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Neutral Differential Equations of Second-Order: Iterative Monotonic Properties

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Abstract: In this work, we investigate the oscillatory properties of the neutral differential equation $(r(l)[(s(l)+p(l)s(g(l)))']^{\mathfrak{v}})' + \sum_{i=1}^{n} q_i(l)s^{\mathfrak{v}}(h_i(l)) = 0$, where $s \geq s_0$. We first present new monotonic properties for the solutions of this equation, and these properties are characterized by an iterative nature. Using these new properties, we obtain new oscillation conditions that guarantee that all solutions are oscillate. Our results are a complement and extension to the relevant results in the literature. We test the significance of the results by applying them to special cases of the studied equation.

Keywords: Emden–Fowler; neutral differential equations; oscillation; non-canonical

MSC: 34C10; 34K11

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1. Introduction

It is natural to find an increasing interest in studying the qualitative behavior of solutions of second-order neutral differential equations (NDE), due to the use of this equation in the modeling of many important issues in engineering and physical sciences, such as problems involving lossless transmission lines (as in high-speed computers where such lines are used to interconnect switching circuits), population dynamics, automatic control, mixing liquids, and vibrating masses attached to an elastic bar; see Hale [1].

The objective of this paper is to develop a new criterion for determining whether there are solutions of the second-order NDE with several delays

$$\left(r \cdot \left[(s + p \cdot (s \circ g))'\right]^{\mathfrak{v}}\right)' + \sum_{i=1}^{n} q_i \cdot (s^{\mathfrak{v}} \circ h_i) = 0, \;\; l \geq l_0. \tag{1}$$

Throughout the paper, we assume that:

Hypothesis 1 (H1). $\mathfrak{v}$ *is a ratio of odd positive integer;*

Hypothesis 2 (H2). *p, $q_i \in C([l_0,\infty),[0,\infty))$, $p(l) < \mathfrak{y}(l)/\mathfrak{y}(g(l))$, and q_i does not vanish identically, $i = 1,2,\ldots,n$;*

Hypothesis 3 (H3). *$r \in C([l_0,\infty),\mathbb{R}^+)$, $\mathfrak{y}(l_0) < \infty$, where*

$$\mathfrak{y}(l) := \int_l^\infty r^{-1/\mathfrak{v}}(\varsigma)\mathrm{d}\varsigma;$$

Hypothesis 4 (H4). *$g \in C([l_0,\infty),\mathbb{R})$, $h_i \in C^1([l_0,\infty),\mathbb{R})$, $g(l) \le l$, $h_i(l) \le l$, $h_i'(l) > 0$ and $\lim_{l\to\infty} g(l) = \lim_{l\to\infty} h_i(l) = \infty$.*

First, let us assume that the corresponding function of the solution s is $\mathfrak{z} := s + p \cdot s \circ g$. By a solution of (1), we mean a real-valued function $s \in C([l_s,\infty),\mathbb{R})$, $l_s \geqslant l_0$, with $r(l)(\mathfrak{z}'(l))^{\mathfrak{v}} \in C^1[l_s,\infty)$, and which satisfies Equation (1) on $[l_s,\infty)$. We consider only those solutions s of (1) which satisfy

$$\sup\{|s(l)| : l \geqslant L\} > 0, \text{ for all } L \ge l_s.$$

As usual, a solution of (1) is called oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory.

Recently, a research movement was launched that aims to improve and develop methods for studying the oscillation of solutions of NDEs. These research works can be categorized in several ways. According to the order, equations of the second-order have received the most attention, because the development in this type is in turn reflected in the higher-orders. According to the operators, studies are divided into those with canonical or non-canonical operators.

Agarwal et al. [2,3] Győri and Ladas [4] presented and summarized many methods, techniques, and results about the oscillation of solutions of NDEs. Furthermore, the results in [5–8] contributed to the development of oscillation theory for solutions of second-order NDEs. The development of the study of oscillations for solutions of equations of higher orders can be traced through works [9–12]. On the other hand, the development in the study of the qualitative behavior of solutions of differential equations is reflected in the study of the qualitative properties of fractional and difference equations, see [13–17].

For second-order equations with $\mathfrak{y}(l_0) = \infty$, Baculikova and Dzurina [18], Grace et al. [19], and Moaaz et al. [20,21] developed and improved many results and techniques for studying the oscillation of certain classes of NDEs. Very recently, Jadlovská [22] introduced more efficient and effective criteria for oscillation of second order NDEs.

For second-order equations with non-canonical operator $\mathfrak{y}(l_0) < \infty$, Baculikova [5] studied the oscillatory properties of a delay equation

$$(r \cdot s')' + q_i \cdot (s \circ h) = 0.$$

The results in [5] completed and improved the results in [23,24]. Han et al. [25], Agarwal et al. [26] and Bohner et al. [27] discussed the issue of the oscillation of the solutions

$$\left(r \cdot (\mathfrak{z}')^{\mathfrak{v}}\right)' + q_i \cdot (s^{\mathfrak{v}} \circ h) = 0,$$

where Bohner's results improved Agarwal's results, as did Agarwal's results with Han's results.

In this paper, we obtain new oscillation conditions for the second-order NDE (1) in the non-canonical case. We derive some qualitative features of the positive solutions of (1). Moreover, we use the new features to obtain criteria that are of an iterative nature so that they can be applied more than once while the relevant results fail.

2. Main Results

For the sake of brevity, we define the class $\mathfrak{S}$ of all eventually positive solutions (1). We also define $h(l) := \max\{h_i(l), i = 1, \ldots, n,\}$ and

$$Q(l) := \sum_{i=1}^{n} q_i(l) \left(1 - p(h_i(l)) \frac{\mathfrak{y}(g(h_i(l)))}{\mathfrak{y}(h_i(l))}\right)^{\mathfrak{v}}.$$

Lemma 1. *Suppose that $s \in \mathfrak{S}$. Then $\mathfrak{z}$ satisfies one of the following two cases, eventually:*

(P_1) *$\mathfrak{z}$ and $\mathfrak{z}'$ are positive, and $\left(r(\mathfrak{z}')^{\mathfrak{v}}\right)'$ is negative;*
(P_2) *$\mathfrak{z}$ is positive, $\mathfrak{z}'$ and $\left(r(\mathfrak{z}')^{\mathfrak{v}}\right)'$ are negative.*

Proof. Let $s \in \mathfrak{S}$. In view of (H_4), there is a $l_1 \geqslant l_0$ with $s \circ h_i > 0$ and $s \circ g > 0$, for $l \geqslant l_1$. Then, $\mathfrak{z}(l) > 0$, for all $l \geqslant l_1$. From (1), we have

$$\left(r \cdot (\mathfrak{z}')^{\mathfrak{v}}\right)' = -\sum_{i=1}^{n} q_i \cdot (s^{\mathfrak{v}} \circ h_i) < 0.$$

Hence, we get that $r(\mathfrak{z}')^{\mathfrak{v}}$ is decreasing, and so $r(\mathfrak{z}')^{\mathfrak{v}} > 0$ or $r(\mathfrak{z}')^{\mathfrak{v}} < 0$, eventually. Thus, the proof is complete. □

Lemma 2. *Suppose that $s \in \mathfrak{S}$. If*

$$\int_{l_0}^{\infty} \frac{1}{r^{1/\mathfrak{v}}(u)} \left(\int_{l_0}^{u} Q(\varsigma) \mathrm{d}\varsigma\right)^{1/\mathfrak{v}} \mathrm{d}u = \infty, \tag{2}$$

then, $\mathfrak{z}$ is decreasing and satisfies the following:

(a) $r^{1/\mathfrak{v}}(l)\mathfrak{z}'(l)\mathfrak{y}(l) + \mathfrak{z}(l) \geqslant 0$;
(b) $\mathfrak{z}(l)/\mathfrak{y}(l)$ *is increasing;*
(c) $\left(r(l)(\mathfrak{z}'(l))^{\mathfrak{v}}\right)' \leq -Q(l)\mathfrak{z}^{\mathfrak{v}}(h(l))$;
(d) $\lim_{l\to\infty} \mathfrak{z}(l) = 0$.

Proof. Let $s \in \mathfrak{S}$. Suppose the contrary that $\mathfrak{z}$ is an increasing function for $l \geq l_1 \geq l_0$. Then there is a constant $k > 0$ with $\mathfrak{z}(l) \geq k$ and $\mathfrak{z}(h_i(l)) \geq k$, eventually, for $i = 1, 2, \ldots, n$. In view of the definition of $\mathfrak{z}$, we have

$$\begin{aligned} s &= \mathfrak{z} - p \cdot (s \circ g) \geqslant \mathfrak{z} - p \cdot (\mathfrak{z} \circ g) \\ &\geqslant (1 - p)\mathfrak{z}. \end{aligned}$$

Then, (1) becomes

$$\left(r \cdot (\mathfrak{z}')^{\mathfrak{v}}\right)' \leq -\sum_{i=1}^{n} q_i \cdot (1 - (p \circ h_i))^{\mathfrak{v}} \cdot (\mathfrak{z}^{\mathfrak{v}} \circ h_i). \tag{3}$$

Since $\mathfrak{y}'(l) < 0$ and $g(l) \leq l$, we find

$$\frac{\mathfrak{y}(g(h_i(l)))}{\mathfrak{y}(h_i(l))} \geq 1,$$

and then

$$1 - p(h_i(l)) \geq 1 - \frac{\mathfrak{y}(g(h_i(l)))}{\mathfrak{y}(h_i(l))} p(h_i(l)). \tag{4}$$

Combining (3) and (4), and integrating the inferring inequality from l_1 to ∞, we conclude that

$$\begin{aligned} r(l_1)\left(\mathfrak{z}'(l_1)\right)^{\mathfrak{v}} &\geq \int_{l_1}^{\infty} \sum_{i=1}^{n} q_i(\varsigma)\left(1 - p(h_i(\varsigma))\frac{\mathfrak{y}(g(h_i(\varsigma)))}{\mathfrak{y}(h_i(\varsigma))}\right)^{\mathfrak{v}} \mathfrak{z}^{\mathfrak{v}}(h_i(\varsigma))\mathrm{d}\varsigma \\ &\geq k^{\mathfrak{v}} \int_{l_1}^{\infty} \sum_{i=1}^{n} q_i(\varsigma)\left(1 - p(h_i(\varsigma))\frac{\mathfrak{y}(g(h_i(\varsigma)))}{\mathfrak{y}(h_i(\varsigma))}\right)^{\mathfrak{v}} \mathrm{d}\varsigma \\ &\geq k^{\mathfrak{v}} \int_{l_1}^{\infty} Q(\varsigma)\mathrm{d}\varsigma, \end{aligned} \tag{5}$$

It follows from (2) and (H_3) that $\int_{l_1}^{l} Q(\varsigma)\mathrm{d}\varsigma$ must be unbounded. Thus,

$$\int_{l_1}^{\infty} Q(\varsigma)\mathrm{d}\varsigma = \infty. \tag{6}$$

which with (5) gives a contradiction.

(a): Next, we have

$$\begin{aligned} -\mathfrak{z}(t) &\leq \int_{l}^{\infty} \frac{1}{r^{1/\mathfrak{v}}(\varsigma)} r^{1/\mathfrak{v}}(\varsigma)\mathfrak{z}'(\varsigma)\mathrm{d}\varsigma \\ &\leq r^{1/\mathfrak{v}}(l)\mathfrak{z}'(l) \int_{l}^{\infty} \frac{1}{r^{1/\mathfrak{v}}(\varsigma)}\mathrm{d}\varsigma \\ &= r^{1/\mathfrak{v}}(l)\mathfrak{z}'(l)\mathfrak{y}(l). \end{aligned}$$

(b): From the last inequality, we obtain

$$\left(\frac{\mathfrak{z}}{\mathfrak{y}}\right)' = \frac{r^{1/\mathfrak{v}}\mathfrak{z}'\mathfrak{y} + \mathfrak{z}}{r^{1/\mathfrak{v}}\mathfrak{y}^2} \geqslant 0.$$

(c): Since $\mathfrak{z}(l)/\mathfrak{y}(l)$ is increasing, we find

$$\mathfrak{z}(g(l)) \leq \frac{\mathfrak{y}(g(l))}{\mathfrak{y}(l)}\mathfrak{z}(l).$$

Hence, we arrive at

$$\begin{aligned} s &= \mathfrak{z} - p \cdot (s \circ g) \geqslant \mathfrak{z} - p \cdot (\mathfrak{z} \circ g) \\ &\geqslant \mathfrak{z}\left(1 - p \cdot \frac{(\mathfrak{y} \circ g)}{\mathfrak{y}}\right). \end{aligned}$$

Thus, from (1), we arrive at

$$\begin{aligned} \left(r(l)\left(\mathfrak{z}'(l)\right)^{\mathfrak{v}}\right)' &\leq -\sum_{i=1}^{n} q_i(l)\left(1 - p(h_i(l))\frac{\mathfrak{y}(g(h_i(l)))}{\mathfrak{y}(h_i(l))}\right)^{\mathfrak{v}} \mathfrak{z}^{\mathfrak{v}}(h_i(l)) \\ &\leq -\mathfrak{z}^{\mathfrak{v}}(h(l)) \sum_{i=1}^{n} q_i(l)\left(1 - p(h_i(l))\frac{\mathfrak{y}(g(h_i(l)))}{\mathfrak{y}(h_i(l))}\right)^{\mathfrak{v}} \\ &\leq -Q(l)\mathfrak{z}^{\mathfrak{v}}(h(l)). \end{aligned} \tag{7}$$

(d) : Now, since $\mathfrak{z}$ is positive and decreasing for $l \geq l_1$, we have that $\lim_{l\to\infty} \mathfrak{z}(l) = \kappa \geq 0$. If $\kappa \neq 0$, then $\mathfrak{z}(l) \geq \kappa$, for $t \geq t_2 \geq t_1$. Then, integrating (a) from l_1 to l, we arrive at

$$\begin{aligned} r(l)\left(\mathfrak{z}'(l)\right)^{\mathfrak{v}} &\leq r(l_1)\left(\mathfrak{z}'(l_1)\right)^{\mathfrak{v}} - \int_{l_1}^{l} Q(\varsigma)\mathfrak{z}^{\mathfrak{v}}(h(\varsigma))\mathrm{d}\varsigma \\ &\leq -\kappa^{\mathfrak{v}} \int_{l_1}^{l} Q(\varsigma)\mathrm{d}\varsigma, \end{aligned}$$

and so

$$\mathfrak{Z}'(l) \leq -\kappa^{\mathfrak{v}} \frac{1}{r^{1/\mathfrak{v}}(l)} \left(\int_{l_1}^{l} Q(\varsigma) \mathrm{d}\varsigma \right)^{1/\mathfrak{v}}.$$

Integrating this inequality from l_1 to ∞, we get

$$\mathfrak{Z}(l_1) \geq \kappa^{\mathfrak{v}} \int_{l_1}^{\infty} \frac{1}{r^{1/\mathfrak{v}}(u)} \left(\int_{l_1}^{u} Q(\varsigma) \mathrm{d}\varsigma \right)^{1/\mathfrak{v}} \mathrm{d}u,$$

which contradicts to (2). Therefore, $\kappa = 0$.

Hence, the proof is complete. □

Next, we show new monotonic properties for solutions of (1).

Lemma 3. *Suppose that $s \in \mathfrak{S}$. If there is a $\delta \in (0,1)$ with*

$$r^{1/\mathfrak{v}}(l) \mathfrak{y}^{\mathfrak{v}+1}(l) Q(l) \geqslant \mathfrak{v} \delta^{\mathfrak{v}}, \tag{8}$$

then

(a_0) $\mathfrak{Z}(l)/\mathfrak{y}^{\delta}(l)$ *is decreasing;*

(b_0) $\lim_{l\to\infty} \mathfrak{Z}(l)/\mathfrak{y}^{\delta}(l) = 0$;

(c_0) $\mathfrak{Z}(l)/\mathfrak{y}^{1-\delta}(l)$ *is increasing.*

Proof. Let $s \in \mathfrak{S}$. From (8), we note that

$$\begin{aligned} \int_{l_0}^{\infty} \frac{1}{r^{1/\mathfrak{v}}(u)} \left(\int_{l_0}^{u} Q(\varsigma) \mathrm{d}\varsigma \right)^{1/\mathfrak{v}} \mathrm{d}u &\geq \mathfrak{v}\delta^{\mathfrak{v}} \int_{l_0}^{\infty} \frac{1}{r^{1/\mathfrak{v}}(u)} \left(\int_{l_1}^{u} \frac{\mathrm{d}\varsigma}{r^{1/\mathfrak{v}}(\varsigma) \mathfrak{y}^{\mathfrak{v}+1}(\varsigma)} \right)^{1/\mathfrak{v}} \mathrm{d}u \\ &= \delta^{\mathfrak{v}} \int_{l_0}^{\infty} \frac{1}{r^{1/\mathfrak{v}}(u)} \left(\mathfrak{y}^{-\mathfrak{v}}(u) - \mathfrak{y}^{\mathfrak{v}}(l_1) \right)^{1/\mathfrak{v}} \mathrm{d}u. \end{aligned}$$

From the fact that $\lim_{l\to\infty} \mathfrak{y}(l) = 0$, there exists a $t_1 \geq t_0$ such that $\mathfrak{y}^{-\mathfrak{v}}(u) - \mathfrak{y}^{\mathfrak{v}}(l_1) \geq \lambda \mathfrak{y}^{-\mathfrak{v}}(u)$ for $\lambda \in (0,1)$. Thus,

$$\begin{aligned} \int_{l_0}^{\infty} \frac{1}{r^{1/\mathfrak{v}}(u)} \left(\int_{l_0}^{u} Q(\varsigma) \mathrm{d}\varsigma \right)^{1/\mathfrak{v}} \mathrm{d}u &\geq \lambda \delta^{\mathfrak{v}} \int_{l_0}^{\infty} \frac{1}{r^{1/\mathfrak{v}}(u) \mathfrak{y}(u)} \mathrm{d}u \\ &= \lambda \delta^{\mathfrak{v}} \lim_{u\to\infty} \ln \frac{\mathfrak{y}(l_0)}{\mathfrak{y}(u)} \to \infty. \end{aligned}$$

Hence, From Lemma 2, we have (a), (b), (c) and (d) hold.

(a_0): Integrating (c) from l_1 to l, we obtain

$$\begin{aligned} -r(l) \left(\mathfrak{Z}'(l) \right)^{\mathfrak{v}} &\geqslant -r(l_1) \left(\mathfrak{Z}'(l_1) \right)^{\mathfrak{v}} + \int_{l_1}^{l} Q(\varsigma) \mathfrak{Z}^{\mathfrak{v}}(h(\varsigma)) \mathrm{d}\varsigma \\ &\geqslant -r(l_1) \left(\mathfrak{Z}'(l_1) \right)^{\mathfrak{v}} + \mathfrak{Z}^{\mathfrak{v}}(l) \int_{l_1}^{l} Q(\varsigma) \mathrm{d}\varsigma. \end{aligned}$$

Using (8), we get

$$\begin{aligned} -r(l) \left(\mathfrak{Z}'(l) \right)^{\mathfrak{v}} &\geqslant -r(l_1) \left(\mathfrak{Z}'(l_1) \right)^{\mathfrak{v}} + \mathfrak{v}\delta^{\mathfrak{v}} \mathfrak{Z}^{\mathfrak{v}}(l) \int_{l_1}^{l} \frac{1}{r^{1/\mathfrak{v}}(\varsigma) \mathfrak{y}^{\mathfrak{v}+1}(\varsigma)} \mathrm{d}\varsigma \\ &\geq -r(l_1) \left(\mathfrak{Z}'(l_1) \right)^{\mathfrak{v}} + \delta^{\mathfrak{v}} \frac{\mathfrak{Z}^{\mathfrak{v}}(l)}{\mathfrak{y}^{\mathfrak{v}}(l)} - \delta^{\mathfrak{v}} \frac{\mathfrak{Z}^{\mathfrak{v}}(l)}{\mathfrak{y}^{\mathfrak{v}}(l_1)}. \end{aligned} \tag{9}$$

From (d), there is a $l_2 \in [l_1, \infty)$ such that

$$-r(l_1)\left(\mathfrak{z}'(l_1)\right)^{\mathfrak{v}} - \delta^{\mathfrak{v}} \frac{\mathfrak{z}^{\mathfrak{v}}(l)}{\mathfrak{y}^{\mathfrak{v}}(l_1)} \geq 0,\ l \geq l_2,$$

and so, (9) becomes

$$-r^{1/\mathfrak{v}}(l)\mathfrak{z}'(l) \geq \delta \frac{\mathfrak{z}(l)}{\mathfrak{y}(l)}. \tag{10}$$

Consequently,

$$\left(\frac{\mathfrak{z}(l)}{\mathfrak{y}^{\delta}(l)}\right)' = \frac{\mathfrak{y}^{\delta-1}(l)\left[r^{1/\mathfrak{v}}(l)\mathfrak{y}(l)\mathfrak{z}'(l) + \delta\mathfrak{z}(l)\right]}{r^{1/\mathfrak{v}}(l)\mathfrak{y}^{2\delta}(l)} \leq 0.$$

(b_0): Since $\mathfrak{z}/\mathfrak{y}^{\delta}$ is positive and decreasing, $\lim_{l\to\infty} \mathfrak{z}(l)/\mathfrak{y}^{\delta}(l) = \kappa \geqslant 0$. If $\kappa \neq 0$, then $\mathfrak{z}(l)/\mathfrak{y}^{\delta}(l) \geqslant \kappa > 0$ eventually. Now, we define

$$\Omega(l) = \left(r^{1/\mathfrak{v}}(l)\mathfrak{z}'(l)\mathfrak{y}(l) + \mathfrak{z}(l)\right)\mathfrak{y}^{-\delta}(l).$$

In view of (a), we note that $\Omega(l) > 0$ and

$$\begin{aligned}
\Omega' &= \left(r^{1/\mathfrak{v}}\mathfrak{z}'\right)'\mathfrak{y}^{1-\delta} - (1-\delta)\mathfrak{z}'\mathfrak{y}^{-\delta}(l) + \mathfrak{z}'\mathfrak{y}^{-\delta} + \delta\mathfrak{z}\frac{\mathfrak{y}^{-1-\delta}}{r^{1/\mathfrak{v}}} \\
&= \frac{1}{\mathfrak{v}}\left(r(\mathfrak{z}')^{\mathfrak{v}}\right)'\left(r^{1/\mathfrak{v}}\mathfrak{z}'\right)^{1-\mathfrak{v}}\mathfrak{y}^{1-\delta} + \delta\mathfrak{z}'\mathfrak{y}^{-\delta}(l) + \delta\mathfrak{z}\frac{\mathfrak{y}^{-1-\delta}}{r^{1/\mathfrak{v}}} \\
&\leq -\frac{1}{\mathfrak{v}}\left(r^{1/\mathfrak{v}}\mathfrak{z}'\right)^{1-\mathfrak{v}}\mathfrak{y}^{1-\delta}Q(\varsigma)\mathfrak{z}^{\mathfrak{v}}(h(\varsigma)) + \delta\mathfrak{z}'\mathfrak{y}^{-\delta}(l) + \delta\mathfrak{z}\frac{\mathfrak{y}^{-1-\delta}}{r^{1/\mathfrak{v}}} \\
&\leq -\frac{1}{\mathfrak{v}}\left(r^{1/\mathfrak{v}}\mathfrak{z}'\right)^{1-\mathfrak{v}}\mathfrak{y}^{1-\delta}\mathfrak{z}^{\mathfrak{v}}(l)Q(l) + \delta\mathfrak{z}'\mathfrak{y}^{-\delta} + \delta\mathfrak{z}\frac{\mathfrak{y}^{-1-\delta}}{r^{1/\mathfrak{v}}}.
\end{aligned}$$

Using (8) and (10), we get

$$\begin{aligned}
\Omega' &\leq -\left(\frac{\delta\mathfrak{z}}{\mathfrak{y}}\right)^{1-\mathfrak{v}}\mathfrak{y}^{1-\delta}\frac{\delta^{\mathfrak{v}}}{r^{1/\mathfrak{v}}\mathfrak{y}^{\mathfrak{v}+1}}\mathfrak{z}^{\mathfrak{v}}(l) + \delta\mathfrak{z}'\mathfrak{y}^{-\delta} + \delta\mathfrak{z}\frac{\mathfrak{y}^{-1-\delta}}{r^{1/\mathfrak{v}}} \\
&\leq \delta\mathfrak{z}'\mathfrak{y}^{-\delta}(l) \\
&\leq -\frac{\delta^{3}}{r^{1/\mathfrak{v}}\mathfrak{y}}\frac{\mathfrak{z}}{\mathfrak{y}^{\delta}(l)}.
\end{aligned}$$

Using the fact that $\mathfrak{z}(l) \geqslant \mathfrak{y}^{\delta}(l)\kappa$, we have

$$\Omega'(l) \leqslant -\frac{\delta^{2}\kappa}{r^{1/\mathfrak{v}}(l)\mathfrak{y}(l)} < 0.$$

Integrating this inequality from l_1 to l, we find

$$\Omega(l_1) \geq \delta\kappa \ln\frac{\mathfrak{y}(l_1)}{\mathfrak{y}(l)} \to \infty \text{ as } l \to \infty,$$

which is a contradiction. Thus, $\kappa = 0$.

(c_0): Finally, we have

$$\begin{aligned}
\left(r^{1/\mathfrak{v}}(l)\mathfrak{z}'(l)\mathfrak{y}(l)+\mathfrak{z}(l)\right)' &= \left(r^{1/\mathfrak{v}}(l)\mathfrak{z}'(l)\right)'\mathfrak{y}(l) \\
&= \frac{1}{\mathfrak{v}}\left(r(l)(\mathfrak{z}'(l))^{\mathfrak{v}}\right)'\left(r^{1/\mathfrak{v}}(l)\mathfrak{z}'(l)\right)^{1-\mathfrak{v}}\mathfrak{y}(l) \\
&\leq -\frac{1}{\mathfrak{v}}Q(l)\mathfrak{z}^{\mathfrak{v}}(l)\left(r^{1/\mathfrak{v}}(t)\mathfrak{z}'(t)\right)^{1-\mathfrak{v}}\mathfrak{y}(t) \\
&\leq -\beta_0^{\mathfrak{v}}\frac{1}{r^{1/\mathfrak{v}}(l)\mathfrak{y}^{1+\mathfrak{v}}(l)}\mathfrak{z}^{\mathfrak{v}}(l)\left(-\beta_0\frac{\mathfrak{z}(t)}{\mathfrak{y}(t)}\right)^{1-\mathfrak{v}}\mathfrak{y}(t) \\
&\leq -\beta_0^{\mathfrak{v}}\frac{1}{r^{1/\mathfrak{v}}(t)\mathfrak{y}^{\mathfrak{v}}(l)}(\mathfrak{z}(l))^{\mathfrak{v}}\left(\beta_0\frac{\mathfrak{z}(l)}{\mathfrak{y}(l)}\right)^{1-\mathfrak{v}} \\
&\leq \frac{-\beta_0}{r^{1/\mathfrak{v}}(l)\mathfrak{y}(l)}\mathfrak{z}(l).
\end{aligned}$$

Integrating the last inequality from l to ∞, we get

$$\begin{aligned}
r^{1/\mathfrak{v}}(t)\mathfrak{z}'(t)\mathfrak{y}(t)+\mathfrak{z}(t) &\geq \beta_0\int_l^\infty \frac{1}{r^{1/\mathfrak{v}}(\varsigma)}\frac{\mathfrak{z}(\varsigma)}{\mathfrak{y}(\varsigma)}\mathrm{d}\varsigma \\
&\geq \beta_0\frac{\mathfrak{z}(l)}{\mathfrak{y}(l)}\int_l^\infty \frac{1}{r^{1/\mathfrak{v}}(\varsigma)}\mathrm{d}\varsigma \\
&\geq \beta_0\mathfrak{z}(l).
\end{aligned}$$

Thus,

$$r^{1/\mathfrak{v}}(l)\mathfrak{y}(l)\mathfrak{z}'(l)+(1-\delta)\mathfrak{z}(l)\geqslant 0,$$

and hence

$$\left(\frac{\mathfrak{z}(l)}{\mathfrak{y}^{1-\delta}(l)}\right)' = \frac{\mathfrak{y}^{-\delta}(l)\left[r^{1/\mathfrak{v}}(l)\mathfrak{y}(l)\mathfrak{z}'(l)+(1-\delta)\mathfrak{z}(l)\right]}{r^{1/\mathfrak{v}}(l)\mathfrak{y}^{2-2\delta}(l)}\geqslant 0.$$

Hence, the proof is complete. □

Theorem 1. *Suppose that there is a $\delta\in(0,1)$ with (8) holds. If*

$$\delta>\frac{1}{2}, \tag{11}$$

then, (1) is oscillatory.

If $\delta\leq\frac{1}{2}$, then we can improve the results given in Lemma 3. Since $\mathfrak{y}$ is decreasing, there is a constant $\lambda\geq 1$ with

$$\frac{\mathfrak{y}(h(l))}{\mathfrak{y}(l)}\geq\lambda. \tag{12}$$

We introduce the constant $\delta_1>\delta$ as follows:

$$\delta_1=\lambda^{\delta}\sqrt[\mathfrak{v}]{\frac{\delta}{1-\delta}}. \tag{13}$$

Lemma 4. *Suppose that $s\in\mathfrak{S}$ and there is a $\delta\in(0,1)$ with (8) holds. If (12) holds, then*

(a_1) *$\mathfrak{z}(l)/\mathfrak{y}^{\delta_1}(l)$ is decreasing;*
(b_1) $\lim_{l\to\infty}\mathfrak{z}(l)/\mathfrak{y}^{\delta_1}(l)=0$;
(c_1) *$\mathfrak{z}(l)/\mathfrak{y}^{1-\delta_1}(l)$ is increasing.*

Proof. Let $s\in\mathfrak{S}$. From Lemma 2, we have (a), (b), (c) and (d) hold. Furthermore, it follows from Lemma 3 that (a_0), (b_0) and (c_0) hold.

(a_1): Integrating (1) from l_1 to l, we get

$$-r(l)\left(\mathfrak{z}'(l)\right)^{\mathfrak{v}} \geqslant -r(l_1)\left(\mathfrak{z}'(l_1)\right)^{\mathfrak{v}} + \int_{l_1}^{l} Q(\varsigma)\mathfrak{z}^{\mathfrak{v}}(h(\varsigma))\mathrm{d}\varsigma.$$

By using the fact that $\mathfrak{z}(l)/\eta^{\delta}(l)$ is decreasing, we have

$$\begin{aligned}-r(l)\left(\mathfrak{z}'(l)\right)^{\mathfrak{v}} &\geqslant -r(l_1)\left(\mathfrak{z}'(l_1)\right)^{\mathfrak{v}} + \int_{l_1}^{l}\left(\frac{\mathfrak{z}(\varsigma)}{\eta^{\delta}(\varsigma)}\right)^{\mathfrak{v}}\eta^{\mathfrak{v}\delta}(h(\varsigma))Q(\varsigma)\mathrm{d}\varsigma \\ &\geqslant -r(l_1)\left(\mathfrak{z}'(l_1)\right)^{\mathfrak{v}} + \left(\frac{\mathfrak{z}(l)}{\eta^{\delta}(l)}\right)^{\mathfrak{v}}\int_{l_1}^{l}\eta^{\mathfrak{v}\delta}(h(\varsigma))Q(\varsigma)\mathrm{d}\varsigma.\end{aligned}$$

From (8) and (12), $\mathfrak{z}e$ get

$$\begin{aligned}-r(l)\left(\mathfrak{z}'(l)\right)^{\mathfrak{v}} &\geqslant -r(l_1)\left(\mathfrak{z}'(l_1)\right)^{\mathfrak{v}} + \mathfrak{v}\delta\lambda^{\mathfrak{v}\delta}\left(\frac{\mathfrak{z}(l)}{\eta^{\delta}(l)}\right)^{\mathfrak{v}}\int_{l_1}^{l}\frac{1}{r^{1/\mathfrak{v}}(\varsigma)\eta^{\mathfrak{v}+1}(\varsigma)}\eta^{\mathfrak{v}\delta}(\varsigma)\mathrm{d}\varsigma \\ &\geqslant -r(l_1)\left(\mathfrak{z}'(l_1)\right)^{\mathfrak{v}} + \mathfrak{v}\delta\lambda^{\mathfrak{v}\delta}\left(\frac{\mathfrak{z}(l)}{\eta^{\delta}(l)}\right)^{\mathfrak{v}}\int_{l_1}^{l}\frac{\eta^{-1-\mathfrak{v}+\mathfrak{v}\delta}(\varsigma)}{r^{1/\mathfrak{v}}(\varsigma)}\mathrm{d}\varsigma \\ &\geqslant -r(l_1)\left(\mathfrak{z}'(l_1)\right)^{\mathfrak{v}} + \frac{\delta\lambda^{\mathfrak{v}\delta}}{(1-\delta)}\left(\frac{\mathfrak{z}(l)}{\eta^{\delta}(l)}\right)^{\mathfrak{v}}\left[\eta^{\mathfrak{v}\delta-\mathfrak{v}}(l) - \eta^{\mathfrak{v}\delta-\mathfrak{v}}(l_1)\right] \\ &\geqslant -r(l_1)\left(\mathfrak{z}'(l_1)\right)^{\mathfrak{v}} - \delta_1^{\mathfrak{v}}\eta^{\mathfrak{v}\delta-\mathfrak{v}}(l_1)\left(\frac{\mathfrak{z}(l)}{\eta^{\delta}(l)}\right)^{\mathfrak{v}} + \delta_1^{\mathfrak{v}}\frac{\mathfrak{z}^{\mathfrak{v}}(l)}{\eta^{\mathfrak{v}}(l)}.\end{aligned}$$

Using (b_0), there is $l_2 \in [l_1, \infty)$ such that

$$-r(l_1)\left(\mathfrak{z}'(l_1)\right)^{\mathfrak{v}} - \delta_1^{\mathfrak{v}}\eta^{\mathfrak{v}\delta-\mathfrak{v}}(l_1)\left(\frac{\mathfrak{z}(l)}{\eta^{\delta}(l)}\right)^{\mathfrak{v}} \geq 0, \text{ for } l \geq l_2,$$

and so

$$-r(l)\left(\mathfrak{z}'(l)\right)^{\mathfrak{v}} \geqslant \delta_1^{\mathfrak{v}}\frac{\mathfrak{z}^{\mathfrak{v}}(l)}{\eta^{\mathfrak{v}}(l)},$$

or

$$r^{1/\mathfrak{v}}(l)\eta(l)\mathfrak{z}'(l) + \delta_1\mathfrak{z}(l) \leq 0. \tag{14}$$

Consequently

$$\left(\frac{\mathfrak{z}(l)}{\eta^{\delta_1}(l)}\right)' = \frac{\eta^{\delta_1-1}(l)\left[r^{1/\mathfrak{v}}(l)\eta(l)\mathfrak{z}'(l) + \delta_1\mathfrak{z}(l)\right]}{r^{1/\mathfrak{v}}(l)\eta^{2\delta_1}(l)} \leqslant 0,$$

so $\mathfrak{z}(l)/\eta^{\delta_1}(l)$ is decreasing.

Proceeding exactly as in the proof of Lemma 3, we can verify that (b_1) and (c_1) hold. □

If $\delta_1 \leq 1/2$, we can repeat the above process and define δ_2 as follows

$$\delta_2 = \lambda^{\delta_1}\sqrt[\mathfrak{v}]{\frac{\delta}{1-\delta_1}}.$$

In general, if $\delta_i \leq 1/2$ for $i = 1, 2, \ldots, n-1$, we can define

$$\delta_n = \lambda^{\delta_{n-1}}\sqrt[\mathfrak{v}]{\frac{\delta}{1-\delta_{n-1}}}. \tag{15}$$

Moreover, proceeding exactly as in the proof of Lemma 4, we can verify that

(a_n) $\mathfrak{z}(l)/\eta^{\delta_n}(l)$ is decreasing;

($\mathrm{b_n}$) $\lim_{l\to\infty} \mathfrak{z}(l)/\mathfrak{y}^{\delta_n}(l) = 0$;
($\mathrm{c_n}$) $\mathfrak{z}(l)/\mathfrak{y}^{1-\delta_n}(l)$ is increasing.

Theorem 2. *Suppose that there exists a $\delta \in (0,1)$ such that (8) holds. If there is a $n \in \mathbb{N}$ such that*

$$\delta_n > \frac{1}{2}, \tag{16}$$

then (1) is oscillatory.

Example 1. *Consider the following NDE*

$$\left(l^{4/3}\left(\mathfrak{z}'(l)\right)^{1/3}\right)' + q_0\left(s^{1/3}\left(\frac{1}{3}l\right) + s^{1/3}\left(\frac{1}{4}l\right) + s^{1/3}\left(\frac{1}{5}l\right)\right) = 0,\ l \geq 1, \tag{17}$$

where $\mathfrak{z}(l) = s(l) + \frac{1}{16}s\left(\frac{1}{2}l\right)$. It is easy to see that $\mathfrak{v} = \frac{1}{3}$, $p(l) = \frac{1}{16}, r(l) = l^{4/3}$, $q_i = q_0$, $g(l) = \frac{1}{2}l$, $h_1(l) = \frac{1}{3}l$, $h_2(l) = \frac{1}{4}l$, $h_3(l) = \frac{1}{5}l$, and $h(l) = \frac{1}{3}l$. Then, we have $\mathfrak{y}(l) = \frac{1}{3l^3}$, $Q(l) = \frac{3q_0}{2^{1/3}}$, $\delta = \frac{9}{2}q_0^3$, and $\lambda = 27$.

If we let $q_0 = 0.5$, then $\delta = 0.5625$ and condition (11) holds.

For $q_0 = 0.3$, we have

$$\delta = 0.1215,\ \delta_1 = 0.26746,\ \text{and}\ \delta_2 = 0.74630,$$

and (16) holds for $n = 2$, that is for $a = 0.3$ (17) is oscillatory.

For $a = 0.25$, we have

$$\delta = 0.0703125,\ \text{and}\ \delta_6 = 0.69141,$$

and (16) holds for $n = 6$, that is for $a = 0.3$ (17) is oscillatory.

Example 2. *Consider the following NDE*

$$\left(l^{6/5}\left[\left(s(l) + p_0 s\left(\frac{1}{2}l\right)\right)'\right]^{1/5}\right)' + \sum_{i=1}^{n}\gamma_i s^{1/5}(h_i(l)) = 0,\ l \geq 1. \tag{18}$$

where $\gamma_i < 0$ and $p_0 < 1/32$. Clearly $\mathfrak{v} = \frac{1}{5}$, $p(l) = p_0$, $q_i(l) = q_i > 0$, $r(l) = l^{6/5}$, $g(l) = \frac{1}{2}l$. Then, we have $\mathfrak{y}(l) = \frac{1}{5l^5}$,

$$Q(l) = (1 - 32p_0)^{1/5}\sum_{i=1}^{n}\gamma_i,$$

and

$$\delta = \frac{1}{5}(1 - 32p_0)\left(\sum_{i=1}^{n}\gamma_i\right)^5.$$

By Theorem 1 we have that Equation (18) is oscillatory if

$$\frac{2}{5}(1 - 32p_0)\left(\sum_{i=1}^{n}\gamma_i\right)^5 > 1.$$

3. Conclusions

In this paper, in the non-canonical case, we investigated the oscillatory behavior of a class of second-order NDEs. We obtained a new condition $\beta_0 > 1/2$ that guarantees the oscillation of all solutions. In addition, we used an iterative approach to improve this condition if it is not met. Finally, we applied our results to some special cases of the studied equation. As future work, it would be interesting to extend the results obtained to even-order equations, as well as to the advanced cases.

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