Article

On the De Blasi Measure of Noncompactness and Solvability of a Delay Quadratic Functional Integro-Differential Equation

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Abstract: Quadratic integro-differential equations have been discussed in many works, for instance. Some analytic results on the existence and the uniqueness of problem solutions to quadratic integro-differential equations have been investigated in different classes. Various techniques have been applied such as measure of noncompactness, Schauder's fixed point theorem and Banach contraction mapping. Here, we shall investigate quadratic functional integro-differential equations with delay. To prove the existence of solutions of the quadratic integro-differential equations, we use the technique of De Blasi measure of noncompactness. Moreover, we study some uniqueness results and continuous dependence of the solution on the initial condition and on the delay function. Some examples are presented to verify our results.

Keywords: quadratic integro-differential equation; measure of noncompactness; existence of monotonic integrable solutions

MSC: 34L30; 34K06

1. Introduction

Quadratic integral equations have gained much attention [1–4] because of their application of the real world. The existence of solutions of those equations have been studied in different classes of function spaces (see [1,2,5–17] and the references therein). For the theoretical results concerning the existence of solutions, in the classes of continuous or integrable functions, you can see Banaś [18–21].

Each of these monographs contains some existence results, and the main objective is to present a technique to obtain some results concerning various quadratic integral equations.

In this paper, we study the quadratic integro-differential equations by considering the initial value problem of the implicit quadratic integro-differential equation with delay.

$$\frac{dx}{dt} = f\left(t, \frac{dx}{dt} \cdot \int_0^{\phi(t)} g(s, x(s))ds\right), \quad a.e. \ t \in (0,1] \tag{1}$$

satisfying an initial condition

$$x(0) = x_0. \tag{2}$$

We present a new quadratic integro-differential, where the derivative of the function x is multiplied by an integral term involving the function x.

Let $\frac{dx}{dt} = y(t)$ then we can deduce that

$$x(t) = x_0 + \int_0^t y(s)ds \tag{3}$$

and (1) can be written as

$$y(t) = f\left(t, y(t) \cdot \int_0^{\phi(t)} g\left(s, x_0 + \int_0^s y(\theta) d\theta\right) ds\right). \tag{4}$$

The existence of non-decreasing solutions $y \in L_1[0,1]$ of (4) will be studied by the De Blasi measure of non-compactness. Additionally, we shall prove the continuous dependence of the solution of the problems (1) and (2) on the delay function and on the functions g. Consequently, the existence of a solution $x \in AC[0,1]$ of the problems (1) and (2) will be studied.

2. Research Methods

Let $I = [0,1]$ and suppose that:

(i)　$\phi : I \to I$, $\phi(t) \leq t$ is continuous and increasing.

(ii)　$f, g : I \times R \to R^+$ are Carathéodory functions, which are measurable in $t \in I \; \forall \, x \in R$ and continuous in $x \in R \; \forall \, t \in I$, and there exist $m_i : I \to R$, $m_i \in L_1(I)$, $i = 1, 2$ and $b_i \in R^+$ where

$$|f(t,x)| \leq |m_1(t)| + b_1|x| \leq \|m_1\| + b_1|x|, \quad \|m_1\| = \int_0^1 |m_1(t)| dt;$$

$$|g(t,x)| \leq |m_2(t)| + b_2|x| \leq \|m_2\| + b_2|x|, \quad \|m_2\| = \int_0^1 |m_2(t)| dt.$$

Moreover, f is non-decreasing for every non-decreasing x, i.e., for almost all $t_1, t_2 \in I$ satisfying $t_1 \leq t_2$ and for all $x(t_1) \leq x(t_2)$ implies $f(t_1, x(t_1)) \leq f(t_2, x(t_2))$.

(iii)　Let r_α be a positive root of the following equation

$$b_1 b_2 r^2 + (\|m_2\|b_1 + b_1 b_2 |x_0| - 1)r + \|m_1\| = 0.$$

Now, the following lemma can be proved.

Lemma 1. *The problems (1) and (2) is equivalent to the coupled system of integral Equations (3) and (4).*

Proof. It is clear that the solution of the problems (1) and (2) is given by the coupled system of integral Equations (3) and (4).

Conversely, let the solution $y \in L_1(I)$ of (4) exist, then from (3) and $\frac{d}{dt}x(t)$, the solution $x \in AC[0,1]$ of the problems (1) and (2) will exist. □

Now, we have the following existences theorem.

Theorem 1. *Suppose the conditions (i)–(iii) hold. If $b_1(\|m_2\| + b_2|x_0| + b_2 r) < 1$, then the Equation (4) has a solution $y \in L_1(I)$, which is non-decreasing.*

Proof. Let Q_r be a closed ball containing all non-decreasing functions on I by

$$Q_r = \{y \in L_1(I) : \|y\| \leq r\}, \quad r = \|m_1\| + \|m_2\|b_1 r + b_1 b_2 |x_0| r + b_1 b_2 r^2$$

and define the supper position operator F

$$Fy(t) = f\left(t, y(t) \cdot \int_0^{\phi(t)} g\left(s, x_0 + \int_0^s y(\theta) d\theta\right) ds\right), \; y \in Q_r.$$

Now, let $y \in Q_r$, then

$$|Fy(t)| = \left| f\left(t, y(t). \int_0^{\phi(t)} g(s, x_0 + \int_0^s y(\theta)d\theta)ds\right)\right|$$

$$\leq |m_1(t)| + b_1|y(t). \int_0^{\phi(t)} g(s, x_0 + \int_0^s y(\theta)d\theta)ds|$$

$$\leq |m_1(t)| + b_1|y(t)| \left(\int_0^{\phi(t)} |m_2(s)|ds + b_2 \int_0^{\phi(t)} \left| x_0 + \int_0^s y(\theta)d\theta \right| ds \right)$$

$$\leq |m_1(t)| + b_1|y(t)| \left(\int_0^{\phi(t)} |m_2(s)|ds + b_2|x_0| \int_0^{\phi(t)} ds + b_2 \int_0^{\phi(t)} \int_0^s |y(\theta)|d\theta ds \right)$$

$$\leq |m_1(t)| + b_1|y(t)|(\|m_2\| + b_2|x_0| + b_2\|y\|)$$

and

$$\int_0^1 |Fy(t)|dt \leq \int_0^1 |m_1(t)|dt + b_1(\|m_2\| + b_2|x_0| + b_2 r). \int_0^1 |y(t)|dt$$
$$\|Fy\| \leq \|m_1\| + b_1 r(\|m_2\| + b_2|x_0| + b_2 r)$$
$$\leq \|m_1\| + \|m_2\|b_1 r + b_1 b_2|x_0|r + b_1 b_2 r^2 = r.$$

Now, let $\{y_n\} \subset Q_r$, and $y_n \to y$, then

$$Fy_n(t) = f\left(t, y_n(t). \int_0^{\phi(t)} g(s, x_0 + \int_0^s y_n(\theta)d\theta)ds\right)$$

and

$$\lim_{n\to\infty} Fy_n(t) = \lim_{n\to\infty} f\left(t, y_n(t). \int_0^{\phi(t)} g(s, x_0 + \int_0^s y_n(\theta)d\theta)ds\right).$$

Applying Lebesgue dominated convergence Theorem [22], then from our assumptions we get

$$\lim_{n\to\infty} Fy_n(t) = f\left(t, \lim_{n\to\infty} y_n(t). \int_0^{\phi(t)} g(s, x_0 + \int_0^s \lim_{n\to\infty} y_n(\theta)d\theta)ds\right)$$
$$= f\left(t, y(t). \int_0^{\phi(t)} g(s, x_0 + \int_0^s y(\theta)d\theta)ds\right) = Fy(t).$$

i.e., $Fy_n(t) \to Fy(t)$ implies the operator F is continuous.

Taking Ω be a non empty subset of Q_r. Let $\epsilon > 0$ be fixed number and take a measurable set $D \subset I$ such that measure $D \leq \epsilon$. Then, for any $y \in \Omega$,

$$\|Fy\|_{L_1(D)} \leq \int_D |Fy(t)|dt \leq \int_D |m_1(t)|dt + b_1(\|m_2\| + b_2|x_0| + b_2 r). \int_D |y(t)|dt.$$

But

$$\lim_{\epsilon\to 0}\{sup\int_D |m_1(t)|dt : D \subset I, meas D < \epsilon\} = 0,$$

then applying the De Blasi measure of noncompactness [23–26].

$$\beta(X) = \lim_{\epsilon\to 0}\left(\sup_{x\in X}\left(sup\left[\int_D |x(t)|dt : D \subset [a,b], \ meas \ D \leq \epsilon\right]\right)\right), \tag{5}$$

we obtain

$$\beta(Fy(t)) \leq 0 + b_1\beta(y(t)).(\|m_2\| + b_2|x_0| + b_2 r)$$

and

$$\beta(F\Omega) \le b_1 \beta(y(t)).(\|m_2\| + b_2|x_0| + b_2r).$$

Then implies

$$\chi(F\Omega) \le b_1(\|m_2\| + b_2|x_0| + b_2r)\chi(\Omega),$$

where χ is the Hausdorff measure of noncompactness [23–26], which is defined by

$$\chi(X) = inf(r > 0; \text{ there exist a finite subset } Y \text{ of } E \text{ such that } X \subset Y + B_r). \qquad (6)$$

Since $b_1(\|m_2\| + b_2|x_0| + b_2r) < 1$, then F is a contraction with regard to χ [26] and has a fixed point $y \in Q_r$. Then, there exists a solution $y \in L_1(I)$ for (4). Hence, $\exists$ a solution $x \in AC(I)$ of the problems (1) and (2). □

2.1. Uniqueness of the Solution

Now, assume that:

$(ii)^*$ $f, g : I \times R \to R$ are measurable in $t \in I$ $\forall x \in R$ and satisfy

$$|f(t,x) - f(t,y)| \le b_1|x - y|, \ t \in I, \ x, y \in R.$$

$$|g(t,x) - g(t,y)| \le b_2|x - y|, \ t \in I, \ x, y \in R.$$

Moreover, f is non-decreasing for every non-decreasing x, i.e., for almost all $t_1, t_2 \in I$ satisfying $t_1 \le t_2$ and for all $x(t_1) \le x(t_2)$ implies $f(t_1, x(t_1)) \le f(t_2, x(t_2))$.

From the assumption $(ii)^*$ we have

$$|f(t,x)| \le |f(t,0)| + b_1|x|$$

and

$$|f(t,x)| \le \|m_1\| + b_1|x|, \quad \text{where } \|m_1\| = \int_0^1 |m_1(t)| dt.$$

Additionally, we get

$$|g(t,x)| \le |g(t,0)| + b_2|x|$$

and

$$|g(t,x)| \le \|m_2\| + b_2|x|, \quad \text{where } \|m_2\| = \int_0^1 |m_2(t)| dt.$$

Remark 1. *The assumption $(ii)^*$ implies the assumptions (ii).*

Theorem 2. *Assume that (i) and $(ii)^*$ are satisfied. Moreover,*

$$2b_1b_2r + \|m_2\|b_1 + b_1b_2|x_0| < 1.$$

Then, there exists a unique solution of (1) and (2).

Proof. From Remark 1, the assumptions of Theorem 1 are satisfied and the solution of (4) exists. Let y_1, y_2 be two solutions in Q_r of the integral Equation (4), then

$$
\begin{aligned}
|y_2(t) - y_1(t)| \quad = \quad & \left| f(t, y_2(t). \int_0^{\phi(t)} g(s, x_0 + \int_0^s y_2(\theta)d\theta)ds) \right. \\
- \quad & \left. f(t, y_1(t). \int_0^{\phi(t)} g(s, x_0 + \int_0^s y_1(\theta)d\theta)ds) \right| \\
\leq \quad & b_1 \left| y_2(t). \int_0^{\phi(t)} g(s, x_0 + \int_0^s y_2(\theta)d\theta)ds - y_1(t). \int_0^{\phi(t)} g(s, x_0 + \int_0^s y_1(\theta)d\theta)ds \right| \\
\leq \quad & b_1 \left| y_2(t). \int_0^{\phi(t)} g(s, x_0 + \int_0^s y_2(\theta)d\theta)ds - y_2(t). \int_0^{\phi(t)} g(s, x_0 + \int_0^s y_1(\theta)d\theta)ds \right. \\
+ \quad & \left. y_2(t). \int_0^{\phi(t)} g(s, x_0 + \int_0^s y_1(\theta)d\theta)ds - y_1(t). \int_0^{\phi(t)} g(s, x_0 + \int_0^s y_1(\theta)d\theta)ds \right| \\
\leq \quad & b_1 b_2 |y_2(t)|. \int_0^{\phi(t)} \int_0^s |y_2(\theta) - y_1(\theta)|d\theta ds + b_1 |y_2(t) - y_1(t)|.(\|m_2\| + b_2|x_0| + b_2\|y_1\|).
\end{aligned}
$$

Then,

$$
\int_0^1 |y_2(t) - y_1(t)|dt \quad \leq \quad b_1 b_2 \|y_2 - y_1\| \int_0^1 |y_2(t)|dt + b_1(\|m_2\| + b_2|x_0| + b_2 r). \int_0^1 |y_2(t) - y_1(t)|dt
$$

and

$$
\|y_2 - y_1\| \quad \leq \quad b_1 b_2 \|y_2 - y_1\| r + (\|m_2\| b_1 + b_1 b_2 |x_0| + b_1 b_2 r) \|y_2 - y_1\|.
$$

Hence,

$$
\|y_2 - y_1\|(1 - (2b_1 b_2 r + \|m_2\| b_1 + b_1 b_2 |x_0|)) \leq 0,
$$

then $y_1 = y_2$ and the solution of (4) is unique. As a result, the uniqueness of the solution of (1) and (2) is proved. $\square$

2.2. Continuous Dependence

Theorem 3. *Suppose the conditions of Theorem 2 hold, then the unique solution of the problems (1) and (2) depends continuously on the parameter x_0.*

Proof. Given $\delta > 0$ and $|x_0 - x_0^*| \leq \delta$ and let x^* be the solution of (1) and (2) corresponding to initial value x_0^*, then

$$
\begin{aligned}
|x(t) - x^*(t)| \quad = \quad & |x_0 + \int_0^t y(s)ds - x_0^* - \int_0^t y^*(s)ds| \\
\leq \quad & |x_0 - x_0^*| + \int_0^t |y(s) - y^*(s)| \leq \delta + \|y - y^*\|.
\end{aligned}
$$

However,

$$
\begin{aligned}
|y(t) - y^*(t)| \quad &= \quad \left| f\left(t, y(t). \int_0^{\phi(t)} g(s, x_0 + \int_0^s y(\theta)d\theta)ds\right) \right. \\
&\quad - \left. f\left(t, y^*(t). \int_0^{\phi(t)} g(s, x_0^* + \int_0^s y^*(\theta)d\theta)ds\right) \right| \\
&\leq \quad b_1\left(|y(t)|. \int_0^{\phi(t)} g(s, x_0 + \int_0^s |y(\theta)|d\theta) - |y^*(t)|. \int_0^{\phi(t)} g(s, x_0^* + \int_0^s |y^*(\theta)|d\theta)\right) \\
&\leq \quad b_1\left(|y(t)|. \int_0^{\phi(t)} g(s, x_0 + \int_0^s |y(\theta)|d\theta) - |y(t)|. \int_0^{\phi(t)} g(s, x_0^* + \int_0^s |y^*(\theta)|d\theta)\right. \\
&\quad + \left. |y(t)|. \int_0^{\phi(t)} g(s, x_0^* + \int_0^s |y^*(\theta)|d\theta) - |y^*(t)|. \int_0^{\phi(t)} g(s, x_0^* + \int_0^s |y^*(\theta)|d\theta)\right) \\
&\leq \quad b_1 b_2 |y(t)|\left(\int_0^{\phi(t)} |x_0 - x_0^*|ds + \int_0^{\phi(t)} \int_0^s |y(\theta) - y^*(\theta)|d\theta\right) \\
&\quad + \quad b_1 |y(t) - y^*(t)|. \int_0^{\phi(t)} (m_2(s) + b_2|x_0^* + \|y^*\||)ds \\
&\leq \quad b_1 b_2 |y(t)|(\delta + \|y - y^*\|) + b_1|y(t) - y^*(t)|(\|m_2\| + b_2|x_0^*| + b_2 r).
\end{aligned}
$$

Then,

$$
\int_0^1 |y(t) - y^*(t)|dt \quad \leq \quad b_1 b_2(\delta + \|y - y^*\|). \int_0^1 |y(t)|dt + b_1(\|m_2\| + b_2|x_0^*| + b_2 r). \int_0^1 |y(t) - y^*(t)|dt
$$

and

$$
\|y - y^*\| \quad \leq \quad b_1 b_2 r \delta + b_1 b_2 r\|y - y^*\| + (\|m_2\|b_1 + b_1 b_2|x_0^*| + b_1 b_2 r)\|y - y^*\|.
$$

Hence,

$$
\|y - y^*\|\left(1 - (2b_1 b_2 r + \|m_2\|b_1 + b_1 b_2|x_0^*|)\right) \leq b_1 b_2 r \delta,
$$

then

$$
\|y - y^*\| \quad \leq \quad \frac{b_1 b_2 r \delta}{1 - (2b_1 b_2 r + \|m_2\|b_1 + b_1 b_2|x_0^*|)} = \epsilon_1
$$

and

$$
\|x - x^*\|_c \quad \leq \quad \delta + \epsilon_1 = \epsilon.
$$

□

Theorem 4. *Suppose that the conditions of Theorem 2 hold, then the unique solution of the problems (1) and (2) depends continuously on g.*

Proof. Given $\delta > 0$ and $|g(t, x(t)) - g^*(t, x(t))| \leq \delta$ and let x^* be the solution of (1) and (2) corresponding to $g^*(t, x(t))$, then

$$
\begin{aligned}
|x(t) - x^*(t)| \quad &= \quad \left| x_0 + \int_0^t y(s)ds - x_0 - \int_0^t y^*(s)ds \right| \\
&\leq \quad |y(t) - y^*(t)| \leq \|y - y^*\|.
\end{aligned}
$$

However,

$$
\begin{aligned}
|y(t) - y^*(t)| \;=\; & \left| f\left(t, y(t). \int_0^{\phi(t)} g(s, x_0 + \int_0^s y(\theta)d\theta)ds \right) \right. \\
& \left. - f\left(t, y^*(t). \int_0^{\phi(t)} g^*(s, x_0 + \int_0^s y^*(\theta)d\theta)ds \right) \right| \\
\leq \; & b_1\left(|y(t)|. \int_0^{\phi(t)} g(s, x_0 + \int_0^s |y(\theta)|d\theta)ds - |y^*(t)|. \int_0^{\phi(t)} g^*(s, x_0 + \int_0^s |y^*(\theta)|d\theta)ds \right) \\
\leq \; & b_1\left(|y(t)|. \int_0^{\phi(t)} g(s, x_0 + \int_0^s |y(\theta)|d\theta)ds - |y(t)|. \int_0^{\phi(t)} g^*(s, x_0 + \int_0^s |y^*(\theta)|d\theta)ds \right. \\
+ \; & \left. |y(t)|. \int_0^{\phi(t)} g^*(s, x_0 + \int_0^s |y^*(\theta)|d\theta)ds - |y^*(t)|. \int_0^{\phi(t)} g^*(s, x_0 + \int_0^s |y^*(\theta)|d\theta)ds \right| \\
\leq \; & b_1\left(|y(t)|. \int_0^{\phi(t)} \left(g(s, x_0 + \int_0^s |y(\theta)|d\theta) - g^*(s, x_0 + \int_0^s |y^*(\theta)|d\theta) \right) ds \right) \\
+ \; & |y(t) - y^*(t)|. \int_0^{\phi(t)} g^*(s, x_0 + \int_0^s |y^*(\theta)|d\theta)ds \\
\leq \; & b_1|y(t)|\delta + b_1|y(t) - y^*(t)|.(\|m_2\| + b_2|x_0| + b_2\|y^*\|)
\end{aligned}
$$

Then,

$$
\int_0^1 |y(t) - y^*(t)|dt \;\leq\; b_1\delta \int_0^1 |y(t)|dt + b_1(\|m_2\| + b_2|x_0| + b_2 r). \int_0^1 |y(t) - y^*(t)|dt
$$

and

$$
\begin{aligned}
\|y - y^*\| \;\leq\; & b_1\delta r + b_1(\|m_2\| + b_2|x_0| + b_2 r)\|y - y^*\| \\
\leq\; & b_1 r\delta + (\|m_2\|b_1 + b_1 b_2|x_0| + b_1 b_2 r)\|y - y^*\| + b_1 b_2 r\|y - y^*\|.
\end{aligned}
$$

Hence,

$$
\|y - y^*\|\left(1 - (2b_1 b_2 r + \|m_2\|b_1 + b_1 b_2|x_0|) \right) \leq b_1 r\delta,
$$

then

$$
\|y - y^*\| \;\leq\; \frac{b_1 r\delta}{1 - (2b_1 b_2 r + \|m_2\|b_1 + b_1 b_2|x_0|)} = \epsilon
$$

and

$$
\|x - x^*\|_c \;\leq\; \epsilon.
$$

$\square$

Theorem 5. *Suppose that the conditions of Theorem 2 hold, then the unique solution of the problems (1) and (2) depends continuously on the delay function ϕ.*

Proof. Given $\delta > 0$ and $|\phi(t) - \phi^*(t)| \leq \delta$ and let x^* satisfies

$$
x^*(t) = x_0 + \int_0^t y^*(s)ds,
$$

then

$$
\begin{aligned}
|x(t) - x^*(t)| &= \left| x_0 + \int_0^t y(s)ds - x_0 - \int_0^t y^*(s)ds \right| \\
&\leq |y(t) - y^*(t)| \leq \|y - y^*\|.
\end{aligned}
$$

However,

$$
\begin{aligned}
|y(t) - y^*(t)| &= \left| f\left(t, y(t).\int_0^{\phi(t)} g(s, x_0 + \int_0^s y(\theta)d\theta)ds\right) \right. \\
&\quad \left. - f\left(t, y^*(t).\int_0^{\phi^*(t)} g(s, x_0 + \int_0^s y^*(\theta)d\theta)ds\right) \right| \\
&\leq b_1 \left| y(t).\int_0^{\phi(t)} g(s, x_0 + \int_0^s y(\theta)d\theta)ds - y^*(t).\int_0^{\phi^*(t)} g(s, x_0 + \int_0^s y^*(\theta)d\theta)ds \right| \\
&\leq b_1\left(|y(t)|.\int_0^{\phi(t)} g(s, x_0 + \int_0^s |y(\theta)|d\theta)ds - |y(t)|.\int_0^{\phi^*(t)} g(s, x_0 + \int_0^s |y^*(\theta)|d\theta)ds \right. \\
&\quad + \left. |y(t)|.\int_0^{\phi^*(t)} g(s, x_0 + \int_0^s |y^*(\theta)|d\theta)ds - |y^*(t)|.\int_0^{\phi^*(t)} g(s, x_0 + \int_0^s |y^*(\theta)|d\theta) \right) \\
&\leq b_1\left(|y(t)|.\left(\int_0^{\phi(t)} g(s, x_0 + \int_0^s |y(\theta)|d\theta)ds - \int_0^{\phi^*(t)} g(s, x_0 + \int_0^s |y^*(\theta)|d\theta)ds \right) \right. \\
&\quad + \left. |y(t) - y^*(t)|.\int_0^{\phi^*(t)} g(s, x_0 + \int_0^s |y^*(\theta)|d\theta)ds \right) \\
&\leq b_1 |y(t)|.\left(\int_0^{\phi(t)} g(s, x_0 + \int_0^s |y(\theta)|d\theta)ds - \int_0^{\phi(t)} g(s, x_0 + \int_0^s |y^*(\theta)|d\theta)ds \right. \\
&\quad + \left. \int_0^{\phi(t)} g(s, x_0 + \int_0^s |y^*(\theta)|d\theta)ds - \int_0^{\phi^*(t)} g(s, x_0 + \int_0^s |y^*(\theta)|d\theta)ds \right) \\
&\quad + b_1 |y(t) - y^*(t)|.(\|a\| + b_2|x_0| + b_2\|y^*\|) \Big) \\
&\leq b_1 |y(t)|\left(b_2 \int_0^{\phi(t)}\int_0^s |y(\theta) - y^*(\theta)|d\theta ds + \int_{\phi^*(t)}^{\phi(t)} g(s, x_0 + \int_0^s |y^*(\theta)|d\theta)ds \right) \\
&\quad + b_1 |y(t) - y^*(t)|(\|m_2\| + b_2|x_0| + b_2 r) \\
&\leq b_1 |y(t)|\left(b_2\|y - y^*\| + (\|m_2\| + b_2|x_0| + b_2\|y^*\|)|\phi - \phi^*| \right) \\
&\quad + b_1 |y(t) - y^*(t)|(\|m_2\| + b_2|x_0| + b_2 r) \\
&\leq b_1 b_2 \|y - y^*\||y(t)| + |y(t)|(\|m_2\|b_1 + b_1 b_2|x_0| + b_1 b_2 r)\delta \\
&\quad + |y(t) - y^*(t)|(\|m_2\|b_1 + b_1 b_2|x_0| + b_1 b_2 r).
\end{aligned}
$$

Then

$$
\begin{aligned}
\int_0^1 |y(t) - y^*(t)|dt &\leq b_1 b_2 \|y - y^*\| \int_0^1 |y(t)|dt \\
&\quad + (\|m_2\|b_1 + b_1 b_2|x_0| + b_1 b_2 r)\delta \int_0^1 |y(t)|dt \\
&\quad + (\|m_2\|b_1 + b_1 b_2|x_0| + b_1 b_2 r)\int_0^1 |y(t) - y^*(t)|dt
\end{aligned}
$$

and

$$
\|y - y^*\| \leq b_1 b_2 r \|y - y^*\| + (\|m_2\|b_1 + b_1 b_2|x_0| + b_1 b_2 r)r\delta + \|y - y^*\|(\|m_2\|b_1 + b_1 b_2|x_0| + b_1 b_2 r).
$$

Hence,

$$\|y - y^*\| \left(1 - (2b_1 b_2 r + \|m_2\| b_1 + b_1 b_2 |x_0|)\right) \leq (\|m_2\| b_1 + b_1 b_2 |x_0| + b_1 b_2 r) r \delta,$$

then

$$\|y - y^*\| \leq \frac{(\|m_2\| b_1 + b_1 b_2 |x_0| + b_1 b_2 r) r \delta}{1 - (2b_1 b_2 r + \|m_2\| b_1 + b_1 b_2 |x_0|)} = \epsilon$$

and

$$\|x - x^*\|_c \leq \epsilon.$$

$\square$

3. Examples

Example 1. *Let the following differential equation*

$$\frac{dx}{dt} = \frac{t^3}{96} + \frac{1}{2}\frac{dx}{dt} \cdot \int_0^{t^\beta} \left(\frac{s}{4} + \frac{1}{2}|x(s)|\right) ds. \quad t \in (0, 1]. \tag{7}$$

satisfying the initial data

$$x(0) = 1. \tag{8}$$

Then

$$f(t, x) = m_1(t) + b_1 \left(\frac{dx}{dt} \cdot \int_0^{t^\beta} (m_{(s)} + b_2 |x(s)|) ds\right) = \frac{t^3}{96} + \frac{1}{2}\frac{dx}{dt} \cdot \int_0^{t^\beta} \left(\frac{s}{4} + \frac{1}{2}|x(s)|\right) ds. \quad t \in I, \ \beta \geq 1$$

$$g(t, x(t)) = m_2(t) + b_2(|x(t)|) = \frac{t}{4} + \frac{1}{2}|x(t)|, \ \phi(t) = t^\beta \quad t \in I, \ \beta \geq 1.$$

Easily we can verify all conditions of Theorem 1. Then, the initial value problems (7) and (8) has a solution.

Example 2. *Let the following differential equation*

$$\frac{dx}{dt} = \frac{3t}{40} + \frac{1}{4}\frac{dx}{dt} \int_0^{t^\beta} \left(\frac{s}{3} + \frac{1}{2}|x(s)|\right) ds. \quad t \in (0, 1]. \tag{9}$$

with initial data

$$x(0) = 1. \tag{10}$$

Then,

$$f(t, x) = \frac{3t}{40} + \frac{1}{4}\frac{dx}{dt} \int_0^{t^\beta} \left(\frac{s}{3} + \frac{1}{2}|x(s)|\right) ds. \quad t \in I, \ \beta \geq 1$$

$$g(t, x(t)) = \frac{t}{3} + \frac{1}{2}x(t), \ \phi(t) = t^\beta \quad t \in I, \ \beta \geq 1.$$

Obviously we can verify all conditions of Theorem 1. Then, the initial value problems (9) and (10) has a solution.

4. Conclusions

In this paper, we have studied a delay quadratic functional integro-differential Equation (1). We have investigated the solvability of the problem (1) and (2) by applying the

technique of measure of non-compactness. Then, we have established some uniqueness results and continuous dependence of solution on some initial data and the functions g, ϕ. Finally, two examples have been introduced to demonstrate our results.

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