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Algebraic Systems with Positive Coefficients and Positive Solutions

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Abstract: The paper is devoted to the existence, uniqueness and nonuniqueness of positive solutions to nonlinear algebraic systems of equations with positive coefficients. Such systems appear in large numbers of applications, such as steady-state equations in continuous and discrete dynamical models, Dirichlet problems, difference equations, boundary value problems, periodic solutions and numerical solutions for differential equations. We apply Brouwer's fixed point theorem, Krasnoselskii's fixed point theorem and monotone iterative methods in order to extend some known results and to obtain new results. We relax some hypotheses used in the literature concerning the strict monotonicity of the involved functions. We show that, in some cases, the unique positive solution can be obtained by a monotone increasing iterative method or by a monotone decreasing iterative method. As a consequence of one of our results, we recover the existence of a non-negative solution of the Leontief system and describe a monotone iterative method to find it.

Keywords: Brouwer's fixed point theorem; Krasnoselskii's fixed point theorem; nonlinear algebraic systems of equations

MSC: 39A10; 65H10; 47J05; 65H20

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1. Introduction

Algebraic systems with positive coefficients and positive solutions have many applications: see, e.g., [1] and the references therein. Several techniques are used for studying the existence, uniqueness and nonuniqueness of a positive solution to such a system. Fixed point theorems (such as Brouwer's and Krasnoselskii's theorems) and monotone iterative methods are often applied, depending on the nature of the system. In this paper, we use these tools in order to extend some known results and to get new results concerning systems related to (S_0) below.

Brouwer's fixed point theorem was used in [2] to prove the existence and uniqueness of a positive solution to a system of the form (S_0) under specific assumptions on the strict monotonicity of the functions f_i. In Section 2, we relax the assumptions and prove the existence of a positive solution by using a monotone iterative method. The results are related to those of [1]. In particular, we show that under suitable hypotheses, the solution can be obtained by a monotone increasing iterative method and/or by a monotone decreasing iterative method. See Theorems 1–6.

Example 2 is related to the Leontief model (see [3]). It is well-known (Theorem 10.5 of [3]) that a Leontief system has a unique non-negative solution. The existence of such a solution and a monotone iterative method to find it can be proved by using Theorem 3. Section 2 ends with some examples.

In Section 3, we use Brouwer's theorem in order to study the existence of a non-negative solution to the system (S_1), related to (S_0) (see Theorem 8). For the functions $f_i(t) = t^{c_i}$, (S_0) was investigated in the cases $c_i > 1$, $c_i < -1$, $0 < c_i \leq 1$, $i = 1, \ldots, n$. The case $-1 \leq c_i < 0$, $i = 1, \ldots, n$, is settled in Corollary 1 as an application of Theorem 8.

In Section 4, we use Krasnoselskii's fixed point theorem in order to provide examples of systems of the above type with no positive solution or with two positive solutions.

Related results can be found in the papers in the bibliography. General methods for solving systems are described in [4–16]. For applications to discrete inclusions, see [17]. Many applications to difference equations can be found in [18] and the references therein. In [19], applications to extremum problems are presented, while [20] describes applications to parameter estimation. Linear systems with positive coefficients and positive solutions are studied in [21,22], while special classes of systems are investigated in [23–25]. Applications to boundary value problems can be found in [26,27] and the references therein. Applied boundary value problems and nonlinear quantum integro-difference boundary value problems are addressed in [28,29] with Krasnoselskii's fixed point theorem as the main theoretical tool.

General methods and applications of positive matrices are presented, e.g., in [30–32].

We end this section by mentioning several areas of applications where systems such as (S_0) below play a significant role. Such applications are described, e.g., in [1], where the reader can find several types of steady-state equations in continuous and discrete dynamical models. For other applications, see [33]: Dirichlet problems, difference equations, boundary value problems, periodic solutions and numerical solutions for differential equations. In all these applications, the study of the system (S_0) is an essential step. In our article, we present new results about such systems, which could be useful for applications.

2. The Monotone Iterative Method

Consider the system

$$(S_0) \begin{cases} f_1(x_1) = a_{11}x_1 + \cdots + a_{1n}x_n + p_1, \\ \ldots \\ f_n(x_n) = a_{n1}x_1 + \cdots + a_{nn}x_n + p_n, \end{cases} \tag{1}$$

where $f_i : [0, \infty) \to [0, \infty)$ is continuous and strictly increasing, $f_i(0) = 0$, $a_{ij} \geq 0$, $p_i \geq 0$, $i, j = 1, \ldots, n$. A non-negative solution is a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $x_1 \geq 0, \ldots, x_n \geq 0$ and x satisfies (S_0). If $x_1 > 0, \ldots, x_n > 0$, we say that the solution is positive. A sequence $x^k = (x_1^k, \ldots, x_n^k) \in \mathbb{R}^n$, $k \geq 1$, is called increasing if each sequence $(x_i^k)_{k \geq 1}$ is increasing.

Let $s_i := a_{i1} + \cdots + a_{in}$, $i = 1, \ldots, n$. We denote by $R(f_i)$ the range of f_i, which is an interval of the form $[0, c_i), c_i \leq \infty$.

Theorem 1. *Assume that there exists $t_0 > 0$ such that*

$$f_i(t_0) \geq s_i t_0 + p_i, \; i = 1, \ldots, n. \tag{2}$$

Then, the sequence $(x^k)_{k \geq 0}$ defined by

$$x_i^0 := t_0, \; i = 1, 2, \ldots, n, \tag{3}$$

$$x_i^{k+1} := f_i^{-1}(a_{i1}x_1^k + \cdots + a_{in}x_n^k + p_i), \; k \geq 0, \; i = 1, \ldots, n, \tag{4}$$

is decreasing and convergent to a non-negative solution x^* of (S_0).

Proof. We prove by induction the statement

$$P(k) : a_{i1}x_1^k + \cdots + a_{in}x_n^k + p_i \in R(f_i) \text{ and } x_i^{k+1} \leq x_i^k, \, i = 1, \ldots, n.$$

To prove $P(0)$, let us remark that

$$a_{i1}x_1^0 + \cdots + a_{in}x_n^0 + p_i = s_i t_0 + p_i,$$

which is in $R(f_i)$, using $f_i(0) = 0$ and (2). Moreover,

$$x_i^1 = f_i^{-1}(a_{i1}x_1^0 + \cdots + a_{in}x_n^0 + p_i) \le f_i^{-1}(s_i t_0 + p_i) \le t_0 = x_i^0, \ i = 1, \ldots, n.$$

Now suppose that $P(k)$ is true. Then,

$$0 \le a_{i1}x_1^{k+1} + \cdots + a_{in}x_n^{k+1} + p_i \le a_{i1}x_1^k + \cdots + a_{in}x_n^k + p_i \in R(f_i),$$

and so $a_{i1}x_1^{k+1} + \cdots + a_{in}x_n^{k+1} + p_i \in R(f_i)$, $i = 1, \ldots, n$.

Therefore,

$$\begin{aligned} x_i^{k+2} &= f_i^{-1}(a_{i1}x_1^{k+1} + \cdots + a_{in}x_n^{k+1} + p_i) \le f_i^{-1}(a_{i1}x_1^k + \cdots + a_{in}x_n^k + p_i) \\ &= x_i^{k+1}, \ i = 1, \ldots, n. \end{aligned}$$

This shows that $P(k)$ is true for all $k \ge 0$. Consequently, each sequence $(x_i^k)_{k\ge 0}$ is decreasing and bounded, hence convergent to a certain $x_i^* \ge 0$. From (4), we see that

$$f_i(x_i^{k+1}) = a_{i1}x_1^k + \cdots + a_{in}x_n^k + p_i, \ i = 1, \ldots, n,$$

and so $f_i(x_i^*) = a_{i1}x_1^* + \cdots + a_{in}x_n^* + p_i$.

This concludes the proof. □

Theorem 2. *Assume that there exists $t_0 > 0$ satisfying (2), and $t_1 \in [0, t_0)$ such that*

$$f_i(t_1) \le s_i t_1 + p_i, \ i = 1, \ldots, n. \tag{5}$$

Then, the sequence $(x^k)_{k\ge 0}$ defined by

$$x_i^0 := t_1, \ i = 1, 2, \ldots, n,$$

and by (4) is increasing and convergent to a non-negative solution $\tilde{x}$ of (S_0).

Proof. Let us prove by induction the statement

$$Q(k): \ a_{i1}x_1^k + \cdots + a_{in}x_n^k + p_i \in R(f_i) \text{ and } x_i^k \le x_i^{k+1} \le t_0, \ i = 1, \ldots, n.$$

First, $0 \le a_{i1}x_1^0 + \cdots + a_{in}x_n^0 + p_i = s_i t_1 + p_i \le s_i t_0 + p_i \in R(f_i)$.

Thus, $a_{i1}x_1^0 + \cdots + a_{in}x_n^0 + p_i \in R(f_i)$, and

$$x_i^0 = t_1 \le f_i^{-1}(s_i t_1 + p_i) = f_i^{-1}(a_{i1}x_1^0 + \cdots + a_{in}x_n^0 + p_i) = x_i^1.$$

Moreover, $x_i^1 = f_i^{-1}(s_i t_1 + p_i) \le f_i^{-1}(s_i t_0 + p_i) \le t_0$. This proves $Q(0)$.

Now, suppose that $Q(k)$ is true. Then,

$$s_i t_0 + p_i \ge a_{i1}x_1^{k+1} + \cdots + a_{in}x_n^{k+1} + p_i \ge a_{i1}x_1^k + \cdots + a_{in}x_n^k + p_i,$$

so that $a_{i1}x_1^{k+1} + \cdots + a_{in}x_n^{k+1} + p_i \in R(f_i)$.

Moreover,

$$x_i^{k+1} = f_i^{-1}\left(a_{i1}x_1^k + \cdots + a_{in}x_n^k + p_i\right) \le f_i^{-1}(a_{i1}x_1^{k+1} + \cdots + a_{in}x_n^{k+1} + p_i) = x_i^{k+2},$$

and

$$x_i^{k+2} = f_i^{-1}\left(a_{i1}x_1^{k+1} + \cdots + a_{in}x_n^{k+1} + p_i\right) \le f_i^{-1}(s_i t_0 + p_i) \le t_0,$$

for all $i = 1, \ldots, n$.

So $Q(k)$ is true for all $k \geq 0$, which shows that each sequence $(x_i^k)_{k \geq 0}$ is increasing and bounded and hence convergent to a certain $\tilde{x}_i > 0$.

It follows that $\tilde{x}$ is a positive solution of (S_0). □

Remark 1. *It is easy to see that if $a_{ij} > 0$, $i, j = 1, \ldots, n$, and $\sum_{i=1}^{n} p_i > 0$, then the solution x^* from Theorem 1 is a positive solution. Moreover, if $p_i > 0$, $i = 1, \ldots, n$, then $\tilde{x}$ from Theorem 2 is also positive.*

Theorem 3. *Suppose that there exist $x_1^0 > 0, \ldots, x_n^0 > 0$ such that*

$$f_i(x_i^0) > a_{i1}x_1^0 + \cdots + a_{in}x_n^0 + p_i, \ i = 1, \ldots, n. \tag{6}$$

Then, the sequence $(x^k)_{k \geq 0}$ defined by (4) is decreasing and convergent to a non-negative solution x^* of (S_0).

The proof is similar to that of Theorem 1 and will be omitted.

Example 1. *Consider the system*

$$\begin{cases} \sqrt{x_1} = \dfrac{1}{100}x_1 + \dfrac{1}{100}x_2 + 8.9, \\ x_2 = \dfrac{1}{200}x_1 + \dfrac{1}{10}x_2 + 8.5. \end{cases}$$

It has the positive solution $x_1 = 100$, $x_2 = 10$, which can be obtained using

(1) Theorem 1 with $x_1^0 = x_2^0 = t_0 = 144$,
(2) Theorem 2 with $x_1^0 = x_2^0 = t_1 = 0$,
(3) Theorem 3 with $x_1^0 = 144$, $x_2^0 = 100$.

The system has also the positive solution

$$x_1 = 8000.85\ldots, \ x_2 = 53.89\ldots$$

Example 2. *Let $f_i(t) = t$, $i = 1, \ldots, n$, $t \geq 0$. Then, (S_0) can be written as*

$$(\Sigma): \ X = AX + P,$$

where $X = col(x_1, \ldots, x_n)$, $P = col(p_1, \ldots, p_n)$, and A is the matrix with entries a_{ij}. Suppose that $a_{ij} \geq 0$, $i, j = 1, \ldots, n$, and for each j, there exists some i such that $a_{ij} > 0$. Moreover, suppose that A is productive (see [3], p. 172); i.e., there exists $\tilde{X} > 0$ such that

$$\tilde{X} > A\tilde{X}. \tag{7}$$

This is the Leontief model (see [3], p. 172). Theorem 10.5 in [3] shows that under these assumptions, the system (Σ) has a unique non-negative solution. Let us remark that Theorem 2.3 implies the existence of a non-negative solution and provides a monotone iterative method to find it. Indeed, if $c > 0$ is sufficiently large, then $X^0 := c\tilde{X}$ will satisfy $X^0 > AX^0 + P$, which is (6). Now, according to Theorem 3, the sequence $(X^k)_{k \geq 0}$ defined by

$$X^{k+1} = AX^k + P, k \geq 0,$$

is decreasing and convergent to the non-negative solution of (Σ). For a related result, see [1] (Corollary 3.2).

Theorem 4. *Suppose that* $\lim_{t\to\infty} f_i(t) = \infty$, $i = 1, \ldots, n$, *and there exists* $t_0 > 0$ *satisfying (2). Then, the sequence* $(x^k)_{k\geq 0}$ *defined by* $x_i^0 = 0$, $i = 1, \ldots, n$, *and by (4) is increasing and convergent to a non-negative solution* $\overline{x}$ *of* (S_0).

Proof. For each $i = 1, \ldots, n$ we have $R(f_i) = [0, \infty)$. Let us prove by induction the statement

$$L(k): \ x_i^k \leq x_i^{k+1} \leq t_0, \ i = 1, \ldots, n.$$

From (2) we see that $f_i(t_0) \geq p_i$, and so

$$x_i^o = 0 \leq x_i^1 = f_i^{-1}(p_i) \leq t_0, \ i = 1, \ldots, n.$$

This proves $L(0)$. Now suppose that $L(k)$ is true. Then

$$\begin{aligned} x_i^{k+1} &= f_i^{-1}(a_{i1}x_1^k + \cdots + a_{in}x_n^k + p_i) \\ &\leq f_i^{-1}\left(a_{i1}x_1^{k+1} + \cdots + a_{in}x_n^{k+1} + p_i\right) = x_i^{k+2}, \ i = 1, \ldots, n. \end{aligned}$$

Moreover,

$$\begin{aligned} x_i^{k+2} &= f_i^{-1}(a_{i1}x_1^{k+1} + \cdots + a_{in}x_n^{k+1} + p_i) \\ &\leq f_i^{-1}(s_i t_0 + p_i) \leq t_0, \ i = 1, \ldots, n. \end{aligned}$$

Thus, $L(k+1)$ is also true, which means that $(x_i^k)_{k\geq 0}$ is increasing and bounded, $i = 1, \ldots, n$. Its limit $\overline{x}$ will be a solution to (S_0). □

Let us recall a result from [2].

Theorem 5 (Theorem 1 of [2])**.** *Assume that* $a_{ij} > 0$, $p_i \geq 0$, $i, j = 1, \ldots, n$. *Let* $f_i(t) = t g_i(t)$, $t > 0$, *where* $g_i : (0, \infty) \to (0, \infty)$ *is continuous and strictly increasing,* $i = 1, \ldots, n$. *Suppose that for each* $i = 1, \ldots, n$ *there exists* $t_i > 0$ *such that* $g_i(t_i) = s_i$. *Then, the system* (S_0) *has a unique positive solution* z^*.

In what follows, we show that z^* can be approached by an increasing iterative method and also by a decreasing iterative method.

Take a number u such that $0 < u < \min\{t_i : i = 1, \ldots, n\}$. Then, $g_i(u) < g_i(t_i) = s_i$, hence $f_i(u) = u g_i(u) < u s_i$, and finally

$$f_i(u) < \sum_{j=1}^{n} a_{ij} u + p_i, \ i = 1, \ldots, n. \tag{8}$$

Let us remark that if $v > \max\{t_i : i = 1, \ldots, n\}$ then $g_i(v) > g_i(t_i) = s_i$, $i = 1, \ldots, n$. Consequently, we can choose $v > 0$ such that $v(g_i(v) - s_i) > p_i$, $i = 1, \ldots, n$. It follows that

$$f_i(v) = v g_i(v) > v s_i + p_i, \text{ and so}$$

$$f_i(v) > \sum_{j=1}^{n} a_{ij} v + p_i, \ i = 1, \ldots, n. \tag{9}$$

Theorem 6. *Under the hypotheses of Theorem 5, consider the sequence* $(x^k)_{k\geq 0}$ *defined by*

$$x_i^0 := u, \ i = 1, \ldots, n, \tag{10}$$

$$x_i^{k+1} := f_i^{-1}\left(\sum_{j=1}^{n} a_{ij} x_j^k + p_i\right), \ i = 1, \ldots, n, \tag{11}$$

and the sequence $(y^k)_{k\geq 0}$ *defined by*

$$y_i^0 := v,\ i = 1,\ldots,n, \tag{12}$$

$$y_i^{k+1} := f_i^{-1}\left(\sum_{j=1}^n a_{ij}y_j^k + p_i\right),\ i = 1,\ldots,n. \tag{13}$$

Then, $(x^k)_{k\geq 0}$ *is increasing,* $(y^k)_{k\geq 0}$ *is decreasing, and the limit of both of them is* z^**—the unique positive solution of* (S_0)*.*

Proof. We prove by induction the statement

$$M(k):\ x_i^k \leq x_i^{k+1} \leq y_i^{k+1} \leq y_i^k,\ i = 1,\ldots,n.$$

To prove $M(0)$, let us start by using (8) and (10)

$$x_i^0 = u < f_i^{-1}\left(\sum_{j=1}^n a_{ij}u + p_i\right) = f_i^{-1}\left(\sum_{j=1}^n a_{ij}x_j^0 + p_i\right) = x_i^1.$$

Moreover,

$$\begin{aligned} x_i^1 &= f_i^{-1}\left(\sum_{j=1}^n a_{ij}x_j^0 + p_i\right) = f_i^{-1}\left(\sum_{j=1}^n a_{ij}u + p_i\right) \\ &\leq f_i^{-1}\left(\sum_{j=1}^n a_{ij}v + p_i\right) = f_i^{-1}\left(\sum_{j=1}^n a_{ij}y_j^0 + p_i\right) = y_i^1. \end{aligned}$$

Finally, $y_i^1 = f_i^{-1}\left(\sum_{j=1}^n a_{ij}y_j^0 + p_i\right) = f_i^{-1}\left(\sum_{j=1}^n a_{ij}v + p_i\right)$. Using (9), we obtain $y_i^1 \leq v = y_i^0$, and $M(0)$ is proved.

Now, suppose that $M(k)$ is true. Then,

$$\begin{aligned} x_i^{k+1} &= f_i^{-1}\left(\sum_{j=1}^n a_{ij}x_j^k + p_i\right) \leq f_i^{-1}\left(\sum_{j=1}^n a_{ij}x_j^{k+1} + p_i\right) = x_i^{k+2}, \\ x_i^{k+2} &= f_i^{-1}\left(\sum_{j=1}^n a_{ij}x_j^{k+1} + p_i\right) \leq f_i^{-1}\left(\sum_{j=1}^n a_{ij}y_j^{k+1} + p_i\right) = y_i^{k+2}, \\ y_i^{k+2} &= f_i^{-1}\left(\sum_{j=1}^n a_{ij}y_j^{k+1} + p_i\right) \leq f_i^{-1}\left(\sum_{j=1}^n a_{ij}y_j^k + p_i\right) = y_i^{k+1}. \end{aligned}$$

So, for each $i = 1,\ldots,n$, $(x_i^k)_{k\geq 0}$ is increasing, $(y_i^k)_{k\geq 0}$ is decreasing, and $x_i^k \leq y_i^k$. Let $x_i := \lim_{k\to\infty} x_i^k$, $y_i := \lim_{k\to\infty} y_i^k$. Then, $(x_1,\ldots,x_n)$ and $(y_1,\ldots,y_n)$ are positive solutions to (S_0). However, according to Theorem 5, (S_0) has a unique positive solution z^*, and we conclude that $z^* = \lim_{k\to\infty} x^k = \lim_{k\to\infty} y^k$. □

Remark 2. *The system* (S_0) *is described with the functions* f_i*, but the proofs and the iterative methods are formulated in terms of the inverse functions* f_i^{-1}*. In some places,* $f_i(x) = xg_i(x)$*.*

We close this section with some examples of continuous, strictly increasing, surjective functions $g : (0,\infty) \to (0,\infty)$ *such that* $f(x) = xg(x)$ *can be explicitly inverted.*

Example 3. *Let* $q \geq 1$, $c \geq 0$, $g(x) = x^{2q-1} + cx^{q-1}$. *Then* $f^{-1}(x) = \left(\frac{\sqrt{4x+c^2}-c}{2}\right)^{1/q}$, $x \geq 0$.

Example 4. *If* $q \geq 1$ *and* $g(x) = \frac{1}{x}(\exp(x^q) - 1)$, *then* $f^{-1}(x) = (\log(1+x))^{1/q}$, $x \geq 0$.

Example 5. *Let* $q > 1$, $c \geq 1$, $g(x) = \frac{(1+x^c)^q - 1}{x}$. *Then* $f^{-1}(x) = \left((1+x)^{1/q} - 1\right)^{1/c}$, $x \geq 0$.

3. Applications of Brouwer's Fixed Point Theorem

The above Theorem 5 was proved by using Brouwer's fixed point theorem. The same classical result was used in proving

Theorem 7 (Theorem 2 of [2])**.** *Suppose that* $a_{ij} > 0$, $p_i \geq 0$, $i, j = 1, \ldots, n$. *Let* $f_i(t) = \frac{h_i(t)}{t}$, $t > 0$, *where* $h_i : (0, \infty) \to (0, \infty)$ *is strictly decreasing and continuous. Assume that for each* $i = 1, \ldots, n$, *there exists* $t_i > 0$ *such that* $h_i(t_i) = s_i$. *Then,* (S_0) *has a unique positive solution.*

To prove the next result, we again use Brouwer's fixed point theorem.

Theorem 8. *Let* $M_i > 0$, $\varphi_i : [0, \infty) \to [0, \infty)$ *continuous and bounded by* M_i, $a_{ij} \geq 0$, $q_i \geq 0$, $i, j = 1, \ldots, n$. *Then,*

$$(S_1) \begin{cases} x_1 = \varphi_1(a_{11}x_1 + \cdots + a_{1n}x_n + q_1), \\ \cdots \\ x_n = \varphi_n(a_{n1}x_1 + \cdots + a_{nn}x_n + q_n), \end{cases}$$

has a solution in $[0, M_1] \times \cdots \times [0, M_n]$.

Proof. Let $x = (x_1, \ldots, x_n)$ and

$$F(x) = (\varphi_1(a_{11}x_1 + a_{1n}x_n + q_1), \ldots, \varphi_n(a_{n1}x_1 + \cdots + a_{nn}x_n + q_n)),$$

where $x \in [0, M_1] \times \cdots \times [0, M_n]$.

The function $F : [0, M_1] \times \cdots \times [0, M_n] \to [0, M_1] \times \cdots \times [0, M_n]$ is continuous on the compact and convex set $[0, M_1] \times \cdots \times [0, M_n]$. According to Brouwer's fixed point theorem, it has a fixed point, which is a solution to (S_1). □

Let us return to the system (S_0) and take $f_i(t) = t^{c_i}$, $t > 0$. If $c_i > 1$, we can apply Theorem 5 or Theorem 6. If $c_i < -1$, Theorem 3 can be applied. If $0 < c_i \leq 1$, Theorems 1–4 could be useful.

When $-1 \leq c_i < 0$, we are dealing with the system

$$(S_2) \begin{cases} x_1 = (a_{11}x_1 + \cdots + a_{1n}x_n + p_1)^{1/c_1}, \\ \cdots \\ x_n = (a_{n1}x_1 + \cdots + a_{nn}x_n + p_n)^{1/c_n}. \end{cases}$$

Corollary 1. *Let* $-1 \leq c_i < 0$, $a_{ij} \geq 0$, $p_i > 0$, $i, j = 1, \ldots, n$. *Then* (S_2) *has a solution in* $\prod_{i=1}^n [0, p_i^{1/c_i}]$.

Proof. In Theorem 8, choose $q_i = 0$ and $\varphi_i(t) = (t + p_i)^{1/c_i}$, $t \geq 0$. Then, $\varphi_i : [0, +\infty) \to [0, +\infty)$ is continuous and bounded by $M_i = p_i^{1/c_i}$. With these choices, (S_1) becomes (S_2) and has a solution in $\prod_{i=1}^n [0, p_i^{1/c_i}]$. □

Example 6. *Let $a, b, c, d > 0$, $p, q \geq 0$, $\varphi, \psi : [0, \infty) \to [0, \infty)$ strictly decreasing. The system*

$$\begin{cases} x = \varphi(ax + by + p), \\ y = \psi(cx + dy + q), \end{cases}$$

is of the form (S_1). It is equivalent to

$$\begin{cases} y = \dfrac{1}{b}(\varphi^{-1}(x) - ax - p), \\ x = \dfrac{1}{c}(\psi^{-1}(y) - dy - q). \end{cases}$$

It is easy to check that these two curves intersect exactly once in the first quadrant; hence, our system has exactly one positive solution.

4. Aplications of Krasnoselskii's Fixed Point Theorem

The Krasnoselskii's fixed point theorem on the compression and expansion of a cone can be employed to prove the existence of one or two positive solutions. In [28], the authors use the Krasnoselskii's fixed point theorem to prove the existence of solutions for a system of nonlinear differential equations defined on the graph representation of the ethane. In fact, this is an application of such a fixed-point theorem in the context of an applied boundary value problem. Similarly, there is another application of the mentioned theorem to another boundary value problem given in [29], where the existence results are proved for a nonlinear quantum integro-difference boundary value problem.

The version of the Krasnoselskii's theorem in conical shells is used in [33,34]. Moreover, with a clever combination of ideas, the authors of these papers derive results concerning the non-existence of solutions. Here, we use two results from [33,34] in order to complement Theorems 1 and 5.

Remember that under the hypotheses of Theorem 1, (S_0) has a non-negative solution, and under those of Theorem 5, (S_0) has exactly one positive solution. An important condition in Theorem 5 is that each function $f_i(t)/t$ is strictly increasing on $(0, \infty)$. We show that without this condition, the uniqueness of the positive solution can be lost. We present also an example where (S_0) has no positive solution.

Let $\alpha_i > 1$, $a_{ij} > 0$, $i, j = 1, \ldots, n$, $\lambda > 0$. Consider the system

$$(S_3) \quad \begin{cases} f_1(x_1) = \lambda(a_{11}x_1 + \cdots + a_{1n}x_n), \\ \cdots \\ f_n(x_n) = \lambda(a_{n1}x_1 + \cdots + a_{nn}x_n), \end{cases}$$

where $f_i(t) = (e^t - 1)^{1/\alpha_i},\ t \geq 0$.

It is of the form (S_0). Setting $\varphi_i(t) := f_i^{-1}(t) = \log(1 + t^{\alpha_i})$, $t \geq 0$, $i = 1, \ldots, n$, (S_3) can be written as

$$(S_3') \quad \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \varphi(x_1) \\ \cdots \\ \varphi(x_n) \end{pmatrix}.$$

Systems of this form are investigated in [34,35].

Proposition 1. *(i) There exists a $\lambda_0 > 0$ such that for each $\lambda > \lambda_0$, (S_3') has two positive solutions.*
(ii) There exists a $\lambda_1 > 0$ such that for each $0 < \lambda < \lambda_1$, (S_3') has no positive solutions.

Proof. Let us remark that

$$\lim_{t \to 0} \frac{\varphi_i(t)}{t} = 0, \ \lim_{t \to \infty} \frac{\varphi_i(t)}{t} = 0, \text{ and } \varphi_i(t) > 0 \text{ if } t > 0,\ i = 1, \ldots, n.$$

Now, (i) is a consequence of Theorem 4.1 in [33] (which gives sufficient conditions for the existence of two positive solutions) and (ii) a consequence of Theorem 4.7 in [33] (which gives sufficient conditions under which the system has no positive solution); see also Example 2 in [34]. □

5. Conclusions and Further Work

In many application-oriented problems, an important step is represented by the study of an algebraic system with positive coefficients. In such a case, only positive solutions are of interest. Therefore, a considerable number of papers have been devoted to such systems. Fixed point theorems and iterative methods are useful tools. In our paper, we obtain new results related to several classes of systems. We relax some hypotheses used in the literature concerning the strict monotonicity of the involved functions. We show that in some cases, the unique positive solution can be obtained by a monotone increasing iterative method and/or by a monotone decreasing iterative method. As a consequence of one of our results (Theorem 3), we recover the existence of a non-negative solution of the Leontief system and describe a monotone iterative method to find it. In Corollary 1, we fill in a gap in a series of results from the literature concerning a special system. Examples of positive systems having no positive solution or having two positive solutions are provided in Section 4 with Krasnoselskii's fixed point theorem as the main tool. Some applications of this theorem are also mentioned.

The above-mentioned results can be starting points for new investigations. We presented iterative methods to get a solution, and we will be interested in estimating the rate of convergence to the solution. Usually, there are several iterative methods, and a comparison of their speeds of convergence will be one significant direction of study. Our efforts will be also directed toward new applications.

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