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Gradient and Parameter Dependent Dirichlet $(p(x), q(x))$-Laplace Type Problem

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Abstract: We analyze a Dirichlet $(p(x), \mu q(x))$-Laplace problem. For a gradient dependent non-linearity of Carathéodory type, we discuss the existence, uniqueness and asymptotic behavior of weak solutions, as the parameter μ varies on the non-negative real axis. The results are obtained by applying the properties of pseudomonotone operators, jointly with certain a priori estimates.

Keywords: Lebesgue and Sobolev spaces with variable exponents; parametric problems; gradient dependent term; Nemitsky map; pseudomonotone operator.

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1. Introduction

We study a inhomogeneous equation with Dirichlet boundary condition of the form

$$-\Delta_{p(x)}u(x) - \mu\Delta_{q(x)}u(x) = f(x, u(x), \nabla u(x)) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \tag{1}$$

on a bounded domain $\Omega \subseteq \mathbb{R}^N$, with smooth boundary $\partial\Omega$. On the left-hand side, we find the sum of two $m(x)$-Laplace differential operators with $m \in C(\overline{\Omega})$, whose combined effects are related to the values of a non-negative real number μ. In details, we recall that the notation $\Delta_{m(x)}$ corresponds to the following largely investigated operator

$$\Delta_{m(x)}u = \operatorname{div}(|\nabla u|^{m(x)-2}\nabla u) \quad \text{for all } u \in W_0^{1,m(x)}(\Omega),$$

where

$$1 < m^- := \min_{x\in\overline{\Omega}} m(x) \le m(x) \le m^+ := \max_{x\in\overline{\Omega}} m(x) < +\infty.$$

The $m(x)$-Laplace equation $-\Delta_{m(x)}u = f$ arises naturally in the analysis of nonlinear phenomena of physical interest, as in the study of rheological fluids and elasticity of materials. For pure mathematicians, the interest for this equation originates from the (Dirichlet) variational integral

$$I(u) = \int_\Omega |\nabla u(x)|^{m(x)}dx, \quad 1 < m(x) < +\infty.$$

Indeed, this variational integral is related to the total energy of the equation and its manipulation leads to the proper definition of a weak solution to the same equation. This is done according to John Ball's total energy theorem, and a clear introduction to these arguments is the monography of Lindqvist [1]. Turning to the right-hand side of Equation (1), we find a gradient-dependent function, whose regularity and growth conditions will be given in Section 2 (see assumptions A_1, A_2) and in Section 5 (see assumptions A_3, A_4, A_5).

We point out that the presence of the gradient-dependence is crucial in the choice of a working strategy, as it inhibits the use of variational methods. Consequently, we establish our results by using the properties of pseudomonotone operators.

Briefly, we give some comments over the existing literature. A special form of the m-Laplace equation in the case $m(x) = m =$ constant was given attention by de Figueiredo–Girardi–Matzeu [2], Fragnelli–Papageorgiou–Mugnai [3] and Ruiz [4]. These papers deal respectively with mountain-pass techniques [2], the Leray–Schauder alternative principle [3], the blow-up argument and a Liouville-type theorem to obtain a priori estimates [4].

Later, Equation $-\Delta_{p(x)}u - \Delta_{q(x)}u = f(x, u, \nabla u)$ was studied by Faria–Miyagaki–Motreanu [5] and Papageorgiou–Vetro–Vetro [6] (special case with both p and q constant exponents), Liu–Papageorgiou [7] (where f is also resonant) and Gasiński–Winkert [8] (double phase operator). These papers use respectively a comparison principle together with an approximation reasoning [5], Leray–Schauder principle and method of freezing variables [7], and surjectivity results of suitable operators [8].

A feature of Equation (1) is the presence of a parameter μ acting on the $q(x)$-Laplace differential operator. In the case $\mu = 0$, (1) reduces to the $p(x)$-Laplace equation, as it is given in Wang–Hou–Ge [9] (existence and uniqueness of weak solution). Similarly, Vetro [10] deals with the case $\mu = 0$, but in the presence of a Kirchhoff term weighting the $p(x)$-Laplace differential operator (both degenerate and non-degenerate Kirchhoff type problems are considered). Dealing with the case $\mu \neq 0$, we will analyze the asymptotic behavior of weak solutions to (1). The results are obtained working in the context of the variable exponent Lebesgue space $L^{m(x)}(\Omega)$ and the variable exponent Sobolev spaces $W^{1,m(x)}(\Omega)$, $W_0^{1,m(x)}(\Omega)$ (where $W_0^{1,p(x)}(\Omega)$ is the $W^{1,p(x)}$-norm closure of $C_0^\infty(\Omega)$). The required notions and notation are given in Section 2, but the readers can consult the books by Diening–Harjulehto–Hästö–Růžička [11] and by Rădulescu–Repovš [12], for details. A discussion about the uniqueness of weak solution will conclude the work herein, using certain additional assumptions on the nonlinearity. For additional problems involving different $p(x)$-Laplace type differential operators, we suggest the works by Ekincioglu and co-workers [13–17]. Finally, we mention the recent work by Bahrouni-Repovš [18] dealing with the existence and the nonexistence of solutions for a new class of $p(x)$-curl systems arising in electromagnetism.

2. Functional Framework

We give some notions involving a reflexive Banach space $(X, \|\cdot\|)$ with topological dual X^*. By $\langle\cdot,\cdot\rangle$, we mean the duality brackets of (X^*, X). According to Gasiński–Papageorgiou [19], we recall the following concept and lemmas of a generalized pseudomonotone operator.

Definition 1. *For a generalized pseudomonotone operator, we mean an operator $T : X \to X^*$ such that, for every $\{x_n\}_{n\in\mathbb{N}} \subseteq X$, with*

$$\begin{aligned} &x_n \xrightarrow{w} x \text{ in } X, \text{ for some } x \in X, \\ &T(x_n) \xrightarrow{w} x^* \text{ in } X^*, \text{ for some } x^* \in X^*, \\ &\limsup_{n\to+\infty} \langle T(x_n), x_n - x\rangle \le 0, \end{aligned}$$

we obtain

$$\begin{aligned} &x^* = T(x), \\ &\langle T(x_n), x_n\rangle \to \langle T(x), x\rangle. \end{aligned}$$

Lemma 1. *Each bounded generalized pseudomonotone operator $T : X \to X^*$ is also pseudomonotone.*

We recall that an operator $T : X \to X^*$ is strongly coercive if $\dfrac{\langle T(u), u\rangle}{\|u\|}$ goes to $+\infty$, as $\|u\|$ goes to $+\infty$ too. This property leads to the following surjectivity lemma.

Lemma 2. *Each pseudomonotone, bounded and strongly coercive operator $T : X \to X^*$ is surjective (hence range$(T) = X^*$).*

We also recall the following Lemma 2.2.27, p. 141, of Gasiński–Papageorgiou [19].

Lemma 3. *Given two Banach spaces X and Y with $X \subseteq Y$, we have that:*

(a) if the embedding is continuous and X is dense in Y, then the embedding $Y^* \subseteq X^*$ is continuous;

(b) in addition to (a), if X is reflexive, then Y^* is dense in X^*.

Now, we focus on the Lebesgue space $L^{m(x)}(\Omega)$ and the Sobolev space $W^{1,m(x)}(\Omega)$, where the study of Equation (1) will be developed. Precisely, we consider

$$L^{m(x)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ such that } u \text{ is measurable with } \int_\Omega |u(x)|^{m(x)} dx < +\infty\right\},$$

with norm

$$\|u\|_{L^{m(x)}(\Omega)} := \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u}{\lambda}\right|^{m(x)} dx \le 1\right\},$$

and

$$W^{1,m(x)}(\Omega) := \{u \in L^{m(x)}(\Omega) : |\nabla u| \in L^{m(x)}(\Omega)\},$$

with norm

$$\|u\|_{W^{1,m(x)}(\Omega)} = \|u\|_{L^{m(x)}(\Omega)} + \|\nabla u\|_{L^{m(x)}(\Omega)}$$
$$(\text{recall } \|\nabla u\|_{L^{m(x)}(\Omega)} = \||\nabla u|\|_{L^{m(x)}(\Omega)}).$$

About these norms, from [11], we know that

$$\|u\|_{L^{m(x)}(\Omega)} \le c_1 \|\nabla u\|_{L^{m(x)}(\Omega)} \quad \text{for all } u \in W_0^{1,m(x)}(\Omega), \text{ some } c_1 > 0.$$

This means that there is equivalence between $\|u\|_{W^{1,m(x)}(\Omega)}$ and $\|\nabla u\|_{L^{m(x)}(\Omega)}$ on $W_0^{1,m(x)}(\Omega)$. Consequently, $\|\nabla u\|_{L^{m(x)}(\Omega)}$ can be used in place of $\|u\|_{W^{1,m(x)}(\Omega)}$, and $\|u\| = \|\nabla u\|_{L^{m(x)}(\Omega)}$ in $W_0^{1,m(x)}(\Omega)$.

Let $L^{m'(x)}(\Omega)$ denote the conjugate space of $L^{m(x)}(\Omega)$, where $\dfrac{1}{m(x)} + \dfrac{1}{m'(x)} = 1$. For any $u \in L^{m(x)}(\Omega)$ and $v \in L^{m'(x)}(\Omega)$, the Hölder type inequality

$$\int_\Omega uv dx \le 2\|u\|_{L^{m(x)}(\Omega)} \|v\|_{L^{m'(x)}(\Omega)}$$

holds true.

Fan–Zhao [20] gives us that $L^{m(x)}(\Omega)$, $W^{1,m(x)}(\Omega)$ and $W_0^{1,m(x)}(\Omega)$, equipped with these norms, are separable, reflexive, and uniformly convex Banach spaces. In the same paper [20], some Sobolev embedding results are given. We recall them in the following lemma.

Lemma 4. *Let $m_1, m_2 \in C(\overline{\Omega})$ be such that $m_1(x), m_2(x) > 1$ for all $x \in \overline{\Omega}$. Then, we have:*

(a) $W_0^{1,m_1(x)}(\Omega) \hookrightarrow L^{m_2(x)}(\Omega)$ is compact, provided that $m_2(x) < m_1^*(x)$ for all $x \in \overline{\Omega}$, where $m_1^*(x) = \frac{Nm_1(x)}{N-m_1(x)}$ if $m_1(x) < N$ or $m_1^*(x) = +\infty$ if $m_1(x) \ge N$;

(b) $L^{m_1(z)}(\Omega) \hookrightarrow L^{m_2(x)}(\Omega)$ is continuous, provided that $m_2(x) \le m_1(x)$ for all $x \in \overline{\Omega}$;

(c) $W_0^{1,m_1(x)}(\Omega) \hookrightarrow W_0^{m_2(x)}(\Omega)$ *is continuous, provided that* $m_2(x) \le m_1(x)$ *for all* $x \in \overline{\Omega}$

Another significant result for our analysis is the following theorem of [20].

Theorem 1. *Let* $u \in L^{m(x)}(\Omega)$ *and* $\rho_m(u) := \int_\Omega |u(x)|^{m(x)} dx$. *Then, the following relations hold:*

(a) $\|u\|_{L^{m(x)}(\Omega)} < 1\ (=1,\ >1) \Leftrightarrow \rho_m(u) < 1\ (=1,\ >1)$;

(b) *if* $\|u\|_{L^{m(x)}(\Omega)} > 1$, *then* $\|u\|^{m^-}_{L^{m(x)}(\Omega)} \le \rho_m(u) \le \|u\|^{m^+}_{L^{m(x)}(\Omega)}$;

(c) *if* $\|u\|_{L^{m(x)}(\Omega)} < 1$, *then* $\|u\|^{m^+}_{L^{r(x)}(\Omega)} \le \rho_m(u) \le \|u\|^{m^-}_{L^{m(x)}(\Omega)}$.

Remark 1. *The inequalities in Theorem 1 can be used to obtain some a priori estimates. For further use, starting from*

$$\|u\|^{m^-}_{L^{m(x)}(\Omega)} - 1 \le \rho_{m(x)}(u) \le \|u\|^{m^+}_{L^{m(x)}(\Omega)} + 1, \tag{2}$$

we can deduce that, if $u \in L^{m(x)}(\Omega)$, *then* $|u|^{m(x)-1} \in L^{m'(x)}(\Omega)$ *and*

$$\||u|^{m(x)-1}\|_{L^{m'(x)}(\Omega)} \le 2 + \|u\|^{m^+}_{L^{m(x)}(\Omega)}. \tag{3}$$

Precisely, we observe that

$$\begin{aligned}
\||u|^{m(x)-1}\|^{(m')^-}_{L^{m'(x)}(\Omega)} &\le 1 + \int_\Omega (|u|^{m(x)-1})^{\frac{m(x)}{m(x)-1}} dx \quad \textit{(by (2))} \\
&= 1 + \int_\Omega |u|^{m(x)} dx \\
&\le 1 + 1 + \|u\|^{m^+}_{L^{m(x)}(\Omega)},
\end{aligned}$$

which establishes (3). Following a similar argument, one can derive the inequality

$$\||\nabla u|^{\frac{p(x)}{\alpha'(x)}}\|_{L^{\alpha'(x)}(\Omega)} \le 2 + \|\nabla u\|^{p^+}_{L^{p(x)}(\Omega)}, \quad \alpha \in C(\overline{\Omega}) \textit{ with } \alpha(x) > 1 \textit{ for all } x \in \overline{\Omega}. \tag{4}$$

We will work with the integral operator

$$\langle T_m(u), h\rangle = \int_\Omega |\nabla u|^{m(x)-2} (\nabla u, \nabla h)_{\mathbb{R}^N} dx \quad \text{for all } u, h \in W_0^{1,m(x)}(\Omega),$$

with $range(T_m) = W^{-1,m'(x)}(\Omega) = W^{1,m(x)}(\Omega)^*$ and possessing the following features:

(i) boundedness, that is, T_m maps bounded sets to bounded sets;

(ii) continuity;

(iii) monotonicity, and hence maximal monotonicity;

(iv) $(S)_+$-property, that is, if $u_n \xrightarrow{w} u$ in $W_0^{1,m(x)}(\Omega)$ and $\limsup_{n\to+\infty} \langle T_m(u_n), u_n - u\rangle \le 0$, then $u_n \to u$ in $W_0^{1,m(x)}(\Omega)$.

Since we know that there is absence of homogeneity in T_m, we will impose the following assumptions for the exponents:

Let $\alpha \in C(\overline{\Omega})$ be such that

$$1 < q(x) \le \min_{x\in\overline{\Omega}} \alpha(x) \le \max_{x\in\overline{\Omega}} \alpha(x) \le p(x) \quad \text{for all } x \in \overline{\Omega}, \tag{5}$$

and

$$\max_{x\in\overline{\Omega}} q(x) = q^+ < p^- = \min_{x\in\overline{\Omega}} p(x). \tag{6}$$

Therefore, the inequality (5) leads to

$$\lambda_1 := \inf_{u \in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\nabla u|^{p(x)} + |\nabla u|^{q(x)})dx}{\int_\Omega |u|^{\alpha(x)}dx} > 0. \tag{7}$$

Observe that, for any $x \in \overline{\Omega}$, we have $p(x) \geq \alpha^+ \geq \alpha(x) \geq \alpha^- \geq q(x)$. Thus, we deduce that, for all $u \in W_0^{1,p(x)}(\Omega)$, the following inequalities hold

$$|\nabla u(x)|^{\alpha^+} + |\nabla u(x)|^{\alpha^-} \leq 2(|\nabla u(x)|^{p(x)} + |\nabla u(x)|^{q(x)}),$$

and

$$|u(x)|^{\alpha(x)} \leq |u(x)|^{\alpha^+} + |u(x)|^{\alpha^-}.$$

Integrating the above inequalities, we find

$$\int_\Omega (|\nabla u|^{\alpha^+} + |\nabla u|^{\alpha^-})dx \leq 2\int_\Omega (|\nabla u|^{p(x)} + |\nabla u|^{q(x)})dx \tag{8}$$

for all $u \in W_0^{1,p(x)}(\Omega)$, and

$$\int_\Omega |u|^{\alpha(x)}dx \leq \int_\Omega (|u|^{\alpha^+} + |u|^{\alpha^-})dx \quad \text{for all } u \in W_0^{1,p(x)}(\Omega). \tag{9}$$

By Sobolev embeddings, there exist positive constants C_{α^+} and C_{α^-} such that

$$C_{\alpha^+}\int_\Omega |u|^{\alpha^+}dx \leq \int_\Omega |\nabla u|^{\alpha^+}dx \quad \text{for all } u \in W_0^{1,\alpha^+}(\Omega), \tag{10}$$

and

$$C_{\alpha^-}\int_\Omega |u|^{\alpha^-}dx \leq \int_\Omega |\nabla u|^{\alpha^-}dx \quad \text{for all } u \in W_0^{1,\alpha^-}(\Omega). \tag{11}$$

Using again the fact that $\alpha^- \leq \alpha^+ \leq p(x)$ for any $x \in \overline{\Omega}$, we deduce that $W_0^{1,p(x)}(\Omega)$ is continuously embedded in $W_0^{1,\alpha^+}(\Omega)$ and in $W_0^{1,\alpha^-}(\Omega)$. Thus, inequalities (10) and (11) hold true for any $u \in W_0^{1,p(x)}(\Omega)$. Using inequalities (9)–(11), it is clear that there exists a positive constant λ such that

$$\lambda\int_\Omega |u|^{\alpha(x)}dx \leq \int_\Omega (|\nabla u|^{\alpha^+} + |\nabla u|^{\alpha^-})dx \quad \text{for all } u \in W_0^{1,p(x)}(\Omega). \tag{12}$$

Next, inequalities (8) and (12) yield

$$\lambda\int_\Omega |u|^{\alpha(x)}dx \leq 2\int_\Omega (|\nabla u|^{p(x)} + |\nabla u|^{q(x)})dx \quad \text{for all } u \in W_0^{1,p(x)}(\Omega).$$

This establishes (7).

Before stating the assumptions on the nonlinearity, we recall that a function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is said to be "Carathéodory" provided that:

(i) for all $(z, y) \in \mathbb{R} \times \mathbb{R}^N$, $x \to f(x, z, y)$ is measurable;
(ii) for almost all $x \in \Omega$, $(z, y) \to f(x, z, y)$ is continuous.

Therefore, f is jointly measurable (see Hu-Papageorgiou [21], p. 142). We impose the following assumptions on the Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$:

A_1 : there exist $\sigma \in L^{\alpha'(x)}(\Omega)$, $\alpha \in C(\overline{\Omega})$ satisfying (5) and $c > 0$ such that

$$|f(x, z, y)| \leq c(\sigma(x) + |z|^{\alpha(x)-1} + |y|^{\frac{p(x)}{\alpha'(x)}}) \quad \text{for a.a. } x \in \Omega, \text{ all } z \in \mathbb{R}, \text{ all } y \in \mathbb{R}^N;$$

$A_2:$ there exist $a_0 \in L^1(\Omega)$ and $b_1, b_2 \geq 0$ with $b_1\lambda_1^{-1} + b_2 < 1$ such that

$$f(x,z,y)z \leq a_0(x) + b_1|z|^{\alpha(x)} + b_2|y|^{p(x)} \quad \text{for a.a. } x \in \Omega, \text{ all } z \in \mathbb{R}, \text{ all } y \in \mathbb{R}^N.$$

We note that the interest for equations subject to $m(x)$-growth conditions (and hence the significance of assumptions as A_1 and A_2) is supported by their applications. For instance, there are fluids that start flowing only after a certain threshold/strength is overcome, but the same fluids freeze as soon as the forcing factor leaves (that is, the typical behavior of certain oil paints (Bingham fluids)). The study of these phenomena requires variable exponents spaces and variable exponents growth conditions (see again [11,12]).

Example 1. *A nonlinearity satisfying the assumptions A_1 and A_2 is obtained combining two power terms in the form*

$$f(x,z,y) = -b_1|z|^{\alpha(x)-2}z + b_2|y|^{\frac{p(x)}{\alpha'(x)}} \quad \textit{for a.a. } x \in \Omega, \textit{ all } z \in \mathbb{R}, \textit{ all } y \in \mathbb{R}^N.$$

Here, $b_1, b_2 \geq 0$ satisfy the inequality $(b_1 + \frac{b_2}{\alpha^-})\lambda_1^{-1} + \frac{b_2}{(\alpha')^-}) < 1$.

To check A_1 and A_2, we recall that both

$$\begin{aligned}|f(x,z,y)| &= |-b_1|z|^{\alpha(x)-2}z + b_2|y|^{\frac{p(x)}{\alpha'(x)}}| \\ &\leq b_1|z|^{\alpha(x)-1} + b_2|y|^{\frac{p(x)}{\alpha'(x)}},\end{aligned}$$

and

$$\begin{aligned}f(x,z,y)z &= [-b_1|z|^{\alpha(x)-2}z + b_2|y|^{\frac{p(x)}{\alpha'(x)}}]z \\ &\leq b_1|z|^{\alpha(x)} + b_2|y|^{\frac{p(x)}{\alpha'(x)}}|z| \\ &\leq (b_1 + \frac{b_2}{\alpha^-})|z|^{\alpha(x)} + \frac{b_2}{(\alpha')^-}|y|^{p(x)},\end{aligned}$$

hold for a.a. $x \in \Omega$, all $z \in \mathbb{R}$, all $y \in \mathbb{R}^N$.

3. Existence and Asymptotic Results

Before establishing the existence of a weak solution to (1), we define the Nemitsky map $N_f^* : W_0^{1,p(x)}(\Omega) \subset L^{\alpha(x)}(\Omega) \to L^{\alpha'(x)}(\Omega)$ associated with the nonlinearity. Precisely, we have

$$N_f^*(u)(\cdot) = f(\cdot, u(\cdot), \nabla u(\cdot)) \quad \text{for all } u \in W_0^{1,p(x)}(\Omega).$$

Such map possesses some regularities. Indeed, referring to the work of Galewski [22], A_1 ensures the boundedness and continuity of $N_f^*(\cdot)$.

With respect to the embedding $i^* : L^{\alpha'(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$, we deduce by Lemma 3 that i^* is continuous. This fact leads to the boundedness and continuity of the operator $N_f : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ given as $N_f = i^* \circ N_f^*$.

Now, we say in which sense the solutions to (1) are considered here. By Lemma 4, a solution will be sought in the variable exponent space $W_0^{1,p(x)}(\Omega)$. Precisely, $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of Equation (1) if

$$\langle T_p(u), h\rangle + \mu\langle T_q(u), h\rangle = \int_\Omega f(x,u,\nabla u)h dx \quad \text{for all } h \in W_0^{1,p(x)}(\Omega). \tag{13}$$

These notions will be used to construct the following result, along with the theory of pseudomonotone operators.

Theorem 2. *Assume that A_1, A_2 and (6) are satisfied, then Equation (1) has at least one weak solution for all $\mu \geq 0$.*

Proof. Let $\mu \geq 0$ be fixed. We consider the operator $T : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ given as

$$T(u) = T_p(u) + \mu T_q(u) - N_f(u) \quad \text{for all } u \in W_0^{1,p(x)}(\Omega).$$

This operator possesses some regularities. Indeed, boundedness and continuity can be deduced easily by definition. Thus, we focus on the pseudo-monotonicity of $T(\cdot)$. We observe that $T(\cdot)$ is everywhere defined and bounded, and hence, with respect to ([19], Proposition 3.2.49), we remain to prove that $T(\cdot)$ is generalized pseudomonotone. Thus, we assume it satisfies the hypotheses $u_n \xrightarrow{w} u$ in $W_0^{1,p(x)}(\Omega)$, $T(u_n) \xrightarrow{w} u^*$ in $W^{-1,p'(x)}(\Omega)$ and

$$\limsup_{n\to+\infty} \langle T(u_n), u_n - u\rangle \leq 0. \tag{14}$$

From (14), we have

$$\limsup_{n\to+\infty} [\langle T_p(u_n), u_n - u\rangle + \mu\langle T_q(u_n), u_n - u\rangle - \int_\Omega f(x, u_n, \nabla u_n)(u_n - u)dx] \leq 0. \tag{15}$$

Now, assumption A_1 leads to the following estimate

$$\begin{aligned}
&\left|\int_\Omega f(x, u_n, \nabla u_n)(u_n - u)dx\right| \leq c\int_\Omega [|\sigma(x)| + |u_n|^{\alpha(x)-1} + |\nabla u_n|^{\frac{p(x)}{\alpha'(x)}}]\,|u_n - u|dx \\
&\leq 2c\|u_n - u\|_{L^{\alpha(x)}(\Omega)}\left[\|\sigma\|_{L^{\alpha'(x)}(\Omega)} + \left\||u_n|^{\alpha(x)-1}\right\|_{L^{\alpha'(x)}} + \left\||\nabla u_n|^{\frac{p(x)}{\alpha'(x)}}\right\|_{L^{\alpha'(x)}(\Omega)}\right] \\
&\qquad\qquad \text{(by Hölder inequality)} \\
&\leq 2c\|u_n - u\|_{L^{\alpha(x)}(\Omega)}\left[\|\sigma\|_{L^{\alpha'(x)}(\Omega)} + 2 + \|u_n\|^{\alpha^+}_{L^{\alpha(x)}(\Omega)} + 2 + \|\nabla u_n\|^{p^+}_{p(x)}\right] \\
&\qquad\qquad \text{(by (3) and (4))}.
\end{aligned} \tag{16}$$

The importance of this estimate lays in the fact that, along with the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $W_0^{1,p(x)}(\Omega)$ and the convergence $u_n \to u$ in $L^{\alpha(x)}(\Omega)$, we obtain

$$\int_\Omega f(x, u_n, \nabla u_n)(u_n - u)dx \to 0 \quad \text{as } n \to +\infty.$$

On the other side, (15) leads to

$$\begin{aligned}
&\limsup_{n\to+\infty}\big[\langle T_p(u_n), u_n - u\rangle + \mu\langle T_q(u_n), u_n - u\rangle\big] \leq 0, \\
\Rightarrow\ &\limsup_{n\to+\infty}\big[\langle T_p(u_n), u_n - u\rangle + \mu\langle T_q(u), u_n - u\rangle\big] \leq 0, \\
&\text{(recall the monotonicity of } T_q(\cdot)), \\
\Rightarrow\ &\limsup_{n\to+\infty}\langle T_p(u_n), u_n - u\rangle \leq 0, \\
\Rightarrow\ &u_n \to u \text{ in } W_0^{1,p(x)}(\Omega) \quad (\text{use the } (S)_+\text{-property of } T_p(\cdot)).
\end{aligned}$$

Now, $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, together with the fact that $T(\cdot)$ is continuous, give us

$$u^* = T(u), \quad \langle T(u_n), u_n\rangle \to \langle T(u), u\rangle.$$

The proof of general pseudo-monotonicity of $T(\cdot)$ is completed, and hence also the pseudo-monotonicity of $T(\cdot)$ is established.

Next, we show the strong coercivity of $T(\cdot)$, using assumption A_2. Precisely, we have

$$\begin{aligned}
\langle T(u), u\rangle &= \int_\Omega |\nabla u|^{p(x)}dx + \mu\int_\Omega |\nabla u|^{q(x)}dx - \int_\Omega f(x,u,\nabla u)u dx \\
&\geq \int_\Omega |\nabla u|^{p(x)}dx + \mu\int_\Omega |\nabla u|^{q(x)}dx - \int_\Omega |a_0(x)|dx - b_1\int_\Omega |u|^{\alpha(x)}dx \\
&\qquad - b_2\int_\Omega |\nabla u|^{p(x)}dx \qquad \text{(see assumption } A_2) \\
&\geq \int_\Omega |\nabla u|^{p(x)}dx + \mu\int_\Omega |\nabla u|^{q(x)}dx - b_1\lambda_1^{-1}\Big(\int_\Omega |\nabla u|^{p(x)}dx + \int_\Omega |\nabla u|^{q(x)}dx\Big) \\
&\qquad - b_2\int_\Omega |\nabla u|^{p(x)}dx - \|a_0\|_{L^1(\Omega)} \\
&= (1 - b_1\lambda_1^{-1} - b_2)\int_\Omega |\nabla u|^{p(x)}dx + (\mu - b_1\lambda_1^{-1})\int_\Omega |\nabla u|^{q(x)}dx - \|a_0\|_{L^1(\Omega)}, \\
\Rightarrow \quad \langle T(u), u\rangle &\geq (1 - b_1\lambda_1^{-1} - b_2)(\|u\|^{p^-} - 1) + (\mu - b_1\lambda_1^{-1})g(\|\nabla u\|) - \|a_0\|_{L^1(\Omega)} \\
&\qquad \text{(by (2))},
\end{aligned}$$

where

$$g(\|\nabla u\|) = \|\nabla u\|^{q^-}_{L^{q(x)}(\Omega)} - 1 \quad \text{if } \mu - b_1\lambda_1^{-1} > 0,$$

and

$$g(\|\nabla u\|) = \|\nabla u\|^{q^+}_{L^{q(x)}(\Omega)} + 1 \quad \text{if } \mu - b_1\lambda_1^{-1} < 0.$$

As $q^+ < p^-$ by (6), we deduce the strong coercivity of $T(\cdot)$.

By Lemma 2, every pseudomonotone strongly coercive operator is surjective. Consequently, there exists $\widehat{u} \in W_0^{1,p(x)}(\Omega)$ such that $T(\widehat{u}) = 0$. We conclude that Equation (1) has at least one weak solution for all $\mu \geq 0$. □

Next, we will analyze the asymptotic behavior of weak solutions to (1). We indicate some of the notations used throughout this section. Let

$$\mathcal{S}_\mu = \text{set of solutions to Equation (1), fixed } \mu \geq 0,$$

$$\mathcal{S} = \cup_{\mu\geq 0}\, \mathcal{S}_\mu = \text{set of solutions to Equation (1).}$$

We observe that these two sets are bounded in $W_0^{1,p(x)}(\Omega)$. We give the proof in the following lemma.

Lemma 5. *Assume that A_1, A_2, and (6) are satisfied, then $\mathcal{S}_\mu$ is a bounded set in $W_0^{1,p(x)}(\Omega)$ for all $\mu \geq 0$. Moreover, $\mathcal{S} = \cup_{\mu\geq 0}\, \mathcal{S}_\mu$ is also bounded in $W_0^{1,p(x)}(\Omega)$.*

Proof. We first establish the boundedness of $\mathcal{S}_\mu$ in $W_0^{1,p(x)}(\Omega)$ for a fixed $\mu \geq 0$. Thus, without loss of generality, we consider a solution to (1), namely $u \in W_0^{1,p(x)}(\Omega)$, such that $\|u\| > 1$. From the definition of weak solution (see (13)), choosing the test function $h = u$, we deduce that

$$
\begin{aligned}
\int_\Omega |\nabla u|^{p(x)}dx &\le \langle T_p(u),u\rangle + \mu\langle T_q(u),u\rangle \\
&= \int_\Omega f(x,u,\nabla u)u dx \\
&\le \int_\Omega (a_0(x) + b_1|u|^{\alpha(x)} + b_2|\nabla u|^{p(x)})dx \quad \text{(see assumption } A_2\text{)} \\
&\le \|a_0\|_{L^1(\Omega)} + b_1\lambda_1^{-1}\Big(\int_\Omega |\nabla u|^{p(x)}dx + \int_\Omega |\nabla u|^{q(x)}dx\Big) \\
&\quad + b_2\int_\Omega |\nabla u|^{p(x)}dx \\
\Rightarrow \quad \int_\Omega |\nabla u|^{p(x)}dx &\le \frac{\|a_0\|_{L^1(\Omega)} + b_1\lambda_1^{-1}\int_\Omega |\nabla u|^{q(x)}dx}{1-b_1\lambda_1^{-1}-b_2}, \\
\Rightarrow \quad \|\nabla u\|^{p^-}_{L^{p(x)}(\Omega)} &\le \frac{\|a_0\|_{L^1(\Omega)} + b_1\lambda_1^{-1}(\|\nabla u\|^{q^+}_{L^{q(x)}(\Omega)}+1)}{1-b_1\lambda_1^{-1}-b_2} + 1 \quad \text{(by (2))}.
\end{aligned} \tag{17}
$$

Since $q^+ < p^-$ by (6) and the continuity of $L^{p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, we conclude that $\mathcal{S}_\mu$ is bounded in $W_0^{1,p(x)}(\Omega)$.

We remain to prove that $\mathcal{S} = \cup_{\mu\ge 0}\mathcal{S}_\mu$ is bounded in $W_0^{1,p(x)}(\Omega)$ too. Observe that (17) is independent from μ, and hence holds for each $u \in \mathcal{S}$. Consequently, $\mathcal{S}$ is bounded in $W_0^{1,p(x)}(\Omega)$. $\square$

Before stating our next lemma, we remark that, throughout this paper, given a sequence $\{u_n\}_{n\in\mathbb{N}}$, we denote every relabeled subsequence again with $\{u_n\}_{n\in\mathbb{N}}$.

The first lemma concerns the behavior of (1) in the case $\mu \to 0^+$.

Lemma 6. *Assume that A_1, A_2 and (6) are satisfied. Given a sequence of parameters $\{\mu_n\}_{n\in\mathbb{N}}$ converging to 0^+, and a sequence $\{u_n\}_{n\in\mathbb{N}}$ of solutions to Equation (1) such that $u_n \in \mathcal{S}_{\mu_n}$ for all $n \in \mathbb{N}$, then there is a relabeled subsequence of $\{u_n\}_{n\in\mathbb{N}}$ such that $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$ with $u \in W_0^{1,p(x)}(\Omega)$ solution to (1).*

Proof. Let $u_n \in \mathcal{S}_{\mu_n}$ for all $n \in \mathbb{N}$. The proof of the boundedness of $\mathcal{S} = \cup_\mu \mathcal{S}_\mu$ in $W_0^{1,p(x)}(\Omega)$ in Lemma 5 gives us that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Thus, we can find a relabeled subsequence of $\{u_n\}_{n\in\mathbb{N}}$ such that $u_n \xrightarrow{w} u$ in $W_0^{1,p(x)}(\Omega)$ and $u_n \to u$ in $L^{\alpha(x)}(\Omega)$, for some $u \in W_0^{1,p(x)}(\Omega)$. By (16), we derive that

$$\int_\Omega f(x,u_n,\nabla u_n)(u_n-u)dx \to 0 \quad \text{as } n \to +\infty,$$

whenever $u_n \to u$ in $L^{\alpha(x)}(\Omega)$ (by assumption A_1). From $u_n \in \mathcal{S}_{\mu_n}$ for all $n \in \mathbb{N}$, we obtain

$$\langle T_p(u_n),h\rangle + \mu_n\langle T_q(u_n),h\rangle = \int_\Omega f(x,u_n,\nabla u_n)h dx \quad \text{for all } h \in W_0^{1,p(x)}(\Omega). \tag{18}$$

Putting $h = u_n - u \in W_0^{1,p(x)}(\Omega)$ in (18), we have

$$\langle T_p(u_n),u_n-u\rangle + \mu_n\langle T_q(u_n),u_n-u\rangle = \int_\Omega f(x,u_n,\nabla u_n)(u_n-u)dx \tag{19}$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to +\infty$ in (19), since $\mu_n \to 0^+$, we deduce that

$$
\begin{aligned}
&\lim_{n\to+\infty}\langle T_p(u_n),u_n-u\rangle = 0, \\
\Rightarrow \quad & u_n \to u \text{ in } W_0^{1,p(x)}(\Omega) \text{ (by } (S)_+\text{-property of } T_p).
\end{aligned}
$$

Assumption A_1 ensures that $N_f : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ defined by $N_f = i^* \circ N_f^*$ is bounded and continuous. It follows that

$$\langle N_f(u_n), h \rangle \to \langle N_f(u), h \rangle \quad \text{in } W^{-1,p'(x)}(\Omega).$$

Moreover, $\langle T_p(u_n), h \rangle \to \langle T_p(u), h \rangle$ in $W^{1,-p'(x)}(\Omega)$ and $\langle T_q(u_n), h \rangle$ is bounded. Combining these informations and passing to the limit in (18) for $n \to +\infty$, we obtain that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of Equation (1) in the case $\mu = 0$. Formally, $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of the $p(x)$-Laplace equation

$$-\Delta_{p(x)} u(x) = f(x, u(x), \nabla u(x))$$

subject to the Dirichlet boundary condition $u|_{\partial\Omega} = 0$. □

In a similar fashion, the following lemma deals with the case $\mu \to +\infty$.

Lemma 7. *Assume that A_1, A_2 and (6) are satisfied. Given a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of parameters diverging to $+\infty$, then every $\{u_n\}_{n\in\mathbb{N}}$ such that $u_n \in \mathcal{S}_{\mu_n}$ for all $n \in \mathbb{N}$ converges to zero in $W_0^{1,q(x)}(\Omega)$.*

Proof. Following the proof of Lemma 6 and using Lemma 5, $\{u_n\}_{n\in\mathbb{N}}$ bounded in $W_0^{1,p(x)}(\Omega)$ ensures that we can find a relabeled subsequence of $\{u_n\}_{n\in\mathbb{N}}$ such that $u_n \xrightarrow{w} u$ in $W_0^{1,p(x)}(\Omega)$, for certain $u \in W_0^{1,p(x)}(\Omega)$. Since $\mu_n \to +\infty$ here, the Equation (18) remains well posed, dividing both its members by μ_n, that is,

$$\frac{1}{\mu_n}\langle T_p(u_n), h \rangle + \langle T_q(u_n), h \rangle = \frac{1}{\mu_n}\int_\Omega f(x, u_n, \nabla u_n) h dx \quad \text{for all } h \in W_0^{1,p(x)}(\Omega). \tag{20}$$

Clearly, the asymptotic behavior of (20) can be established on the similar lines as in the proof of Lemma 6. Indeed, interchanging $T_p(\cdot)$ with $T_q(\cdot)$, we obtain easily that $u_n \to u$ in $W_0^{1,q(x)}(\Omega)$. Note that the limit of the right-hand side in (20) as $n \to +\infty$ is equal to zero. Thus, for $n \to +\infty$, (20) reduces to the $q(x)$-Laplace equation

$$-\Delta_{q(x)} u(x) = 0,$$

which gives us the solution $u = 0$. Since this result does not depend on the choice of the subsequence of $\{u_n\}_{n\in\mathbb{N}}$, we conclude that, for the whole sequence, we have $u_n \to 0$. □

4. Compactness Results

In this section, we discuss compactness (hence closedness), of $\mathcal{S}_\mu$ and $\mathcal{S}$. The starting point is the boundedness of $\mathcal{S}_\mu$ and $\mathcal{S}$ in $W_0^{1,p(x)}(\Omega)$, given in Lemma 5.

Proposition 1. *Assume that A_1, A_2 and (6) are satisfied, then $\mathcal{S}_\mu$ is compact in $W_0^{1,p(x)}(\Omega)$ for all $\mu \geq 0$.*

Proof. Consider $u \in \overline{\mathcal{S}_\mu} \setminus \mathcal{S}_\mu$ for some $\mu \geq 0$ fixed. This means that we can find $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{S}_\mu$ such that $u_n \to u$. We give the proof in two steps.

Claim 1: We show that $\mathcal{S}_\mu$ is closed for all parameter values $0 \leq \mu < +\infty$.

In a similar fashion as in the proof of Lemma 6 (recall (18)), $u_n \in \mathcal{S}_\mu$ for all $n \in \mathbb{N}$ means that

$$\langle T_p(u_n), h \rangle + \mu\langle T_q(u_n), h \rangle = \int_\Omega f(x, u_n, \nabla u_n) h dx \quad \text{for all } h \in W_0^{1,p(x)}(\Omega). \tag{21}$$

If we take the limit in (21) for $n \to +\infty$, we obtain that

$$\langle T_p(u), h\rangle + \mu\langle T_q(u), h\rangle = \int_\Omega f(x, u, \nabla u)h dx \quad \text{for all } h \in W_0^{1,p(x)}(\Omega).$$

This implies that $u \in \mathcal{S}_\mu$, and hence $\mathcal{S}_\mu$ is closed in $W_0^{1,p(x)}(\Omega)$. This concludes the proof of Claim 1.

Claim 2: We show that every $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{S}_\mu$ has a convergent subsequence to certain $u \in \mathcal{S}_\mu$.

From Lemma 5, we know that every sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{S}_\mu$ is bounded in $W_0^{1,p(x)}(\Omega)$. Therefore, we can find a relabeled subsequence of $\{u_n\}_{n\in\mathbb{N}}$ satisfying

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p(x)}(\Omega) \text{ and } u_n \to u \text{ in } L^{\alpha(x)}(\Omega), \text{ for some } u \in W_0^{1,p(x)}(\Omega).$$

We know from the a priori estimate in (16) that

$$\int_\Omega f(x, u_n, \nabla u_n)(u_n - u)dx \to 0 \quad \text{as } n \to +\infty$$

in the case that $u_n \to u$ in $L^{\alpha(x)}(\Omega)$ (recall assumption A_1). Putting $h = u_n - u \in W_0^{1,p(x)}(\Omega)$ in (21), we obtain that

$$\langle T_p(u_n), u_n - u\rangle + \mu\langle T_q(u_n), u_n - u\rangle = \int_\Omega f(x, u_n, \nabla u_n)(u_n - u)dx \quad \text{for all } n \in \mathbb{N}. \tag{22}$$

From the monotonicity of $T_q(\cdot)$, taking the limit as $n \to +\infty$ in (22), we obtain

$$\limsup_{n\to+\infty}\langle T_p(u_n), u_n - u\rangle \le 0,$$
$$\Rightarrow \quad u_n \to u \text{ in } W_0^{1,p(x)}(\Omega) \text{ (by } (S_+)\text{-property of } T_p).$$

Thus, we conclude that $u \in \mathcal{S}_\mu$, and hence Claim 2 is established.

We observe that the two claims together give us the compactness of $\mathcal{S}_\mu$ in $W_0^{1,p(x)}(\Omega)$. □

Proposition 2. *Assume that A_1, A_2 and (6) are satisfied, then $\mathcal{S}$ is closed whenever $0 \in \mathcal{S}$. Thus, $\mathcal{S} \cup \{0\}$ is a closed subset of $W_0^{1,p(x)}(\Omega)$.*

Proof. Lemma 7 plays a crucial role in establishing our result here. Observe that Lemma 7 leads to $0 \in \overline{\mathcal{S}}$. Now, we assume that $u \in \overline{\mathcal{S}} \setminus (\mathcal{S} \cup \{0\})$ and prove that $u \in \mathcal{S}$. Since $u \in \overline{\mathcal{S}} \setminus (\mathcal{S} \cup \{0\})$, then we can find $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ with $u_n \to u$. Moreover, to each $n \in \mathbb{N}$ corresponds a parameter value μ_n so that $u_n \in \mathcal{S}_{\mu_n}$. From $u_n \in \mathcal{S}_{\mu_n}$, we obtain

$$\langle T_p(u_n), h\rangle + \mu_n\langle T_q(u_n), h\rangle = \int_\Omega f(x, u_n, \nabla u_n)h dx \quad \text{for all } h \in W_0^{1,p(x)}(\Omega). \tag{23}$$

Now, Lemma 7 gives us the boundedness of $\{\mu_n\}_{n\in\mathbb{N}}$. Hence, we suppose $\mu_n \to \mu$ for a certain $\mu \in [0, +\infty)$. The convergence $u_n \to u$ ensures that

$$\langle N_f(u_n), h\rangle \to \langle N_f(u), h\rangle, \quad \langle A_p(u_n), h\rangle \to \langle A_p(u), h\rangle, \quad \langle A_q(u_n), h\rangle \to \langle A_q(u), h\rangle$$

in $W^{-1,p'(x)}(\Omega)$. Passing to the limit in (23) as $n \to +\infty$, we obtain

$$\langle T_p(u), h\rangle + \mu\langle T_q(u), h\rangle = \int_\Omega f(x, u(x), \nabla u(x))h dx \quad \text{for all } h \in W_0^{1,p(x)}(\Omega),$$

which implies $u \in \mathcal{S}_\mu \subset \mathcal{S}$. We conclude that $\mathcal{S}$ is closed whenever $0 \in \mathcal{S}$. In addition, $\mathcal{S} \cup \{0\}$ is in any case closed in $W_0^{1,p(x)}(\Omega)$. □

Let $F : [0, +\infty) \to 2^{W_0^{1,p(x)}(\Omega)}$ be the multivalued mapping defined by

$$F(\mu) = \mathcal{S}_\mu \quad \text{for all } 0 \le \mu < +\infty. \tag{24}$$

This mapping represents the solution mapping of Equation (1). We show that F possesses some regularities.

Proposition 3. *Assume that A_1, A_2, and (6) are satisfied; then, the multivalued mapping F defined by (24) is upper semicontinuous.*

Proof. Observe that the upper semicontinuity of (24) means that, for every closed subset C of $W_0^{1,p(x)}(\Omega)$,

$$F^-(C) := \{\mu \in [0, +\infty) : F(\mu) \cap C \neq \emptyset\}$$

is a closed set in $[0, +\infty)$.

Consider $\{\mu_n\}_{n\in\mathbb{N}} \subset F^-(C)$ satisfying $\mu_n \to \mu$ in $[0, +\infty)$. Clearly, for every $n \in \mathbb{N}$, there exists $u_n \in F(\mu_n) \cap C$. From the last sentence in the proof of Lemma 5 (boundedness of $\mathcal{S}$), we know that $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence. Moreover, from the proof of Lemma 6, we know that $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$.

Using the similar arguments as in the proof of Proposition 2 (recall $u_n \in \mathcal{S}_{\mu_n}$), we obtain $u \in \mathcal{S}_\mu = F(\mu)$. Since we know that $u \in C$ as C is closed, then $\mu \in F^-(C)$. □

Proposition 4. *Assume that A_1, A_2, and (6) are satisfied, then the multivalued mapping F defined by (24) is compact (that is, F maps the bounded sets in $[0, +\infty)$ into relatively compact subsets of $W_0^{1,p(x)}(\Omega)$).*

Proof. Consider a bounded set $\Lambda \subset [0, +\infty)$, $\{u_n\}_{n\in\mathbb{N}} \subset F(\Lambda)$, and $\mu_n \in \Lambda$ satisfying $u_n \in \mathcal{S}_{\mu_n}$ for all $n \in \mathbb{N}$.

To establish the assertion, we discuss separately two situations. We distinguish the following two cases:

Case 1. If $\{\mu_n : n \in \mathbb{N}\}$ is a finite set, then we can find $\mu \in \Lambda$ with $\mu = \mu_n$ for infinite values of n. It follows that

$$\begin{aligned} & \{u_n\}_{n\in\mathbb{N}} \text{ has a subsequence } \{u_{n_k}\}_{k\in\mathbb{N}} \subset \mathcal{S}_\mu, \\ \Rightarrow\ & \{u_{n_k}\}_{k\in\mathbb{N}} \text{ has a subsequence converging to some } u \in \mathcal{S}_\mu \subset F(\Lambda) \\ & \text{(by compactness of } \mathcal{S}_\mu). \end{aligned}$$

Case 2. If $\{\mu_n : n \in \mathbb{N}\}$ is not a finite set, then $\{\mu_n\}_{n\in\mathbb{N}}$ admits a convergent subsequence (without loss of generality, we continue to call it $\{\mu_n\}_{n\in\mathbb{N}}$). Now, if $\mu_n \to \mu$ for certain $\mu \in \overline{\Lambda}$, we obtain

$$\begin{aligned} & u_n \xrightarrow{w} u \text{ in } W_0^{1,p(x)}(\Omega) \text{ for some } u \in W_0^{1,p(x)}(\Omega) \\ & \text{(recall that } \{u_n\}_{n\in\mathbb{N}} \text{ is bounded)}, \\ \Rightarrow\ & u_n \to u \text{ in } W_0^{1,p(x)}(\Omega). \end{aligned}$$

We easily obtain $u \in \mathcal{S}_\mu$ and $u \in \overline{F(\Lambda)}$.

Assume $\{u_n\}_{n\in\mathbb{N}}$ in $\overline{F(\Lambda)} \setminus F(\Lambda)$. Since $F(\Lambda) \subset \mathcal{S}$, we deduce that $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ and so it is bounded. Consequently, for a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ (namely, again $\{u_n\}_{n\in\mathbb{N}}$), we obtain

$$u_n \to u \text{ in } W_0^{1,p(x)}(\Omega) \text{ for some } u \in \mathcal{S},$$

and hence $u \in \overline{F(\Lambda)}$. We conclude that $F(\Lambda)$ is a relatively compact subset of $W_0^{1,p(x)}(\Omega)$, and this proves the compactness of F. □

5. A Complete Uniqueness Result

This section is devoted to the study of uniqueness of solution to Equation (1), using some additional assumptions on the nonlinearity. Precisely, we impose the following:
$A_3:$ $(f(x,z,y) - f(x,w,y))(z-w) \leq 0$ for a.a. $x \in \Omega$, all $z, w \in \mathbb{R}$, all $y \in \mathbb{R}^N$;
$A_4:$ there exists $b_3 \geq 0$ such that

$$|f(x,z,\xi) - f(x,z,y)| \leq b_3 |\xi - y|^{\frac{p(x)}{\alpha'(x)}} \quad \text{for a.a. } x \in \Omega, \text{ all } z \in \mathbb{R}, \text{ all } \xi, y \in \mathbb{R}^N.$$

For the sake of clarity, we underline that the above assumptions work in addition to A_1, A_2 and hence α herein has to satisfy A_1, A_2. Moreover, $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is always Carathéodory.

This time, we also have to impose certain restrictions on the exponents, as follows:
A_5: $q(x), p(x) \geq 2$ for all $x \in \Omega$ and $2^{p^+-2}\left(\frac{b_3}{\alpha^-}\lambda_1^{-1} + \frac{b_3}{(\alpha')^-}\right) < 1$.

Assumption A_5 is motivated by technical needs of our proof below, in obtaining certain estimates (see also Lindqvist [1], p. 97).

Theorem 3. *Assume that $A_1 - A_5$ and (6) are satisfied, then Equation (1) has a unique weak solution for all $\mu \in \left[\frac{b_3}{\alpha^-}\lambda_1^{-1}2^{q^+-2}, +\infty\right)$.*

Proof. Suppose that, for certain $\mu \in [\frac{b_3}{\alpha^-}\lambda_1^{-1}2^{q^+-2}, +\infty)$, there exist $u_1, u_2 \in \mathcal{S}_\mu$ with $u_1 \neq u_2$. From (13) putting $u = u_1$ and $u = u_2$, respectively, and $h = (u_1 - u_2) \in W_0^{1,p(x)}(\Omega)$, we obtain

$$\langle T_p(u_1), u_1 - u_2\rangle + \mu\langle T_q(u_1), u_1 - u_2\rangle = \int_\Omega f(x, u_1, \nabla u_1)(u_1 - u_2)dx,$$

and

$$\langle T_p(u_2), u_1 - u_2\rangle + \mu\langle T_q(u_1), u_1 - u_2\rangle = \int_\Omega f(x, u_2, \nabla u_2)(u_1 - u_2)dx.$$

Now subtracting member to member the two equations, we obtain

$$\begin{aligned}&\langle T_p(u_1) - T_p(u_2), u_1 - u_2\rangle + \mu\langle T_q(u_1) - T_q(u_2), u_1 - u_2\rangle \\ &= \int_\Omega [f(x, u_1, \nabla u_1) - f(x, u_2, \nabla u_2)](u_1 - u_2)dx.\end{aligned}$$

We have the following estimate:

$$\begin{aligned}&\langle T_p(u_1) - T_p(u_2), u_1 - u_2\rangle + \mu\langle T_q(u_1) - T_q(u_2), u_1 - u_2\rangle \\ &\leq \int_\Omega |f(x, u_1, \nabla u_1) - f(x, u_2, \nabla u_2)|\,|u_1 - u_2|dx \\ &= \int_\Omega |f(x, u_1, \nabla u_1) - f(x, u_2, \nabla u_1) + f(x, u_2, \nabla u_1) - f(x, u_2, \nabla u_2)|\,|u_1 - u_2|dx \\ &\leq \int_\Omega b_3 |\nabla u_1 - \nabla u_2|^{\frac{p(x)}{\alpha'(x)}} |u_1 - u_2|dx \quad \text{(by assumptions } A_3 \text{ and } A_4) \\ &\leq \int_\Omega b_3 \int_\Omega \frac{|\nabla u_1 - \nabla u_2|^{p(x)}}{\alpha'(x)}dx + b_3 \int_\Omega \frac{|u_1 - u_2|^{\alpha(x)}}{\alpha(x)}dx \\ &\leq \left(\frac{b_3}{\alpha^-}\lambda_1^{-1} + \frac{b_3}{(\alpha')^-}\right)\int_\Omega |\nabla u_1 - \nabla u_2|^{p(x)}dx + \frac{b_3}{\alpha^-}\lambda_1^{-1}\int_\Omega |\nabla u_1 - \nabla u_2|^{q(x)}dx.\end{aligned}$$

On the other hand (see Lindqvist [1]), we know that

$$(|\xi|^{m(x)-2}\xi - |\eta|^{m(x)-2}\eta)(\xi - \eta) \geq \left(\frac{1}{2}\right)^{m(x)-2} |\xi - \eta|^{m(x)},$$

holds for $m(x) \geq 2$ for all $x \in \Omega$, and hence we deduce that

$$\frac{1}{2^{p^+-2}} \int_\Omega |\nabla u_1 - \nabla u_2|^{p(x)} dx > \left(\frac{b_3}{\alpha^-}\lambda_1^{-1} + \frac{b_3}{(\alpha')^-}\right) \int_\Omega |\nabla u_1 - \nabla u_2|^{p(x)} dx,$$

and for each $\mu \in [\frac{b_3}{\alpha^-}\lambda_1^{-1}2^{q^+-2}, +\infty)$, we have

$$\frac{\mu}{2^{q^+-2}} \int_\Omega |\nabla u_1 - \nabla u_2|^{q(x)} dx \geq \frac{b_3}{\alpha^-}\lambda_1^{-1} \int_\Omega |\nabla u_1 - \nabla u_2|^{q(x)} dx.$$

This ensures that

$$\int_\Omega |\nabla u_1 - \nabla u_2|^{p(x)} dx \leq 2^{p^+-2}\left(\frac{b_3}{\alpha^-}\lambda_1^{-1} + \frac{b_3}{(\alpha')^-}\right) \int_\Omega |\nabla u_1 - \nabla u_2|^{p(x)} dx.$$

Thus, we obtain that

$$\int_\Omega |\nabla u_1 - \nabla u_2|^{p(x)} dx = 0 \quad \text{(by assumption } A_5\text{)}.$$

We conclude that $u_1 = u_2$, which contradicts the assumption $u_1 \neq u_2$. We deduce that $\mathcal{S}_\mu$ is singleton, and hence the solution to Equation (1) is unique. □

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