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Article

On Kudriasov Conditions for Univalence of Integral Operators Defined by Generalized Bessel Functions

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Abstract: In this article, we studied the necessary conditions for the univalence of integral operators that involve two functions: the generalized Bessel function and a function from the well-known class of normalized analytic functions in the open unit disk. The main tools for our discussions were the Kudriasov conditions for the univalency of functions, as well as functional inequalities for the generalized Bessel functions. We included the conditions for the univalency of integral operators that involve Bessel, modified Bessel and spherical Bessel functions as special cases. Furthermore, we provided sufficient conditions for the integral operators that involve trigonometric, as well as hyperbolic, functions as an application of our results.

Keywords: Bessel functions; modified Bessel functions; spherical Bessel functions; integral operators; Kudriasov conditions; univalence criteria

MSC: 30C45; 30C50

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1. Introduction and Preliminaries

Special functions are functions that do not have a formal definition but are still widely used in mathematical analysis, physics, functional analysis and some other branches of applied science. Many elementary functions, such as trigonometric and hyperbolic functions, are also treated as special functions. The theory of special functions has earned the attention of many researchers throughout the nineteenth century and has been involved in many emerging fields. In particular, trigonometric functions have been used in astronomy due to their remarkable properties. In the twentieth century, the application of special functions enriched other branches of mathematics, such as topology, algebra, functional and real analysis and differential equations. Among the most popular and widely studied special functions, Bessel functions have a prominent position due to their applications and remarkable properties. Bessel functions and hypergeometric functions have been used in many emerging fields, such as probability, statistics, mathematics, applied physics and engineering science. Watson summarized all of the characteristics and applications of Bessel functions in his book [1]. This book is very important in the theory of special functions and is considered as a classical text on the asymptotic behavior of Bessel functions and their applications. We refer the reader to [2–6] for more information on generalized Bessel functions.

Special functions in general, and Bessel functions and hypergeometric functions in particular, have a wide variety of applications within the theory of analytic functions. Their use in the proof of the landmark result "Bieberbach conjecture"in 1986, later known as "De Branges' theorem" initiated their meaningful involvement in the study of the geometric characteristics of analytic functions. As well as the use of special functions to solve many problems in geometric function theory, the geometric properties of many special functions, such as Bessel functions [7], Mittag Leffler functions [8–10], Dini functions [11–13], Gauss hypergeometric functions [14,15], Struve functions [16,17], Wright functions [18,19] and others, have been studied extensively. In this article, we intended to study the criteria for the univalency of the integral operators that are defined by using Bessel, modified Bessel and spherical Bessel functions.

Let $\mathcal{A}$ denote the class of analytic functions g in the form:

$$g(\zeta) = \zeta + \sum_{m=2}^{\infty} a_m \zeta^m, \tag{1}$$

in the open unit disk $E = \{\zeta : |\zeta| < 1\}$, where $a_m = \frac{g^{(m)}(0)}{m!}$ is a complex constant, and let $\mathcal{S}$ denote the collection of all univalent functions in $\mathcal{A}$. Assuming the following second-order differential equation, which is homogeneous:

$$\zeta^2 s''(\zeta) + c\zeta s'(\zeta) + \left(\eta\zeta^2 - \sigma^2 + (1-c)\sigma\right) s(\zeta) = 0, \tag{2}$$

where $c,\ \eta, \sigma \in \mathbb{C}$ (see [7] for more details), the generalized Bessel function of the first kind and of order σ is defined as follows:

$$s_{\sigma,c,\eta}(\zeta) = \sum_{m=0}^{\infty} \frac{(-\eta)^m (\zeta/2)^{2m+\sigma}}{m!\Gamma(\sigma + m + (c+1)/2)}, \tag{3}$$

where the notation $\Gamma(.)$ represents the gamma function. The function $s_{\sigma,c,\eta}(\zeta)$ describes certain types of Bessel functions, as follows.

Special Cases

(i) For $c = 1, \eta = 1$, the function that was defined in (3) produces the Bessel function of the first kind and of order σ, which are given by:

$$J_\sigma(\zeta) = \sum_{m=0}^{\infty} \frac{(-1)^m (\zeta/2)^{2m+\sigma}}{m!\Gamma(\sigma + m + 1)}. \tag{4}$$

(ii) For $c = 1,\ \eta = -1$, the function that was defined in (3) produces the modified Bessel function of the first kind and of order σ, which are given by:

$$I_\sigma(\zeta) = \sum_{m=0}^{\infty} \frac{(\zeta/2)^{2m+\sigma}}{m!\Gamma(\sigma + m + 1)}. \tag{5}$$

(iii) For $c = 2,\ \eta = 1$, the function that was defined in (3) produces the spherical Bessel function of the first kind and of order σ, which are given by:

$$j_\sigma(\zeta) = \sqrt{\frac{\pi}{2}} \sum_{m=0}^{\infty} \frac{(\zeta/2)^{2m+\sigma}}{m!\Gamma(\sigma + m + 3/2)}. \tag{6}$$

For more details about these functions, see [1,20].

Deniz et al. [21] studied the generalized Bessel functions that were defined in (3) by discussing the specific geometric properties of the following function:

$$\psi_{\sigma,c,\eta}(\zeta) = 2^\sigma \Gamma\left(\sigma + \frac{c+1}{2}\right) \zeta^{1-\sigma/2} w_{\sigma,c,\eta}\left(\zeta^{1/2}\right),$$

where $c, \eta, \sigma \in \mathbb{C}$. With the help of the renowned Pochhammer symbol, which is defined as:

$$(\beta)_m = \frac{\Gamma(\beta+m)}{\Gamma(\beta)} = \begin{cases} 1, & m=0,\ \beta \in \mathbb{C}\backslash\{0\}, \\ \beta(\beta+1)\ldots(\beta+m-1), & m \in \mathbb{N},\ \beta \in \mathbb{C}, \end{cases}$$

we obtain the following series form of $\psi_{\sigma,c,\eta}(\zeta)$:

$$\psi_{\sigma,c,\eta}(\zeta) = \zeta + \sum_{m=1}^{\infty} \frac{(-\eta)^m}{4^m (l)_m m!} \zeta^{m+1}, \tag{7}$$

where $l = \sigma + (c+1)/2 \neq 0, -1, -2, \cdots$.

For the univalency, convexity, functional inequalities, starlikeness, integral representations, uniform convexity and some other geometric characteristics of $\psi_{\sigma,c,\eta}$, we refer the readers to [7,20,22–28].

A significant area within function theory is the study of integral operators in the class of analytic functions ([29]). Alexander was the first mathematician to introduce and study an integral operator in a family of univalent functions within this area of research. R. Libera, S. Bernardi, S. S. Miller, P.T. Mocanu, M. O. Reade, R. Singh, N. N. Pascu and D. Breaz are among the major contributors to the study of the univalency criteria for integral operators. Nowadays, new frontiers for integral operators are designed to stimulate interest among young researchers within the field of geometric function theory. For more information on the integral operators of analytic functions, see references [30–34].

Baricz and Frasin [35] first used a special function (the Bessel function) to introduce a single family integral operator and studied its univalency conditions. This operator was further studied by Frasin [36], Ularu [37] and Arif and Raza [38]. By using generalized Bessel functions, Deniz et al. and Deniz [21,39] studied the univalency and convexity properties of the following integral operators:

$$G_{\sigma_1,\ldots\sigma_m,c,\eta,\alpha_1,\ldots\alpha_m,\beta}(\zeta) = \left\{ \beta \int_0^{\zeta} t^{\beta-1} \prod_{i=1}^{m} \left(\frac{\psi_{\sigma_i,c,\eta}(t)}{t} \right)^{\frac{1}{\alpha_i}} dt \right\}^{1/\beta}, \tag{8}$$

$$G_{\sigma_1,\ldots\sigma_m,c,\eta,m,\mu}(\zeta) = \left\{ (m\mu+1) \int_0^{\zeta} \prod_{i=1}^{m} (\psi_{\sigma_i,c,\eta}(t))^{\mu} dt \right\}^{1/(m\mu+1)}, \tag{9}$$

$$H_{\sigma_1,\ldots\sigma_m,c,\eta,\delta_1,\ldots\delta_m,\mu}(\zeta) = \left\{ \mu \int_0^{\zeta} t^{\mu-1} \prod_{i=1}^{m} \left(\psi'_{\sigma_i,c,\eta}(t) \right)^{\delta_j} dt \right\}^{1/\mu}, \tag{10}$$

$$Q_{\sigma,c,\eta,\lambda}(\zeta) = \left\{ \lambda \int_0^{\zeta} t^{\overset{m}{\lambda-1}} e^{(\psi_{\sigma_i,c,\eta}(t))^{\lambda}} dt \right\}^{1/\lambda}. \tag{11}$$

Recently, some authors have studied the families of one parameter integral operator using certain special functions, such as Mittag–Leffler functions [40], Struve functions [41], Lommel functions [42] and Dini functions [43]. The geometric properties of these one parameter families of integral operators have been extensively studied by various authors. For details, we refer the reader to [44–50].

The main aim of this article is to introduce and study the univalence criteria for integral operators that involve two functions: the generalized Bessel function and the function of normalized analytic functions. These integral operators are defined as follows:

$$G_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m,\beta}(\zeta) = \left\{ \beta \int_0^{\zeta} t^{\beta-1} \prod_{i=1}^{m} \left(\frac{\psi_{\sigma_i,c,\eta}(t)}{g_i(t)} \right)^{\mu_i} dt \right\}^{1/\beta}, \tag{12}$$

$$H_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m,\beta}(\zeta) = \left\{ \beta \int_0^\zeta t^{\beta-1} \prod_{i=1}^m \left(\frac{\psi'_{\sigma i_i,c,\eta}(t)}{g'_i(t)} \right)^{\mu_i} dt \right\}^{1/\beta}, \tag{13}$$

and

$$I_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m,\beta}(\zeta) = \left\{ \beta \int_0^\zeta t^{\beta-1} \prod_{i=1}^m \left(\frac{\psi_{\sigma_i,c,\eta}(t)}{t} \right)^{\mu_i} (g'_i(t))^{\delta_i} dt \right\}^{1/\beta}. \tag{14}$$

To proceed further, we need the following results, which are also helpful in proving our consequent results:

Lemma 1. *[51] Let $g \in \mathcal{A}$ satisfies the following inequality:*

$$\left(\frac{1 - |\zeta|^{2Re(\nu)}}{Re(\nu)} \right) \left| \frac{\zeta g''(\zeta)}{g'(\zeta)} \right| \le 1, \;\; Re(\nu) > 0.$$

Then, for every complex number β, $Re(\beta) \ge Re(\nu)$ produces the function:

$$G_\beta(\zeta) = \left(\beta \int_0^\zeta t^{\beta-1} g'(t) dt \right)^{\frac{1}{\beta}} \in S.$$

Lemma 2. *[52] Let g be a regular function in E and $g(\zeta) = \zeta + a_2\zeta^2 + \ldots$. When:*

$$\left| \frac{g''(\zeta)}{g'(\zeta)} \right| \le \xi, \;\; \zeta \in E$$

where $\xi \simeq 3.05$, then g is univalent.

Remark 1. *The constant ξ is the solution for:*

$$8\left[y(y-2)^3 \right]^{1/2} - 3(4-y)^2 - 12 = 0.$$

The approximated solution of the above equation is 3.03902118847875. Kudriasov used 3.05 as this approximated value.

Lemma 3. *[21] Let σ, $c \in \mathbb{R}$ and $\eta \in \mathbb{C}$ be restricted so that $l > \max\{0, (|\eta| - 2)/4\}$. Then, the function $\psi_{\sigma,c,\eta} : E \to \mathbb{C}$ is given by (6) such that:*

(i)

$$\left| \frac{\zeta \psi'_{\sigma,c,\eta}(\zeta)}{\psi_{\sigma,c,\eta}(\zeta)} - 1 \right| \le \frac{8(l+1)|\eta|}{32l(l+1) - 8(2l+1)|\eta| + |\eta|^2}, \; \zeta \in E,$$

(ii)

$$\left| \frac{\zeta \psi''_{\sigma,c,\eta}(\zeta)}{\psi'_{\sigma,c,\eta}(\zeta)} \right| \le \frac{4(l+1)|\eta| + |\eta|^2}{8l(l+1) - 2(3l+2)|\eta|}, \; \zeta \in E.$$

2. Main Results

Our first main result provided sufficient univalence conditions for the integral operators of the type in (12) when the function $g_i \in \mathcal{A}$ $(i = 1, \cdots, n)$ and the generalized Bessel function $\psi_{\sigma_i,c,\eta}$ involved some parameters. We used Lemma 3, the univalence criteria for integral operators due to Pascu [51] and the Kudriasov conditions for the univalence of the normalized analytic functions.

Theorem 1. *Let* $\sigma_1, \cdots, \sigma_m$, $c \in \mathbb{R}$, $\mu_i, \eta \in \mathbb{C}$ *and* $l_i > (|\eta| - 2)/4$ *where* $l_i = \sigma_i + (c+1)/2$, $i = 1, \ldots, m$. *Let* $\psi_{\sigma_i,c,\eta} : E \to \mathbb{C}$ *be given by:*

$$\psi_{\sigma_i,c,\eta}(\zeta) = 2^{\sigma_i}\Gamma\left(\sigma_i + \frac{(c+1)}{2}\right)\zeta^{1-\sigma_i/2}\psi_{\sigma_i,c,\eta}\left(\sqrt{\zeta}\right).$$

Suppose $l = \min\{l_1, l_2, \ldots l_m\}$ *and when* $g_i \in \mathcal{A}$ *with:*

$$\left|\frac{g_i''(\zeta)}{g_i'(\zeta)}\right| \leq \xi, \qquad \zeta \in E,$$

where $\xi \simeq 3.05$, *such that:*

$$\frac{1}{Re(\nu)}\left(5 + \frac{8(l+1)|\eta|}{32l(l+1) - 8(2l+1)|\eta| + |\eta|^2}\right)\sum_{i=1}^{m}|\mu_i| < 1, \tag{15}$$

when $0 < Re(\nu) < 1$ *and:*

$$\left[\frac{1}{Re(\nu)}\left(1 + \frac{8(l+1)|\eta|}{32l(l+1) - 8(2l+1)|\eta| + |\eta|^2}\right) + 4\right]\sum_{i=1}^{m}|\mu_i| < 1 \tag{16}$$

for $Re(\nu) \geq 1$, *then, for* $\beta \in \mathbb{C}$ *and* $Re(\beta) \geq Re(\nu) > 0$, *the function* $G_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m,\beta} : E \to \mathbb{C}$ *that is given by* (12) *is in* $\mathcal{S}$.

Proof. Consider:

$$G_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta) = \int_0^{\zeta}\prod_{i=1}^{m}\left(\frac{\psi_{\sigma_i,c,\eta}(t)}{g_i(t)}\right)^{\mu_i}dt. \tag{17}$$

It is clear that $G_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(0) = G'_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m} - 1 = 0$. So, it follows easily that:

$$\frac{G''_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}{G'_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)} = \sum_{i=1}^{m}\mu_i\left\{\frac{\psi'_{\sigma_i,c,\eta}(\zeta)}{\psi_{\sigma_i,c,\eta}(\zeta)} - \frac{g_i'(\zeta)}{g_i(\zeta)}\right\}.$$

Therefore, we obtain:

$$\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\left|\frac{\zeta G''_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}{G'_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}\right| \leq \frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\sum_{i=1}^{m}|\mu_i|\left\{\left|\frac{\psi'_{\sigma_i,c,\eta}(\zeta)}{\psi_{\sigma_i,c,\eta}(\zeta)}\right| + \left|\frac{\zeta g_i'(\zeta)}{g_i(\zeta)}\right|\right\}. \tag{18}$$

Now, using Lemma 2, we have $g_i \in \mathcal{S}$, $i = 1 \ldots m$, and:

$$\left|\frac{\zeta g_i'(\zeta)}{g_i(\zeta)}\right| \leq \frac{1+|\zeta|}{1-|\zeta|}. \tag{19}$$

By virtue of (18) and (19), we find that:

$$\begin{aligned}
\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\left|\frac{\zeta G''_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}{G'_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}\right| &\leq \frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\sum_{i=1}^{m}|\mu_i|\left\{\left|\frac{\zeta\psi'_{\sigma_i,c,\eta}(\zeta)}{\psi_{\sigma_i,c,\eta}(\zeta)}\right| + \frac{1+|\zeta|}{1-|\zeta|}\right\} \\
&\leq \frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\sum_{i=1}^{m}|\mu_i|\left|\frac{\zeta\psi'_{\sigma_i,c,\eta}(\zeta)}{\psi_{\sigma_i,c,\eta}(\zeta)}\right| \\
&\quad + \frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\frac{2}{1-|\zeta|}\sum_{i=1}^{m}|\mu_i|.
\end{aligned}$$

First, we consider:

$$\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\sum_{i=1}^{m}|\mu_i|\left|\frac{\zeta\psi\prime_{\sigma_i,c,\eta}(\zeta)}{\psi_{\sigma_i,c,\eta}(\zeta)}\right|.$$

This implies that:

$$\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\sum_{i=1}^{m}|\mu_i|\left|\frac{\zeta\psi'_{\sigma_i,c,\eta}(\zeta)}{\psi_{\sigma_i,c,\eta}(\zeta)}\right| \leq \frac{1}{Re(\nu)}\sum_{i=1}^{m}|\mu_i|\left|\frac{\zeta\psi'_{\sigma_i,c,\eta}(\zeta)}{\psi_{\sigma_i,c,\eta}(\zeta)}\right|.$$

Using Lemma 3 (i), we have:

$$\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\sum_{i=1}^{m}|\mu_i|\left|\frac{\zeta\psi\prime_{\sigma_i,c,\eta}(\zeta)}{\psi_{\sigma_i,c,\eta}(\zeta)}\right| \leq \frac{1}{Re(\nu)}\sum_{i=1}^{m}|\mu_i|\left\{1+\frac{8(l_i+1)|\eta|}{32l_i(2l_i+1)-8(2l_i+1)|\eta|+|\eta|^2}\right\}.$$

Now, we take the function $u : \left(\frac{|\eta|-2}{4},\infty\right) \to \mathbb{R}$, $u(y) = \frac{8|\eta|(y+1)}{8y(y+1)-32|\eta|(2y+1)+|\eta|^2}$. It is a decreasing function, therefore:

$$\frac{8(l_i+1)|\eta|}{32l_i(2l_i+1)-8(2l_i+1)|\eta|+|\eta|^2} \leq \frac{8(l+1)|\eta|}{32l(l+1)-8(2l+1)|\eta|+|\eta|^2}.$$

Hence:

$$\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\sum_{i=1}^{m}|\mu_i|\left|\frac{\zeta\psi'_{\sigma_i,c,\eta}(\zeta)}{\psi_{\sigma_i,c,\eta}(\zeta)}\right| \leq$$
$$\frac{1}{Re(\nu)}\left\{1+\frac{8(l+1)|\eta|}{32l(l+1)-8(2l+1)|\eta|+|\eta|^2}\right\}\sum_{i=1}^{m}|\mu_i|. \tag{20}$$

Consider:

$$\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\frac{2}{1-|\zeta|}\sum_{i=1}^{m}|\mu_i|.$$

For this, we have the following cases:

(1) For $0 < Re(\nu) < 1$, the function $\sigma : (0,1) \to \mathbb{R}$ is defined by:

$$\sigma(y) = 1-a^{2y},\ y = Re(\nu),\ |\zeta| = a$$

is increasing and:

$$1-|\zeta|^{2Re(\nu)} \leq 1-|\zeta|^2,$$

therefore:

$$\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\frac{2}{1-|\zeta|}\sum_{i=1}^{m}|\mu_i| \leq \frac{4}{Re(\nu)}\sum_{i=1}^{m}|\mu_i|. \tag{21}$$

From (20) and (21) and for $0 < Re(\nu) < 1$, we have:

$$\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\left|\frac{\zeta G''_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}{G'_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}\right| \leq$$
$$\frac{1}{Re(\nu)}\left(5+\frac{8(l+1)|\eta|}{32l(l+1)-8(2l+1)|\eta|+|\eta|^2}\right)\sum_{i=1}^{m}|\mu_i|. \tag{22}$$

(2) For $Re(\nu) \geq 1$, we define the function $w : [1,\infty) \to \mathbb{R}$, $w(y) = \frac{1-a^{2y}}{y}$, $y = Re(\nu)$ and $|\zeta| = a$ as a decreasing function and

$$\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)} \leq 1-|\zeta|^2,$$

therefore:

$$\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)} \frac{2}{1-|\zeta|}\sum_{i=1}^{m}|\mu_i| \leq 4\sum_{i=1}^{m}|\mu_i|. \tag{23}$$

Combining (22) and (23) for $Re(\nu) \geq 1$, we obtain:

$$\left(\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\right)\left|\frac{\zeta G''_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}{G'_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}\right|$$

$$\leq \frac{1}{Re(\nu)}\left(1+4Re(\nu)+\frac{8(l+1)|\eta|}{32l(l+1)-8(2l+1)|\eta|+|\eta|^2}\right)\sum_{i=1}^{m}|\mu_i|. \tag{24}$$

From (15), (22), (16) and (24), we obtain:

$$\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\left|\frac{\zeta G''_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}{G'_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}\right| < 1.$$

Now, from (17), we have $G'_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta) = \prod_{i=1}^{m}\left(\frac{\psi_{\sigma_i,c,\eta}(t)}{g_i(t)}\right)^{\mu_i}$. Therefore, using Lemma 1, we can obtain the required result. □

Our second main result provided sufficient univalence conditions for the integral operators of the type in (13) when the function $g_i \in \mathcal{A}$ $(i = 1, \cdots, n)$ and the generalized Bessel function $\psi_{\sigma_i,c,\eta}$ involved some parameters. We used Lemma 3, the univalence criteria for integral operators due to Pascu [51] and the Kudriasov conditions for the univalence of the normalized analytic functions.

Theorem 2. *Let $\sigma_1, \ldots \sigma_m, c \in \mathbb{R}$, $\mu_i, \eta \in \mathbb{C}$ and $l_i > (|\eta|-2)/4$ where $l_i = \sigma_i + (c+1)/2$, $i = 1, \ldots, m$. Let $\psi_{\sigma_i,c,\eta} : E \to \mathbb{C}$ be given by:*

$$\psi_{\sigma_i,c,\eta}(\zeta) = 2^{\sigma_i}\Gamma\left(\sigma_i + \frac{(c+1)}{2}\right)\zeta^{1-\sigma_i/2}\psi_{\sigma_i,c,\eta}\left(\sqrt{\zeta}\right).$$

Suppose $l = \min\{l_1, l_2, \ldots l_m\}$ and when $g_i \in A$ with:

$$\left|\frac{g''_i(\zeta)}{g'_i(\zeta)}\right| \leq \xi,$$

for all $\zeta \in E$, where $\xi \simeq 3.05$ and these numbers satisfy the relation:

$$\left\{\frac{1}{Re(\nu)}\frac{4(l+1)|\eta|+|\eta|^2}{8l(l+1)-2(3l+2)|\eta|} + \frac{2\xi}{(2Re(\nu)+1)^{(2Re(\nu)+1)/2Re(\nu)}}\right\}\sum_{i=1}^{m}|\mu_i| < 1. \tag{25}$$

Then, for $\beta \in \mathbb{C}$ and $Re(\beta) \geq Re(\nu)$, the function $H_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m,\beta} : E \to \mathbb{C}$ that is given by (13) is in $\mathcal{S}$.

Proof. Consider the function:

$$H_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta) = \int_0^\zeta \prod_{i=1}^{m}\left(\frac{\psi'_{\sigma i_i,c,\eta}(t)}{g'_i(t)}\right)^{\mu_i} dt.$$

Clearly, $H_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m} \in \mathcal{A}$, i.e., $H_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(0) = H'_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m} - 1 = 0$. On the other hand, it is easy to see that:

$$\frac{H''_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}{H'_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)} = \sum_{i=1}^{m}\mu_i\left\{\frac{\psi''_{\sigma_i,c,\eta}(\zeta)}{\psi'_{\sigma_i,c,\eta}(\zeta)} - \frac{g''_i(\zeta)}{g'_i(\zeta)}\right\}$$

Therefore, we obtain:

$$\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\left|\frac{\zeta H''_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}{H'_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}\right| \leq \frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\sum_{i=1}^{m}|\mu_i|\left\{\left|\frac{\psi''_{\sigma_i,c,\eta}(\zeta)}{\psi'_{\sigma_i,c,\eta}(\zeta)}\right| + |\zeta|\left|\frac{g''_i(\zeta)}{g'_i(\zeta)}\right|\right\}. \tag{26}$$

This implies that:

$$\begin{aligned}\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\left|\frac{\zeta H''_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}{H'_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}\right| \quad &\leq \quad \frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\sum_{i=1}^{m}|\mu_i|\left|\frac{\zeta\psi''_{\sigma_i,c,\eta}(\zeta)}{\psi'_{\sigma_i,c,\eta}(\zeta)}\right| \\ &+\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}|\zeta|\sum_{i=1}^{m}|\mu_i|\left|\frac{g''_i(\zeta)}{g'_i(\zeta)}\right|.\end{aligned}$$

Using the Lemmas 2 and 3 (ii), we obtain:

$$\begin{aligned}\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\left|\frac{\zeta H''_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}{H'_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}\right| \quad &\leq \quad \frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\sum_{i=1}^{m}|\mu_i|\frac{4(l_i+1)|\eta|+|\eta|^2}{8l_i(l_i+1)-2(3l_i+2)|\eta|} \\ &+\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}|\zeta|\xi\sum_{i=1}^{m}|\mu_i|.\end{aligned}$$

Define $h:[0,1]\to\mathbb{R},\ j(y) = y\left(1-y^{2a}\right)/a,\ y=|\zeta|,\ a=Re(\nu)$. Then:

$$\max_{y\in[0,1]} h(y) = \frac{2}{(2a+1)^{(2a+1)/2a}}.$$

Additionally, define $l:\left(\frac{|\eta|-2}{4},\infty\right)\to\mathbb{R},\ l(y) = \frac{4|\eta|(y+1)+|\eta|^2}{8y(y+1)-2|\eta|(3y+2)}$. The function l is decreasing, therefore:

$$\frac{4(l_i+1)|\eta|+|\eta|^2}{8l_i(1+l_i)-2|\eta|(2+3l_i)} \leq \frac{4|\eta|(l+1)+|\eta|^2}{8l(1+l)-2|\eta|(2+3l)}.$$

This implies that:

$$\begin{aligned}\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\left|\frac{\zeta H''_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}{H'_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}\right| \quad &\leq \quad \frac{1}{Re(\nu)}\frac{4(l+1)|\eta|+|\eta|^2}{8l(l+1)-2(3l+2)|\eta|}\sum_{i=1}^{m}|\mu_i| \\ &+\frac{2\xi}{(2Re(\nu)+1)^{(2Re(\nu)+1)/2Re(\nu)}}\sum_{i=1}^{m}|\mu_i|.\end{aligned}$$

Using (25) and Lemma 1, we can obtain the required result. □

Our third main result provided sufficient univalence conditions for the integral operators of the type in (14) when the function $g_i \in \mathcal{A}$ $(i=1,\cdots,n)$ and the generalized Bessel function $\psi_{\sigma_i,c,\eta}$ involved some parameters. We used Lemma 3, the univalence criteria for integral operators due to Pascu [51] and the Kudriasov conditions for the univalence of the normalized analytic functions.

Theorem 3. *Let $\sigma_1, \ldots \sigma_m, c \in \mathbb{R}$, μ_i, δ_i $\eta \in \mathbb{C}$ and $l_i > (|\eta| - 2)/4$ where $l_i = \sigma_i + (c+1)/2$, $i = 1, \ldots, m$. Let $\psi_{\sigma_i,c,\eta} : E \to \mathbb{C}$ be given as:*

$$\psi_{\sigma_i,c,\eta}(\zeta) = 2^{\sigma_i}\Gamma\left(\sigma_i + \frac{(c+1)}{2}\right)\zeta^{1-\sigma_i/2}\psi_{\sigma_i,c,\eta}\left(\sqrt{\zeta}\right).$$

Suppose $l = \min\{l_1, l_2, \ldots l_m\}$ and when $g_i \in A$ with:

$$\left|\frac{g_i''(\zeta)}{g_i'(\zeta)}\right| \leq \xi, \ \ \zeta \in E$$

where $\xi \simeq 3.05$, such that:

$$\frac{1}{Re(\nu)}\frac{8(l+1)|\eta|}{32l(l+1) - 8(2l+1)|\eta| + |\eta|^2}\sum_{i=1}^{m}|\mu_i| + \frac{2\xi}{(1+2Re(\nu))^{(1+2Re(\nu))/2Re(\nu)}}\sum_{i=1}^{m}|\delta_i| < 1. \tag{27}$$

Then, for $\beta \in \mathbb{C}$ and $Re(\beta) \geq Re(\nu) > 0$, the function $I_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m,\delta,\beta} : E \to \mathbb{C}$ that is given by (14) is in $\mathcal{S}$.

Proof. Consider the function:

$$I_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m,\delta}(\zeta) = \int_0^{\zeta}\prod_{i=1}^{m}\left(\frac{\psi_{\sigma_i,c,\eta}(t)}{t}\right)^{\mu_i}\left(g_i'(t)\right)^{\delta_i}dt. \tag{28}$$

Clearly, $I_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m,\delta} \in \mathcal{A}$, i.e., $I_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m,\delta}(0) = I'_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m,\delta} - 1 = 0$. On the other hand, it is easy to see that:

$$\frac{\zeta I''_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}{I'_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)} = \sum_{i=1}^{m}\mu_i\left(\frac{\psi'_{\sigma_i,c,\eta}(\zeta)}{\psi_{\sigma_i,c,\eta}(\zeta)} - 1\right) + \sum_{i=1}^{m}\delta_i\left\{\frac{\zeta g_i''(\zeta)}{g_i'(\zeta)}\right\}.$$

This implies that:

$$\begin{aligned}&\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\left|\frac{\zeta I''_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}{I'_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}\right| \\ \leq\ &\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\sum_{i=1}^{m}\left\{|\mu_i|\left|\frac{\psi'_{\sigma_i,c,\eta}(\zeta)}{\psi_{\sigma_i,c,\eta}(\zeta)} - 1\right| + |\zeta||\delta_i|\left|\frac{g_i''(\zeta)}{g_i'(\zeta)}\right|\right\}.\end{aligned} \tag{29}$$

Hence:

$$\begin{aligned}\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\left|\frac{\zeta I''_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}{I'_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}\right| \ &\leq\ \frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\sum_{i=1}^{m}|\mu_i|\left|\frac{\psi'_{\sigma_i,c,\eta}(\zeta)}{\psi_{\sigma_i,c,\eta}(\zeta)} - 1\right| \\ &+\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}|\zeta|\sum_{i=1}^{m}|\delta_i|\left|\frac{g_i''(\zeta)}{g_i'(\zeta)}\right|.\end{aligned}$$

Using the Lemmas 2 and 3 (i), we obtain:

$$\begin{aligned}&\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\left|\frac{\zeta I''_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}{I'_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}\right| \leq \\ &\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\sum_{i=1}^{m}|\mu_i|\frac{8(l_i+1)|\eta|}{32l_i(2l_i+1) - 8(2l_i+1)|\eta| + |\eta|^2} + \frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}|\zeta|\xi\sum_{i=1}^{m}|\delta_i|.\end{aligned}$$

We also see that:

$$\max_{x\in[0,1]} h(x) = \frac{2}{(2a+1)^{(2a+1)/2a}},$$

and
$$\frac{8(l_i+1)|\eta|}{32l_i(2l_i+1)-8(2l_i+1)|\eta|+|\eta|^2} \le \frac{8(l+1)|\eta|}{32l(l+1)-8(2l+1)|\eta|+|\eta|^2}.$$
Therefore:
$$\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\left|\frac{\zeta I''_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}{I'_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}\right| \le \frac{1}{Re(\nu)}\frac{8(l+1)|\eta|}{32l(l+1)-8(2l+1)|\eta|+|\eta|^2}\sum_{i=1}^{m}|\mu_i| + \frac{2\xi}{(2Re(\nu)+1)^{(2Re(\nu)+1)/2Re(\nu)}}\sum_{i=1}^{m}|\delta_i|. \tag{30}$$

Using (27) and (30), we obtain:
$$\frac{1-|\zeta|^{2Re(\nu)}}{Re(\nu)}\left|\frac{\zeta I''_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}{I'_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m}(\zeta)}\right| < 1.$$

Now, from (28), it is clear that $I'_{\sigma_1,\ldots\sigma_m,c,\eta,\mu_1,\ldots\mu_m,\delta}(\zeta) = \prod_{i=1}^{m}\left(\frac{\varphi_{\sigma_i,c,\eta}(t)}{t}\right)^{\mu_i}(g_i'(t))^{\delta_i}$. The result can be obtained using Lemma 1. □

3. Applications

Now, we provide some applications for our results in terms of the univalence of integral operators that involve Bessel, modified Bessel and spherical Bessel functions. We also present particular examples for trignometric and hyperbolic functions.

3.1. Bessel Functions

By choosing $c = 1$ and $\eta = 1$ in (2) and (3), we obtain the Bessel functions of the first kind and of order σ that are defined by (4). Further, we have $\mathcal{J}_{3/2}(\zeta) = \frac{3\sin\sqrt{\zeta}}{\sqrt{\zeta}} - 3\cos\sqrt{\zeta}$, $\mathcal{J}_{1/2}(\zeta) = \sqrt{\zeta}\sin\sqrt{\zeta}$ and $\mathcal{J}_{-1/2}(\zeta) = \sqrt{\zeta}\cos\sqrt{\zeta}$.

Corollary 1. *Let $\mathcal{J}_\sigma : E \to \mathbb{C}$ be introduced as $\mathcal{J}_\sigma(\zeta) = 2^\sigma\Gamma(\sigma+1)\zeta^{1-\sigma/2}J_\sigma(\sqrt{\zeta})$. Let $\sigma_1,\ldots,\sigma_m > -1.25$. Additionally consider that $\sigma = \min\{\sigma_1,\ldots,\sigma_m\}$ and $\mu_1,\ldots\mu_m$, as in Theorem 1, and when $g_i \in \mathcal{A}$ with:*
$$\left|\frac{g_i''(\zeta)}{g_i'(\zeta)}\right| \le \xi,\ \ \zeta \in E.$$
These numbers satisfy the relations:
$$\frac{1}{Re(\nu)}\left(5+\frac{\sigma+2}{4\sigma^2+10\sigma+41/8}\right)\sum_{i=1}^{m}|\mu_i| < 1,$$
then, when $0 < Re(\nu) < 1$ and for $Re(\nu) \ge 1$:
$$\left[\frac{1}{Re(\nu)}\left(1+\frac{\sigma+2}{4\sigma^2+10\sigma+41/8}\right)+4\right]\sum_{i=1}^{m}|\mu_i| < 1.$$
Then, for $\beta \in \mathbb{C}$ and $Re(\beta) \ge Re(\nu) > 0$, the function $G_{\sigma_1,\ldots\sigma_m,\mu_1,\ldots\mu_m,\beta} : E \to \mathbb{C}$ that is given by:
$$G_{\sigma_1,\ldots\sigma_m,\mu_1,\ldots\mu_m,\beta}(\zeta) = \left\{\beta\int_0^\zeta t^{\beta-1}\prod_{i=1}^{m}\left(\frac{\mathcal{J}_{\sigma_i}(\zeta)(t)}{g_i(t)}\right)^{\mu_i}dt\right\}^{1/\beta},$$
is in $\mathcal{S}$.

1. In particular, for:

$$\frac{|\mu|}{Re(\nu)}\frac{960}{233}<1,$$

when $0<Re(\nu)<1$ and for $Re(\nu)\geq 1$:

$$\left[\frac{1}{Re(\nu)}\left(\frac{261}{233}\right)+4\right]|\mu|<1.$$

Then, for $\beta\in\mathbb{C}$ and $Re(\beta)\geq Re(\nu)>0$, the function $G_{3/2,\mu,\beta}:E\to\mathbb{C}$ that is given by:

$$G_{3/2,\mu,\beta}(\zeta)=\left\{\beta\int_0^{\zeta}t^{\beta-1}\left(\frac{3\sin\sqrt{t}-3\sqrt{t}\cos\sqrt{t}}{t\sqrt{t}}\right)^{\mu}dt\right\}^{1/\beta}$$

is in $\mathcal{S}$.

2. For:

$$\frac{|\mu|}{Re(\nu)}\frac{376}{89}<1,$$

when $0<Re(\nu)<1$ and for $Re(\nu)\geq 1$:

$$\left[\frac{1}{Re(\nu)}\left(\frac{109}{89}\right)+4\right]|\mu|<1.$$

Then, for $\beta\in\mathbb{C}$ and $Re(\beta)\geq Re(\nu)>0$, the function $G_{1/2,\mu,\beta}:E\to\mathbb{C}$ that is given by:

$$G_{1/2,\mu,\beta}(\zeta)=\left\{\beta\int_0^{\zeta}t^{\beta-1}\left(\frac{\sqrt{t}\sin\sqrt{t}}{t}\right)^{\mu}dt\right\}^{1/\beta}$$

is in $\mathcal{S}$.

3. For:

$$\frac{|\mu|}{Re(\nu)}\frac{16}{3}<1,$$

when $0<Re(\nu)<1$ and for $Re(\nu)\geq 1$:

$$\left[\frac{1}{Re(\nu)}\left(\frac{7}{3}\right)+4\right]|\mu|<1.$$

Then, for $\beta\in\mathbb{C}$ and $Re(\beta)\geq Re(\nu)>0$, the function $G_{-1/2,\mu,\beta}:E\to\mathbb{C}$ that is given by:

$$G_{-1/2,\mu,\beta}(\zeta)=\left\{\beta\int_0^{\zeta}t^{\beta-1}\left(\frac{\sqrt{t}\cos\sqrt{t}}{t}\right)^{\mu}dt\right\}^{1/\beta}$$

is in $\mathcal{S}$.

Corollary 2. *Consider $\mathcal{J}_\sigma:E\to\mathbb{C}$ to be defined as $\mathcal{J}_\sigma(\zeta)=2^{\sigma}\Gamma(\sigma+1)\zeta^{1-\sigma/2}J_\sigma(\sqrt{\zeta})$. Let $\sigma_1,\ldots,\sigma_m>-1.25$. Additionally consider that $\sigma=\min\{\sigma_1,\ldots,\sigma_m\}$ and $\mu_1,\ldots\mu_m$, as in Theorem 2, and when $g_i\in A$ with:*

$$\left|\frac{g_i''(\zeta)}{g_i'(\zeta)}\right|\leq\xi,\ \ \zeta\in E.$$

These numbers satisfy the relations:

$$\left\{\frac{1}{Re(\nu)}\frac{4\sigma+9}{8\sigma^2+18\sigma+6}+\frac{2\xi}{(2Re(\nu)+1)^{(2Re(\nu)+1)/2Re(\nu)}}\right\}\sum_{i=1}^{m}|\mu_i|<1.$$

Then, for $\beta\in\mathbb{C}$ and $Re(\beta)\geq Re(\nu)>0$, the function $H_{\sigma_1,\dots\sigma_m,\mu_1,\dots\mu_m,\beta}:E\to\mathbb{C}$ that is given by:

$$H_{\sigma_1,\dots\sigma_m,\mu_1,\dots\mu_m,\beta}(\zeta)=\left\{\beta\int_0^{\zeta}t^{\beta-1}\prod_{i=1}^{m}\left(\frac{\mathcal{J}'_{\sigma_i}(t)}{g'_i(t)}\right)^{\mu_i}dt\right\}^{1/\beta},$$

is in $\mathcal{S}$.

1. In particular, when $\frac{5|\mu|}{17Re(\nu)}<1$, then, for $\beta\in\mathbb{C}$ and $Re(\beta)\geq Re(\nu)>0$, the function $H_{3/2,\mu,\beta}:E\to\mathbb{C}$ that is given by:

$$H_{3/2,\mu,\beta}(\zeta)=\left\{\beta\int_0^{\zeta}t^{\beta-1}\left(\frac{3t(t-1)\sin\sqrt{t}-3t\sqrt{t}\cos\sqrt{t}}{2t^2\sqrt{t}}\right)^{\mu}dt\right\}^{1/\beta}$$

is in $\mathcal{S}$.

2. When $\frac{11|\mu|}{17Re(\nu)}<1$, then, for $\beta\in\mathbb{C}$ and $Re(\beta)\geq Re(\nu)>0$, the function $H_{1/2,\mu,\beta}:E\to\mathbb{C}$ that is given by:

$$H_{1/2,\mu,\beta}(\zeta)=\left\{\beta\int_0^{\zeta}t^{\beta-1}\left(\frac{\sin\sqrt{t}}{2\sqrt{t}}-\frac{\cos\sqrt{t}}{2}\right)^{\mu}dt\right\}^{1/\beta}$$

is in $\mathcal{S}$.

Corollary 3. *Consider $\mathcal{J}_\sigma:E\to\mathbb{C}$ to be defined as $\mathcal{J}_\sigma(\zeta)=2^{\sigma}\Gamma(\sigma+1)\zeta^{1-\sigma/2}J_\sigma(\sqrt{\zeta})$. Let $\sigma_1,\dots,\sigma_m>-1.25$. Additionally consider that $\sigma=\min\{\sigma_1,\dots,\sigma_m\}$ and $\mu_1,\dots\mu_m$, $\delta_1,\dots\delta_m$, as in Theorem 3, and when $g_i\in\mathcal{A}$ with:*

$$\left|\frac{g''_i(\zeta)}{g'_i(\zeta)}\right|\leq\xi,\ \ \zeta\in E.$$

These numbers satisfy the relations:

$$\frac{1}{Re(\nu)}\frac{\sigma+2}{4\sigma^2+10\sigma+41/8}\sum_{i=1}^{m}|\mu_i|+\frac{2\xi}{(2Re(\nu)+1)^{(2Re(\nu)+1)/2Re(\nu)}}\sum_{i=1}^{m}|\delta_i|<1.$$

Then, for $\beta\in\mathbb{C}$ and $Re(\beta)\geq Re(\nu)>0$, the function $I_{\sigma_1,\dots\sigma_m,\mu_1,\dots\mu_m,\delta,\beta}:E\to\mathbb{C}$ that is given by:

$$I_{\sigma_1,\dots\sigma_m,c,\eta,\mu_1,\dots\mu_m,\delta,\beta}(\zeta)=\left\{\beta\int_0^{\zeta}t^{\beta-1}\prod_{i=1}^{m}\left(\frac{\mathcal{J}'_{\sigma_i}(t)}{t}\right)^{\mu_i}\left(g'_i(t)\right)^{\delta_i}dt\right\}^{1/\beta},$$

is in $\mathcal{S}$.

1. In particular, when $g\in\mathcal{A}$ with:

$$\left|\frac{g''(\zeta)}{g'(\zeta)}\right|\leq\xi,\ \ \zeta\in E.$$

$$\frac{4}{3Re(\nu)}|\mu| + \frac{2|\delta|\xi}{(2Re(\nu)+1)^{(2Re(\nu)+1)/2Re(\nu)}} < 1.$$

Then, for $\beta \in \mathbb{C}$ and $Re(\beta) \geq Re(\nu) > 0$, the function $I_{-1/2,\mu,\delta,\beta} : E \to \mathbb{C}$ that is given by:

$$I_{-1/2,\mu,\delta,\beta}(\zeta) = \left\{ \beta \int_0^\zeta t^{\beta-1} \left(\frac{\left(\sqrt{\zeta}\cos\sqrt{\zeta}\right)'}{t} \right)^\mu \left(g'(t)\right)^\delta dt \right\}^{1/\beta}$$

is in $\mathcal{S}$.

3.2. Modified Bessel Functions

By choosing $c = 1$ and $\eta = -1$ in (2) and (3), we obtain the Modified Bessel functions that are given by (5). We also see that $\mathcal{I}_{3/2}(\zeta) = 3\cosh\sqrt{\zeta} - \frac{3\sinh\sqrt{\zeta}}{\sqrt{\zeta}}$, $\mathcal{I}_{1/2}(\zeta) = \sqrt{\zeta}\sinh\sqrt{\zeta}$ and $\mathcal{I}_{-1/2}(\zeta) = \sqrt{\zeta}\cosh\sqrt{\zeta}$.

Corollary 4. *Let $\mathcal{I}_\sigma : E \to \mathbb{C}$ be given by $\mathcal{I}_\sigma(\zeta) = 2^\sigma\Gamma(\sigma+1)\zeta^{1-\sigma/2}I_\sigma(\sqrt{\zeta})$. Let $\sigma_1, \ldots, \sigma_m > -1.25$. Additionally consider that $\sigma = \min\{\sigma_1, \ldots, \sigma_m\}$ and $\mu_1, \ldots \mu_m$, as in Theorem 1, and when $g_i \in A$ such that:*

$$\left|\frac{g_i''(\zeta)}{g_i'(\zeta)}\right| \leq \xi, \ \ \zeta \in E.$$

Additionally:

$$\frac{1}{Re(\nu)}\left(5 + \frac{\sigma+2}{4\sigma^2+10\sigma+41/8}\right)\sum_{i=1}^m |\mu_i| < 1,$$

when $0 < Re(\nu) < 1$ *and:*

$$\left[\frac{1}{Re(\nu)}\left(1 + \frac{\sigma+2}{4\sigma^2+10\sigma+41/8}\right) + 4\right]\sum_{i=1}^m |\mu_i| < 1$$

for $Re(\nu) \geq 1$. Then, for $\beta \in \mathbb{C}$ and $Re(\beta) \geq Re(\nu) > 0$, the function $G_{\sigma_1,\ldots\sigma_m,\mu_1,\ldots\mu_m,\beta} : E \to \mathbb{C}$ that is given by:

$$G_{\sigma_1,\ldots\sigma_m,\mu_1,\ldots\mu_m,\beta}(\zeta) = \left\{ \beta \int_0^\zeta t^{\beta-1} \prod_{i=1}^m \left(\frac{\mathcal{I}_{\sigma_i}(t)}{g_i(t)} \right)^{\mu_i} dt \right\}^{1/\beta},$$

is in $\mathcal{S}$.

1. In particular, for:

$$\frac{|\mu|}{Re(\nu)}\frac{960}{233} < 1,$$

where $0 < Re(\nu) < 1$ and for $Re(\nu) \geq 1$:

$$\left[\frac{1}{Re(\nu)}\left(\frac{261}{233}\right) + 4\right]|\mu| < 1.$$

Then, for $\beta \in \mathbb{C}$ and $Re(\beta) \geq Re(\nu) > 0$, the function $G_{3/2,\mu,\beta} : E \to \mathbb{C}$ that is given by:

$$G_{3/2,\mu,\beta}(\zeta) = \left\{ \beta \int_0^\zeta t^{\beta-1} \left(3\cosh\sqrt{t} - \frac{3\sinh\sqrt{t}}{\sqrt{t}} \right)^\mu dt \right\}^{1/\beta}$$

is in $\mathcal{S}$.

Corollary 5. *Let $\mathcal{I}_\sigma : E \to \mathbb{C}$ be given as $\mathcal{I}_\sigma(\zeta) = 2^\sigma \Gamma(\sigma+1)\zeta^{1-\sigma/2} I_\sigma(\sqrt{\zeta})$. Let $\sigma_1, \ldots, \sigma_m > -1.25$. Additionally consider that $\sigma = \min\{\sigma_1, \ldots, \sigma_m\}$ and $\mu_1, \ldots \mu_m$, as in Theorem 2, and when $g_i \in \mathcal{A}$ such that:*

$$\left|\frac{g_i''(\zeta)}{g_i'(\zeta)}\right| \le \xi, \ \ \zeta \in E.$$

Additionally:

$$\left\{\frac{1}{Re(\nu)}\frac{4\sigma+9}{8\sigma^2+18\sigma+6} + \frac{2\xi}{(2Re(\nu)+1)^{(2Re(\nu)+1)/2Re(\nu)}}\right\}\sum_{i=1}^{m}|\mu_i| < 1.$$

Then, for $\beta \in \mathbb{C}$ and $Re(\beta) \ge Re(\nu) > 0$, the function $H_{\sigma_1,\ldots\sigma_m,\mu_1,\ldots\mu_m,\beta} : E \to \mathbb{C}$ that is given by:

$$H_{\sigma_1,\ldots\sigma_m,\mu_1,\ldots\mu_m,\beta}(\zeta) = \left\{\beta\int_0^\zeta t^{\beta-1}\prod_{i=1}^{m}\left(\frac{\mathcal{I}_{\sigma_i}'(t)}{g_i'(t)}\right)^{\mu i} dt\right\}^{1/\beta},$$

is in $\mathcal{S}$.

1. In particular, when $\frac{11}{17Re(\nu)}|\mu| < 1$, then, for $\beta \in \mathbb{C}$ and $Re(\beta) \ge Re(\nu) > 0$, the function $H_{1/2,\mu,\beta} : E \to \mathbb{C}$ that is given by:

$$H_{1/2,\mu,\beta}(\zeta) = \left\{\beta\int_0^\zeta t^{\beta-1}\left(\frac{\sinh\sqrt{t}}{2\sqrt{t}} + \frac{\cosh\sqrt{t}}{2}\right)^{\mu} dt\right\}^{1/\beta}$$

is in $\mathcal{S}$.

Corollary 6. *Consider $\mathcal{J}_\sigma : E \to \mathbb{C}$ to be defined as $\mathcal{J}_\sigma(\zeta) = 2^\sigma\Gamma(\sigma+1)\zeta^{1-\sigma/2}J_\sigma(\sqrt{\zeta})$. Let $\sigma_1, \ldots, \sigma_m > -1.25$. Additionally consider that $\sigma = \min\{\sigma_1, \ldots, \sigma_m\}$ and $\mu_1, \ldots \mu_m$, $\delta_1, \ldots \delta_m$, as in Theorem 3, and when $g_i \in A$ with:*

$$\left|\frac{g_i''(\zeta)}{g_i'(\zeta)}\right| \le \xi, \ \ \zeta \in E.$$

Let:

$$\frac{1}{Re(\nu)}\frac{\sigma+2}{4\sigma^2+10\sigma+41/8}\sum_{i=1}^{m}|\mu_i| + \frac{2\xi}{(1+2Re(\nu))^{(1+2Re(\nu))/2Re(\nu)}}\sum_{i=1}^{m}|\delta_i| < 1.$$

Then, for $\beta \in \mathbb{C}$ and $Re(\beta) \ge Re(\nu) > 0$, the function $I_{\sigma_1,\ldots\sigma_m,\mu_1,\ldots\mu_m,\ \delta_1,\ldots\delta_m,\beta} : E \to \mathbb{C}$ that is given by:

$$I_{\sigma_1,\ldots\sigma_m,\mu_1,\ldots\mu_m,\ \delta_1,\ldots\delta_m,\beta}(\zeta) = \left\{\beta\int_0^\zeta t^{\beta-1}\prod_{i=1}^{m}\left(\frac{\mathcal{J}_{\sigma_i}'(t)}{t}\right)^{\mu_i}(g_i'(t))^{\delta_i}dt\right\}^{1/\beta}$$

is in $\mathcal{S}$.

1. In particular, when $\frac{4|\mu|}{3Re(\nu)} < 1$, then, for $\beta \in \mathbb{C}$ and $Re(\beta) \ge Re(\nu) > 0$, the function $I_{-1/2,\mu,\beta} : E \to \mathbb{C}$ that is given by:

$$I_{-1/2,\mu,\beta}(\zeta) = \left\{\beta\int_0^\zeta t^{\beta-1}\left(\frac{\cosh\sqrt{t}}{2\sqrt{t}} + \frac{\sinh\sqrt{t}}{2}\right)^{\mu} dt\right\}^{1/\beta}$$

is in $\mathcal{S}$.

3.3. Spherical Bessel Functions

By choosing $c = 2$ and $\eta = 1$ in (2) and (3), we obtain the function $\sqrt{2}j_\sigma(\zeta)/\sqrt{\pi}$, where j_σ is the spherical Bessel function that is given by (6).

Corollary 7. *Let $\mathcal{K}_\sigma : E \to \mathbb{C}$ be introduced as $\mathcal{K}_\sigma(\zeta) = \pi^{\frac{-1}{2}} 2^\sigma \Gamma(\sigma + 3/2)\zeta^{1-\sigma/2} j_\sigma(\sqrt{\zeta})$. Let $\sigma_1, \ldots, \sigma_m > -2.25$. Additionally consider that $\sigma = \min\{\sigma_1, \ldots, \sigma_m\}$ and $\mu_1, \ldots \mu_m$, as in Theorem 1, and when $g_i \in A$ such that:*

$$\left|\frac{g_i''(\zeta)}{g_i'(\zeta)}\right| \leq \xi, \ \ \zeta \in E.$$

Additionally:

$$\frac{1}{Re(\nu)}\left(5 + \frac{2\sigma + 5}{8\sigma^2 + 28\sigma + 89/4}\right)\sum_{i=1}^{m}|\mu_i| < 1,$$

when $0 < Re(\nu) < 1$ and:

$$\left[\frac{1}{Re(\nu)}\left(1 + \frac{2\sigma + 5}{8\sigma^2 + 28\sigma + 89/4}\right) + 4\right]\sum_{i=1}^{m}|\mu_i| < 1,$$

for $Re(\nu) \geq 1$. Then, for $\beta \in \mathbb{C}$ and $Re(\beta) \geq Re(\nu) > 0$, the function $G_{\sigma_1,\ldots\sigma_m,\mu_1,\ldots\mu_m,\beta} : E \to \mathbb{C}$ that is given by:

$$G_{\sigma_1,\ldots\sigma_m,\mu_1,\ldots\mu_m,\beta}(\zeta) = \left\{\beta\int_0^\zeta t^{\beta-1}\prod_{i=1}^{m}\left(\frac{\mathcal{K}_{\sigma_i}(t)}{g_i(t)}\right)^{\mu_i} dt\right\}^{1/\beta}$$

is in $\mathcal{S}$.

Corollary 8. *Let $\mathcal{K}_\sigma : E \to \mathbb{C}$ be introduced as $\mathcal{K}_\sigma(\zeta) = \pi^{\frac{-1}{2}} 2^\sigma \Gamma(\sigma + 3/2)\zeta^{1-\sigma/2} j_\sigma(\sqrt{\zeta})$. Let $\sigma_1, \ldots, \sigma_m > -2.25$. Additionally, $\sigma = \min\{\sigma_1, \ldots, \sigma_m\}$ and $\mu_1, \ldots \mu_m$ are as in Theorem 2 and when $g_i \in A$ with:*

$$\left|\frac{g_i''(\zeta)}{g_i'(\zeta)}\right| \leq \xi, \ \ \zeta \in E.$$

Let:

$$\left\{\frac{1}{Re(\nu)}\frac{4\sigma + 11}{8\sigma^2 + 26\sigma + 17} + \frac{2\xi}{(1 + 2Re(\nu))^{(1+2Re(\nu))/2Re(\nu)}}\right\}\sum_{i=1}^{m}|\mu_i| < 1$$

Then, for $\beta \in \mathbb{C}$ and $Re(\beta) \geq Re(\nu) > 0$, the function $H_{\sigma_1,\ldots\sigma_m,\mu_1,\ldots\mu_m,\beta} : E \to \mathbb{C}$ that is given by:

$$H_{\sigma_1,\ldots\sigma_m,\mu_1,\ldots\mu_m,\beta}(\zeta) = \left\{\beta\int_0^\zeta t^{\beta-1}\prod_{i=1}^{m}\left(\frac{\mathcal{K}'_{\sigma_i}(t)}{g_i'(t)}\right)^{\mu i} dt\right\}^{1/\beta}$$

is in $\mathcal{S}$.

Corollary 9. *Consider $\mathcal{K}_\sigma : E \to \mathbb{C}$ to be defined as $\mathcal{K}_\sigma(\zeta) = \pi^{\frac{-1}{2}} 2^\sigma \Gamma(\sigma + 3/2)\zeta^{1-\sigma/2} j_\sigma(\sqrt{\zeta})$. Let $\sigma_1, \ldots, \sigma_m > -2.25$. Additionally consider that $\sigma = \min\{\sigma_1, \ldots, \sigma_m\}$ and $\mu_1, \ldots \mu_m$, $\delta_1, \ldots \delta_m$, as in Theorem 3, and when $g_i \in A$ with:*

$$\left|\frac{g_i''(\zeta)}{g_i'(\zeta)}\right| \leq \xi, \ \ \zeta \in E.$$

These numbers satisfy the relations:

$$\frac{1}{Re(\nu)}\frac{2\sigma+5}{8\sigma^2+28\sigma+89/4}\sum_{i=1}^{m}|\mu_i|+\frac{2\xi}{(1+2Re(\nu))^{(1+2Re(\nu))/2Re(\nu)}}\sum_{i=1}^{m}|\delta_i|<1.$$

Then, for $\beta \in \mathbb{C}$ and $Re(\beta) \geq Re(\nu) > 0$, the function $I_{\sigma_1,\ldots\sigma_m,\mu_1,\ldots\mu_m,\ \delta_1,\ldots\delta_m,\beta} : E \to \mathbb{C}$that is given by:

$$I_{\sigma_1,\ldots\sigma_m,\mu_1,\ldots\mu_m,\ \delta_1,\ldots\delta_m,\beta}(\zeta)=\left\{\beta\int_0^{\zeta}t^{\beta-1}\prod_{i=1}^{m}\left(\frac{\mathcal{K}'_{\sigma_i}(t)}{t}\right)^{\mu_i}\left(g'_i(t)\right)^{\delta_i}dt\right\}^{1/\beta}$$

is in $\mathcal{S}$.

The sharp bounds of the the above results can be obtained using the following results. These results were due to [39].

Lemma 4. *When $\sigma, c \in \mathbb{R}$ and $\eta \in \mathbb{C}$ are such that $l > \max\{0, (|\eta|/8-1)/\}$, then, the function $\psi_{\sigma,c,\eta} : E \to \mathbb{C}$ that is given by (6) satisfies:*

$$\left|\frac{\zeta\psi'_{\sigma,c,\eta}(\zeta)}{\psi_{\sigma,c,\eta}(\zeta)}-1\right| \leq \frac{8(l+1)|\eta|+|\eta|^2}{32l(l+1)-4(2l+3)|\eta|},\ \zeta\in E,$$

and

$$\left|\frac{\zeta\psi''_{\sigma,c,\eta}(\zeta)}{\psi'_{\sigma,c,\eta}(\zeta)}\right| \leq \frac{|\eta|}{2}\left[\frac{64(l+1)^2\{8(l+2)|\eta|\}+128(l+1)(l+2)-\{8(l+2)+|\eta|\}|\eta|^2}{2l(l+1)(8(l+1)-|\eta|)\{16(l+1)(2l-|\eta|)-(4l+|\eta|)|\eta|\}}\right].$$

Using the inequalities $(l)_m > l(l+\lambda_0)^{m-1}$ and $m! > (1+\lambda_0)^{m-1}$ for $l > 0$ and $m > 3$. Here, $\lambda_0 \simeq 1.302775637$ is the largest solution for the equation $\lambda^2+\lambda-3=0$. These results were proved in [20]. By making use of these inequalities, Deniz [39] discussed the following results. Similar results can be proved using the following results:

Lemma 5. *When $\sigma, c \in \mathbb{R}$ and $\eta \in \mathbb{C}$ are such that $l = \sigma_i + (c+1)/2 > \max\{0, |\eta|/4(1+\lambda_0)-\lambda_0\}$, then, the function $\psi_{\sigma,c,\eta} : E \to \mathbb{C}$ that is given by (6) satisfies:*

$$\left|\frac{\zeta\psi'_{\sigma,c,\eta}(\zeta)}{\psi_{\sigma,c,\eta}(\zeta)}-1\right| < \frac{|\eta|(1+\Phi(l+1))}{4l(1+\Phi(l))},\ \zeta\in E,$$

and

$$\left|\frac{\zeta\psi''_{\sigma,c,\eta}(\zeta)}{\psi'_{\sigma,c,\eta}(\zeta)}\right| \leq \frac{(|\eta|/2l)\{|\eta|/8(l+1)[1+\Phi(l+2)]+1+\Phi(l+1)\}}{(|\eta|/4l)[1-\Phi(l+1)]+1-\Phi(l)},$$

where:

$$\Phi(l) = -\frac{|\eta|^2}{16(1+\lambda_0)l(l+\lambda_0)}+\frac{|\eta|^2}{32l(l+1)}+\frac{|\eta|}{l}\left(\frac{(1+\lambda_0)(l+\lambda_0)}{4(1+\lambda_0)(l+\lambda_0)+|\eta|}\right).$$

4. Conclusions

In this paper, we studied the univalence of certain integral operators using generalized Bessel functions. We used the Kudriasov conditions for functions to be univalent in order to derive the univalence criteria for these integral operators. For particular parameters, we obtained the univalence of the integral operators that are defined by Bessel, modified Bessel and spherical Bessel functions.

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