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A Study on Dynamics of CD4⁺ T-Cells under the Effect of HIV-1 Infection Based on a Mathematical Fractal-Fractional Model via the Adams-Bashforth Scheme and Newton Polynomials

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Abstract: In recent decades, AIDS has been one of the main challenges facing the medical community around the world. Due to the large human deaths of this disease, researchers have tried to study the dynamic behaviors of the infectious factor of this disease in the form of mathematical models in addition to clinical trials. In this paper, we study a new mathematical model in which the dynamics of CD4⁺ T-cells under the effect of HIV-1 infection are investigated in the context of a generalized fractal-fractional structure for the first time. The kernel of these new fractal-fractional operators is of the generalized Mittag-Leffler type. From an analytical point of view, we first derive some results on the existence theory and then the uniqueness criterion. After that, the stability of the given fractal-fractional system is reviewed under four different cases. Next, from a numerical point of view, we obtain two numerical algorithms for approximating the solutions of the system via the Adams-Bashforth method and Newton polynomials method. We simulate our results via these two algorithms and compare both of them. The numerical results reveal some stability and a situation of lacking a visible order in the early days of the disease dynamics when one uses the Newton polynomial.

Keywords: existence; fractal-fractional derivative; HIV-1 infection; Newton polynomial; Adams-Bashforth

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1. Introduction

According to medical definitions and clinical findings and virology, human immunodeficiency virus (HIV) is a type of retrovirus that leads to acquired immunodeficiency syndrome (AIDS) in humans [1]. In fact, CD4⁺ T-lymphocytes are the largest number of white blood cells in the human immune system that are attacked by HIV viruses, which attack CD4⁺ T-cells and infect them, reducing their number and efficiency. They disrupt cells. Therefore, this process reduces the resistance of the immune system in the human body and weakens it [1]. Although the HIV virus infects other cells, it causes the most damage to T cells by causing the degradation and destruction of CD4⁺ T-cells. As a result, the affected person's body gradually becomes sensitive to various types of infections and

contaminants and completely loses its immunity. Usually, according to laboratory results, a healthy person has white blood cells with a normal $CD4^+$ T-lymphocyte count of 800 to 1200/ mm^3 . The decrease in $CD4^+$ T-lymphocyte count may be due to thymus insufficiency or bone defects. One way to determine whether a person is infected with the virus is recognized through very small number of $CD4^+$ T-lymphocytes (less than 200/ mm^3).

Currently, AIDS is one of the most unknown and dangerous diseases of our time. According to UNAIDS 2017, which is published annually, about 36.7 million people worldwide are living with HIV, and nearly 1.8 million people have recently been infected with HIV, and almost one million people worldwide are affected by various complication, and died of AIDS in 2016. There are even several countries, especially in Africa and some less developed countries, where up to thirty-five percent of the population between the ages of fifteen and fifty are infected with the HIV virus [2]. Despite significant advances in the control of the disease, no vaccine for HIV has yet been found.

In recent decades, many attempts have been made to design, analyze, simulate and solve mathematical dynamic models, which include the basic rules in the analysis of control and prevention of the spread of various diseases and infections. In this regard, biology and engineering are among the most widely used fields for various types of mathematical modeling, which are particularly popular among researchers. For such a purpose, derivation operators and differential equations play an important role. To demonstrate such high efficiency, we can refer to a sample of articles in which various types of mathematical models are observed, including time delaying model of COVID-19 [3], anthrax model for animals [4], Hepatitis C model [5], memristor chaotic model of circuit [6], Lorenz-Stenflo hyperchaotic model [7], dynamics of environmental persistence of infections [8], Langevin model [9], Mump virus [10], Zika virus [11], mosaic disease [12], Computer viruses [13], thermostat control model [14,15], pantograph model [16,17], canine distemper virus [18], Lassa fever [19], hybrid model of p-Laplacian operators [20], co-dynamics of COVID-19 and diabetes [21], chemical modeling of cyclohexane [22,23], Navier systems [24], etc.

Designing mathematical models to analyze the dynamics of HIV infection is also known as a valuable and efficient measure in this regard [25–27]. Usually in almost all of these dynamic mathematical models, we can see the association between HIV viruses and non-infected $CD4^+$ T-cells and the effects of a variety of drug therapies and clinical therapies on reducing and controlling infected cells. Perlson [28] designed and presented a simple model for primary HIV infection in 1989. This model, which is related to HIV infection, is important, and many other models have been developed later, which are inspired by this model. Four years later, Perlson et al. generalized the model and examined some of the behaviors and characteristics of the new model [29]. They considered the model by defining four different state functions: virus population, non-infected $CD4^+$ T-cells, productive infected $CD4^+$ T-cells, and latent infected $CD4^+$ T-cells.

In recent years, some other mathematicians developed the basic integer order models of HIV infection in the context of the fractional order systems. In [30], Ding et al. introduced a fractional version of the infection of HIV for $CD4^+$ T-cells and analyzed the non-negative solutions of this system. Arafa et al. [31] used the generalized Euler technique for his model to obtain solutions and investigated the impact of changes of viral particles on the blood. After that, Bulut et al. [32] combined the homotopy and sumudu techniques to study the dynamics of their fractional model of the infection of HIV on $CD4^+$ T-cells. Lichae et al. [33] extended their fractional HIV model based on the effect of drug therapy and solved three-compartmental model of $CD4^+$ with the help of LADM (Laplace Adomian decomposition method). In 2020, Nazir et al. [34] investigated not only existence theory, but also stability criteria for the Caputo-Fabrizio HIV model of $CD4^+$ T-cells. Wang et al. [35] studied the time periodic reaction-diffusion equations for modeling 2-LTR dynamics in HIV-infected patients.

In 2017, Atangana [36] presented a generalized kind of operators entitled fractal-fractional operators. This definition used the existing notions of fractional calculus and fractal calculus with together. These operators are the convolution of the power-law,

exponential law and generalized Mittag–Leffler law with fractal derivatives. There exist two components for such fractal-fractional operators: the fractional order and fractal dimension (order). Fractal-fractional differential equations transfer the order and dimension of every dynamical system into a rational order one. In fact, we are able to generalize each standard differential equation to the generalized systems having arbitrary order and dimension of derivatives. The main goal of such a combination is to analyze a vast range of nonlocal BVPs or IVPs that possess fractal behaviors. In this direction, a limited researchers obtained some results in which we see that the generalized fractal-fractional operators give accurate and more exact simulations for describing mathematical models of real-world phenomena. Some of new works in this regard are [37–41].

Due to the novelty and efficacy of these new fractal-fractional operators, in this paper, we aim to design a mathematical model of CD4⁺ T-cells under the effect of HIV-1 infection in which derivatives are fractal-fractional operators in the sense of Atangana-Baleanu. It is notable that in 2021, Ahmad et al. [42] studied the dynamics of HIV primary infection in the context of a model designed by the fractal-fractional operators. The main contribution and novelty of our work in comparison to their paper [42] is that we analyze all qualitative behaviors of such a fractal-fractional system. In other words, we first review the existence and uniqueness of solutions and further, we complete our study by giving new results about the stability (Ulam-Hyers-Rassias) of solutions. Also, for the first time, in this paper, we derive numerical schemes for the fractal-fractional CD4⁺-HIV-1 model with the help of the Newton polynomials and by applying some real data, we compare our results with the Adams-Bashforth simulations. In this direction, we can see some dynamical behaviors of the solutions in our simulations.

The arrangement of the paper is as follows: we introduce our fractal-fractional model in Section 2 and describe its parameters and coefficients. The existence results are given in Section 3 and further, in Section 4, we investigate the uniqueness. The stability criterion are implemented in Section 5. In the sequel, we derive two numerical schemes. In other words, in Section 6, the Adams-Bashforth method are done and then, we derive another algorithm in Section 7 by using the Newton polynomials. We present some simulations and discussions about both numerical methods in Section 8. We end the paper by giving conclusions in Section 9.

2. The Structure of the Model for CD4⁺ T-Cells and HIV-1

In 2006, Wang and Li [1] formulated an integer-order classical mathematical structure of dynamics of CD4⁺ T-cells under the HIV-1 infection in three-compartmental model as

$$\begin{cases} \mathbb{T}'(s) = \theta - q\mathbb{V}\mathbb{T} - \rho\mathbb{T} + \zeta\mathbb{U}, \\ \mathbb{U}'(s) = q\mathbb{V}\mathbb{T} - (\zeta + \kappa)\mathbb{U}, \\ \mathbb{V}'(s) = \kappa N\mathbb{U} - \vartheta\mathbb{V}, \end{cases} \quad (1)$$

via the initial values $\mathbb{T}(0) = \mathbb{T}_0$, $\mathbb{U}(0) = \mathbb{U}_0$, and $\mathbb{V}(0) = \mathbb{V}_0$, and also the state functions $\mathbb{T}(s)$, $\mathbb{U}(s)$, $\mathbb{V}(s)$ are an amount of susceptible CD4⁺ T-cells, an amount of infectious CD4⁺ T-cells, and the free particles of the infection of the HIV virus in the blood at the time $s \in \mathbb{J} := [0, S]$, ($S > 0$), respectively. Moreover, the parameter N stands for the average number of infected particles by an existing infected cell, ϑ is the natural rate of death for the virus, ζ is the return rate of infected cells to susceptible compartment, κ is the rate of death for infected T-cells, q is the rate of infection T-cells, ρ is the natural rate of death, and θ shows the supply rate for new T-cells. They considered all parameters as positive values and $\mathbb{T}_0, \mathbb{U}_0, \mathbb{V}_0 \geq 0$.

To upgrade and improve the exact results, inspired by the standard model (1), we present a mathematical fractal-fractional model on dynamics of CD4⁺ T-cells under the effect of HIV-1 infection via the generalized Mittag-Leffler-type kernel (fractal-fractional CD4⁺-HIV-1-model) as

$$\begin{cases} \text{FFML} \mathcal{D}_{0,s}^{(\delta,\sigma)} \mathbb{T}(s) = \theta - q\mathbb{V}(s)\mathbb{T}(s) - \rho\mathbb{T}(s) + \zeta\mathbb{U}(s), \\ \text{FFML} \mathcal{D}_{0,s}^{(\delta,\sigma)} \mathbb{U}(s) = q\mathbb{V}(s)\mathbb{T}(s) - (\zeta + \kappa)\mathbb{U}(s), \\ \text{FFML} \mathcal{D}_{0,s}^{(\delta,\sigma)} \mathbb{V}(s) = \kappa N\mathbb{U}(s) - \vartheta\mathbb{V}(s), \end{cases} \tag{2}$$

subject to

$$\mathbb{T}(0) = \mathbb{T}_0 \geq 0, \quad \mathbb{U}(0) = \mathbb{U}_0 \geq 0, \quad \mathbb{V}(0) = \mathbb{V}_0 \geq 0,$$

where all assumptions and parameters are similar to above classical model (1). Also, $\text{FFML} \mathcal{D}_{0,s}^{(\delta,\sigma)}$ is the (δ, σ) -fractal-fractional derivative with the fractional order $\delta \in (0, 1]$ and the fractal order $\sigma \in (0, 1]$ via the Mittag-Leffler-type kernel.

In other words, let a continuous map $\Psi : (a, b) \rightarrow [0, \infty)$ be fractal differentiable of dimension σ . In this case, the (δ, σ) -fractal-fractional derivative of Ψ of the generalized Mittag-Leffler-type kernel of order δ in the Riemann-Liouville sense is defined as

$$\text{FFML} \mathcal{D}_{a,s}^{(\delta,\sigma)} \Psi(s) = \frac{\mathcal{AB}(\delta)}{1 - \delta} \frac{d}{ds^\sigma} \int_a^s \mathbb{E}_\delta \left[-\frac{\delta}{1 - \delta} (s - \mathfrak{w})^\delta \right] \Psi(\mathfrak{w}) d\mathfrak{w}, \quad 0 < \delta, \sigma \leq 1, \tag{3}$$

where

$$\frac{d\Psi(\mathfrak{w})}{d\mathfrak{w}^\sigma} = \lim_{s \rightarrow \mathfrak{w}} \frac{\Psi(s) - \Psi(\mathfrak{w})}{s^\sigma - \mathfrak{w}^\sigma},$$

is the fractal derivative and $\mathcal{AB}(\delta) = 1 - \delta + \frac{\delta}{\Gamma(\delta)}$ and $\mathcal{AB}(0) = \mathcal{AB}(1) = 1$ [36].

Moreover, accordingly, for such a function Ψ , the (δ, σ) -fractal-fractional integral via the Mittag-Leffler-type kernel is given by

$$\text{FFML} \mathcal{I}_{a,s}^{(\delta,\sigma)} \Psi(s) = \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_a^s \mathfrak{w}^{\sigma-1} (s - \mathfrak{w})^{\delta-1} \Psi(\mathfrak{w}) d\mathfrak{w} + \frac{(1 - \delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} \Psi(s), \tag{4}$$

if it exists, where $\delta, \sigma > 0$ [36].

3. Existence Property

In this section, the existence property is investigated based on fixed point theory. For the qualitative analysis, make the Banach space $\mathbb{X} = \mathbb{M}^3$, where $\mathbb{M} = C(\mathbb{J}, \mathbb{R})$ with

$$\|\mathbb{K}\|_{\mathbb{X}} = \|(\mathbb{T}, \mathbb{U}, \mathbb{V})\|_{\mathbb{X}} = \max \{ |W(s)| : s \in \mathbb{J} \},$$

for $|W| := |\mathbb{T}| + |\mathbb{U}| + |\mathbb{V}|$. We reformulate the R.H.S. of the fractal-fractional CD4^+ -HIV-1-model (2) as:

$$\begin{cases} \mathbb{Q}_1(s, \mathbb{T}(s), \mathbb{U}(s), \mathbb{V}(s)) = \theta - q\mathbb{V}(s)\mathbb{T}(s) - \rho\mathbb{T}(s) + \zeta\mathbb{U}(s), \\ \mathbb{Q}_2(s, \mathbb{T}(s), \mathbb{U}(s), \mathbb{V}(s)) = q\mathbb{V}(s)\mathbb{T}(s) - (\zeta + \kappa)\mathbb{U}(s), \\ \mathbb{Q}_3(s, \mathbb{T}(s), \mathbb{U}(s), \mathbb{V}(s)) = \kappa N\mathbb{U}(s) - \vartheta\mathbb{V}(s). \end{cases} \tag{5}$$

In this case, the fractal-fractional CD4^+ -HIV-1-model (2) is transformed into the following system

$$\begin{cases} \text{ABR} \mathcal{D}_{0,s}^\delta \mathbb{T}(s) = \sigma s^{\sigma-1} \mathbb{Q}_1(s, \mathbb{T}(s), \mathbb{U}(s), \mathbb{V}(s)), \\ \text{ABR} \mathcal{D}_{0,s}^\delta \mathbb{U}(s) = \sigma s^{\sigma-1} \mathbb{Q}_2(s, \mathbb{T}(s), \mathbb{U}(s), \mathbb{V}(s)), \\ \text{ABR} \mathcal{D}_{0,s}^\delta \mathbb{V}(s) = \sigma s^{\sigma-1} \mathbb{Q}_3(s, \mathbb{T}(s), \mathbb{U}(s), \mathbb{V}(s)). \end{cases} \tag{6}$$

In view of (6), we rewrite the developed tripled-system with the compact IVP which takes the form

$$\begin{cases} \text{ABR}\mathcal{D}_{0,s}^\delta \mathbb{K}(s) = \sigma s^{\sigma-1} \mathbb{Q}(s, \mathbb{K}(s)), \\ \mathbb{K}(0) = \mathbb{K}_0, \end{cases} \tag{7}$$

where

$$\mathbb{K}(s) = (\mathbb{T}(s), \mathbb{U}(s), \mathbb{V}(s))^T, \quad \mathbb{K}_0 = (\mathbb{T}_0, \mathbb{U}_0, \mathbb{V}_0)^T, \quad \delta, \sigma \in (0, 1], \tag{8}$$

and

$$\mathbb{Q}(s, \mathbb{K}(s)) = \begin{cases} \mathbb{Q}_1(s, \mathbb{T}(s), \mathbb{U}(s), \mathbb{V}(s)), \\ \mathbb{Q}_2(s, \mathbb{T}(s), \mathbb{U}(s), \mathbb{V}(s)), \\ \mathbb{Q}_3(s, \mathbb{T}(s), \mathbb{U}(s), \mathbb{V}(s)), \quad s \in \mathbb{J}. \end{cases} \tag{9}$$

By the definition and by (7), we have

$$\frac{\mathcal{AB}(\delta)}{1-\delta} \frac{d}{ds} \int_0^s \mathbb{E}_\delta \left[-\frac{\delta}{1-\delta} (s-\mathfrak{w})^\delta \right] \mathbb{K}(\mathfrak{w}) \, d\mathfrak{w} = \sigma s^{\sigma-1} \mathbb{Q}(s, \mathbb{K}(s)). \tag{10}$$

In the sequel, operating the fractal-fractional Atangana-Baleanu integral on (10), we get

$$\mathbb{K}(s) = \mathbb{K}(0) + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1} (s-\mathfrak{w})^{\delta-1} \mathbb{Q}(\mathfrak{w}, \mathbb{K}(\mathfrak{w})) \, d\mathfrak{w} + \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}(s, \mathbb{K}(s)). \tag{11}$$

Due to the above compact form of the fractal-fractional integral equation, the extended representation of it is illustrated as

$$\left\{ \begin{aligned} \mathbb{T}(s) &= \mathbb{T}_0 + \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}_1(s, \mathbb{T}(s), \mathbb{U}(s), \mathbb{V}(s)) \\ &\quad + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1} (s-\mathfrak{w})^{\delta-1} \mathbb{Q}_1(\mathfrak{w}, \mathbb{T}(\mathfrak{w}), \mathbb{U}(\mathfrak{w}), \mathbb{V}(\mathfrak{w})) \, d\mathfrak{w}, \\ \mathbb{U}(s) &= \mathbb{U}_0 + \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}_2(s, \mathbb{T}(s), \mathbb{U}(s), \mathbb{V}(s)) \\ &\quad + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1} (s-\mathfrak{w})^{\delta-1} \mathbb{Q}_2(\mathfrak{w}, \mathbb{T}(\mathfrak{w}), \mathbb{U}(\mathfrak{w}), \mathbb{V}(\mathfrak{w})) \, d\mathfrak{w}, \\ \mathbb{V}(s) &= \mathbb{V}_0 + \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}_3(s, \mathbb{T}(s), \mathbb{U}(s), \mathbb{V}(s)) \\ &\quad + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1} (s-\mathfrak{w})^{\delta-1} \mathbb{Q}_3(\mathfrak{w}, \mathbb{T}(\mathfrak{w}), \mathbb{U}(\mathfrak{w}), \mathbb{V}(\mathfrak{w})) \, d\mathfrak{w}. \end{aligned} \right. \tag{12}$$

We consider a self-map to derive a fixed-point problem, by defining $F : \mathbb{X} \rightarrow \mathbb{X}$ as

$$\begin{aligned} F(\mathbb{K}(s)) &= \mathbb{K}(0) + \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}(s, \mathbb{K}(s)) \\ &\quad + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1} (s-\mathfrak{w})^{\delta-1} \mathbb{Q}(\mathfrak{w}, \mathbb{K}(\mathfrak{w})) \, d\mathfrak{w}. \end{aligned} \tag{13}$$

The Leray-Schauder theorem is utilized to prove the existence property in relation to the fractal-fractional CD4⁺-HIV-1-model (2).

Theorem 1. [43] Let \mathbb{X} be a Banach space, $\mathbb{E} \subset \mathbb{X}$ a convex closed bounded set, $\mathbb{O} \subset \mathbb{E}$ an open set, and $0 \in \mathbb{O}$. Then for the continuous and compact map $F : \mathbb{O} \rightarrow \mathbb{E}$, either:

- (HY1) $\exists y \in \mathbb{O}$ s.t. $y = F(y)$, or
- (HY2) $\exists y \in \partial\mathbb{O}$ and $0 < \mu < 1$ s.t. $y = \mu F(y)$.

Theorem 2. Let $\mathbb{Q} \in C(\mathbb{J} \times \mathbb{X}, \mathbb{X})$. Assume:

- (P1) $\exists \varphi \in L^1(\mathbb{J}, \mathbb{R}^+)$ and $\exists A \in C([0, \infty), (0, \infty))$ (A is non-decreasing) s.t. $\forall s \in \mathbb{J}$ and $\mathbb{K} \in \mathbb{X}$,

$$|\mathbb{Q}(s, \mathbb{K}(s))| \leq \varphi(s)A(|\mathbb{K}(s)|);$$

- (P2) $\exists \omega > 0$ s.t.

$$\frac{\omega}{\mathbb{K}_0 + \left[\frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \right] \varphi_0^* A(\omega)} > 1, \tag{14}$$

with $\varphi_0^* = \sup_{s \in \mathbb{J}} |\varphi(s)|$.

Then there is a solution for the fractal-fractional problem (7) and accordingly, for the fractal-fractional CD4⁺-HIV-1-model (2) on \mathbb{J} .

Proof. First, consider $F : \mathbb{X} \rightarrow \mathbb{X}$ which is defined by (13) and assume

$$N_r = \{ \mathbb{K} \in \mathbb{X} : \|\mathbb{K}\|_{\mathbb{X}} \leq r \},$$

for some $r > 0$. Clearly, as \mathbb{Q} is continuous, thus F is so. From (P1), we get

$$\begin{aligned} |F(\mathbb{K}(s))| &\leq |\mathbb{K}(0)| + \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} |\mathbb{Q}(s, \mathbb{K}(s))| \\ &\quad + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1} (s-\mathfrak{w})^{\delta-1} |\mathbb{Q}(\mathfrak{w}, \mathbb{K}(\mathfrak{w}))| \, d\mathfrak{w} \\ &\leq \mathbb{K}_0 + \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} \varphi(s)A(|\mathbb{K}(s)|) \\ &\quad + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1} (s-\mathfrak{w})^{\delta-1} \varphi(\mathfrak{w})A(|\mathbb{K}(\mathfrak{w})|) \, d\mathfrak{w} \\ &\leq \mathbb{K}_0 + \frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} \varphi_0^* A(r) + \frac{\delta\sigma S^{\delta+\sigma-1} B(\delta, \sigma)}{\mathcal{AB}(\delta)\Gamma(\delta)} \varphi_0^* A(r) \\ &= \mathbb{K}_0 + \frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} \varphi_0^* A(r) + \frac{\delta\sigma S^{\delta+\sigma-1} \Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \varphi_0^* A(r), \end{aligned}$$

for $\mathbb{K} \in N_r$. Hence

$$\|F\mathbb{K}\|_{\mathbb{X}} \leq \mathbb{K}_0 + \left[\frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \right] \varphi_0^* A(r) < \infty. \tag{15}$$

Thus F is uniformly bounded on \mathbb{X} . Now, take $s, v \in [0, S]$ s.t. $s < v$ and $\mathbb{K} \in N_r$. By denoting

$$\sup_{(s, \mathbb{K}) \in \mathbb{J} \times N_r} |\mathbb{Q}(s, \mathbb{K}(s))| = \mathbb{Q}^* < \infty,$$

we estimate

$$\begin{aligned}
 |F(\mathbb{K}(v)) - F(\mathbb{K}(s))| &\leq \left| \frac{(1-\delta)\sigma v^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}(v, \mathbb{K}(v)) - \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}(s, \mathbb{K}(s)) \right. \\
 &\quad + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^v \mathfrak{w}^{\sigma-1} (v-\mathfrak{w})^{\delta-1} \mathbb{Q}(\mathfrak{w}, \mathbb{K}(\mathfrak{w})) \, d\mathfrak{w} \\
 &\quad \left. - \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1} (s-\mathfrak{w})^{\delta-1} \mathbb{Q}(\mathfrak{w}, \mathbb{K}(\mathfrak{w})) \, d\mathfrak{w} \right| \\
 &\leq \frac{(1-\delta)\sigma \mathbb{Q}^*}{\mathcal{AB}(\delta)} (v^{\sigma-1} - s^{\sigma-1}) \\
 &\quad + \frac{\delta\sigma \mathbb{Q}^*}{\mathcal{AB}(\delta)\Gamma(\delta)} \left| \int_0^v \mathfrak{w}^{\sigma-1} (v-\mathfrak{w})^{\delta-1} \, d\mathfrak{w} - \int_0^s \mathfrak{w}^{\sigma-1} (s-\mathfrak{w})^{\delta-1} \, d\mathfrak{w} \right| \\
 &\leq \frac{(1-\delta)\sigma \mathbb{Q}^*}{\mathcal{AB}(\delta)} (v^{\sigma-1} - s^{\sigma-1}) + \frac{\delta\sigma \mathbb{Q}^* B(\delta, \sigma)}{\mathcal{AB}(\delta)\Gamma(\delta)} [v^{\delta+\sigma-1} - s^{\delta+\sigma-1}] \\
 &= \frac{(1-\delta)\sigma \mathbb{Q}^*}{\mathcal{AB}(\delta)} (v^{\sigma-1} - s^{\sigma-1}) + \frac{\delta\sigma \mathbb{Q}^* \Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} [v^{\delta+\sigma-1} - s^{\delta+\sigma-1}].
 \end{aligned} \tag{16}$$

We see that (16) approaches to 0 independent of \mathbb{K} , as $v \rightarrow s$. Consequently

$$\|F(\mathbb{K}(v)) - F(\mathbb{K}(s))\|_{\mathbb{X}} \rightarrow 0,$$

when $v \rightarrow s$. This gives the equicontinuity of F , and accordingly the compactness of F on N_r by the Arzelá–Ascoli theorem. Since all conditions of Theorem 1 are fulfilled on F , we have one of (HY1) or (HY2). From (P2), we set

$$\Phi := \{ \mathbb{K} \in \mathbb{X} : \|\mathbb{K}\|_{\mathbb{X}} < \omega \},$$

for some $\omega > 0$ s.t.

$$\mathbb{K}_0 + \left[\frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \right] \varphi_0^* A(\omega) < \omega.$$

From (P1) and (15), we have

$$\|F\mathbb{K}\|_{\mathbb{X}} \leq \mathbb{K}_0 + \left[\frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \right] \varphi_0^* A(\|\mathbb{K}\|_{\mathbb{X}}). \tag{17}$$

Suppose that there are $\mathbb{K} \in \partial\Phi$ and $0 < \mu < 1$ s.t. $\mathbb{K} = \mu F(\mathbb{K})$. Then by (17), we write

$$\begin{aligned}
 \omega = \|\mathbb{K}\|_{\mathbb{X}} = \mu \|F\mathbb{K}\|_{\mathbb{X}} &< \mathbb{K}_0 + \left[\frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \right] \varphi_0^* A(\|\mathbb{K}\|_{\mathbb{X}}) \\
 &< \mathbb{K}_0 + \left[\frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \right] \varphi_0^* A(\omega) < \omega,
 \end{aligned}$$

which cannot be held. Therefore, (HY2) does not hold and F possesses a fixed-point in $\bar{\Phi}$ by Theorem 1. This confirms the existence of a solution to the fractal-fractional CD4⁺-HIV-1-model (2). □

4. Uniqueness Property

Here, on the fractal-fractional CD4⁺-HIV-1-model (2), we investigate the Lipschitz property in the first step and further, the uniqueness property.

Lemma 1. Consider $\mathbb{T}, \mathbb{U}, \mathbb{V}, \mathbb{T}^*, \mathbb{U}^*, \mathbb{V}^* \in \mathbb{M} := C(\mathbb{J}, \mathbb{R})$, and let

(C1) $\|\mathbb{T}\| \leq \beta_1, \|\mathbb{U}\| \leq \beta_2, \|\mathbb{V}\| \leq \beta_3$ for some constants $\beta_1, \beta_2, \beta_3 > 0$.

Then the kernels $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3$ defined in (5) are fulfilled the Lipschitz property with constants $\alpha_1, \alpha_2, \alpha_3 > 0$ w.r.t. the relevant components, where

$$\alpha_1 = \rho + q\beta_3, \quad \alpha_2 = \zeta + \kappa, \quad \alpha_3 = \vartheta. \tag{18}$$

Proof. For \mathbb{Q}_1 , we take $\mathbb{T}, \mathbb{T}^* \in \mathbb{M} := C(\mathbb{J}, \mathbb{R})$ arbitrarily, and we have

$$\begin{aligned} & \|\mathbb{Q}_1(s, \mathbb{T}(s), \mathbb{U}(s), \mathbb{V}(s)) - \mathbb{Q}_1(s, \mathbb{T}^*(s), \mathbb{U}(s), \mathbb{V}(s))\| \\ &= \|(\vartheta - q\mathbb{V}(s)\mathbb{T}(s) - \rho\mathbb{T}(s) + \zeta\mathbb{U}(s)) \\ & \quad - (\vartheta - q\mathbb{V}(s)\mathbb{T}^*(s) - \rho\mathbb{T}^*(s) + \zeta\mathbb{U}(s))\| \\ & \leq [\rho + q\|\mathbb{V}(s)\|] \|\mathbb{T}(s) - \mathbb{T}^*(s)\| \\ & \leq [\rho + q\beta_3] \|\mathbb{T}(s) - \mathbb{T}^*(s)\| \\ &= \alpha_1 \|\mathbb{T}(s) - \mathbb{T}^*(s)\|. \end{aligned}$$

From the above, we find out that \mathbb{Q}_1 is Lipschitz w.r.t. \mathbb{T} under the constant $\alpha_1 > 0$. For \mathbb{Q}_2 , we choose two arbitrary elements $\mathbb{U}, \mathbb{U}^* \in \mathbb{M} := C(\mathbb{J}, \mathbb{R})$, and estimate

$$\begin{aligned} & \|\mathbb{Q}_2(s, \mathbb{T}(s), \mathbb{U}(s), \mathbb{V}(s)) - \mathbb{Q}_2(s, \mathbb{T}(s), \mathbb{U}^*(s), \mathbb{V}(s))\| \\ &= \|(q\mathbb{V}(s)\mathbb{T}(s) - (\zeta + \kappa)\mathbb{U}(s)) - (q\mathbb{V}(s)\mathbb{T}(s) - (\zeta + \kappa)\mathbb{U}^*(s))\| \\ & \leq [\zeta + \kappa] \|\mathbb{U}(s) - \mathbb{U}^*(s)\| \\ &= \alpha_2 \|\mathbb{U}(s) - \mathbb{U}^*(s)\|. \end{aligned}$$

The above inequality means that \mathbb{Q}_2 is Lipschitz w.r.t. \mathbb{U} under the constant $\alpha_2 > 0$. Finally, for both arbitrary elements $\mathbb{V}, \mathbb{V}^* \in \mathbb{M} := C(\mathbb{J}, \mathbb{R})$, we have

$$\begin{aligned} & \|\mathbb{Q}_3(s, \mathbb{T}(s), \mathbb{U}(s), \mathbb{V}(s)) - \mathbb{Q}_3(s, \mathbb{T}(s), \mathbb{U}(s), \mathbb{V}^*(s))\| \\ &= \|(\kappa N\mathbb{U}(s) - \vartheta\mathbb{V}(s)) - (\kappa N\mathbb{U}(s) - \vartheta\mathbb{V}^*(s))\| \\ & \leq \vartheta \|\mathbb{V}(s) - \mathbb{V}^*(s)\| \\ &= \alpha_3 \|\mathbb{V}(s) - \mathbb{V}^*(s)\|. \end{aligned}$$

This confirms that \mathbb{Q}_3 is Lipschitz w.r.t. \mathbb{V} under the constant $\alpha_3 > 0$. Therefore kernel functions $\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3$ are Lipschitz w.r.t. under the constants $\alpha_1, \alpha_2, \alpha_3 > 0$, respectively. \square

Now, by invoking the above lemma, we are able to prove the uniqueness property for solutions of the fractal-fractional system (2).

Theorem 3. Consider (C1) to be held. Then the fractal-fractional CD4⁺-HIV-1-model (2) possesses exactly one solution if

$$\left[\frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \right] \alpha_j < 1, \tag{19}$$

for $j \in \{1, 2, 3\}$, and the Lipschitz constants $\alpha_j > 0$ introduced by (18).

Proof. To prove the desired result, we consider the contrary of the conclusion of theorem. That is, consider the existence of another solution for the fractal-fractional CD4⁺-HIV-1-model (2), namely $(\mathbb{T}^*(s), \mathbb{U}^*(s), \mathbb{V}^*(s))$ under the initial conditions

$$(\mathbb{T}^*(0) = \mathbb{T}_0, \mathbb{U}^*(0) = \mathbb{U}_0, \mathbb{V}^*(0) = \mathbb{V}_0).$$

From (12), we have

$$\begin{aligned} \mathbb{T}^*(s) &= \mathbb{T}_0 + \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}_1(s, \mathbb{T}^*(s), \mathbb{U}^*(s), \mathbb{V}^*(s)) \\ &\quad + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1} (s-\mathfrak{w})^{\delta-1} \mathbb{Q}_1(\mathfrak{w}, \mathbb{T}^*(\mathfrak{w}), \mathbb{U}^*(\mathfrak{w}), \mathbb{V}^*(\mathfrak{w})) \, d\mathfrak{w}, \end{aligned}$$

$$\begin{aligned} \mathbb{U}^*(s) &= \mathbb{U}_0 + \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}_2(s, \mathbb{T}^*(s), \mathbb{U}^*(s), \mathbb{V}^*(s)) \\ &\quad + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1} (s-\mathfrak{w})^{\delta-1} \mathbb{Q}_2(\mathfrak{w}, \mathbb{T}^*(\mathfrak{w}), \mathbb{U}^*(\mathfrak{w}), \mathbb{V}^*(\mathfrak{w})) \, d\mathfrak{w}, \end{aligned}$$

and

$$\begin{aligned} \mathbb{V}^*(s) &= \mathbb{V}_0 + \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}_3(s, \mathbb{T}^*(s), \mathbb{U}^*(s), \mathbb{V}^*(s)) \\ &\quad + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1} (s-\mathfrak{w})^{\delta-1} \mathbb{Q}_3(\mathfrak{w}, \mathbb{T}^*(\mathfrak{w}), \mathbb{U}^*(\mathfrak{w}), \mathbb{V}^*(\mathfrak{w})) \, d\mathfrak{w}. \end{aligned}$$

In this case, we estimate

$$\begin{aligned} |\mathbb{T}(s) - \mathbb{T}^*(s)| &\leq \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} \left| \mathbb{Q}_1(s, \mathbb{T}(s), \mathbb{U}(s), \mathbb{V}(s)) - \mathbb{Q}_1(s, \mathbb{T}^*(s), \mathbb{U}^*(s), \mathbb{V}^*(s)) \right| \\ &\quad + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1} (s-\mathfrak{w})^{\delta-1} \\ &\quad \times \left| \mathbb{Q}_1(\mathfrak{w}, \mathbb{T}(\mathfrak{w}), \mathbb{U}(\mathfrak{w}), \mathbb{V}(\mathfrak{w})) - \mathbb{Q}_1(\mathfrak{w}, \mathbb{T}^*(\mathfrak{w}), \mathbb{U}^*(\mathfrak{w}), \mathbb{V}^*(\mathfrak{w})) \right| \, d\mathfrak{w} \\ &\leq \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} \alpha_1 \|\mathbb{T} - \mathbb{T}^*\| + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1} (s-\mathfrak{w})^{\delta-1} \alpha_1 \|\mathbb{T} - \mathbb{T}^*\| \, d\mathfrak{w} \\ &\leq \left[\frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \right] \alpha_1 \|\mathbb{T} - \mathbb{T}^*\|, \end{aligned}$$

and so

$$\left(1 - \left[\frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \right] \alpha_1 \right) \|\mathbb{T} - \mathbb{T}^*\| \leq 0.$$

From (19), we know that the above inequality holds if $\|\mathbb{T} - \mathbb{T}^*\| = 0$, or $\mathbb{T} = \mathbb{T}^*$. In the similar manner, from

$$\|\mathbb{U} - \mathbb{U}^*\| \leq \left[\frac{(1 - \delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta + \sigma)} \right] \alpha_2 \|\mathbb{U} - \mathbb{U}^*\|,$$

we obtain

$$\left(1 - \left[\frac{(1 - \delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta + \sigma)} \right] \alpha_2 \right) \|\mathbb{U} - \mathbb{U}^*\| \leq 0,$$

and this gives $\|\mathbb{U} - \mathbb{U}^*\| = 0$ or $\mathbb{U} = \mathbb{U}^*$. Further,

$$\|\mathbb{V} - \mathbb{V}^*\| \leq \left[\frac{(1 - \delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta + \sigma)} \right] \alpha_3 \|\mathbb{V} - \mathbb{V}^*\|,$$

yields

$$\left(1 - \left[\frac{(1 - \delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta + \sigma)} \right] \alpha_3 \right) \|\mathbb{V} - \mathbb{V}^*\| \leq 0.$$

Hence $\mathbb{V} = \mathbb{V}^*$. In consequence,

$$(\mathbb{T}(s), \mathbb{U}(s), \mathbb{V}(s)) = (\mathbb{T}^*(s), \mathbb{U}^*(s), \mathbb{V}^*(s)).$$

The latter equality confirms that the fractal-fractional CD4⁺-HIV-1-model (2) possesses a unique solution. □

5. Ulam-Hyers-Rassias Stability

In this part, the stability result of solutions in the context of four types of the Ulam–Hyers, Ulam–Hyers–Rassias and their generalizations are proved on the tripled system of the fractal-fractional CD4⁺-HIV-1-model (2).

Definition 1. The fractal-fractional CD4⁺-HIV-1-model (2) is Ulam–Hyers stable if $\exists a_{\mathbb{Q}_1}, a_{\mathbb{Q}_2}, a_{\mathbb{Q}_3} \in \mathbb{R}^+$ such that $\forall r_j > 0, j = 1, 2, 3$, and $\forall (\mathbb{T}^*, \mathbb{U}^*, \mathbb{V}^*) \in \mathbb{X}$ satisfying

$$\begin{cases} \left| \text{FFML} \mathcal{D}_{0,s}^{(\delta,\sigma)} \mathbb{T}^*(s) - \mathbb{Q}_1(s, \mathbb{T}^*(s), \mathbb{U}^*(s), \mathbb{V}^*(s)) \right| < r_1, \\ \left| \text{FFML} \mathcal{D}_{0,s}^{(\delta,\sigma)} \mathbb{U}^*(s) - \mathbb{Q}_2(s, \mathbb{T}^*(s), \mathbb{U}^*(s), \mathbb{V}^*(s)) \right| < r_2, \\ \left| \text{FFML} \mathcal{D}_{0,s}^{(\delta,\sigma)} \mathbb{V}^*(s) - \mathbb{Q}_3(s, \mathbb{T}^*(s), \mathbb{U}^*(s), \mathbb{V}^*(s)) \right| < r_3, \end{cases} \tag{20}$$

$\exists (\mathbb{T}, \mathbb{U}, \mathbb{V}) \in \mathbb{X}$ satisfying the fractal-fractional CD4⁺-HIV-1-model (2) s.t.

$$\begin{cases} |\mathbb{T}^*(s) - \mathbb{T}(s)| \leq a_{\mathbb{Q}_1} r_1, \\ |\mathbb{U}^*(s) - \mathbb{U}(s)| \leq a_{\mathbb{Q}_2} r_2, \\ |\mathbb{V}^*(s) - \mathbb{V}(s)| \leq a_{\mathbb{Q}_3} r_3. \end{cases}$$

Definition 2. The fractal-fractional CD4⁺-HIV-1-model (2) is generalized Ulam–Hyers stable if $\exists a_{\mathbb{Q}_j} \in C(\mathbb{R}^+, \mathbb{R}^+), (j \in \{1, 2, 3\})$ with $a_{\mathbb{Q}_j}(0) = 0$ s.t. $\forall r_j > 0$ and $\forall (\mathbb{T}^*, \mathbb{U}^*, \mathbb{V}^*) \in \mathbb{X}$

fulfilling (20), there is $(T, U, V) \in \mathbb{X}$ as a solution of the given fractal-fractional CD4⁺-HIV-1-model (2) s.t.

$$\begin{cases} |T^*(s) - T(s)| \leq a_{Q_1}(r_1), \\ |U^*(s) - U(s)| \leq a_{Q_2}(r_2), \\ |V^*(s) - V(s)| \leq a_{Q_3}(r_3). \end{cases}$$

Remark 1. $(T^*, U^*, V^*) \in \mathbb{X}$ is a solution for (20) iff $\exists z_1, z_2, z_3 \in C([0, S], \mathbb{R})$ (each of them depend on T^*, U^*, V^* , respectively) s.t. $\forall s \in \mathbb{J}$,

- (i) $|z_j(s)| < r_j$;
- (ii) We have

$$\begin{cases} \text{FFML} \mathcal{D}_{0,s}^{(\delta,\sigma)} T^*(s) = Q_1(s, T^*(s), U^*(s), V^*(s)) + z_1(s), \\ \text{FFML} \mathcal{D}_{0,s}^{(\delta,\sigma)} U^*(s) = Q_2(s, T^*(s), U^*(s), V^*(s)) + z_2(s), \\ \text{FFML} \mathcal{D}_{0,s}^{(\delta,\sigma)} V^*(s) = Q_3(s, T^*(s), U^*(s), V^*(s)) + z_3(s). \end{cases}$$

Definition 3. The fractal-fractional CD4⁺-HIV-1-model (2) is Ulam–Hyers–Rassias stable w.r.t. functions $h_j, (j \in \{1, 2, 3\})$ if $\exists 0 < a_{(Q_j, h_j)} \in \mathbb{R}$ s.t. $\forall r_j > 0$ and $\forall (T^*, U^*, V^*) \in \mathbb{X}$ fulfilling

$$\begin{cases} \left| \text{FFML} \mathcal{D}_{0,s}^{(\delta,\sigma)} T^*(s) - Q_1(s, T^*(s), U^*(s), V^*(s)) \right| < r_1 h_1(s), \\ \left| \text{FFML} \mathcal{D}_{0,s}^{(\delta,\sigma)} U^*(s) - Q_2(s, T^*(s), U^*(s), V^*(s)) \right| < r_2 h_2(s), \\ \left| \text{FFML} \mathcal{D}_{0,s}^{(\delta,\sigma)} V^*(s) - Q_3(s, T^*(s), U^*(s), V^*(s)) \right| < r_3 h_3(s), \end{cases} \tag{21}$$

$\exists (T, U, V) \in \mathbb{X}$ as a solution of the fractal-fractional CD4⁺-HIV-1-model (2) s.t.

$$\begin{cases} |T^*(s) - T(s)| \leq r_1 a_{(Q_1, h_1)} h_1(s), \quad \forall s \in \mathbb{J}, \\ |U^*(s) - U(s)| \leq r_2 a_{(Q_2, h_2)} h_2(s), \quad \forall s \in \mathbb{J}, \\ |V^*(s) - V(s)| \leq r_3 a_{(Q_3, h_3)} h_3(s), \quad \forall s \in \mathbb{J}, \end{cases}$$

in which $h_1, h_2, h_3 \in C([0, S], \mathbb{R}^+)$.

Definition 4. The fractal-fractional CD4⁺-HIV-1-model (2) is generalized Ulam–Hyers–Rassias stable w.r.t. functions h_j if $\exists 0 < a_{(Q_j, h_j)} \in \mathbb{R}$ s.t. $\forall (T^*, U^*, V^*) \in \mathbb{X}$ satisfying

$$\begin{cases} \left| \text{FFML} \mathcal{D}_{0,s}^{(\delta,\sigma)} T^*(s) - Q_1(s, T^*(s), U^*(s), V^*(s)) \right| < h_1(s), \\ \left| \text{FFML} \mathcal{D}_{0,s}^{(\delta,\sigma)} U^*(s) - Q_2(s, T^*(s), U^*(s), V^*(s)) \right| < h_2(s), \\ \left| \text{FFML} \mathcal{D}_{0,s}^{(\delta,\sigma)} V^*(s) - Q_3(s, T^*(s), U^*(s), V^*(s)) \right| < h_3(s), \end{cases}$$

$\exists (T, U, V) \in \mathbb{X}$ as a solution of the fractal-fractional CD4⁺-HIV-1-model (2) s.t.

$$\begin{cases} |T^*(s) - T(s)| \leq a_{(Q_1, h_1)} h_1(s), \\ |U^*(s) - U(s)| \leq a_{(Q_2, h_2)} h_2(s), \\ |V^*(s) - V(s)| \leq a_{(Q_3, h_3)} h_3(s). \end{cases}$$

If we take $h_j(s) = 1$, in this case Definition 3 yields the Ulam-Hyers criterion for the stability of solutions.

Remark 2. $(T^*, U^*, V^*) \in \mathbb{X}$ is a solution for (21) iff $\exists z_1, z_2, z_3 \in C([0, S], \mathbb{R})$ (each of them depend on T^*, U^*, V^* , respectively) s.t. $\forall s \in \mathbb{J}$,

- (i) $|z_j(s)| < r_j h_j(s)$;
- (ii) We have

$$\begin{cases} \text{FFML } \mathcal{D}_{0,s}^{(\delta,\sigma)} T^*(s) = Q_1(s, T^*(s), U^*(s), V^*(s)) + z_1(s), \\ \text{FFML } \mathcal{D}_{0,s}^{(\delta,\sigma)} U^*(s) = Q_2(s, T^*(s), U^*(s), V^*(s)) + z_2(s), \\ \text{FFML } \mathcal{D}_{0,s}^{(\delta,\sigma)} V^*(s) = Q_3(s, T^*(s), U^*(s), V^*(s)) + z_3(s). \end{cases}$$

The following lemmas are useful for our main theorems.

Lemma 2. For each $r_1, r_2, r_3 > 0$, suppose that $(T^*, U^*, V^*) \in \mathbb{X}$ is considered as a solution of (20). Then the functions $T^*, U^*, V^* \in \mathbb{M}$ fulfill the inequalities

$$\left| T^*(s) - \left(T_0 + \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} Q_1(s, T^*(s), U^*(s), V^*(s)) + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1} (s-\mathfrak{w})^{\delta-1} \times Q_1(\mathfrak{w}, T^*(\mathfrak{w}), U^*(\mathfrak{w}), V^*(\mathfrak{w})) d\mathfrak{w} \right) \right| \leq \left[\frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \right] r_1, \tag{22}$$

and

$$\left| U^*(s) - \left(U_0 + \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} Q_2(s, T^*(s), U^*(s), V^*(s)) + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1} (s-\mathfrak{w})^{\delta-1} \times Q_2(\mathfrak{w}, T^*(\mathfrak{w}), U^*(\mathfrak{w}), V^*(\mathfrak{w})) d\mathfrak{w} \right) \right| \leq \left[\frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \right] r_2, \tag{23}$$

and

$$\left| V^*(s) - \left(V_0 + \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} Q_3(s, T^*(s), U^*(s), V^*(s)) + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1} (s-\mathfrak{w})^{\delta-1} \times Q_3(\mathfrak{w}, T^*(\mathfrak{w}), U^*(\mathfrak{w}), V^*(\mathfrak{w})) d\mathfrak{w} \right) \right| \leq \left[\frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \right] r_3. \tag{24}$$

Proof. Let $r_1 > 0$ be arbitrary. Since $T^* \in \mathbb{M}$ satisfies

$$\left| \text{FFML } \mathcal{D}_{0,s}^{(\delta,\sigma)} T^*(s) - Q_1(s, T^*(s), U^*(s), V^*(s)) \right| < r_1,$$

so, by Remark 1, we are allowed to take a function $z_1(s)$ s.t.

$$\text{FFML } \mathcal{D}_{0,s}^{(\delta,\sigma)} T^*(s) = Q_1(s, T^*(s), U^*(s), V^*(s)) + z_1(s),$$

and $|z_1(s)| \leq r_1$. Clearly,

$$\begin{aligned} \mathbb{T}^*(s) &= \mathbb{T}_0 + \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} \left[\mathbb{Q}_1(s, \mathbb{T}^*(s), \mathbb{U}^*(s), \mathbb{V}^*(s)) + z_1(s) \right] \\ &\quad + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathbb{w}^{\sigma-1} (s-\mathbb{w})^{\delta-1} \left[\mathbb{Q}_1(\mathbb{w}, \mathbb{T}^*(\mathbb{w}), \mathbb{U}^*(\mathbb{w}), \mathbb{V}^*(\mathbb{w})) + z_1(\mathbb{w}) \right] d\mathbb{w}. \end{aligned}$$

In this case, we estimate

$$\begin{aligned} &\left| \mathbb{T}^*(s) - \left(\mathbb{T}_0 + \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}_1(s, \mathbb{T}^*(s), \mathbb{U}^*(s), \mathbb{V}^*(s)) \right. \right. \\ &\quad \left. \left. + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathbb{w}^{\sigma-1} (s-\mathbb{w})^{\delta-1} \mathbb{Q}_1(\mathbb{w}, \mathbb{T}^*(\mathbb{w}), \mathbb{U}^*(\mathbb{w}), \mathbb{V}^*(\mathbb{w})) d\mathbb{w} \right) \right| \\ &\leq \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} |z_1(s)| + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathbb{w}^{\sigma-1} (s-\mathbb{w})^{\delta-1} |z_1(\mathbb{w})| d\mathbb{w} \\ &\leq \frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} r_1 + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} r_1 \\ &= \left[\frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \right] r_1. \end{aligned}$$

This states that the inequality (22) is fulfilled. Similarly, we can obtain the inequalities (23) and (24). □

To prove the next result, we regard the following:

(C2) \exists increasing mappings $\mathfrak{h}_j \in C([0, S], \mathbb{R}^+)$, ($j \in \{1, 2, 3\}$) and $\exists \Delta_{\mathfrak{h}_j} > 0$ provided that

$$\text{FFML} \mathcal{I}_{0,s}^{(\delta,\sigma)} \mathfrak{h}_j(s) < \Delta_{\mathfrak{h}_j} \mathfrak{h}_j(s), \quad (j \in \{1, 2, 3\}), \forall s \in \mathbb{J}. \tag{25}$$

Lemma 3. Let (C2) to be held. For each $r_1, r_2, r_3 > 0$, suppose that $(\mathbb{T}^*, \mathbb{U}^*, \mathbb{V}^*) \in \mathbb{X}$ is considered as a solution of (21). Then the functions $\mathbb{T}^*, \mathbb{U}^*, \mathbb{V}^* \in \mathbb{M}$ fulfill the inequalities

$$\left| \mathbb{T}^*(s) - \left(\mathbb{T}_0 + \text{FFML} \mathcal{I}_{0,s}^{(\delta,\sigma)} \mathbb{Q}_1(s, \mathbb{T}^*(s), \mathbb{U}^*(s), \mathbb{V}^*(s)) \right) \right| \leq r_1 \Delta_{\mathfrak{h}_1} \mathfrak{h}_1(s), \tag{26}$$

$$\left| \mathbb{U}^*(s) - \left(\mathbb{U}_0 + \text{FFML} \mathcal{I}_{0,s}^{(\delta,\sigma)} \mathbb{Q}_2(s, \mathbb{T}^*(s), \mathbb{U}^*(s), \mathbb{V}^*(s)) \right) \right| \leq r_2 \Delta_{\mathfrak{h}_2} \mathfrak{h}_2(s), \tag{27}$$

$$\left| \mathbb{V}^*(s) - \left(\mathbb{V}_0 + \text{FFML} \mathcal{I}_{0,s}^{(\delta,\sigma)} \mathbb{Q}_3(s, \mathbb{T}^*(s), \mathbb{U}^*(s), \mathbb{V}^*(s)) \right) \right| \leq r_3 \Delta_{\mathfrak{h}_3} \mathfrak{h}_3(s). \tag{28}$$

Proof. Let $r_1 > 0$. Since $\mathbb{T}^* \in \mathbb{M}$ satisfies

$$\left| \text{FFML} \mathcal{D}_{0,s}^{(\delta,\sigma)} \mathbb{T}^*(s) - \mathbb{Q}_1(s, \mathbb{T}^*(s), \mathbb{U}^*(s), \mathbb{V}^*(s)) \right| < r_1 \mathfrak{h}_1(s),$$

thus, from Remark 2, we are allowed to take a function $z_1(s)$ s.t.

$$\text{FFML} \mathcal{D}_{0,s}^{(\delta,\sigma)} \mathbb{T}^*(s) = \mathbb{Q}_1(s, \mathbb{T}^*(s), \mathbb{U}^*(s), \mathbb{V}^*(s)) + z_1(s),$$

and $|z_1(s)| \leq r_1 \hat{h}_1(s)$. Evidently,

$$\begin{aligned} \mathbb{T}^*(s) &= \mathbb{T}_0 + \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} \left[\mathbb{Q}_1(s, \mathbb{T}^*(s), \mathbb{U}^*(s), \mathbb{V}^*(s)) + z_1(s) \right] \\ &\quad + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1} (s-\mathfrak{w})^{\delta-1} \left[\mathbb{Q}_1(\mathfrak{w}, \mathbb{T}^*(\mathfrak{w}), \mathbb{U}^*(\mathfrak{w}), \mathbb{V}^*(\mathfrak{w})) + z_1(\mathfrak{w}) \right] d\mathfrak{w}. \end{aligned}$$

Then, we estimate

$$\begin{aligned} &\left| \mathbb{T}^*(s) - \left(\mathbb{T}_0 + \text{FFML} \mathcal{I}_{0,s}^{(\delta,\sigma)} \mathbb{Q}_1(s, \mathbb{T}^*(s), \mathbb{U}^*(s), \mathbb{V}^*(s)) \right) \right| = \\ &\left| \mathbb{T}^*(s) - \left(\mathbb{T}_0 + \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}_1(s, \mathbb{T}^*(s), \mathbb{U}^*(s), \mathbb{V}^*(s)) \right. \right. \\ &\quad \left. \left. + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1} (s-\mathfrak{w})^{\delta-1} \mathbb{Q}_1(\mathfrak{w}, \mathbb{T}^*(\mathfrak{w}), \mathbb{U}^*(\mathfrak{w}), \mathbb{V}^*(\mathfrak{w})) d\mathfrak{w} \right) \right| \\ &\leq \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} |z_1(s)| + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1} (s-\mathfrak{w})^{\delta-1} |z_1(\mathfrak{w})| d\mathfrak{w} \\ &= \text{FFML} \mathcal{I}_{0,s}^{(\delta,\sigma)} |z_1(s)| \\ &\leq \text{FFML} \mathcal{I}_{0,s}^{(\delta,\sigma)} r_1 \hat{h}_1(s) \\ &\leq r_1 \Delta_{\hat{h}_1} \hat{h}_1(s). \end{aligned}$$

Similarly, we can obtain the remaining inequalities. \square

The Ulam–Hyers stability is checked about the fractal-fractional CD4⁺-HIV-1-model (2).

Theorem 4. Let (C1) be fulfilled. Then the fractal-fractional CD4⁺-HIV-1-model (2) is Ulam–Hyers stable on $\mathbb{J} := [0, S]$ and also is generalized Ulam–Hyers stable s.t.

$$\left[\frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \right] \alpha_j < 1, \quad j \in \{1, 2, 3\},$$

in which $\alpha_1, \alpha_2, \alpha_3 > 0$ are given by (18).

Proof. Let $r_1 > 0$ and $\mathbb{T}^* \in \mathbb{M}$ be an arbitrary solution of (20). Also, from Theorem 3, we assume $\mathbb{T} \in \mathbb{M}$ as a unique solution of the fractal-fractional CD4⁺-HIV-1-model (2). Then $\mathbb{T}(s)$ is defined as

$$\begin{aligned} \mathbb{T}(s) &= \mathbb{T}_0 + \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}_1(s, \mathbb{T}(s), \mathbb{U}(s), \mathbb{V}(s)) \\ &\quad + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1} (s-\mathfrak{w})^{\delta-1} \mathbb{Q}_1(\mathfrak{w}, \mathbb{T}(\mathfrak{w}), \mathbb{U}(\mathfrak{w}), \mathbb{V}(\mathfrak{w})) d\mathfrak{w}. \end{aligned}$$

Therefore, by Lemma 2 and with the help of the triangle inequality, we estimate

$$|\mathbb{T}^*(s) - \mathbb{T}(s)| \leq \left| \mathbb{T}^*(s) - \mathbb{T}_0 - \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}_1(s, \mathbb{T}(s), \mathbb{U}(s), \mathbb{V}(s)) \right|$$

$$\begin{aligned}
 & - \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1}(s-\mathfrak{w})^{\delta-1} \mathbb{Q}_1(\mathfrak{w}, \mathbb{T}(\mathfrak{w}), \mathbb{U}(\mathfrak{w}), \mathbb{V}(\mathfrak{w})) \, d\mathfrak{w} \Big| \\
 & \leq \left| \mathbb{T}^*(s) - \left(\mathbb{T}_0 + \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}_1(s, \mathbb{T}^*(s), \mathbb{U}^*(s), \mathbb{V}^*(s)) \right. \right. \\
 & \left. \left. + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1}(s-\mathfrak{w})^{\delta-1} \mathbb{Q}_1(\mathfrak{w}, \mathbb{T}^*(\mathfrak{w}), \mathbb{U}^*(\mathfrak{w}), \mathbb{V}^*(\mathfrak{w})) \, d\mathfrak{w} \right) \right| \\
 & + \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} \left| \mathbb{Q}_1(s, \mathbb{T}^*(s), \mathbb{U}^*(s), \mathbb{V}^*(s)) - \mathbb{Q}_1(s, \mathbb{T}(s), \mathbb{U}(s), \mathbb{V}(s)) \right| \\
 & + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1}(s-\mathfrak{w})^{\delta-1} \left| \mathbb{Q}_1(\mathfrak{w}, \mathbb{T}^*(\mathfrak{w}), \mathbb{U}^*(\mathfrak{w}), \mathbb{V}^*(\mathfrak{w})) - \mathbb{Q}_1(\mathfrak{w}, \mathbb{T}(\mathfrak{w}), \mathbb{U}(\mathfrak{w}), \mathbb{V}(\mathfrak{w})) \right| \, d\mathfrak{w} \\
 & \leq \left[\frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \right] r_1 + \frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} \alpha_1 \|\mathbb{T}^* - \mathbb{T}\| \\
 & + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \alpha_1 \|\mathbb{T}^* - \mathbb{T}\| \\
 & \leq \left[\frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \right] r_1 + \left[\frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \right] \alpha_1 \|\mathbb{T} - \mathbb{T}^*\|.
 \end{aligned}$$

Hence,

$$\|\mathbb{T}^* - \mathbb{T}\| \leq \frac{\left[\frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \right] r_1}{1 - \left[\frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \right] \alpha_1}.$$

We set $a_{\mathbb{Q}_1} = \frac{\left[\frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \right]}{1 - \left[\frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \right] \alpha_1}$. In this case, $\|\mathbb{T}^* - \mathbb{T}\| \leq a_{\mathbb{Q}_1} r_1$.

Similarly, we obtain

$$\|\mathbb{U}^* - \mathbb{U}\| \leq a_{\mathbb{Q}_2} r_2,$$

$$\|\mathbb{V}^* - \mathbb{V}\| \leq a_{\mathbb{Q}_3} r_3,$$

where

$$a_{\mathbb{Q}_j} = \frac{\left[\frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \right]}{1 - \left[\frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \right] \alpha_j}, \quad (j \in \{2, 3\}).$$

Hence, the Ulam–Hyers stability of the fractal-fractional CD4⁺-HIV-1-model (2) is fulfilled. On the other hand, if we take

$$a_{\mathbb{Q}_j}(r_j) = \frac{\left[\frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \right] r_j}{1 - \left[\frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \right] \alpha_j}, \quad (j \in \{1, 2, 3\}),$$

then $a_{\mathbb{Q}_j}(0) = 0$, and the generalized Ulam–Hyers stability is simply proved. \square

The Ulam–Hyers–Rassias stability is checked about the fractal-fractional CD4⁺-HIV-1-model (2) in the next theorem.

Theorem 5. *The hypotheses (C1) and (C2) are considered to be held. Then the given fractal-fractional CD4⁺-HIV-1-model (2) is the Ulam–Hyers–Rassias and generalized Ulam–Hyers–Rassias stable.*

Proof. Let $r_1 > 0$, and $\mathbb{T}^* \in \mathbb{M}$ satisfying (21). By Theorem 3, let $\mathbb{T} \in \mathbb{M}$ be the unique solution of the given fractal-fractional CD4⁺-HIV-1-model (2). Then $\mathbb{T}(s)$ becomes

$$\begin{aligned} \mathbb{T}(s) &= \mathbb{T}_0 + \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}_1(s, \mathbb{T}(s), \mathbb{U}(s), \mathbb{V}(s)) \\ &\quad + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1} (s-\mathfrak{w})^{\delta-1} \mathbb{Q}_1(\mathfrak{w}, \mathbb{T}(\mathfrak{w}), \mathbb{U}(\mathfrak{w}), \mathbb{V}(\mathfrak{w})) \, d\mathfrak{w}. \end{aligned}$$

Therefore, by Lemma 3 and with the help of the triangle inequality, we estimate

$$\begin{aligned} |\mathbb{T}^*(s) - \mathbb{T}(s)| &\leq \left| \mathbb{T}^*(s) - \mathbb{T}_0 - \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}_1(s, \mathbb{T}(s), \mathbb{U}(s), \mathbb{V}(s)) \right. \\ &\quad \left. - \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1} (s-\mathfrak{w})^{\delta-1} \mathbb{Q}_1(\mathfrak{w}, \mathbb{T}(\mathfrak{w}), \mathbb{U}(\mathfrak{w}), \mathbb{V}(\mathfrak{w})) \, d\mathfrak{w} \right| \\ &\leq \left| \mathbb{T}^*(s) - \left(\mathbb{T}_0 + \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}_1(s, \mathbb{T}^*(s), \mathbb{U}^*(s), \mathbb{V}^*(s)) \right) \right. \\ &\quad \left. + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1} (s-\mathfrak{w})^{\delta-1} \mathbb{Q}_1(\mathfrak{w}, \mathbb{T}^*(\mathfrak{w}), \mathbb{U}^*(\mathfrak{w}), \mathbb{V}^*(\mathfrak{w})) \, d\mathfrak{w} \right| \\ &\quad + \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} |\mathbb{Q}_1(s, \mathbb{T}^*(s), \mathbb{U}^*(s), \mathbb{V}^*(s)) - \mathbb{Q}_1(s, \mathbb{T}(s), \mathbb{U}(s), \mathbb{V}(s))| \\ &\quad + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1} (s-\mathfrak{w})^{\delta-1} |\mathbb{Q}_1(\mathfrak{w}, \mathbb{T}^*(\mathfrak{w}), \mathbb{U}^*(\mathfrak{w}), \mathbb{V}^*(\mathfrak{w})) - \mathbb{Q}_1(\mathfrak{w}, \mathbb{T}(\mathfrak{w}), \mathbb{U}(\mathfrak{w}), \mathbb{V}(\mathfrak{w}))| \, d\mathfrak{w} \\ &\leq \left| \mathbb{T}^*(s) - \left(\mathbb{T}_0 + \text{FFML} \mathcal{I}_{0,s}^{(\delta,\sigma)} \mathbb{Q}_1(s, \mathbb{T}^*(s), \mathbb{U}^*(s), \mathbb{V}^*(s)) \right) \right| \\ &\quad + \frac{(1-\delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} |\mathbb{Q}_1(s, \mathbb{T}^*(s), \mathbb{U}^*(s), \mathbb{V}^*(s)) - \mathbb{Q}_1(s, \mathbb{T}(s), \mathbb{U}(s), \mathbb{V}(s))| \\ &\quad + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1} (s-\mathfrak{w})^{\delta-1} |\mathbb{Q}_1(\mathfrak{w}, \mathbb{T}^*(\mathfrak{w}), \mathbb{U}^*(\mathfrak{w}), \mathbb{V}^*(\mathfrak{w})) - \mathbb{Q}_1(\mathfrak{w}, \mathbb{T}(\mathfrak{w}), \mathbb{U}(\mathfrak{w}), \mathbb{V}(\mathfrak{w}))| \, d\mathfrak{w} \\ &\leq r_1 \Delta_{\mathfrak{h}_1} \mathfrak{h}_1(s) + \frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} \alpha_1 \|\mathbb{T}^* - \mathbb{T}\| + \frac{\delta\sigma S^{\delta+\sigma-1} \Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \alpha_1 \|\mathbb{T}^* - \mathbb{T}\| \\ &\leq r_1 \Delta_{\mathfrak{h}_1} \mathfrak{h}_1(s) + \left[\frac{(1-\delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1} \Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta+\sigma)} \right] \alpha_1 \|\mathbb{T} - \mathbb{T}^*\|. \end{aligned}$$

Accordingly, it gives

$$\|T^* - T\| \leq \frac{r_1 \Delta_{\hbar_1} \hbar_1(s)}{1 - \left[\frac{(1 - \delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta + \sigma)} \right] \alpha_1}.$$

Set

$$a_{(\mathbb{Q}_1, \hbar_1)} = \frac{\Delta_{\hbar_1}}{1 - \left[\frac{(1 - \delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta + \sigma)} \right] \alpha_1}.$$

Then $\|T^* - T\| \leq r_1 a_{(\mathbb{Q}_1, \hbar_1)} \hbar_1(s)$. Similarly,

$$\|U^* - U\| \leq r_2 a_{(\mathbb{Q}_2, \hbar_2)} \hbar_2(s),$$

$$\|V^* - V\| \leq r_3 a_{(\mathbb{Q}_3, \hbar_3)} \hbar_3(s),$$

where

$$a_{(\mathbb{Q}_j, \hbar_j)} = \frac{\Delta_{\hbar_j}}{1 - \left[\frac{(1 - \delta)\sigma S^{\sigma-1}}{\mathcal{AB}(\delta)} + \frac{\delta\sigma S^{\delta+\sigma-1}\Gamma(\sigma)}{\mathcal{AB}(\delta)\Gamma(\delta + \sigma)} \right] \alpha_j}, \quad (j \in \{2, 3\}).$$

In consequence, the fractal-fractional CD4⁺-HIV-1-model (2) is stable in the context of the Ulam–Hyers–Rassias criterion. Along with this, by defining $r_j = 1$, ($j \in \{1, 2, 3\}$), the mentioned fractal-fractional CD4⁺-HIV-1-model (2) is generalized Ulam–Hyers–Rassias stable. □

6. Numerical Scheme via Adams-Bashforth Method

In the present section of the paper, we aim to derive a numerical algorithm for the fractal-fractional CD4⁺-HIV-1-model (2). To do this, we apply a technique based on two-step Lagrange polynomials entitled fractional Adams-Bashforth method. We re-define fractal-fractional integral equations (12) at s_{k+1} . In fact, we discretize these integral equations (12) for $s = s_{k+1}$ as follows

$$\left\{ \begin{aligned} T(s_{k+1}) &= T_0 + \frac{(1 - \delta)\sigma s_k^{\sigma-1}}{\mathcal{AB}(\delta)} Q_1(s_k, T(s_k), U(s_k), V(s_k)) \\ &\quad + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^{s_{k+1}} w^{\sigma-1} (s_{k+1} - w)^{\delta-1} Q_1(w, T(w), U(w), V(w)) dw, \\ U(s_{k+1}) &= U_0 + \frac{(1 - \delta)\sigma s_k^{\sigma-1}}{\mathcal{AB}(\delta)} Q_2(s_k, T(s_k), U(s_k), V(s_k)) \\ &\quad + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^{s_{k+1}} w^{\sigma-1} (s_{k+1} - w)^{\delta-1} Q_2(w, T(w), U(w), V(w)) dw, \\ V(s_{k+1}) &= V_0 + \frac{(1 - \delta)\sigma s_k^{\sigma-1}}{\mathcal{AB}(\delta)} Q_3(s_k, T(s_k), U(s_k), V(s_k)) \\ &\quad + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^{s_{k+1}} w^{\sigma-1} (s_{k+1} - w)^{\delta-1} Q_3(w, T(w), U(w), V(w)) dw. \end{aligned} \right.$$

The approximation of above integrals are formulated by

$$\left\{ \begin{aligned} \mathbb{T}(s_{k+1}) &= \mathbb{T}_0 + \frac{(1-\delta)\sigma s_k^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}_1(s_k, \mathbb{T}(s_k), \mathbb{U}(s_k), \mathbb{V}(s_k)) \\ &\quad + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \sum_{\ell=1}^k \int_{s_\ell}^{s_{\ell+1}} \mathfrak{w}^{\sigma-1} (s_{k+1} - \mathfrak{w})^{\delta-1} \mathbb{Q}_1(\mathfrak{w}, \mathbb{T}(\mathfrak{w}), \mathbb{U}(\mathfrak{w}), \mathbb{V}(\mathfrak{w})) \, d\mathfrak{w}, \\ \mathbb{U}(s_{k+1}) &= \mathbb{U}_0 + \frac{(1-\delta)\sigma s_k^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}_2(s_k, \mathbb{T}(s_k), \mathbb{U}(s_k), \mathbb{V}(s_k)) \\ &\quad + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \sum_{\ell=1}^k \int_{s_\ell}^{s_{\ell+1}} \mathfrak{w}^{\sigma-1} (s_{k+1} - \mathfrak{w})^{\delta-1} \mathbb{Q}_2(\mathfrak{w}, \mathbb{T}(\mathfrak{w}), \mathbb{U}(\mathfrak{w}), \mathbb{V}(\mathfrak{w})) \, d\mathfrak{w}, \\ \mathbb{V}(s_{k+1}) &= \mathbb{V}_0 + \frac{(1-\delta)\sigma s_k^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}_3(s_k, \mathbb{T}(s_k), \mathbb{U}(s_k), \mathbb{V}(s_k)) \\ &\quad + \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \sum_{\ell=1}^k \int_{s_\ell}^{s_{\ell+1}} \mathfrak{w}^{\sigma-1} (s_{k+1} - \mathfrak{w})^{\delta-1} \mathbb{Q}_3(\mathfrak{w}, \mathbb{T}(\mathfrak{w}), \mathbb{U}(\mathfrak{w}), \mathbb{V}(\mathfrak{w})) \, d\mathfrak{w}. \end{aligned} \right.$$

Next, we approximate three functions $\mathfrak{w}^{\sigma-1}\mathbb{Q}_j(\mathfrak{w}, \mathbb{T}(\mathfrak{w}), \mathbb{U}(\mathfrak{w}), \mathbb{V}(\mathfrak{w}))$, $j = 1, 2, 3$, on the interval $[s_\ell, s_{\ell+1}]$ by applying two-step Lagrange interpolation polynomials under the step size $\mathbf{h} = s_\ell - s_{\ell-1}$. By straightforward computations, we obtain algorithms which yield the numerical solutions to the fractal-fractional CD4⁺-HIV-1-model (2) as

$$\begin{aligned} \mathbb{T}_{k+1} &= \mathbb{T}_0 + \frac{(1-\delta)\sigma s_k^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}_1(s_k, \mathbb{T}_k, \mathbb{U}_k, \mathbb{V}_k) + \frac{\sigma \mathbf{h}^\delta}{\mathcal{AB}(\delta)\Gamma(\delta+2)} \\ &\quad \times \sum_{\ell=1}^k \left[s_\ell^{\sigma-1} \mathbb{Q}_1(s_\ell, \mathbb{T}_\ell, \mathbb{U}_\ell, \mathbb{V}_\ell) \hat{Y}_1(k, \ell) - s_{\ell-1}^{\sigma-1} \mathbb{Q}_1(s_{\ell-1}, \mathbb{T}_{\ell-1}, \mathbb{U}_{\ell-1}, \mathbb{V}_{\ell-1}) \hat{Y}_2(k, \ell) \right], \end{aligned} \tag{29}$$

$$\begin{aligned} \mathbb{U}_{k+1} &= \mathbb{U}_0 + \frac{(1-\delta)\sigma s_k^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}_2(s_k, \mathbb{T}_k, \mathbb{U}_k, \mathbb{V}_k) + \frac{\sigma \mathbf{h}^\delta}{\mathcal{AB}(\delta)\Gamma(\delta+2)} \\ &\quad \times \sum_{\ell=1}^k \left[s_\ell^{\sigma-1} \mathbb{Q}_2(s_\ell, \mathbb{T}_\ell, \mathbb{U}_\ell, \mathbb{V}_\ell) \hat{Y}_1(k, \ell) - s_{\ell-1}^{\sigma-1} \mathbb{Q}_2(s_{\ell-1}, \mathbb{T}_{\ell-1}, \mathbb{U}_{\ell-1}, \mathbb{V}_{\ell-1}) \hat{Y}_2(k, \ell) \right], \end{aligned} \tag{30}$$

$$\begin{aligned} \mathbb{V}_{k+1} &= \mathbb{V}_0 + \frac{(1-\delta)\sigma s_k^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}_3(s_k, \mathbb{T}_k, \mathbb{U}_k, \mathbb{V}_k) + \frac{\sigma \mathbf{h}^\delta}{\mathcal{AB}(\delta)\Gamma(\delta+2)} \\ &\quad \times \sum_{\ell=1}^k \left[s_\ell^{\sigma-1} \mathbb{Q}_3(s_\ell, \mathbb{T}_\ell, \mathbb{U}_\ell, \mathbb{V}_\ell) \hat{Y}_1(k, \ell) - s_{\ell-1}^{\sigma-1} \mathbb{Q}_3(s_{\ell-1}, \mathbb{T}_{\ell-1}, \mathbb{U}_{\ell-1}, \mathbb{V}_{\ell-1}) \hat{Y}_2(k, \ell) \right], \end{aligned} \tag{31}$$

where

$$\hat{Y}_1(k, \ell) = (k+1-\ell)^\delta (k-\ell+2+\delta) - (k-\ell)^\delta (k-\ell+2+2\delta),$$

and

$$\hat{Y}_2(k, \ell) = (k+1-\ell)^{\delta+1} - (k-\ell)^\delta (k-\ell+1+\delta).$$

7. Numerical Scheme via Newton Polynomials Method

In this section, we produce a new numerical scheme for solutions of our fractal-fractional CD4⁺-HIV-1-model (2) which was introduced by Atangana and Araz in the book [44] in 2021. To do this, we again use the compact form of IVP (7) with the conditions (8) and (9). In this case, we have

$$\mathbb{K}(s) - \mathbb{K}(0) = \frac{\delta\sigma}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s \mathfrak{w}^{\sigma-1} (s - \mathfrak{w})^{\delta-1} \mathbb{Q}(\mathfrak{w}, \mathbb{K}(\mathfrak{w})) \, d\mathfrak{w} + \frac{(1 - \delta)\sigma s^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}(s, \mathbb{K}(s)).$$

Set $\mathbb{Q}^*(s, \mathbb{K}(s)) = \sigma s^{\sigma-1} \mathbb{Q}(s, \mathbb{K}(s))$. Then

$$\mathbb{K}(s) - \mathbb{K}(0) = \frac{\delta}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^s (s - \mathfrak{w})^{\delta-1} \mathbb{Q}^*(\mathfrak{w}, \mathbb{K}(\mathfrak{w})) \, d\mathfrak{w} + \frac{(1 - \delta)}{\mathcal{AB}(\delta)} \mathbb{Q}^*(s, \mathbb{K}(s)).$$

By discretizing the above equation at $s = s_{k+1} = (k + 1)\mathbf{h}$, we get

$$\mathbb{K}(s_{k+1}) - \mathbb{K}(0) = \frac{\delta}{\mathcal{AB}(\delta)\Gamma(\delta)} \int_0^{s_{k+1}} (s_{k+1} - \mathfrak{w})^{\delta-1} \mathbb{Q}^*(\mathfrak{w}, \mathbb{K}(\mathfrak{w})) \, d\mathfrak{w} + \frac{(1 - \delta)}{\mathcal{AB}(\delta)} \mathbb{Q}^*(s_k, \mathbb{K}(s_k)).$$

If we approximate the above integral, then it becomes

$$\begin{aligned} \mathbb{K}(s_{k+1}) &= \mathbb{K}_0 + \frac{(1 - \delta)}{\mathcal{AB}(\delta)} \mathbb{Q}^*(s_k, \mathbb{K}(s_k)) \\ &+ \frac{\delta}{\mathcal{AB}(\delta)\Gamma(\delta)} \sum_{\ell=2}^k \int_{s_\ell}^{s_{\ell+1}} (s_{k+1} - \mathfrak{w})^{\delta-1} \mathbb{Q}^*(\mathfrak{w}, \mathbb{K}(\mathfrak{w})) \, d\mathfrak{w}. \end{aligned} \tag{32}$$

In this step, the function $\mathbb{Q}^*(s, \mathbb{K}(s))$ is approximated by the Newton polynomial as

$$\begin{aligned} P_k^*(\mathfrak{w}) &= \mathbb{Q}^*(s_{k-2}, \mathbb{K}(s_{k-2})) + \frac{\mathbb{Q}^*(s_{k-1}, \mathbb{K}(s_{k-1})) - \mathbb{Q}^*(s_{k-2}, \mathbb{K}(s_{k-2}))}{\mathbf{h}} (\mathfrak{w} - s_{k-2}) \\ &+ \frac{\mathbb{Q}^*(s_k, \mathbb{K}(s_k)) - 2\mathbb{Q}^*(s_{k-1}, \mathbb{K}(s_{k-1})) + \mathbb{Q}^*(s_{k-2}, \mathbb{K}(s_{k-2}))}{2\mathbf{h}^2} (\mathfrak{w} - s_{k-2})(\mathfrak{w} - s_{k-1}). \end{aligned} \tag{33}$$

Substitute (33) into (32):

$$\begin{aligned} \mathbb{K}_{k+1} &= \mathbb{K}_0 + \frac{(1 - \delta)}{\mathcal{AB}(\delta)} \mathbb{Q}^*(s_k, \mathbb{K}(s_k)) + \frac{\delta}{\mathcal{AB}(\delta)\Gamma(\delta)} \sum_{\ell=2}^k \int_{s_\ell}^{s_{\ell+1}} \left[\mathbb{Q}^*(s_{\ell-2}, \mathbb{K}(s_{\ell-2})) \right. \\ &+ \frac{\mathbb{Q}^*(s_{\ell-1}, \mathbb{K}(s_{\ell-1})) - \mathbb{Q}^*(s_{\ell-2}, \mathbb{K}(s_{\ell-2}))}{\mathbf{h}} (\mathfrak{w} - s_{\ell-2}) \\ &+ \left. \frac{\mathbb{Q}^*(s_\ell, \mathbb{K}(s_\ell)) - 2\mathbb{Q}^*(s_{\ell-1}, \mathbb{K}(s_{\ell-1})) + \mathbb{Q}^*(s_{\ell-2}, \mathbb{K}(s_{\ell-2}))}{2\mathbf{h}^2} (\mathfrak{w} - s_{\ell-2})(\mathfrak{w} - s_{\ell-1}) \right] \\ &\times (s_{k+1} - \mathfrak{w})^{\delta-1} \, d\mathfrak{w}. \end{aligned}$$

We simplify the above relations, and we get

$$\begin{aligned} \mathbb{K}_{k+1} &= \mathbb{K}_0 + \frac{(1 - \delta)}{\mathcal{AB}(\delta)} \mathbb{Q}^*(s_k, \mathbb{K}(s_k)) \\ &+ \frac{\delta}{\mathcal{AB}(\delta)\Gamma(\delta)} \sum_{\ell=2}^k \left[\int_{s_\ell}^{s_{\ell+1}} \mathbb{Q}^*(s_{\ell-2}, \mathbb{K}(s_{\ell-2})) (s_{k+1} - \mathfrak{w})^{\delta-1} \, d\mathfrak{w} \right. \end{aligned}$$

$$\begin{aligned}
 &+ \int_{s_\ell}^{s_{\ell+1}} \frac{Q^*(s_{\ell-1}, \mathbb{K}_{\ell-1}) - Q^*(s_{\ell-2}, \mathbb{K}_{\ell-2})}{\mathbf{h}} (\mathbf{w} - s_{\ell-2})(s_{k+1} - \mathbf{w})^{\delta-1} d\mathbf{w} \\
 &+ \int_{s_\ell}^{s_{\ell+1}} \frac{Q^*(s_\ell, \mathbb{K}_\ell) - 2Q^*(s_{\ell-1}, \mathbb{K}_{\ell-1}) + Q^*(s_{\ell-2}, \mathbb{K}_{\ell-2})}{2\mathbf{h}^2} (\mathbf{w} - s_{\ell-2})(\mathbf{w} - s_{\ell-1}) \\
 &\times (s_{k+1} - \mathbf{w})^{\delta-1} d\mathbf{w} \Big].
 \end{aligned}$$

In consequence,

$$\begin{aligned}
 \mathbb{K}_{k+1} &= \mathbb{K}_0 + \frac{(1 - \delta)}{\mathcal{AB}(\delta)} Q^*(s_k, \mathbb{K}(s_k)) \\
 &+ \frac{\delta}{\mathcal{AB}(\delta)\Gamma(\delta)} \sum_{\ell=2}^k Q^*(s_{\ell-2}, \mathbb{K}_{\ell-2}) \int_{s_\ell}^{s_{\ell+1}} (s_{k+1} - \mathbf{w})^{\delta-1} d\mathbf{w} \\
 &+ \frac{\delta}{\mathcal{AB}(\delta)\Gamma(\delta)} \sum_{\ell=2}^k \frac{Q^*(s_{\ell-1}, \mathbb{K}_{\ell-1}) - Q^*(s_{\ell-2}, \mathbb{K}_{\ell-2})}{\mathbf{h}} \int_{s_\ell}^{s_{\ell+1}} (\mathbf{w} - s_{\ell-2})(s_{k+1} - \mathbf{w})^{\delta-1} d\mathbf{w} \\
 &+ \frac{\delta}{\mathcal{AB}(\delta)\Gamma(\delta)} \sum_{\ell=2}^k \frac{Q^*(s_\ell, \mathbb{K}_\ell) - 2Q^*(s_{\ell-1}, \mathbb{K}_{\ell-1}) + Q^*(s_{\ell-2}, \mathbb{K}_{\ell-2})}{2\mathbf{h}^2} \\
 &\times \int_{s_\ell}^{s_{\ell+1}} (\mathbf{w} - s_{\ell-2})(\mathbf{w} - s_{\ell-1}) \times (s_{k+1} - \mathbf{w})^{\delta-1} d\mathbf{w} \Big].
 \end{aligned} \tag{34}$$

On the other hand, we compute above three integrals separately, and we get

$$\int_{s_\ell}^{s_{\ell+1}} (s_{k+1} - \mathbf{w})^{\delta-1} d\mathbf{w} = \frac{\mathbf{h}^\delta}{\delta} [(k - \ell + 1)^\delta - (k - \ell)^\delta], \tag{35}$$

and

$$\begin{aligned}
 \int_{s_\ell}^{s_{\ell+1}} (\mathbf{w} - s_{\ell-2})(s_{k+1} - \mathbf{w})^{\delta-1} d\mathbf{w} &= \frac{\mathbf{h}^{\delta+1}}{\delta(\delta + 1)} [(k - \ell + 1)^\delta (k - \ell + 3 + 2\delta) \\
 &- (k - \ell + 1)^\delta (k - \ell + 3 + 3\delta)],
 \end{aligned} \tag{36}$$

and

$$\begin{aligned}
 \int_{s_\ell}^{s_{\ell+1}} (\mathbf{w} - s_{\ell-2})(\mathbf{w} - s_{\ell-1})(s_{k+1} - \mathbf{w})^{\delta-1} d\mathbf{w} &= \frac{\mathbf{h}^{\delta+2}}{\delta(\delta + 1)(\delta + 2)} \left((k - \ell + 1)^\delta [2(k - \ell)^2 \right. \\
 &+ (3\delta + 10)(k - \ell) + 2\delta^2 + 9\delta + 12] - (k - \ell)^\delta [2(k - \ell)^2 \\
 &\left. + (5\delta + 10)(k - \ell) + 6\delta^2 + 18\delta + 12] \right).
 \end{aligned} \tag{37}$$

By putting (35)–(37) in (34), we obtain

$$\begin{aligned}
 \mathbb{K}_{k+1} &= \mathbb{K}_0 + \frac{(1 - \delta)}{\mathcal{AB}(\delta)} Q^*(s_k, \mathbb{K}(s_k)) \\
 &+ \frac{\delta \mathbf{h}^\delta}{\mathcal{AB}(\delta)\Gamma(\delta + 1)} \sum_{\ell=2}^k Q^*(s_{\ell-2}, \mathbb{K}_{\ell-2}) [(k - \ell + 1)^\delta - (k - \ell)^\delta]
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{\delta \mathbf{h}^\delta}{\mathcal{AB}(\delta)\Gamma(\delta+2)} \sum_{\ell=2}^k \left[\mathbb{Q}^*(s_{\ell-1}, \mathbb{K}_{\ell-1}) - \mathbb{Q}^*(s_{\ell-2}, \mathbb{K}_{\ell-2}) \right] \\
 & \times \left[(k-\ell+1)^\delta (k-\ell+3+2\delta) - (k-\ell+1)^\delta (k-\ell+3+3\delta) \right] \\
 & + \frac{\delta \mathbf{h}^\delta}{2\mathcal{AB}(\delta)\Gamma(\delta+3)} \sum_{\ell=2}^k \left[\mathbb{Q}^*(s_\ell, \mathbb{K}_\ell) - 2\mathbb{Q}^*(s_{\ell-1}, \mathbb{K}_{\ell-1}) + \mathbb{Q}^*(s_{\ell-2}, \mathbb{K}_{\ell-2}) \right] \\
 & \times \left[(k-\ell+1)^\delta [2(k-\ell)^2 + (3\delta+10)(k-\ell) + 2\delta^2 + 9\delta + 12] - (k-\ell)^\delta [2(k-\ell)^2 \right. \\
 & \left. + (5\delta+10)(k-\ell) + 6\delta^2 + 18\delta + 12] \right]. \tag{38}
 \end{aligned}$$

Lastly, we replace $\mathbb{Q}^*(s, \mathbb{K}(s)) = \sigma s^{\sigma-1} \mathbb{Q}(s, \mathbb{K}(s))$ in (38), and we get

$$\begin{aligned}
 \mathbb{K}_{k+1} & = \mathbb{K}_0 + \frac{(1-\delta)\sigma s_k^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}(s_k, \mathbb{K}(s_k)) \\
 & + \frac{\delta \sigma \mathbf{h}^\delta}{\mathcal{AB}(\delta)\Gamma(\delta+1)} \sum_{\ell=2}^k s_{\ell-2}^{\sigma-1} \mathbb{Q}(s_{\ell-2}, \mathbb{K}_{\ell-2}) \hat{\Psi}_1(k, \ell, \delta) \\
 & + \frac{\delta \sigma \mathbf{h}^\delta}{\mathcal{AB}(\delta)\Gamma(\delta+2)} \sum_{\ell=2}^k \left[s_{\ell-1}^{\sigma-1} \mathbb{Q}(s_{\ell-1}, \mathbb{K}_{\ell-1}) - s_{\ell-2}^{\sigma-1} \mathbb{Q}(s_{\ell-2}, \mathbb{K}_{\ell-2}) \right] \hat{\Psi}_2(k, \ell, \delta) \\
 & + \frac{\delta \sigma \mathbf{h}^\delta}{2\mathcal{AB}(\delta)\Gamma(\delta+3)} \sum_{\ell=2}^k \left[s_\ell^{\sigma-1} \mathbb{Q}(s_\ell, \mathbb{K}_\ell) - 2s_{\ell-1}^{\sigma-1} \mathbb{Q}(s_{\ell-1}, \mathbb{K}_{\ell-1}) + s_{\ell-2}^{\sigma-1} \mathbb{Q}(s_{\ell-2}, \mathbb{K}_{\ell-2}) \right] \hat{\Psi}_3(k, \ell, \delta), \tag{39}
 \end{aligned}$$

where

$$\begin{aligned}
 \hat{\Psi}_1(k, \ell, \delta) & = (k-\ell+1)^\delta - (k-\ell)^\delta, \\
 \hat{\Psi}_2(k, \ell, \delta) & = (k-\ell+1)^\delta (k-\ell+3+2\delta) - (k-\ell+1)^\delta (k-\ell+3+3\delta), \\
 \hat{\Psi}_3(k, \ell, \delta) & = (k-\ell+1)^\delta [2(k-\ell)^2 + (3\delta+10)(k-\ell) + 2\delta^2 + 9\delta + 12] \\
 & \quad - (k-\ell)^\delta [2(k-\ell)^2 + (5\delta+10)(k-\ell) + 6\delta^2 + 18\delta + 12]. \tag{40}
 \end{aligned}$$

Based on the numerical scheme obtained in (39), numerical solutions of the fractal-fractional CD4⁺-HIV-1-model (2) are given by

$$\begin{aligned}
 \mathbb{T}_{k+1} & = \mathbb{T}_0 + \frac{(1-\delta)\sigma s_k^{\sigma-1}}{\mathcal{AB}(\delta)} \mathbb{Q}_1(s_k, \mathbb{T}(s_k), \mathbb{U}(s_k), \mathbb{V}(s_k)) \\
 & + \frac{\delta \sigma \mathbf{h}^\delta}{\mathcal{AB}(\delta)\Gamma(\delta+1)} \sum_{\ell=2}^k s_{\ell-2}^{\sigma-1} \mathbb{Q}_1(s_{\ell-2}, \mathbb{T}_{\ell-2}, \mathbb{U}_{\ell-2}, \mathbb{V}_{\ell-2}) \hat{\Psi}_1(k, \ell, \delta) \\
 & + \frac{\delta \sigma \mathbf{h}^\delta}{\mathcal{AB}(\delta)\Gamma(\delta+2)} \sum_{\ell=2}^k \left[s_{\ell-1}^{\sigma-1} \mathbb{Q}_1(s_{\ell-1}, \mathbb{T}_{\ell-1}, \mathbb{U}_{\ell-1}, \mathbb{V}_{\ell-1}) - s_{\ell-2}^{\sigma-1} \mathbb{Q}_1(s_{\ell-2}, \mathbb{T}_{\ell-2}, \mathbb{U}_{\ell-2}, \mathbb{V}_{\ell-2}) \right] \hat{\Psi}_2(k, \ell, \delta) \\
 & + \frac{\delta \sigma \mathbf{h}^\delta}{2\mathcal{AB}(\delta)\Gamma(\delta+3)} \sum_{\ell=2}^k \left[s_\ell^{\sigma-1} \mathbb{Q}_1(s_\ell, \mathbb{T}_\ell, \mathbb{U}_\ell, \mathbb{V}_\ell) - 2s_{\ell-1}^{\sigma-1} \mathbb{Q}_1(s_{\ell-1}, \mathbb{T}_{\ell-1}, \mathbb{U}_{\ell-1}, \mathbb{V}_{\ell-1}) \right]
 \end{aligned}$$

$$+ s_{\ell-2}^{\sigma-1} Q_1(s_{\ell-2}, T_{\ell-2}, U_{\ell-2}, V_{\ell-2}) \Psi_3(k, \ell, \delta), \tag{41}$$

and

$$\begin{aligned} U_{k+1} = & U_0 + \frac{(1-\delta)\sigma s_k^{\sigma-1}}{\mathcal{AB}(\delta)} Q_2(s_k, T(s_k), U(s_k), V(s_k)) \\ & + \frac{\delta\sigma h^\delta}{\mathcal{AB}(\delta)\Gamma(\delta+1)} \sum_{\ell=2}^k s_{\ell-2}^{\sigma-1} Q_2(s_{\ell-2}, T_{\ell-2}, U_{\ell-2}, V_{\ell-2}) \Psi_1(k, \ell, \delta) \\ & + \frac{\delta\sigma h^\delta}{\mathcal{AB}(\delta)\Gamma(\delta+2)} \sum_{\ell=2}^k \left[s_{\ell-1}^{\sigma-1} Q_2(s_{\ell-1}, T_{\ell-1}, U_{\ell-1}, V_{\ell-1}) - s_{\ell-2}^{\sigma-1} Q_2(s_{\ell-2}, T_{\ell-2}, U_{\ell-2}, V_{\ell-2}) \right] \Psi_2(k, \ell, \delta) \\ & + \frac{\delta\sigma h^\delta}{2\mathcal{AB}(\delta)\Gamma(\delta+3)} \sum_{\ell=2}^k \left[s_{\ell}^{\sigma-1} Q_2(s_{\ell}, T_{\ell}, U_{\ell}, V_{\ell}) - 2s_{\ell-1}^{\sigma-1} Q_2(s_{\ell-1}, T_{\ell-1}, U_{\ell-1}, V_{\ell-1}) \right. \\ & \left. + s_{\ell-2}^{\sigma-1} Q_2(s_{\ell-2}, T_{\ell-2}, U_{\ell-2}, V_{\ell-2}) \right] \Psi_3(k, \ell, \delta), \tag{42} \end{aligned}$$

and

$$\begin{aligned} V_{k+1} = & V_0 + \frac{(1-\delta)\sigma s_k^{\sigma-1}}{\mathcal{AB}(\delta)} Q_3(s_k, T(s_k), U(s_k), V(s_k)) \\ & + \frac{\delta\sigma h^\delta}{\mathcal{AB}(\delta)\Gamma(\delta+1)} \sum_{\ell=2}^k s_{\ell-2}^{\sigma-1} Q_3(s_{\ell-2}, T_{\ell-2}, U_{\ell-2}, V_{\ell-2}) \Psi_1(k, \ell, \delta) \\ & + \frac{\delta\sigma h^\delta}{\mathcal{AB}(\delta)\Gamma(\delta+2)} \sum_{\ell=2}^k \left[s_{\ell-1}^{\sigma-1} Q_3(s_{\ell-1}, T_{\ell-1}, U_{\ell-1}, V_{\ell-1}) - s_{\ell-2}^{\sigma-1} Q_3(s_{\ell-2}, T_{\ell-2}, U_{\ell-2}, V_{\ell-2}) \right] \Psi_2(k, \ell, \delta) \\ & + \frac{\delta\sigma h^\delta}{2\mathcal{AB}(\delta)\Gamma(\delta+3)} \sum_{\ell=2}^k \left[s_{\ell}^{\sigma-1} Q_3(s_{\ell}, T_{\ell}, U_{\ell}, V_{\ell}) - 2s_{\ell-1}^{\sigma-1} Q_3(s_{\ell-1}, T_{\ell-1}, U_{\ell-1}, V_{\ell-1}) \right. \\ & \left. + s_{\ell-2}^{\sigma-1} Q_3(s_{\ell-2}, T_{\ell-2}, U_{\ell-2}, V_{\ell-2}) \right] \Psi_3(k, \ell, \delta), \tag{43} \end{aligned}$$

where the constants $\Psi_j(k, \ell, \delta)$ are introduced in (40) for $j = 1, 2, 3$.

8. Simulations and Discussions

In this section, we simulate and discuss dynamical behaviors of the fractal-fractional CD4⁺-HIV-1-model (2) via real data for parameters and initial values assumed by [45]. Based on the given data in this source, we let $\theta = 10$, $q = 0.000024$, $\rho = 0.01$, $\zeta = 0.2$, $\kappa = 0.16$, $\vartheta = 3.4$, and $N = 1000$. Moreover, the initial values for three state functions are $T(0) = T_0 = 1000$, $U(0) = U_0 = 0$, $V(0) = V_0 = 0.001$. It is assumed that all parameters are in days.

In the provided figures, we show the behaviors of three state functions T, U, V under the effect of different values for the fractal-fractional orders $\delta = \sigma = 1.00, 0.98, 0.96, 0.94, 0.92, 0.90$ simultaneously. We also compare the results in the graphs with respect to two numerical algorithms given by (29)–(31) and (41)–(43) as shown in Tables 1–3.

In these graphs, the quantity of the time s interprets the number of days. Note that Figures 1 and 2 demonstrate the dynamics of the CD4⁺ T-cells via the Adams-Bashforth method (Section 6). Figure 1a indicates that when the fractal dimensions and fractional

orders decrease, the number of susceptible $CD4^+$ T-cells steadily increases from the 42nd day to the 64th day, and steadily decreases from the 65th day to the 102nd day. Further it increases again from the 103rd day and finally converges to the integer-order at the 250th day to the end of the simulation period.

Figure 1b,c show that when the fractal dimensions and fractional orders move from the integer order, the peak of the amount of infectious $CD4^+$ T-cells and the free particles of the infection of the HIV-1 virus in the blood decrease respectively and also give slight differences in their asymptotic stabilities.

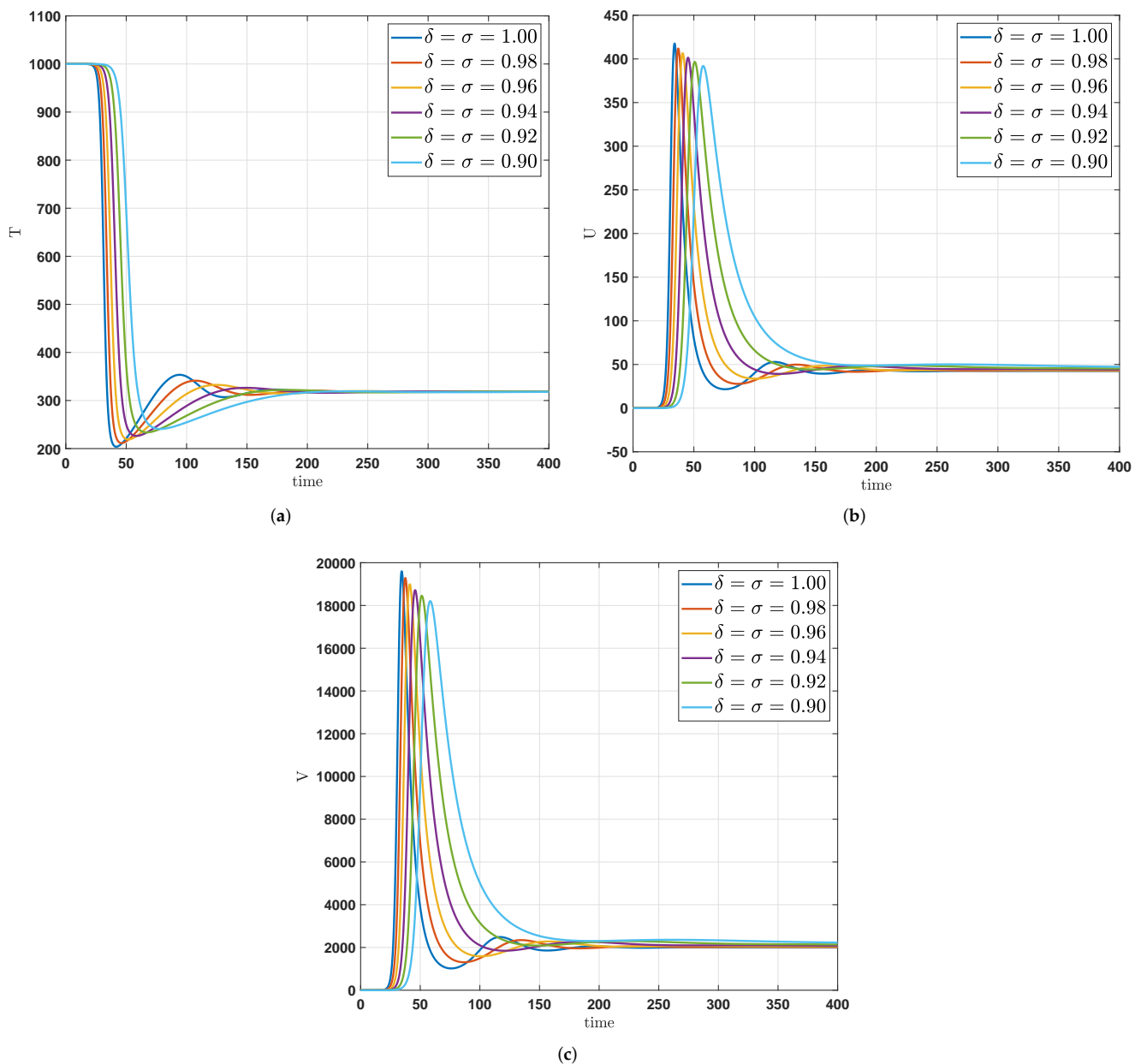


Figure 1. Numerical trajectories by varying both fractal order and fractional order $\delta = \sigma = 1.00, 0.98, 0.96, 0.94, 0.92, 0.90$ via the Adams-Bashforth method. (a) Amount of susceptible $CD4^+$ T-cells. (b) Amount of infectious $CD4^+$ T-cells. (c) Free particles of the infection of the HIV-1 virus in the blood.

In Figure 2a,c, we show a 0.01-variation in fractal-fractional order. In this case, the numerical trajectories show that a slight change in the fractal-fractional order produces a slight changes in the asymptotic behavior of the HIV-1 virus on $CD4^+$ T-cells.

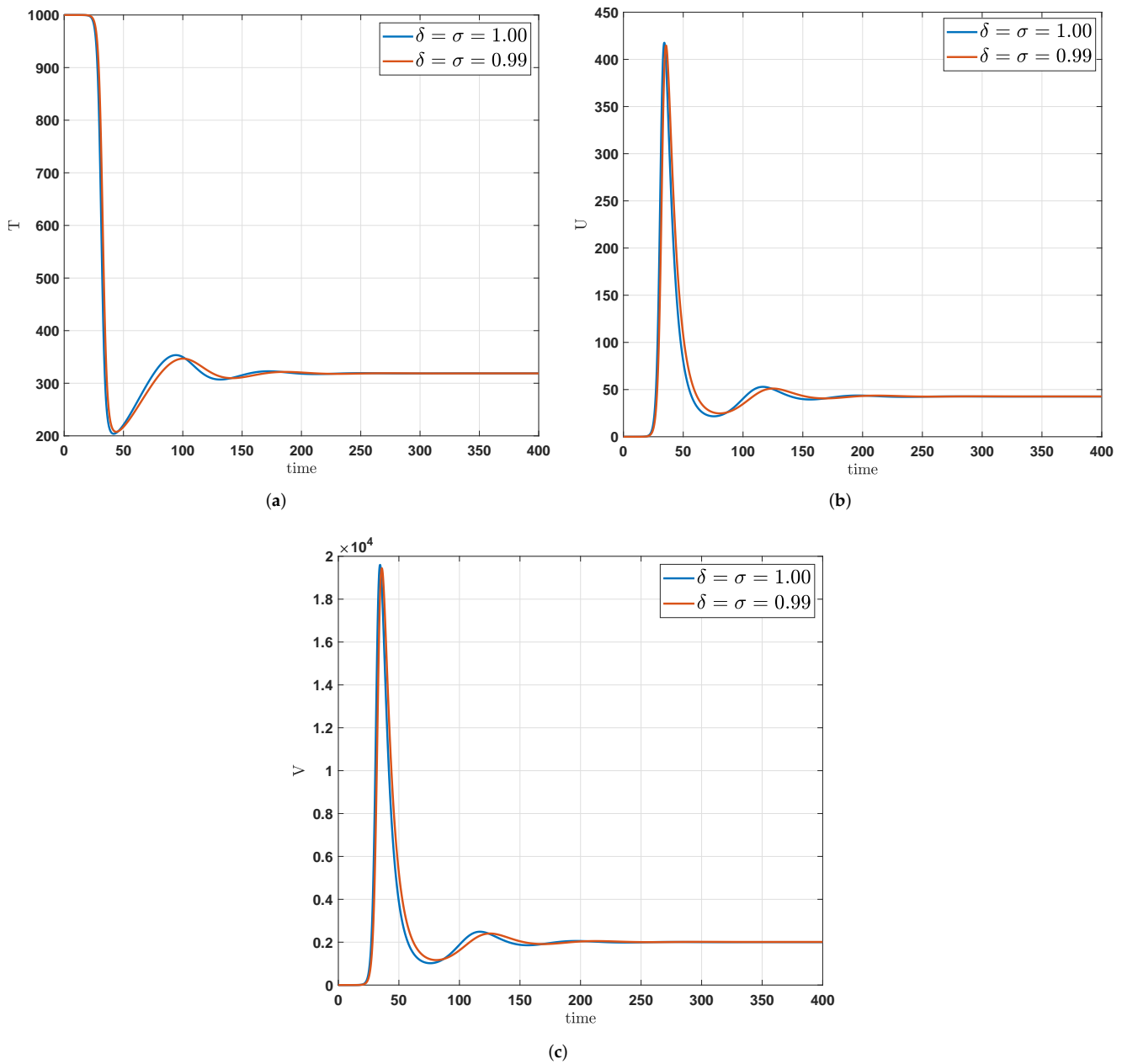


Figure 2. Numerical trajectories by varying both fractal order and fractional orders $\delta = \sigma = 1.00, 0.99$ via the Adams-Bashforth method. (a) A 0.01-variation in fractal-fractional orders of susceptible $CD4^+$ T-cells. (b) A 0.01-variation in fractal-fractional orders of infectious $CD4^+$ T-cells. (c) A 0.01-variation in fractal-fractional orders of free particles of the infection of the HIV-1 virus in the blood.

Figures 3 and 4 demonstrate the dynamics of the $CD4^+$ T-cells via the Newton polynomials method (Section 7). Figure 3a also indicates that when the fractal dimensions and fractional orders decrease, the number of susceptible $CD4^+$ T-cells steadily increases from the 42nd day to the 64th day, and steadily decreases from the 65th day to the 102nd day. Also it increases again from the 103rd day and finally converges to the integer-order at the 250th day to the end of the simulation period, but it shows a disorganized behavior from the 11th day to the 24th day, which indicates the early stages of HIV in the $CD4^+$ T-cells.

Figure 3b,c show that when the fractal dimensions and fractional orders move from the integer order, the peak of the amount of infectious $CD4^+$ T-cells and the free particles of the infection of the HIV-1 virus in the blood decrease, respectively and also give slight

differences in their asymptotic stabilities, but again, they show a disorganized behavior from the 11th day to the 24th day.

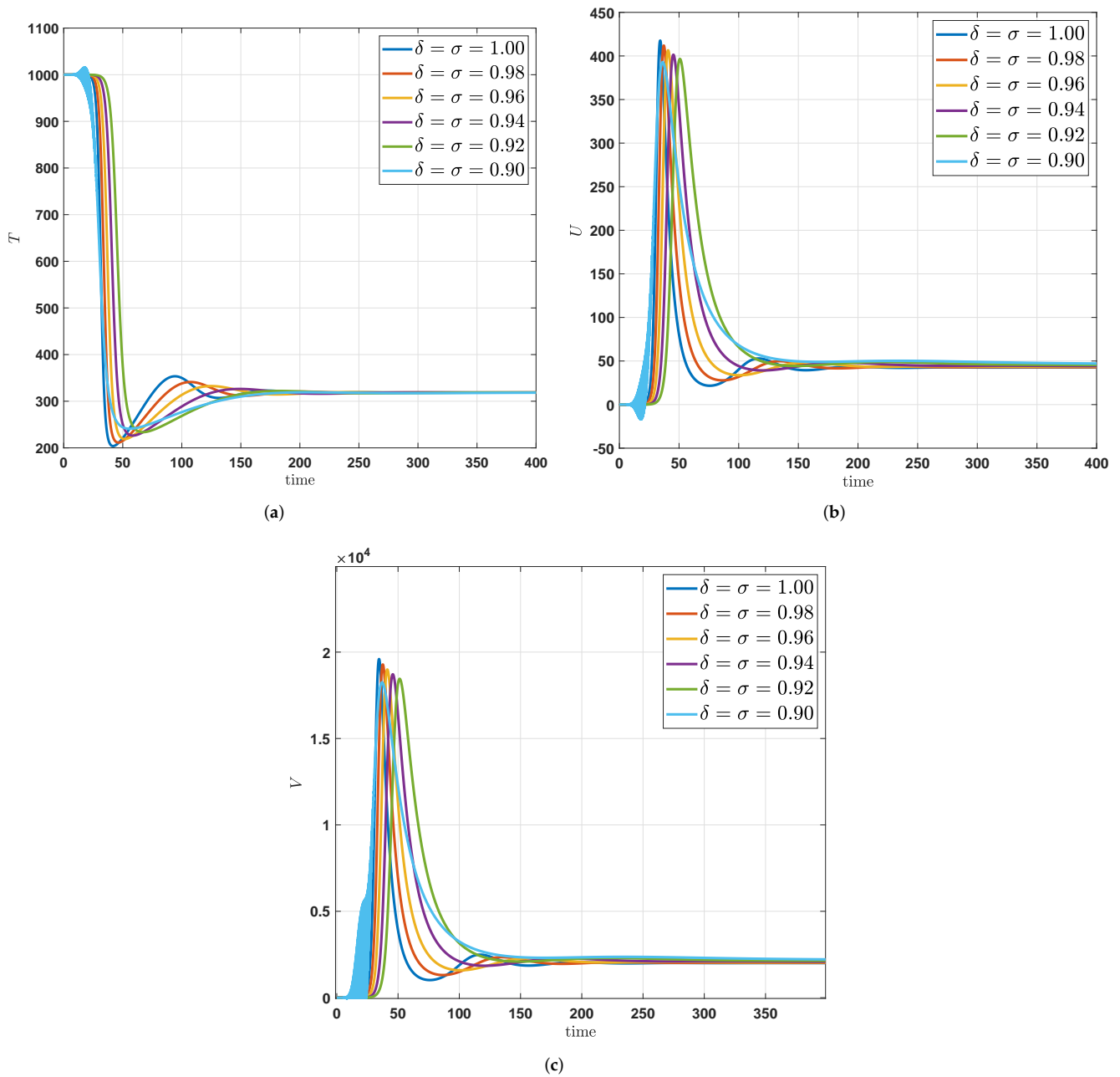


Figure 3. Numerical trajectories by varying both fractal order and fractional order $\delta = \sigma = 1.00, 0.98, 0.96, 0.94, 0.92, 0.90$ via the Newton polynomials method. (a) Amount of susceptible $CD4^+$ T-cells. (b) Amount of infectious $CD4^+$ T-cells. (c) Free particles of the infection of the HIV-1 virus in the blood.

In Figure 4a,c, we show a 0.01-variation in fractal-fractional orders, and the numerical trajectories show that a slight change in the fractal-fractional orders produce slight changes in the asymptotic behavior of the HIV-1 virus on $CD4^+$ T-cells.

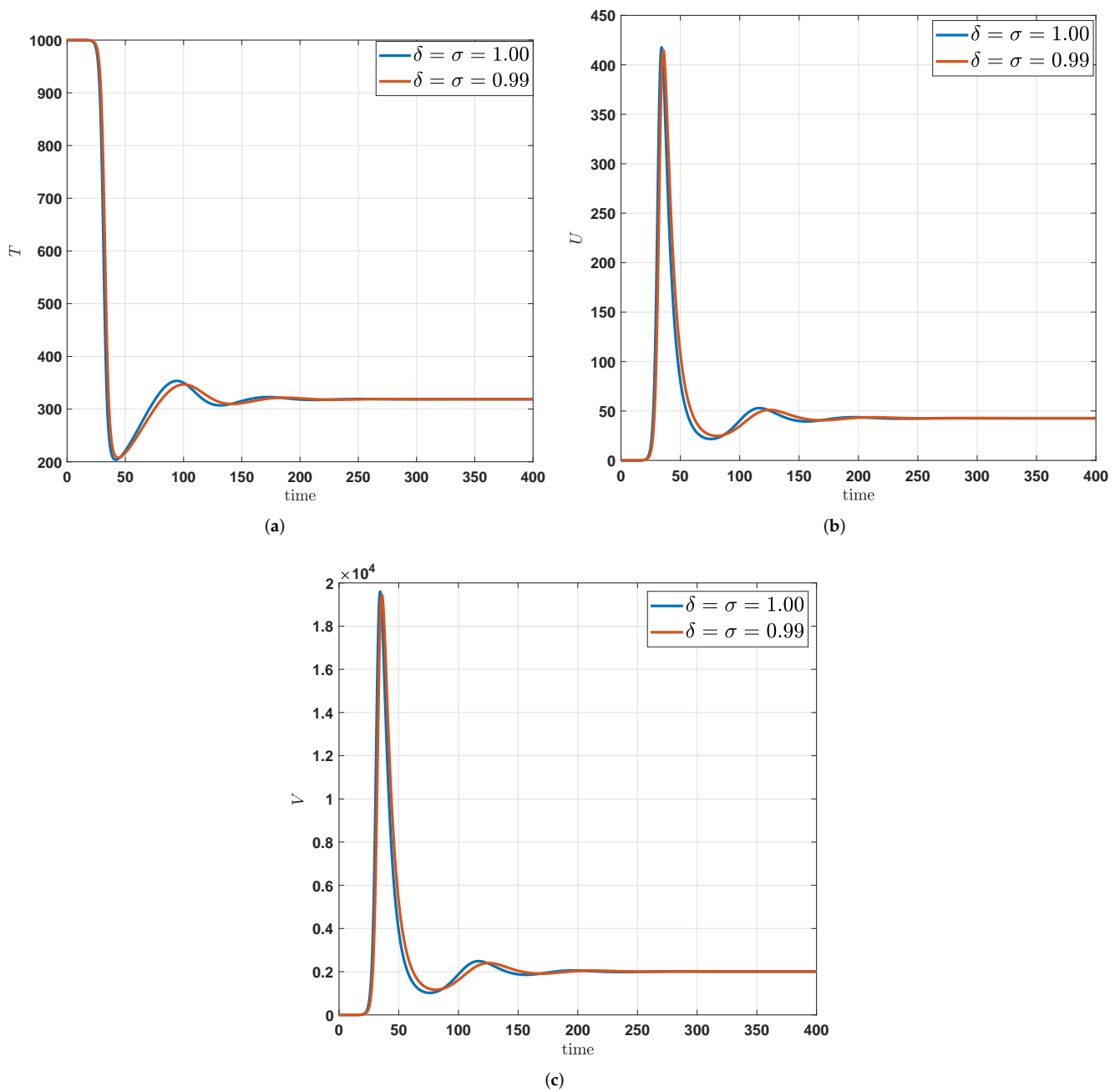


Figure 4. Numerical trajectories by varying both fractal order and fractional order $\delta = \sigma = 1.00, 0.99$ via the Newton polynomials method. (a) A 0.01-variation in fractal-fractional orders of susceptible $CD4^+$ T-cells. (b) A 0.01-variation in fractal-fractional orders of infectious $CD4^+$ T-cells. (c) A 0.01-variation in fractal-fractional orders of free particles of the infection of the HIV-1 virus in the blood.

In Figure 5, based on the Adams-Bashforth method, we see the numerical trajectories of three state functions by varying the average number of infected particles N for the values $N = 500, 600, 700, 800, 900, 1000$ under the fractal-fractional order $\delta = \sigma = 0.95$. Thus, in Figure 5a, it shows that the susceptible $CD4^+$ T-cells increase as the average number of infected particles by an existing infected cell reduces by 10%, and that of the peak of the amount of infectious $CD4^+$ T-cells and the free particles of the infection of the HIV-1 virus in the blood decrease, respectively in Figure 5b,c.

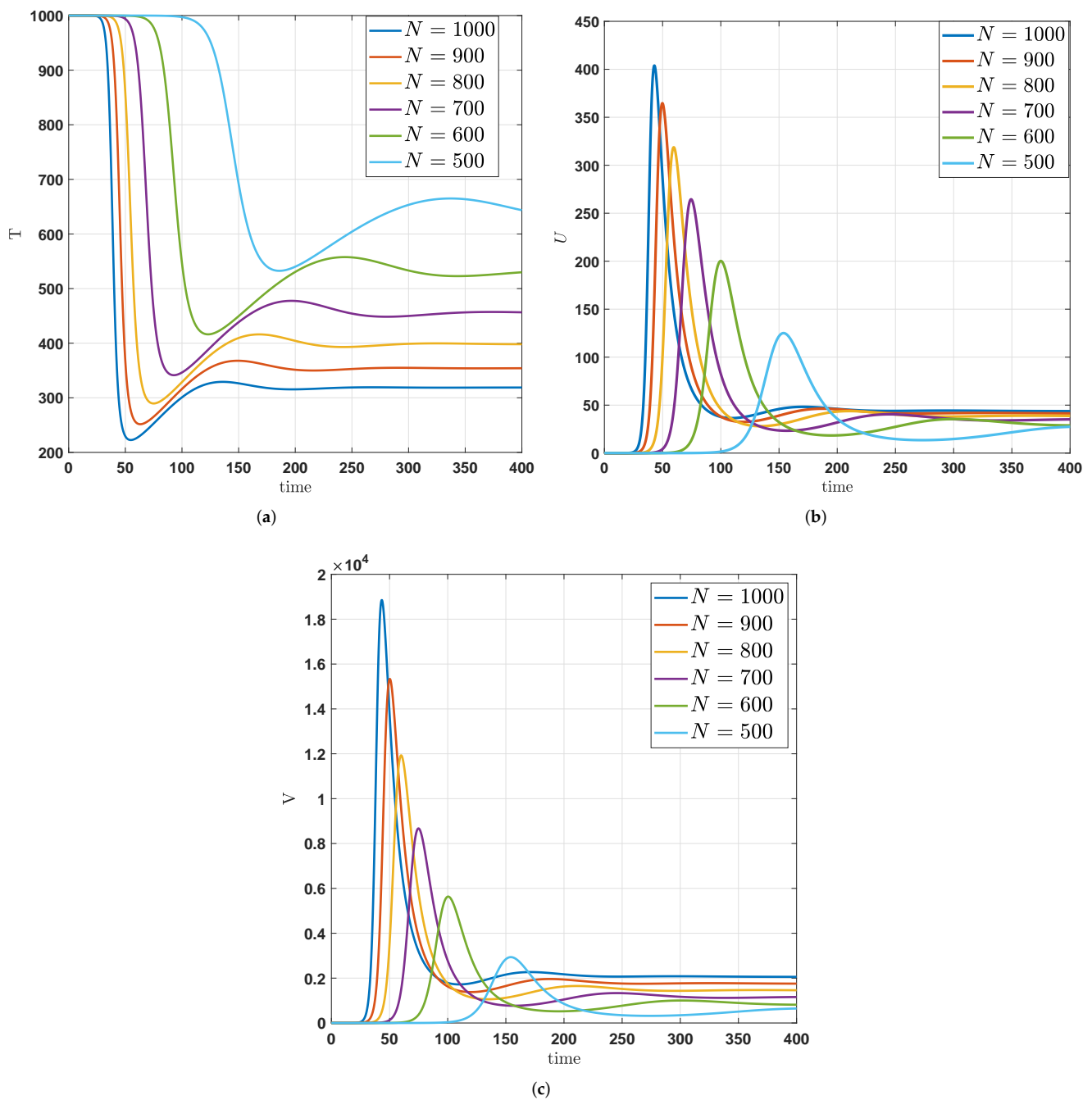


Figure 5. Numerical trajectories by varying the average number of infected particles N under the fractal order and fractional order $\delta = \sigma = 0.95$ via the Adams-Bashforth method. (a) The effects of N on T . (b) The effects of N on U . (c) The effects of N on V .

In Figure 6, based on the Adams-Bashforth method, we see the numerical trajectories by varying the supply rate for new T-cells for $\theta = 10, 15, 20, 25, 30, 35$ under the fractal order and fractional order $\delta = \sigma = 0.95$. Thus, in Figure 6a, it shows that the susceptible $CD4^+$ T-cells increase for a period of time and gradually decrease as the number of the supply rate θ increases before converging at the 150th day.

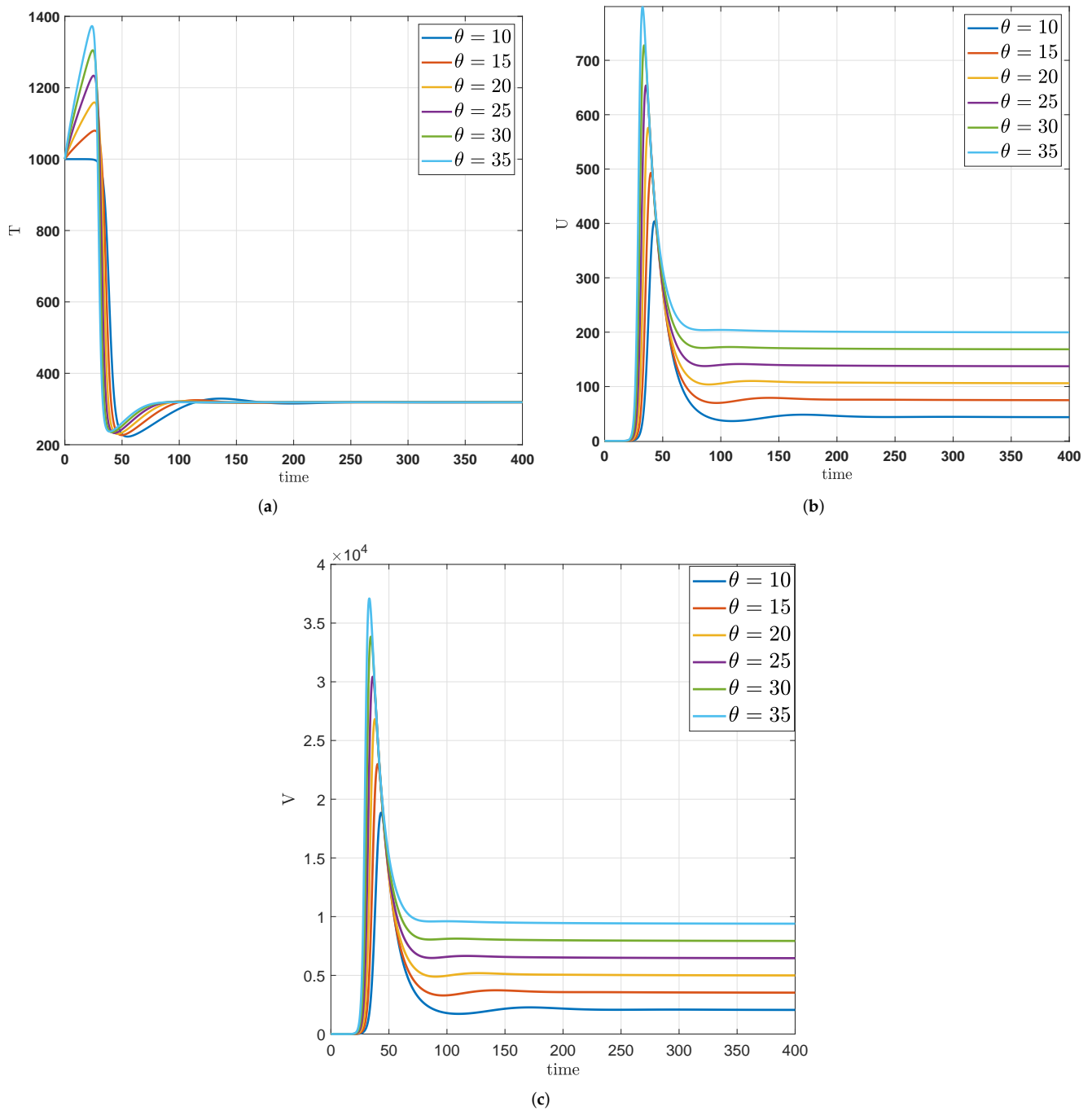


Figure 6. Numerical trajectories by varying the supply rate for new T-cells θ under the fractal order and fractional order $\delta = \sigma = 0.95$ via the Adams-Bashforth method. (a) The effects of θ on \mathbb{T} . (b) The effects of θ on \mathbb{U} . (c) The effects of θ on \mathbb{V} .

In Figure 6b,c, we noticed that the amount of infectious $CD4^+$ T-cells and the free particles of the infection of the HIV-1 virus in the blood increase, respectively. Figure 7a–c display the numerical trajectories when we compare the numerical solutions of the Adams-Bashforth method with the Newton polynomial method under the fractal dimension and fractional order $\delta = \sigma = 0.90$.

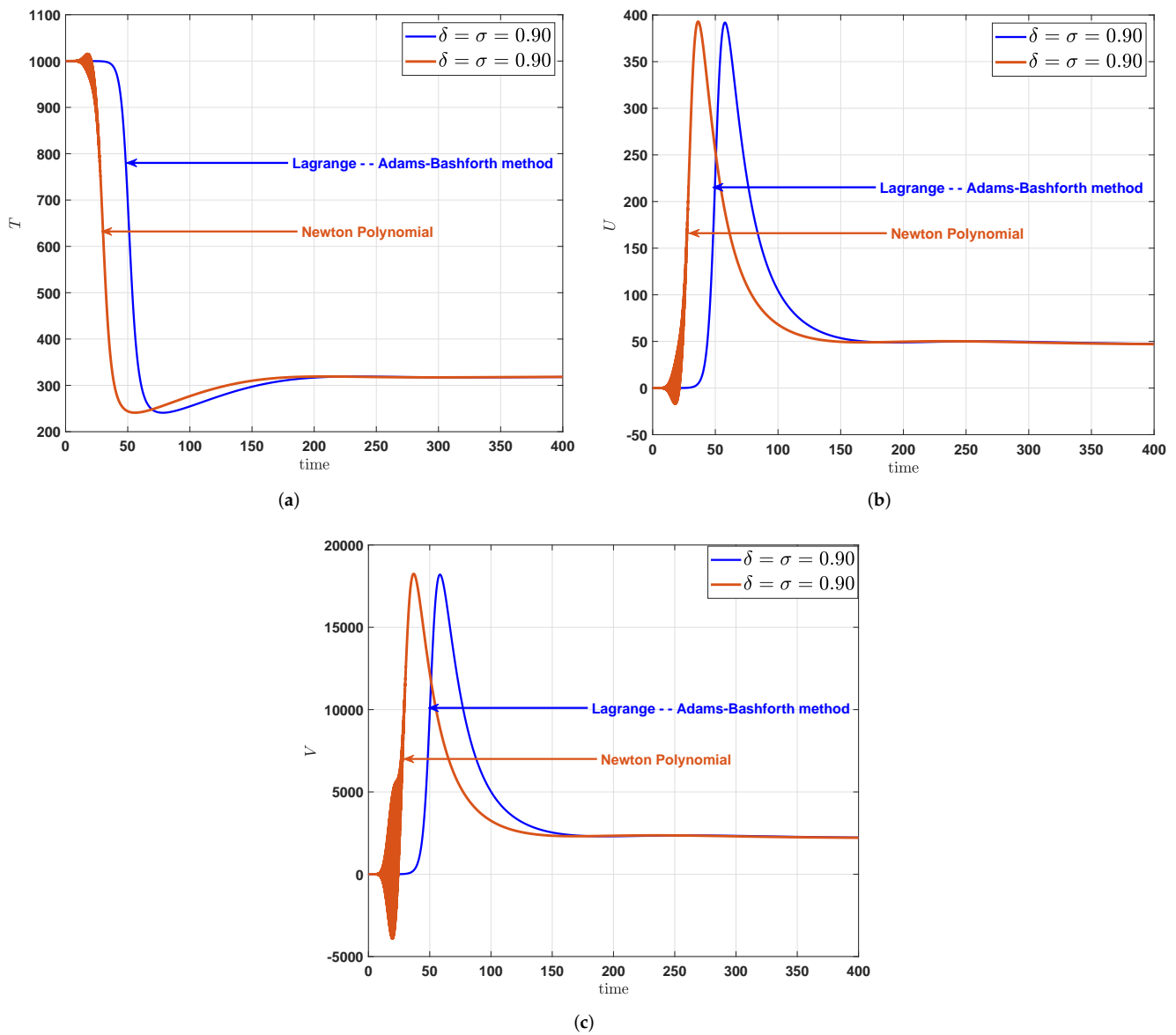


Figure 7. Numerical trajectories in the Comparison of the numerical solutions of the Adams-Bashforth method with Newton polynomials method under the fractal dimension and fractional order $\delta = \sigma = 0.90$. (a) Comparison of the graphs w.r.t. two different numerical algorithms on \mathbb{T} . (b) Comparison of the graphs w.r.t. two different numerical algorithms on \mathbb{U} . (c) Comparison of the graphs w.r.t. two different numerical algorithms on \mathbb{V} .

In Tables 1–3, we see some results of the two numerical schemes (the Adams-Bashforth method and Newton polynomials method) for all of three state functions under the fractal dimension and fractional order $\delta = \sigma = 0.95$ with step size $h = 0.1$.

Table 1. Results of the two numerical schemes for the susceptible $CD4^+$ T-cells \mathbb{T} (s) under the fractal order and fractional order $\delta = \sigma = 0.95$ with the step size $h = 0.1$.

Time: (s)	0.0001	100	200	300	400
Adams-Bashforth method	1000	300.8396	315.4653	318.7946	318.8066
Newton polynomials method	1000	300.7881	315.4643	318.7951	318.8066

Table 2. Results of the two numerical schemes for the infectious CD4⁺ T-cells \mathbb{U} (s) under the fractal order and fractional order $\delta = \sigma = 0.95$ with the step size $h = 0.1$.

Time: (s)	0.0001	100	200	300	400
Adams-Bashforth method	0	37.7693	46.0612	44.2611	43.7066
Newton polynomials method	0	37.7778	46.0645	44.2612	43.7066

Table 3. Results of the two numerical schemes for free particles of the infection of the HIV virus in the blood \mathbb{V} (s) under the fractal order and fractional order $\delta = \sigma = 0.95$ with the step size $h = 0.1$.

Time: (s)	0.0001	100	200	300	400
Adams-Bashforth method	0	1787.4317	2169.7278	2082.8420	2082.8420
Newton polynomials method	0	1787.8611	2169.8866	2082.8597	2082.8472

9. Conclusions

In this paper, we designed a fractal-fractional CD4⁺-HIV-1-model and analyzed the dynamics of CD4⁺ T-cells under the infection of HIV-1 virus. We considered three compartments for this model by defining three state functions \mathbb{T} , \mathbb{U} , and \mathbb{V} for the amount of susceptible CD4⁺ T-cells, amount of infectious CD4⁺ T-cells, and the free particles of the infection of the HIV virus in the blood. We derived three fractal-fractional integral equations and proved that their kernels are Lipschitz. In this direction, we could prove the existence and uniqueness criteria for solutions of the fractal-fractional CD4⁺-HIV-1-model. In the sequel, we investigated four stability results with the help of two auxiliary inequality. We extracted two algorithms via the Adams-Bashforth method and also via the Newton polynomials and simulated our real data in relation to the given fractal-fractional CD4⁺-HIV-1-model. The numerical and graphical results showed that these two numerical algorithms give the same outcomes and differences are small. Also, we investigated the effect of fractal dimensions and fractional orders on these simulations. Also, the effect of different values for the average number of infected particles and the supply rate of new T-cells were simulated in some graphs under the Adams-Bashforth method. This study showed that we can predict the next behavior of the fractal-fractional CD4⁺-HIV-1-model via the two mentioned numerical methods and their results are more accurate and identical. This shows the power of simulation of the fractal-fractional models in comparison to the fractional models. In the next researches, we can develop our numerical methods on different fractal-fractional models of diseases.

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