\phi(t)$, $\overline{\ell}(t)\sin\varphi = -\alpha(t)\cos\phi(t)$.
(3) $\ell(t)\cos\varphi = -\overline{\ell}(t)\sin\phi(t) + \overline{n}(t)\cos\phi(t)$, $\overline{\ell}(t)\cos\varphi = -\ell(t)\sin\phi(t) - m(t)\cos\phi(t)$.
(4) $\alpha(t)\sin\varphi + m(t)\cos\varphi = -\overline{\ell}(t)\cos\phi(t) - \overline{n}(t)\sin\phi(t)$,
$\overline{\alpha}(t)\sin\varphi - \overline{n}(t)\cos\varphi = -\ell(t)\cos\phi(t) + m(t)\sin\phi(t)$.

Proof. By Definition 7 and the proof of Theorem 7, we have

$$\begin{aligned}\overline{\gamma}(t) &= \cos\varphi\gamma(t) - \sin\varphi\nu_1(t),\\ \overline{\nu}_2(t) &= \sin\varphi\gamma(t) + \cos\varphi\nu_1(t),\\ \overline{\mu}(t) &= \cos\phi(t)\nu_2(t) - \sin\phi(t)\mu(t),\\ \overline{\nu}_1(t) &= \sin\phi(t)\nu_2(t) + \cos\phi(t)\mu(t).\end{aligned}$$

We write the moving frame of γ in terms of the moving frame of $\overline{\gamma}$:

$$\begin{aligned}\gamma(t) &= \cos\varphi\overline{\gamma}(t) + \sin\varphi\overline{\nu}_2(t),\\ \nu_1(t) &= -\sin\varphi\overline{\gamma}(t) + \cos\varphi\overline{\nu}_2(t),\\ \nu_2(t) &= \cos\phi(t)\overline{\mu}(t) + \sin\phi(t)\overline{\nu}_1(t),\\ \mu(t) &= -\sin\phi(t)\overline{\mu}(t) + \cos\phi(t)\overline{\nu}_1(t).\end{aligned}$$

By differentiating $\overline{\gamma}(t), \overline{\nu}_1(t), \overline{\nu}_2(t), \overline{\mu}(t)$ and $\gamma(t), \nu_1(t), \nu_2(t), \mu(t)$, we obtain the formulas. □

By Proposition 11, we have the following relations.

Corollary 8. *With the same assumption as in Proposition 11, suppose that* (γ,ν_1,ν_2) *and* $(\overline{\gamma},\overline{\nu}_1,\overline{\nu}_2)$ *are Mannheim mates, then the following relations hold:*

(1) $(\alpha(t)\cos\varphi - m(t)\sin\varphi)(\overline{\alpha}(t)\cos\varphi + \overline{n}(t)\sin\varphi) = \alpha(t)\overline{\alpha}(t)\sin^2\phi(t)$.
(2) $\ell(t)\overline{\ell}(t)\sin^2\varphi = \alpha(t)\overline{\alpha}(t)\cos^2\phi(t)$.
(3) $\ell(t)\overline{\ell}(t)\cos^2\varphi = (\ell(t)\sin\phi(t) + m(t)\cos\phi(t))(\overline{\ell}(t)\sin\phi(t) - \overline{n}(t)\cos\phi(t))$.
(4) $(\alpha(t)\sin\varphi + m(t)\cos\varphi)(\overline{\alpha}(t)\sin\varphi - \overline{n}(t)\cos\varphi)$
$= (\ell(t)\cos\phi(t) - m(t)\sin\phi(t))(\overline{\ell}(t)\cos\phi(t) + \overline{n}(t)\sin\phi(t))$.

Theorem 8. *Let* $(\gamma,\nu_1,\nu_2) : I \to \Delta$ *be a spherical framed curve with curvature* (α,ℓ,m,n)*. Then,* (γ,ν_1,ν_2) *is a Bertrand curve if and only if* (γ,ν_1,ν_2) *is a Mannheim curve.*

Proof. Suppose that (γ,ν_1,ν_2) is a Bertrand curve. By Theorem 6, there exist a constant φ with $\sin\varphi \neq 0$ and a smooth function $\theta : I \to \mathbb{R}$ such that $\ell(t)\sin\varphi\cos\theta(t) + (\alpha(t)\cos\varphi - m(t)\sin\varphi)\sin\theta(t) = 0$ for all $t \in I$. If $\phi(t) = \theta(t) - \pi/2$, then we have

$$-\ell(t)\sin\varphi\sin\phi(t) + (\alpha(t)\cos\varphi - m(t)\sin\varphi)\cos\phi(t) = 0$$

for all $t \in I$. By Theorem 7, (γ,ν_1,ν_2) is a Mannheim curve.

Conversely, suppose that (γ,ν_1,ν_2) is a Mannheim curve. By Theorem 7, there exist a constant φ with $\sin\varphi \neq 0$ and a smooth function $\phi : I \to \mathbb{R}$ such that $-\ell(t)\sin\varphi\sin\phi(t) + (\alpha(t)\cos\varphi - m(t)\sin\varphi)\cos\phi(t) = 0$ for all $t \in I$. If $\theta(t) = \phi(t) + \pi/2$, then we have

$$\ell(t)\sin\varphi\cos\theta(t) + (\alpha(t)\cos\varphi - m(t)\sin\varphi)\sin\theta(t) = 0$$

for all $t \in I$. By Theorem 6, (γ,ν_1,ν_2) is a Bertrand curve. □

Example 3. *Let* $(\gamma,\nu_1,\nu_2) : \mathbb{R} \to \Delta$,

$$\begin{aligned}
\gamma(t) &= \frac{1}{\sqrt{2t^2+\frac{5}{4}}}\Big(t\sin t + \cos t, -t\cos t + \sin t, t\sin 2t + \frac{1}{2}\cos 2t, -t\cos 2t + \frac{1}{2}\sin 2t\Big),\\
\nu_1(t) &= \frac{1}{\sqrt{2}}(-\sin t, \cos t, \sin 2t, -\cos 2t),\\
\nu_2(t) &= \frac{1}{\sqrt{20t^2+\frac{9}{2}}}\Big(-4t\cos t + \frac{3}{2}\sin t, -4t\sin t - \frac{3}{2}\cos t, 2t\cos 2t + \frac{3}{2}\sin 2t,\\
&\quad 2t\sin 2t - \frac{3}{2}\cos 2t\Big).
\end{aligned}$$

Then,

$$\begin{aligned}
\dot{\gamma}(t) &= \frac{t}{\left(2t^2+\frac{5}{4}\right)^{\frac{3}{2}}}\bigg(\Big(2t^2 - \frac{3}{4}\Big)\cos t - 2t\sin t, \Big(2t^2 - \frac{3}{4}\Big)\sin t + 2t\cos t,\\
&\quad 2\Big(2t^2 + \frac{3}{4}\Big)\cos 2t - 2t\sin 2t, 2\Big(2t^2 + \frac{3}{4}\Big)\sin 2t + 2t\cos 2t\bigg),
\end{aligned}$$

we have that $t = 0$ *is a singular point of* γ*. By a direct calculation,* (γ,ν_1,ν_2) *is a spherical framed curve. Then,*

$$
\begin{aligned}
\boldsymbol{\mu}(t) &= \gamma(t) \times \nu_1(t) \times \nu_2(t) \\
&= \frac{1}{\sqrt{20t^4 + 17t^2 + \frac{45}{16}}} \left(\left(2t^2 - \frac{3}{4}\right) \cos t - 2t \sin t, \left(2t^2 - \frac{3}{4}\right) \sin t + 2t \cos t, \right. \\
&\quad \left. 2\left(2t^2 + \frac{3}{4}\right) \cos 2t - 2t \sin 2t, 2\left(2t^2 + \frac{3}{4}\right) \sin 2t + 2t \cos 2t \right)
\end{aligned}
$$

and the curvature is given by

$$
\alpha(t) = \frac{t\sqrt{20t^4 + 17t^2 + \frac{45}{16}}}{\left(2t^2 + \frac{5}{4}\right)^{\frac{3}{2}}}, \ \ell(t) = \frac{8t}{\sqrt{40t^2 + 9}},
$$

$$
m(t) = \frac{24t^2 + 15}{4\sqrt{40t^4 + 34t^2 + \frac{45}{8}}}, \ n(t) = \frac{15(8t^2 + 5)}{8\sqrt{20t^2 + \frac{9}{2}}\sqrt{20t^4 + 17t^2 + \frac{45}{16}}}.
$$

It is easy to see that $m(t) \neq 0$ *for all* $t \in \mathbb{R}$. *Therefore, if we take* $\varphi = \pi/2$, $\sin\theta(t) = \frac{\ell(t)}{\sqrt{\ell^2(t) + m^2(t)}}$ *and* $\cos\theta(t) = \frac{m(t)}{\sqrt{\ell^2(t) + m^2(t)}}$, *then Equation (13) is satisfied. By Theorem 6,* (γ, ν_1, ν_2) *is a Bertrand curve. In fact,* $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) : \mathbb{R} \to \Delta$,

$$
(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2) = \left(-\nu_1, \gamma, \frac{m(t)}{\sqrt{\ell^2(t) + m^2(t)}}\nu_2 - \frac{\ell(t)}{\sqrt{\ell^2(t) + m^2(t)}}\mu \right)
$$

is a spherical framed curve. Hence, (γ, ν_1, ν_2) *and* $(\overline{\gamma}, \overline{\nu}_1, \overline{\nu}_2)$ *are Bertrand mates.*

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