mathematics

MDPI

Article

The Existence and Multiplicity of Homoclinic Solutions for a Fractional Discrete $p-$Laplacian Equation

Yong Wu [1], Bouali Tahar [2,3], Guefaifia Rafik [4], Abita Rahmoune [5] and Libo Yang [6,*]

[1] School of Tourism Data, Guilin Tourism University, Guilin 541006, China; wuyong@gltu.edu.cn
[2] Department of Mathematics, College of Sciences, Jazan University, Jazan 45142, Saudi Arabia; tbouali@jazanu.edu.sa
[3] Department of Mathematics and Computer Science, Larbi Tebessi University, Tebessa 12002, Algeria
[4] Laboratory of Mathematics, Informatics and Systemes (LAMIS), Larbi Tebessi University, Tebessa 12000, Algeria; rafik.guefaifia@univ-tebessa.dz
[5] Laboratory of Pure and Applied Mathematics, University of Laghouat, P.O. Box 37G, Laghouat 03000, Algeria; abitamaths@gmail.com
[6] Faculty of Mathematics and Physics, Huaiyin Institute of Technology, Huai'an 223003, China
* Correspondence: yanglibo80@126.com

Abstract: In this study, we investigate the existence and multiplicity of solutions for a fractional discrete $p-$Laplacian equation on $\mathbb{Z}$. Under suitable hypotheses on the potential function V and the nonlinearity f, with the aid of Ekeland's variational principle, via mountain pass lemma, we obtain that this equation exists at least two nonnegative and nontrivial homoclinic solutions when the real parameter $\lambda > 0$ is large enough.

Keywords: fractional discrete $p-$Laplace equation; mountain pass lemma; homoclinic solutions; Ekeland's variational principle; multiplicity of solutions

MSC: 35J60; 35R11; 35K05; 49M25

check for updates

Citation: Wu, Y.; Tahar, B.; Rafik, G.; Rahmoune, A.; Yang, L. The Existence and Multiplicity of Homoclinic Solutions for a Fractional Discrete $p-$Laplacian Equation. *Mathematics* **2022**, *10*, 1400. https://doi.org/10.3390/math10091400

Academic Editors: Calogero Vetro and Omar Bazighifan

Received: 15 March 2022
Accepted: 19 April 2022
Published: 22 April 2022

1. Introduction

One side, in recent years, lots of researchers pay their attentions on the problem of the second-order difference equation (see [1])

$$-\Delta_T \mathrm{u}(j) + \mathrm{V}(j)\mathrm{u}(j) = f(j, \mathrm{u}(j)) \text{ in } \mathbb{Z}, \tag{1}$$

where $T > 0$ is a real number, $\mathbb{Z}$ is the set of all integers, $V : \mathbb{Z} \to [0, \infty)$ is a potential function, the function $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$, and $-\Delta_T u(j)$ is the discrete Laplace operator, defined as

$$-\Delta_T u(j) = \frac{1}{T^2}[u((j+1)T) - 2u(jT) + u((j-1)T)], \forall u : \mathbb{Z} \to \mathbb{R}.$$

As our known, the famous Schrödinger equation is a widely used equation. It is usually used to solve series of problems of molecules, atoms, nuclei and so on, and the results are very realistic. For the Equation (1), it can be regarded as the discrete version of the famous Schrödinger equation and used to describe an electron in an electromagnetic field or a planetary system. In order to study the dynamics of discrete Schrödinger equation, we need to know the homoclinic orbits, which play a very important role in this area. For more details of second-order difference equations, there are lots of literatures, the interested readers can see for [2–8]. In particular, in [2], by using variational methods, Agarwal, Perera and O'Regan obtained the existence results for second order difference equations like (1) for the first time.

On the other side, recently, nonlocal problems has been received an increasing amount of attentions. There are two very famous pieces of work [9,10] that we highly recommend.

In addition, the fractional Laplacian and related problems are all hot topics for researchers. About fractional Laplace operator and fractional Sobolev Spaces, more details and properties, refer readers to see [11–14]. In many fields, nonlocal fractional problems have very important applications, such as optimization, game theory, quantum mechanics, anomalous diffusion, finance and so forth, readers can see the literatures [3,15–17] and the references cited. For applications of fractional Laplace operators, the literature is very rich, we refer to [8,18–28] and the references therein.

Very recently, in [29], Ciaurri et al., studied an equation as following:

$$(-\Delta_T)^s \mathrm{u} = f, \tag{2}$$

where $s \in (0,1)$, $(-\Delta_T)^s u(j) = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{t\Delta_T} u(j) - u(j)) \frac{dt}{t^{1+2s}}$ is the discrete fractional Laplace operator, $\Gamma(-s)$ is a Gamma function, we denote $e^{t\Delta_T} u(j)$ by $v(t,j)$, which is the solution of the problem as follow:

$$\begin{cases} \partial_t v(t,j) = \Delta_T u(j), \text{ in } \mathbb{Z}_T \times (0, \infty), \\ v(0,j) = u(j), \text{ on } \mathbb{Z}_T, \end{cases}$$

where $\mathbb{Z}_T = \{Tj : j \in \mathbb{Z}\}$.

By the Theorem 1.1 of [29], for any $u \in \mathcal{L}_s := \{v : \mathbb{Z}_T \to \mathbb{R} | \sum_{k\in\mathbb{Z}} \frac{|v(k)|}{(1+|k|)^{1+2s}} < \infty\}$,

$$(-\Delta_T)^s u(j) = \sum_{k\in\mathbb{Z}, k\neq j} (u(j) - u(k)) \mathcal{K}_s^T (j-k),$$

where

$$\mathcal{K}_s^T(k) = \frac{4^s \Gamma(1/2+s)}{\sqrt{\pi} |\Gamma(-s)|} \cdot \frac{\Gamma(|k|-s)}{T^{2s} \Gamma(|k|+1+s)}$$

for any $k \in \mathbb{Z} \backslash \{0\}$ and $\mathcal{K}_S^T(0) = 0$.

When u is bounded, we know that the discrete fractional operator $(-\Delta_T)^s u(j)$ converge to the usual discrete opertor $-\Delta_T u(j)$, as $s \to 1^-$. In addition the solutions of the fractional Laplace equation $(-\Delta)^s u = f$ in $\mathbb{R}$ can be approximated by the solutions of Equation (2).

In [30], Xiang and Zhang first investigated the equation

$$\begin{cases} (-\Delta_1)^s \mathrm{u}(k) + \mathrm{V}(k)|\mathrm{u}(k)| = \lambda f(k, \mathrm{u}(k)), \text{ for } \mathrm{k} \in \mathbb{Z} \\ \mathrm{u}(k) \to 0, \text{as } \quad |k| \to \infty. \end{cases} \tag{3}$$

by using variational principle, the multiplicity results were obtained.

Usually, the solutions of the continuous fractional problems can be approximated by the solutions of the discrete fractional Laplacian equations. However, numerical analysis is difficult for discrete fractional equations, because of the singularity and nonlocality of the discrete fractional Laplace operator, more details see [31] and the reference cited therein.

Motivated by the above literatures, in this study, we investigate the existence and multiplicity of homoclinic solutions of a class of discrete fractional $p-$Laplace difference equation on $\mathbb{Z}$. Specifically speaking, we study

$$\begin{cases} (-\Delta_T)_p^s \mathrm{u}(k) + \mathrm{V}(k)|\mathrm{u}(k)|^{p-2}\mathrm{u}(k) = \lambda f(k, \mathrm{u}(k)), \text{for } \mathrm{k} \in \mathbb{Z} \\ \mathrm{u}(k) \to 0, \text{as } \quad |k| \to \infty, \end{cases} \tag{4}$$

where $T > 0$ is a real number, $s \in (0,1)$, $1 < p < \infty$, $\mathbb{Z}$ denote the set of whole integers, $V : \mathbb{Z} \to (0, \infty)$ is a continuous potential function and the nonlinear term $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$

is a continuous functions too, $(-\Delta_T)^s_p$ is the discrete fractional $p-$Laplace operator, we define it by

$$(-\Delta_T)^s_p u(j) = \sum_{m\in Z, m\neq j} |u(j) - u(m)|^{p-2}(u(j) - u(m))K^T_{s,p}(j-m) \ ,$$

for any $j \in \mathbb{Z}$, $u \in \mathcal{L}_{p,s}$, $\mathcal{L}_{P,s} := \{v : \mathbb{Z}_T \to \mathbb{R} | \sum_{k\in\mathbb{Z}} \frac{|v(k)|}{(1+|k|)^{1+ps}} < \infty\}$. Here, $K^T_{s,p}$ is the discrete kernel, satisfies the following expression:

There exist two constants $c_{s,p}$ and $C_{s,p}$ such that

$$\begin{cases} \frac{c_{s,p}}{T^{ps}|j|^{1+ps}} \leq K^T_{s,p}(j) \leq \frac{C_{s,p}}{T^{ps}|j|^{1+ps}}, \text{ for any } j \in \mathbb{Z}\backslash\{0\} \\ K_{s,p}(0) = 0, \end{cases}$$

where $c_{s,p}$ and $C_{s,p}$ satisfy the condition $0 < c_{s,p} \leq C_{s,p} < \infty$.

When $T = 1$, we have

$$(-\Delta_1)^s_p u(j) = 2 \sum_{m\in Z, m\neq j} |u(j) - u(m)|^{p-2}(u(j) - u(m))K_{s,p}(j-m) \ .$$

Meanwhile, for the fractional discrete $p-$Laplace operator $(-\Delta_T)^s_p u$, when $p = 2$, it is coincide with the usual fractional discrete laplace operator $(-\Delta_T)^s u$, and when $p = 2$ and $d = 1$, then Equation (4) reduces to Equation (3).

As usual, if $u(k) \to 0$ as $|k| \to \infty$, the function $u : \mathbb{Z} \to \mathbb{R}$ is the homoclinic solution of Equation (4).

Next, we give the hypotheses which will be used in this paper. We suppose the continuous potential function $V : \mathbb{Z} \to (0, \infty)$ fulfills

(V) (i) For all $k \in \mathbb{Z}$, there exists a constant $V_0 > 0$ such that $V(k) \geq V_0$;

(ii) $V(k) \to \infty$ as $|k| \to \infty$.

The nonlinear term $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, satisfies

(f_1) $\lim\limits_{t\to 0} \frac{f(k,t)}{t^{p-1}} = 0$ uniformly for all $k \in \mathbb{Z}$;

(f_2) For all $T > 0$, $\sup\limits_{|t|\leq T} |F(\cdot, t)| \in \ell^1$, where $\ell^1 := \{u : \mathbb{Z} \to \mathbb{R} | \sum_{j\in\mathbb{Z}} |u(j)| < \infty\}$, and $F\ (k,t) = \int_0^t f(k,\tau)\ d\tau$;

(f_3) $\lim \sup\limits_{|t|\to\infty} \frac{F(k,t)}{t^p} \leq 0$ uniformly for all $k \in \mathbb{Z}$;

(f_4) $F(h_0, b_0) > 0$ for same $h_0 \in \mathbb{Z}$ and $b_0 \in \mathbb{R}\backslash\{0\}$.

When $1 < q < 2 < p < \infty$, a simple example of f, fulfilling $(f_1) - (f_4)$ is

$$f(k,x) = \begin{cases} |x|^{p-2}x, \text{if } |x| \leq 1 \\ |x|^{q-2}x, \text{if } |x| > 1. \end{cases}$$

Set

$$\lambda^* = \frac{|b_0|^p(pC_{s,p}\sum_{m\neq h_0} \frac{1}{|h_0-m|^{1+ps}} + V(h_0))}{pF(h_0, b_0)}.$$

Theorem 1. *Assume that the potential function satisfies the condition (V) and the function f sitisfies conditions (f_1)–(f_4). Then, for any $\lambda > \lambda^*$, Equation (4) has at least two nontrivial and nonnegative homoclinic solutions.*

To our best knowledge, for fractional discrete $p-$Laplacian, Theorem 1 is the first result established on variational techniques to study the existence of solutions for these type of equation. More precisely, in this paper, when positive constant λ is big enough, we prove the existence of two nontrivial nonnegative homoclinc solutions of Equation (4) by using the mountain pass theorem and Ekeland's variational principle. However, at present, it is still an open problem for all $\lambda > 0$, which can be one of our further research directions.

This paper is composed of three sections in addition to the introduction. In Section 2, a variational framework to Equation (4) and some preliminary outcomes was given. In Section 3, employing critical point theory, two distinct non trivial and nonnegative homoclinic solutions for Equation (4) were gotten.

2. Preliminaries

In this section, we describe the functional setting and some basic definitions in which we shall work and state our main results for more detail see [21,30,32].

Then we give the variational setting to Equation (4) and discuss its properties. For any $1 \le p < \infty$, ℓ^p is defined as

$$\ell^p := \left\{ \mathrm{u} : \mathbb{Z} \to \mathbb{R} : \sum_{j \in \mathbb{Z}} |\mathrm{u}(j)|^p < \infty \right\}$$

with the norm

$$\|\mathrm{u}\|_p = \left(\sum_{j \in \mathbb{Z}} |\mathrm{u}(j)|^p \right)^{1/p}, \text{ for } 1 \le p < \infty,$$

for $p = \infty$

$$\|\mathrm{u}\|_\infty := \sup_{j \in \mathbb{Z}} |\mathrm{u}(j)| < \infty,$$

Define

$$\ell^\infty = \{\mathrm{u} : \mathbb{Z} \to \mathbb{R} : \|\mathrm{u}\|_\infty < \infty\}.$$

We see that $(\ell^p, \|.\|_p)$ and $(\ell^\infty, \|.\|_\infty)$ are Banach spaces, see [32], and $\ell^{p_1} \subset \ell^{p_2}$ for $1 \le p_1 \le p_2 \le \infty$. We denote by $\|.\|_p$ the norm of ℓ^p for all $p \in [1, \infty]$.

For an interval $\mathrm{I} \subset \mathbb{R}$, we clarify ℓ^p_{I} by

$$\ell^p_{\mathrm{I}} = \left\{ \mathrm{u} : \mathrm{I} \to \mathbb{R} : \sum_{j \in \mathrm{I}} |\mathrm{u}(j)|^p < \infty \right\}.$$

Let

$$\mathrm{W} = \left\{ \mathrm{u} : \mathbb{Z} \to \mathbb{R} : \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |\mathrm{u}(j) - \mathrm{u}(k)|^p K_{s,p}(j-k) + \sum_{j \in \mathbb{Z}} \mathrm{V}(j) |\mathrm{u}(j)|^p < \infty \right\}.$$

equipped with the norm

$$\|\mathrm{u}\|_{\mathrm{W}} = \left([\mathrm{u}]^p_{s,p} + \sum_{j \in \mathbb{Z}} \mathrm{V}(j) |\mathrm{u}(j)|^p \right)^{1/p},$$

where

$$[\mathrm{u}]_{s,p} := \left(\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |\mathrm{u}(j) - \mathrm{u}(k)|^p K_{s,p}(j-k) \right)^{1/p}.$$

Lemma 1. *If $u \in \ell^p$, then $[u]_{s,p} < \infty$. Moreover there exists $C(s,p) > 0$, such that $[u]_{s,p} \le C\|u\|_p$ for all $u \in \ell^p$.*

Proof. Let $\mathrm{u} \in \ell^p$. Then

$$
\begin{aligned}
[\mathrm{u}]_{s,p}^p &= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |\mathrm{u}(j)-\mathrm{u}(k)|^p K_{s,p}(j-k) \\
&\le 2^{p-1}C_{s,p}\sum_{j\in\mathbb{Z}}\sum_{k=j}\frac{|\mathrm{u}(j)|^p+|\mathrm{u}(k)|^p}{|j-k|^{1+ps}} = 2^{p-1}C_{s,p}\sum_{k\neq 0}\frac{|\mathrm{u}(0)|^p+|\mathrm{u}(k)|^p}{|k|^{1+ps}} \\
&\quad +2^{p-1}C_{s,p}\sum_{j=0}\sum_{k\neq 0}\frac{|\mathrm{u}(j)|^p}{|k|^{1+ps}} + 2^{p-1}C_{s,p}\sum_{j=0}\sum_{k\neq 0}\frac{|\mathrm{u}(k+j)|^p}{|k|^{1+ps}} \\
&= 2^{p-1}C_{s,p}\sum_{k\neq 0}\frac{|\mathrm{u}(0)|^p+|\mathrm{u}(k)|^p}{|k|^{1+ps}} + 2^{p-1}C_{s,p}\sum_{k\neq 0}\sum_{j=0}\frac{|\mathrm{u}(j)|^p}{|k|^{1+ps}} \\
&\quad +2^{p-1}C_{s,p}\sum_{k\neq 0}\sum_{j\neq 0}\frac{|\mathrm{u}(k+j)|^p}{|k|^{1+ps}} \\
&\le 3\times 2^{p-1}C_{s,p}\sum_{k\neq 0}\frac{1}{|k|^{1+ps}}\sum_{j\in\mathbb{Z}}|\mathrm{u}(j)|^p = C^p\sum_{j\in\mathbb{Z}}|\mathrm{u}(j)|^p,
\end{aligned}
$$

where $0 < C^p = 3\times 2^{p-1}C_{sp}\sum_{k\neq 0}\frac{1}{|k|^{1+ps}} < \infty$. Thus, the proof is completed. □

Besides, the following compactness result holds.

Lemma 2. *If condition* (V) *holds, then embedding* $W \hookrightarrow \ell^q$ *is compact for any* $1 < p \le q < \infty$, *provided the condition* (V) *holds.*

Proof. The proof is similar to papers [21,30].

First, we establish that the result holds for the case $q = p$. According to the hypothesis (V), we have $\|\mathrm{u}\|_p \le \mathrm{V}_0^{-\frac{1}{p}}\|\mathrm{u}\|$ for all $\mathrm{u} \in W$. Indeed, the embedding $W \to \ell^p$ is continuous. Next, we verify that $W \to \ell^p$ is compact. For $\{\mathrm{u}_n\}_n \subset W$, we suppose that there exists $d > 0$ such that $\|\mathrm{u}_n\|_{\mathrm{W}}^p \le d$ for all $n \in \mathbb{N}$. Since W is a reflexive Banach space (see Appendix A), there exist a subsequence of $\{\mathrm{u}_n\}_n$ still denoted by $\{\mathrm{u}_n\}_n$ and a function $\mathrm{u} \in W$ such that $\mathrm{u}_n \rightharpoonup \mathrm{u}$ in W. By hypothesis (V), for any $\varepsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that

$$
\mathrm{V}(j) > \frac{1+d}{\varepsilon}, \text{ for all } |j| > j_0.
$$

For $\mathrm{I} = [-j_0, j_0]$ we define

$$
\mathrm{W_I} := \left\{\mathrm{u} : \mathrm{I} \to \mathbb{R} : \sum_{j\in\mathrm{I}}\sum_{j\neq k\in\mathrm{I}} |\mathrm{u}(j)-\mathrm{u}(k)|^p K_{s,p}(j-k) + \sum_{j\in\mathrm{I}}\mathrm{V}(j)|\mathrm{u}(j)|^p < \infty\right\}.
$$

Because the dimension of W_{I} is finite, we infer that $\{\mathrm{u}_n\}_n$ is a bounded sequence in W_{I}, since $\{\mathrm{u}_n\}_n$ is bounded in ℓ_{I}^p. Thus, up to a subsequence, we conclude that $\mathrm{u}_n \to \mathrm{u}$ on I. Thus there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$

$$
\sum_{j\in\mathrm{I}} |\mathrm{u}_n(j)-\mathrm{u}(j)|^p \le \frac{\delta}{1+d}.
$$

Then

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}} |\mathrm{u}_n(j)-\mathrm{u}(j)|^p &< \frac{\varepsilon}{1+\mathrm{C}} + \frac{\varepsilon}{1+\mathrm{C}}\sum_{|j|>j_0}\mathrm{V}(j)|\mathrm{u}_n(j)-\mathrm{u}(j)|^p \\
&\le \frac{\varepsilon}{1+\mathrm{C}}(1+\|\mathrm{u}_n\|_{\mathrm{W}}^p) \le \delta, \text{ for all } n > n_0.
\end{aligned}
$$

Hence, we conclude that $u_n \to u$ in ℓ^p. Now, we view the case $q > p$. Note that

$$\|u(j)\|_\infty \leq \left(\sum_{j\in\mathbb{Z}} |u(j)|^p \right)^{1/p}$$

for all $u \in \ell^p$. Then

$$\left(\sum_{j\in\mathbb{Z}} |u(j)|^q \right)^{1/q} = \|u\|_\infty \left(\sum_{j\in\mathbb{Z}} \left(\frac{|u(j)|}{||u||_\infty} \right)^p \right)^{1/q} = \|u\|_\infty^{1-\frac{p}{q}} \left(\sum_{j\in\mathbb{Z}} |u(j)|^p \right)^{1/q}$$
$$\leq \|u\|_p^{1-\frac{p}{q}} \|u\|_p^{\frac{p}{q}} = \|u\|_p,$$

with $u \in \ell^p \backslash \{0\}$. Therefore,

$$\|u\|_q \leq \|u\|_p$$

for all $u \in \ell^p$. This inequality jointly with the result of the considered case $q = p$, shows the proof. □

To get some effects of energy functional associated with Equation (4), the following result is required.

Lemma 3. *For any compact subset* U *of* W*, and any* $\varepsilon > 0$*, there is a* $j_0 \in \mathbb{N}$ *such that*

$$\left(\sum_{|j|>j_0} V(j)|u(j)|^p \right)^{1/p} < \varepsilon, \quad u \in U.$$

Proof. We prove it by contradiction, suppose that there exist $\varepsilon > 0$ and a sequence $\{u_n\} \subseteq U$ such that

$$\left(\sum_{|j|>n} V(j)|u_n(j)|^p \right)^{1/p} > \varepsilon \text{ for all } n \in \mathbb{N}.$$

Due to the compactness of U, passing to a subsequence we may assume that $u_n \to u$ in W for some $u \in U$. Thus, there exists $n_0 \in N$, such that $\|u_n - u\| < \frac{\varepsilon}{2}$ for any $n \geq n_0$, moreover, there exists $j_1 \in N$ such that

$$\left(\sum_{|j|>j_1} V(j)|u(j)|^p \right)^{1/p} < \frac{\varepsilon}{2}.$$

Recall the classical Minkowski inequality:

$$\left(\sum_{I=1}^{m} |x_i + y_i|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{m} |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{m} |y_i|^p \right)^{\frac{1}{p}} \text{ for all } m \in \mathbb{Z},\ x_1...x_m.y_1...y_m \in \mathbb{R}. \quad (5)$$

By (5), we have

$$\varepsilon < \left(\sum_{|j|>n} V(j)|u_n(j) - u(j)|^p \right)^{1/p} \leq \left(\sum_{|j|>n} V(j)|u_n(j)|^p \right)^{1/p} + \left(\sum_{|j|>n} V(j)|u(j)|^p \right)^{1/p}$$
$$\leq \|u_n - u\| + \frac{\varepsilon}{2} < \varepsilon,$$

which is a contradiction, and the proof is archived. □

For u $\in W$, we propose the associated energy functional with Equation (4) as

$$\mathrm{I}_\lambda(\mathrm{u}) = \Psi(\mathrm{u}) - \lambda \mathrm{J}(\mathrm{u})\,,$$

where

$$Psi(\mathrm{u}) = \frac{1}{p}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |\mathrm{u}(j)-\mathrm{u}(m)|^p K_{s,p}(j-m) + \frac{1}{p}\sum_{j\in\mathbb{Z}} \mathrm{V}(j)|\mathrm{u}(j)|^p$$

and

$$\mathrm{J}(\mathrm{u}) = \sum_{j\in\mathbb{Z}} F(j,\mathrm{u}(j)).$$

Lemma 4. *If (V) is fulfilled, then Psi is well-defined, of class $C^1(W,\mathbb{R})$ and*

$$\begin{aligned}\langle \Psi'(\mathrm{u}),\, \mathrm{v}\rangle = &\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |\mathrm{u}(j)-\mathrm{u}(m)|^{p-2}(\mathrm{u}(j)-\mathrm{u}(m))(\mathrm{v}(j)-\mathrm{v}(m))K_{s,p}(j-m)\\ &+\sum_{j\in\mathbb{Z}} \mathrm{v}(j)|\mathrm{u}(j)|^{p-2}\mathrm{u}(j)\mathrm{v}(j),\end{aligned}$$

for all u, v $\in W$.

Proof. According to Lemma 1, the functional Psi is well-defined on W. Fix u, v $\in W$. We first prove that

$$\begin{aligned}&\lim_{t\to 0^+}\textstyle\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \frac{|\mathrm{u}(j)+t\mathrm{v}(j)-\mathrm{u}(m)-t\mathrm{v}(m)|^p-|\mathrm{u}(j)-\mathrm{u}(m)|^p}{p}K_{s,p}(j-m) \\ &=\textstyle\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |\mathrm{u}(j)-\mathrm{u}(m)|^{p-2}(\mathrm{u}(j)-\mathrm{u}(m))(\mathrm{v}(j)-\mathrm{v}(m))K_{s,p}(j-m).\end{aligned} \tag{6}$$

Pick $C>0$ such that $\max(\|\mathrm{u}\|_W, \|\mathrm{v}\|_W) \le C$. For all $\varepsilon > 0$ there exists $h_1 \in \mathbb{N}$ such that

$$\sum_{|j|>h}\sum_{|m|>h} |\mathrm{u}(j)-\mathrm{u}(m)|^p K_{s,p}(j-m))^{\frac{1}{p}} < \varepsilon \tag{7}$$

for all $h > h_1$. Indeed, for any $h \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{|j|>h}\sum_{|m|>h} |\mathrm{u}(j)-\mathrm{u}(m)|^p K_{s,p}(j-m) &\le C_{s,p}2^{p-1}\sum_{|j|>h}\sum_{|m|>h,m\ne j}\frac{(|\mathrm{u}(j)|^p+|\mathrm{u}(m)|^p)}{|j-m|^{1+ps}}\\ &\le 2^p C_{s,p}\sum_{|j|>h}\sum_{|m|>h,m=j}\frac{|\mathrm{u}(j)|^p}{|j-m|^{1+ps}}\\ &\le 2^p C_{s,p}(\sum_{k\ne 0}\frac{1}{|k|^{1+ps}})\sum_{|j|>h}|\mathrm{u}(j)|^p,\ \mathrm{u}\in \mathrm{W}.\end{aligned}$$

Therefore (7) holds.

For $h \in \mathbb{N}$, if $|j| \le h$ and $|m| > 2h$, then $|j-m| \ge |m| - |j| \ge |m| - h > \frac{|m|}{2}$. Thus, there exists $h_2 \in N$ such that

$$\left(\sum_{|j|\le h}\sum_{|m|>2h} |\mathrm{u}(j)-\mathrm{u}(m)|^p K_{s,p}(j-m)\right)^{\frac{1}{p}} < \varepsilon \tag{8}$$

for all $h > h_2$. Fix $h > \max\{h_1, h_2\}$. Clearly, there exists $t_0 \in (0,1)$ such that for all $0 < t < t_0$, we get

$$\sum_{|j|\le 2h}\sum_{|m|\le 2h}|\frac{|u(j)+tv(j)-u(m)-tv(m)|^p-|u(j)-u(m)|^p}{p}$$
$$-|u(j)-u(m)|^{p-2}(u(j)-u(m))(v(j)-v(m))|K_{s,p}(j-m)<\varepsilon.$$

Fix $0<t<t_0$, for $j,m\in\mathbb{Z}$, using the mean value theorem, we can find $0<t_{j,m}<t$ such that

$$\begin{aligned}&\frac{(|u(j)+tv(j)-u(m)-tv(m)|^p-|u(j)-u(m)|^p)}{tp}K_{s,p}(j-m) \\ =\;& |y(j)-y(m)|^{p-2}(y(j)-y(m))(v(j)-v(m))K_{s,p}(j-m),\end{aligned} \tag{9}$$

where $y(j)=u(j)+t_{j,m}v(j)$. Evidently $y\in W$ and $\|y\|_W\le 2C$. Observe that

$$\begin{aligned}&\left|\sum_{|j|\le h|}\sum_{m|>2h}|u(j)-u(m)|^{p-2}(u(j)-u(m))(v(j)-v(m))K_{s,p}(j-m)\right| \\ &\le\sum_{|j|\le h|}\sum_{m|>wh}|u(j)-u(m)|^{p-1}|v(j)-v(m)|K_{sp}(j-m) \\ &\le\left(\sum_{|j|\le h|}\sum_{m|>2h}|u(j)-u(m)|^pK_{s,p})^{\frac{p-1}{p}}(\sum_{|j|\le h|}\sum_{m|>2h}|v(j)-v(m)|^pK_{s,p}(j-m)\right)^{\frac{1}{p}}\le C\varepsilon.\end{aligned} \tag{10}$$

From (7) and (10), using Holder's inequality, we infer

$$\begin{aligned}&\left|\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\frac{|u(j)+tv(j)-u(m)-tv(m)|^p-|u(j)-u(m)|^p}{p}K_{s,p}(j-m)\right. \\ &\left.+\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}|u(j)-u(m)|^{p-2}(u(j)-u(m))(v(j)-v(m))K_{s,p}(j-m)\right| \\ &\le\varepsilon+\sum_{|j|\le h|}\sum_{m|>h}+\sum_{|j|>h}\sum_{|m|\le h} \\ &+\sum_{|j|>h}\sum_{|m|>h}|(\phi_p(y(j)-y(m))-\phi_p(u(j)-u(m)))(v(j)-v(m))|K_{s,p}(j-m) \\ &\le C\varepsilon+\sum_{|j|\le h|}\sum_{m|>2h}+\sum_{|j|>2h|}\sum_{m|\le h} \\ &+\sum_{|j|>h}\sum_{|m|>h}|(\phi_p(y(j)-y(m))-\phi_p(u(j)-u(m)))(v(j)-v(m))|K_{s,p}(j-m) \\ &\le C\varepsilon,\end{aligned}$$

where for all $\tau\in\mathbb{R}$, $\phi_p(\tau):=|\tau|^{p-2}\tau$. Consequently (6) holds valid. A similar idea gives

$$\lim_{t\to0^+}\frac{\|u+tv\|^p-\|u\|^p}{pt}=\sum_{j\in\mathbb{Z}}V(j)|u(j)|^{p-2}u(j)v(j).$$

Thus, we get

$$\begin{aligned}\langle\Psi'(u),v\rangle=&\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}|u(j)-u(m)|^{p-2}(u(j)-u(m))(v(j)-v(m))K_{s,p}(j-m) \\ &+\sum_{j\in\mathbb{Z}}V(j)|u(j)|^{p-2}u(j)v(j).\end{aligned}$$

Therefore, Ψ is Gâteaux differentiable in W. In the end, we prove that $\Psi' : \mathrm{W} \to \mathrm{W}^*$ is continuous. To this end, we take $\{\mathrm{u}_n\}_n$ sequence in W with $\mathrm{u}_n \underset{n\to\infty}{\to} \mathrm{u}$ in W. From Lemma 3, for all $\varepsilon > 0$, there exists $h \in \mathbb{N}$ such that

$$\left(\sum_{|j|>h}\sum_{|m|>h}|\mathrm{u}_n(j)-\mathrm{u}_n(m)|^p K_{s,p}(j-m)\right)^{1/p} < \varepsilon \text{ for all } n \in \mathbb{N}$$

and

$$\left(\sum_{|j|>h}\sum_{|m|>h}|\mathrm{u}(j)-\mathrm{u}(m)|^p K_{s,p}(j-m)\right)^{1/p} < \varepsilon.$$

In addition, there exists $n_0 \in \mathbb{N}$ such that

$$\left(\sum_{|j|\le 2h|}\sum_{m|\le 2h}\left|(\phi(\mathrm{u}_n(j)-\mathrm{u}_n(m))-\phi(\mathrm{u}(j)-\mathrm{u}(m)))K_{s,p}^{1/p'}(j-m)\right|^{p'}\right)^{1/p'} < \varepsilon$$

for all $n \ge n_0$, where $p' = \frac{p}{p-1}$. For any $\mathrm{v} \in \mathrm{W}$ with $\|\mathrm{v}\|_{\mathrm{W}} \le 1$, and for any $n \ge n_0$, by the Hölder inequality and a similar argument to above, we conclude

$$\begin{aligned}
&\left|\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}[\phi(\mathrm{u}_n(j)-\mathrm{u}_n(m))-\phi(\mathrm{u}(j)-\mathrm{u}(m))](\mathrm{v}(j)-\mathrm{v}(m))K_{s,p}(j-m)\right| \\
&\le \left(\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\left|(\phi(\mathrm{u}_n(j)-\mathrm{u}_n(m))-\phi(\mathrm{u}(j)-\mathrm{u}(m)))K_{s,p}^{1/p'}(j-m)\right|^{p'}\right)^{1/p'} \\
&\times \left(\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}|\mathrm{v}(j)-\mathrm{v}(m)|^p K_{s,p}(j-m)\right)^{1/p} \le C\varepsilon\|\mathrm{v}\|_{\mathrm{W}}.
\end{aligned}$$

Also, we can show that

$$\left|\sum_{k\in\mathbb{Z}}\mathrm{V}(k)(|\mathrm{u}_n|^{p-2}\mathrm{u}_n-|\mathrm{u}|^{p-2}\mathrm{u})\mathrm{v}\right| \le C\varepsilon\|\mathrm{v}\|_{\mathrm{W}}$$

as $n \to \infty$. Thus,

$$\|\Psi'(\mathrm{u}_n) - = \Psi'(\mathrm{u})\| = \sup_{\|\mathrm{v}\|\le 1}\left|\{\Psi'(\mathrm{u}_n)-\Psi'(\mathrm{u}),\mathrm{v}\}\right| \to 0,$$

which implies that Ψ' is continuous. Hence, we confirm that $\Psi \in C^1(\mathrm{W},\mathbb{R})$. □

Lemma 5. *If conditions* (V) *and* (f_1) *hold, then* $\mathrm{J} \in C^1(\mathrm{W},\mathbb{R})$ *with*

$$\langle \mathrm{J}'(\mathrm{u}),\ \mathrm{v}\rangle = \sum_{j\in\mathbb{Z}} f(j,\mathrm{u}(j))\mathrm{v}(j)$$

for all $\mathrm{u},\mathrm{v} \in \mathrm{W}$.

Proof. By (f_1) , there exists $\delta > 0$ such that $|f(j,t)| \le |t|^{p-1}$ for all $j \in \mathbb{Z}$, $|t| \le \delta$. Integrating we have

$$|F(j,t)| \le \frac{|t|^p}{p} \text{ for all } j \in \mathbb{Z},\ |t| \le \delta \tag{11}$$

for all $u \in W$. There exists $h \in \mathbb{N}$ such that $|u(k)| \leq \delta$ for all $j \in \mathbb{Z}$, $|j| > h$, we obtain

$$\begin{aligned}\left|\sum_{j\in\mathbb{Z}} F(j,u(j))\right| &= \left|\sum_{|j|\leq h} F(j,u(j)) + \sum_{|j|>h} F(j,u(j))\right| \\ &\leq \sum_{|j|\leq h} |F(j,u(j))| + \frac{1}{p}\sum_{|j|>h} |u(j)|^p \\ &\leq \sum_{|j|\leq h} |F(j,u(j))| + \frac{1}{pV_0}\sum_{|j|>h} V(j)|u(j)|^p,\end{aligned}$$

thus J is well defined. Now, fix $u, v \in W$. We show that

$$\lim_{t\to 0^+} \frac{J(u+tv) - J(u)}{t} = \sum_{j\in\mathbb{Z}} f(j,u(j))v(j)), \tag{12}$$

indeed, choose $R > 0$ such that $\max\left(\|u\|_p, \|v\|_p\right) \leq R$. Let $\delta > 0$ be such that (11) holds and

$$\max\{u(j), v(j)\} \leq \frac{\delta}{2} \text{for all } j \in \mathbb{Z},\ |j| > h.$$

For all $\varepsilon > 0$, there exists $h \in \mathbb{N}$ such that

$$\sum_{|j|>h} V(j)|v(j)|^p < \frac{\varepsilon}{6V_0(2R)^{p-1}}.$$

Moreover, we can find $t_0 \in (0,1)$ such that

$$\sum_{|j|\leq h}\left|\frac{F(j,u(j)) + tv(j) - F(j,u(j))}{t} - f(j,u(j))v(j)\right| < \frac{\varepsilon}{3}.$$

Now fix $0 < t \leq t_0$. For all $|j| > h$, there exists $0 \leq t_k \leq t$ such that

$$\frac{F(j,u(j)) + tv(j) - F(j,u(j))}{t} = f(j,u(j) + t_k v(j))v(j).$$

We define $w \in W$ by $w(j) = 0$ for all $|j| \leq h$ and $w(j) = u(j) + t_j v(j)$ for all $|j| > h$. Therefore, $\|w\| \leq \|u\| + \|v\|$ and $|w(j)| \leq \delta$ for all $j \in \mathbb{Z}$. Summarizing what proved above, we have

$$\begin{aligned}&\left|\frac{J(u+tv) - J(u)}{t} - \sum_{j\in\mathbb{Z}} F(j,u(j))v(j)\right| \\ &\leq \frac{\varepsilon}{3} + \sum_{|j|>h} |F(j,w(j))v(j)| + \sum_{|j|>h} |F(j,u(j))v(j)| \\ &\leq \frac{\varepsilon}{3} + \sum_{|j|>h} |w(j)|^{p-1}|v(j)| + \sum_{|j|>h} |u(j)|^{p-1}|v(j)| \\ &\leq \frac{\varepsilon}{3} + \frac{1}{V_0}\left(\left(\sum_{|j|>h} |w(j)|^p\right)^{\frac{1}{q}} + \left(\sum_{|j|>h} |u(j)|^p\right)^{\frac{1}{q}}\right)\left(\sum_{|j|>h} |v(j)|^p\right)^{\frac{1}{q}} \\ &< \frac{\varepsilon}{3} + \frac{1}{V_0}\left((2R)^{p-1} + R^{p-1}\right)\frac{\varepsilon}{6V_0(2R)^{p-1}} < \varepsilon.\end{aligned}$$

Hence, (12) holds. So J is Gâteaux differentiable.

Next, similar to Lemma 4, we can prove that $J \in C^1(W, \mathbb{R})$, combining Lemmas 4 and 5, we see that $I_\lambda \in C^1(W, \mathbb{R})$. □

Lemma 6. *If conditions* (V) *and* (f_1) *hold,* $1 < q < p < \infty$*, then the critical point of* I_λ *is a homoclinic solution of Equation* (4) *for all* $\lambda > 0$.

Proof. Suppose that $u \in W$ is a critical point of I_λ, that is, $I'_\lambda(u) = 0$. Then

$$\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |u(j)-u(m)|^{p-2}(u(j)-u(m))(v(j)-v(m))K_{s,p}(j-m) \\ + \sum_{j\in\mathbb{Z}} V(j)|u(j)|^{p-2}u(j)v(j) = \lambda \sum_{j\in\mathbb{Z}} f(j,u(j))v(j)) \tag{13}$$

for all $v \in W$. For each $k \in \mathbb{Z}$, we define e_k as

$$e_k(j) = \delta_{kj} := \begin{cases} l\ , j = k \\ 0, j \neq k. \end{cases}$$

Obviously, $e_k \in W$. Choosing $v = e_k$ in (13), we get

$$p\sum_{j=k} |u(k)-u(j)|^{p-2}(u(k)-u(j))K_{sp}(k-j) + V(k)|u(k)|^{p-2}u(k) = \lambda f(k,u(k))\ ,$$

which implies that u is a solution of (4). Furthermore, according to $u \in W$ and Lemma 2, we can easily infer that $u(k) \to 0$ as $|k| \to \infty$. Hence u is a homoclinic solution of (4). □

Next, we employ the general mountain pass lemma (see [33]) to prove our main result. we first verify that the functional I_λ possesses the mountain pass geometry.

Lemma 7. *If conditions* (V) *and* (f_1) − (f_4) *hold and*

$$\lambda > \frac{|b_0|^p}{pF(h_0,b_0)}\left(pC_{s,p}\sum_{m\neq h_0}\frac{1}{|h_0-m|^{1+ps}} + V(h_0)\right),$$

then the functional I_λ *fulfills the mountain pass geometry.*

Proof. On the one hand, according to (f_1), for any $0 < \varepsilon < \frac{V_0}{p\lambda}$ there exists $\delta > 0$ such that

$$F(j,t) \le \frac{\varepsilon}{p}|t|^p \text{ for all } |t| < \delta \text{ and } j \in \mathbb{Z}$$

Since $\|u\|_\infty \le \|u\|_p$, we can find $0 < \omega < |b_0|^{p-1}V(h_0)^{\frac{1}{p}}$ such that $\|u\|_\infty < \delta$ for all $u \in W$ with $\|u\| = \omega$. Here, h_0 and b_0 come from assumption (f_4). Then

$$\begin{aligned} I_\lambda(u) = \Psi(u) - \lambda J(u) &\ge \frac{\|u\|^p}{p} - \lambda\varepsilon\|u\|^p \\ &\ge (\frac{1}{p} - \frac{\lambda\varepsilon}{V_0})\|u\|^p \ge (\frac{1}{p} - \frac{\lambda\varepsilon}{V_0})\omega^p > 0. \end{aligned}$$

On the other hand, set $e = b_0 e_{h_0}(j)$ and $e_{h_0}(j) = 1$ if $j = h_0; e_{h_0}(j) = 0$ if $j \neq h_0$. Then $\|e\| = |b_0|^{p-1}\mathrm{V}(h_0)^{\frac{1}{p}} > \omega$ and

$$\begin{aligned} \mathrm{I}_\lambda(e) &= \frac{1}{p}\sum_{j\in\mathbb{Z}}\sum_{m\neq j}|e(j)-e(m)|^p K_{s,p}(j-m) + \frac{1}{p}\sum_{j\in\mathbb{Z}}\mathrm{V}(j)|e(j)|^p - \lambda\sum_{j\in\mathbb{Z}}F(j,e(j)) \\ &= \frac{|b_0|^p}{p}(p\sum_{m\neq h_0}|K_{s,p}(j-m) + \mathrm{V}(h_0)) - \lambda F(h_0,b_0) \\ &\leq \frac{|b_0|^p}{p}(pC_{s,p}\sum_{m\neq h_0}|\frac{1}{|h_0-m|^{1+ps}} + \mathrm{V}(h_0)) - \lambda F(h_0,b_0) < 0 \end{aligned}$$

for all

$$\lambda > \frac{|b_0|^p\left(pC_{s,p}\sum_{m\neq h_0}\frac{1}{|h_0-m|^{1+ps}} + \mathrm{V}(h_0)\right)}{pF(h_0,b_0)}.$$

Therefore, the functional I_λ fulfills the mountain pass geometry. □

Lemma 8. *If conditions (V) and $(\mathrm{f}_1)-(\mathrm{f}_3)$ hold, then for all $\lambda > 0$, the functional I_λ fulfills the $(PS)_c$ condition in* W *for all $c \in \mathbb{R}$ (see [21]).*

Proof. Fix $\lambda > 0$, we first show that I_λ is coercive on W, i.e., $\lim_{\|\mathrm{u}\|\to\infty}\mathrm{I}_\lambda(\mathrm{u}) = +\infty$.

By condition (f_3), for all $\varepsilon \in (0, \frac{\mathrm{V}_0}{p\lambda})$, there exists $T > 0$ such that

$$F(j,t) \leq \varepsilon|t|^p \text{ for all } j \in \mathbb{Z} \text{ and } |t| > T.$$

Again by (f_2), there exists $\theta \in \ell^1$ such that

$$|F(j,t)| \leq \theta(j) \quad \text{for all } j \in \mathbb{Z} \text{ and } |t| \leq T.$$

For all $\mathrm{u} \in \mathrm{W}$, we have

$$\begin{aligned} \mathrm{I}_\lambda(\mathrm{u}) =& \Psi(\mathrm{u}) - \lambda J(\mathrm{u}) \geq \frac{\|\mathrm{u}\|^p}{p} - \lambda\sum_{|\mathrm{u}(j)|\leq T}F(j,\mathrm{u}(j)) - \lambda\sum_{|\mathrm{u}(j)|>T}F(j,\mathrm{u}(j)) \\ \geq& \frac{\|\mathrm{u}\|^p}{p} - \lambda\|\theta\|_1 - \lambda\varepsilon\|\mathrm{u}\|_p^p \geq (\frac{1}{p} - \frac{\lambda\varepsilon}{\mathrm{V}_0})\|\mathrm{u}\|^p - \lambda\|\theta\|_1, \end{aligned} \tag{14}$$

which denotes that coerciveness is valid. Next we prove that I_λ fulfills $(PS)c$ condition. Let $\{\mathrm{u}_n\}_n$ be a sequence in W such that $\mathrm{I}_\lambda(\mathrm{u}_n) \to c$ and $\mathrm{I}'_\lambda(\mathrm{u}_n) \to 0$ in W^*. Because $\{\mathrm{u}_n\}_n$ is bounded due to the coercivity of I_λ, consequently, by Lemma 2, there is a subsequence of $\{\mathrm{u}_n\}_n$, still denoted by $\{\mathrm{u}_n\}_n$, such that $\mathrm{u}_n \rightharpoonup \mathrm{u}$ in W and $\mathrm{u}_n \to \mathrm{u}$ in ℓ^p. Then

$$\lim_{n\to\infty}\langle \mathrm{I}'_\lambda(\mathrm{u}_n - \mathrm{I}'_\lambda(\mathrm{u}), \mathrm{u}_n - \mathrm{u}\rangle = 0. \tag{15}$$

Similar to Lemma 5, it is obvious that

$$\lim_{n\to\infty}\sum_{j\in\mathbb{Z}}(f(j,\mathrm{u}_n(j) - f(j,\mathrm{u}(j))(\mathrm{u}_n(j) - \mathrm{u}(j)) = 0.$$

Combining (15), we know that $\|\mathrm{u}_n - \mathrm{u}\| \to 0$, i.e., $\mathrm{u}_n \to \mathrm{u}$ in W. □

3. Proof of Main Result

Proof of Theorem 1. By Lemmas 7 and 8 and mountain pass lemma, we have that for all

$$\lambda > \frac{|b_0|^p \left(pC_{s,p} \sum_{m \neq h_0} \frac{1}{|h_0-m|^{1+2s}} + \mathrm{V}(h_0)\right)}{pF(h_0, b_0)},$$

there exists a sequence $\{\mathrm{u}_n\}_n \subset \mathrm{W}$ such that

$$\mathrm{I}_\lambda(\mathrm{u}_n) \to c_\lambda > 0 \text{ and } \mathrm{I}'_\lambda(\mathrm{u}_n) \to 0, \text{as } n \to \infty,$$

where

$$c_\lambda = \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} \mathrm{I}_\lambda(\gamma(t))$$

and $\Gamma = \{\gamma \in ([0,1], X) : \gamma(0) = 1,\ \gamma(1) = e\}$.

So there exists a subsequence of $\{\mathrm{u}_n\}_n$ (still denoted by $\{\mathrm{u}_n\}_n$) such that $\mathrm{u}_n \to \mathrm{u}_\lambda^{(1)}$ strongly in W. Furthermore, $\mathrm{I}_\lambda(\mathrm{u}_\lambda^{(1)}) = \alpha \ge 0$ and $\mathrm{I}'_\lambda(\mathrm{u}_\lambda^{(1)}) = 0$. Hence, Lemma 6 implies that $\mathrm{u}_\lambda^{(1)}$ is a homoclinic solution of (4).

Next we prove that Equation (4) has another homoclinic solution. Choose $\omega \in \mathbb{R}$ such that $\mathrm{I}_\lambda(e) < \omega < 0$, where e is given by Lemma 7. Set

$$M = \{\mathrm{u} \in \mathrm{W} : \mathrm{I}_\lambda(\mathrm{u}) \le \omega\}.$$

It is clear that $M \neq 0$. It follows from (14) that M is a bounded subset in W.

Now we infer that I_λ is bounded below on M. If not, we suppose that there exists a sequence $\{\mathrm{u}_n\}_n \subset M$ such that

$$\lim_{n \to \infty} \mathrm{I}_\lambda(\mathrm{u}_n) = -\infty. \tag{16}$$

Since $\{\mathrm{u}_n\}_n$ is bounded, up to a subsequence, we have $\mathrm{u}_n \rightharpoonup \mathrm{u}$ in W and $\mathrm{u}_n \to \mathrm{u}$ in ℓ^p. Similar to Lemma 5, we know that J is continuous in ℓ^p. We obtain that Ψ is weakly lower semi-continuous in W thanks to the convexity of Ψ. Thus,

$$\lim_{n \to \infty} \inf \mathrm{I}_\lambda(\mathrm{u}_n) \ge \mathrm{I}_\lambda(\mathrm{u}) > -\infty,$$

which contradicts (16). So we can define

$$c_\lambda^{\sim} = \inf\{\mathrm{I}_\lambda(\mathrm{u}) :\ \mathrm{u} \in M\} = \inf_{\mathrm{W}} \mathrm{I}_\lambda(\mathrm{u}).$$

Then $c_\lambda^{\sim} < 0$ for all $\lambda > 0$. On basis of Lemma 7 and the Ekeland variational principle, applied in M, there exists a sequence $\{\mathrm{u}_n\}_n$ such that

$$c_\lambda^{\sim} \le \mathrm{I}_\lambda(\mathrm{u}_n) \le c_\lambda^{\sim} + \frac{1}{n} \tag{17}$$

and

$$\mathrm{I}_\lambda(v) \ge \mathrm{I}_\lambda(\mathrm{u}_n) - \frac{\|\mathrm{u}_n - v\|}{n} \tag{18}$$

for all $v \in M$.

It is clear that $\{\mathrm{u}_n\}_n$ is a $(PS)_{c_\lambda^{\sim}}$ sequence for the functional I_λ. Similar to Lemma 8, there exists a subsequence of $\{\mathrm{u}_n\}_n$ (still denoted by $\{\mathrm{u}_n\}_n$) such that $\mathrm{u}_n \to \mathrm{u}_\lambda^{(2)}$ in W. So, we get a nontrivial homoclinic solution $\mathrm{u}_\lambda^{(2)}$ of Equation (4) fulfilling

$$\mathrm{I}_\lambda(\mathrm{u}_\lambda^{(2)}) \le \omega < 0.$$

Furthermore, we have

$$I_\lambda(u_\lambda^{(2)}) = c_\lambda^{\sim} \le \omega < 0 < \alpha < c_\lambda = I_\lambda(u_\lambda^{(1)})$$

for all

$$\lambda > \frac{|b_0|^p \left(pC_{s,p} \sum_{m \neq h_0} \frac{1}{|h_0 - m|^{1+2s}} + V(h_0) \right)}{pF(h_0, b_0)}.$$

Therefore, Equation (4) has at least two nontrivial homoclinic solutions.

Finally, we show that all critical points of the functional I_λ are nonnegative. Let $u \in W \backslash \{0\}$ be a critical point of I_λ. Then, $I'_\lambda(u_n) = 0$ and $u_k \to 0$ as $|k| \to \infty$. Let $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$. We have $\langle I'_\lambda(u_n), -u^- \rangle = 0$, due to $I'_\lambda(u_n) = 0$. It follows from $f(k,t) = 0$ for all $k \in \mathbb{Z}$, $t \le 0$ that

$$\sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} |u(j) - u(m)|^{p-2}(-u^-(j) + u^-(m)) K_{s,p}(j-m) + \sum_{j\in\mathbb{Z}} V(j)|u(j)|^{p-2}(-u^-(j)) = 0,$$

which implies that

$$\sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} |u(j) - u(m)|^{p-2}(-u^-(j) + u^-(m)) K_{s,p}(j-m) \le 0.$$

We know that for all j, $m \in \mathbb{Z}$,

$$\sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} |u^-(j) - u^-(m)|^p K_{s,p}(j-m) \le 0,$$

which means that $u^-(j) = u^-(m)$ for all j, $m \in \mathbb{Z}$. By virtue of $u_k \to 0$ and $u^-(k) \le |u(k)|$, we get that $C = 0$. Hence, we infer that $u^-(k) = 0$, which ends the proof. □

4. Conclusions

Lemmas 1–3 are important contents needed for functional estimation; Lemmas 4 and 5 are important conclusions to ensure the continuous differentiability of functional; Lemma 6 shows that the critical point of functional is a homoclinic solution of Equation (4) for all $\lambda > 0$.; Lemmas 7 and 8 verify the mountain pass geometry and (PS) conditions respectively. Finally, in combination with Ekeland's variational principle, we get two homoclinic solutions. In future studies, we can consider the case of variable order and variable exponent. See for more details (see [34–36]).

Author Contributions: Writing—original draft, B.T.; Writing—review and editing, Y.W., G.R., L.Y. and A.R. All authors have read and agreed to the published version of the manuscript.

Funding: The work is supported by Natural Science Foundational of Huaiyin Institute of Technology (Grant/Award number: 20HGZ002).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors thank the anonymous referees for invaluable comments and insightful suggestions which improved the presentation of this manuscript.

Conflicts of Interest: The authors declare that they have no competing interests.

Appendix A

The pair $(\mathrm{W}, \ \|.\|_{\mathrm{W}})$ equipped with the equivalent norm

$$\|\mathrm{u}\| = \left(\sum_{j\in\mathbb{Z}} \mathrm{V}(j)|\mathrm{u}(j)|^p\right)^{1/p}$$

is a Banach space.

Proof. The proof is similar to [37], for fullness, we provide its facts. Employing hypothesis (V) and Lemma 1, we keep

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \mathrm{V}(j)|\mathrm{u}(j)|^p &\leq \|\mathrm{u}\|_{\mathrm{W}}^p \leq C\sum_{j\in\mathbb{Z}}|\mathrm{u}(j)|^p + \sum_{j\in\mathbb{Z}} \mathrm{V}(j)|\mathrm{u}(j)|^p \\ &\leq C\frac{1}{\mathrm{V}_0}\sum_{j\in\mathbb{Z}} \mathrm{V}(j)|\mathrm{u}(j)|^p + \sum_{j\in\mathbb{Z}} \mathrm{V}(j)|\mathrm{u}(j)|^p = C\sum_{j\in\mathbb{Z}} \mathrm{V}(j)|\mathrm{u}(j)|^p,\end{aligned}$$

which shows that $\|\mathrm{u}\| = \left(\sum_{j\in\mathbb{Z}} \mathrm{V}(j)|\mathrm{u}(j)|^p\right)^{1/p}$ is an equivalent norm of W. Finally, we establish that $(W, \ \|.\|_{\mathrm{W}})$ is complete. Let $\{\mathrm{v}_n\}_n$ be a Cauchy sequence in W.

We point out that

$$\|\mathrm{u}\|_p \leq \mathrm{V}_0^{-\frac{1}{p}}\|\mathrm{u}\|$$

for all u $\in W$. Then, $\{\mathrm{v}_n\}_n$ is even a Cauchy sequence in ℓ^p. By the completeness of ℓ^p, there exists $\mathrm{u} \in \ell^p$ satisfying $\mathrm{u}_n \to \mathrm{u}$ in ℓ^p. In addition, Lemma 1 and hypothesis (V) imply that $.Inaddition u_n \to$ u strongly in W as $n \to \infty$. Thus, we conclude the result. □

References

1. Ye, F. Global structure and one-sign solutions for second-order Sturm-Liouville difference equation with sign-changing weight. *Math. Methods Appl. Sci.* **2022**, *5*, 1176–1188. [CrossRef]
2. Agarwal, R.P.; Perera, K.; O'Regan, D. Multiple positive solutions of singular and nonsingular discrete problems via variational methods. *Nonlinear Anal.* **2004**, *58*, 69–73. [CrossRef]
3. Cabada, A.; Iannizzotto, A.; Tersian, S. Multiple solutions for discrete boundary value problems. *J. Math. Anal. Appl.* **2009**, *356*, 418–428. [CrossRef]
4. Iannizzotto, A.; Rădulescu, V. Positive homoclinic solutions for the discrete $p-$Laplacian with a coercive weight/function. *Differ. Integral Eq.* **2014**, *27*, 35–44.
5. Izydorek, M.; Janczewska, J.; Mawhin, J. Homoclinics for singular strong force Lagrangian systems. *Adv. Nonlinear Anal.* **2020**, *9*, 644–653. [CrossRef]
6. Ma, M.; Guo, Z. Homoclinic orbits for second order self-adjoint difference equations. *J. Math. Anal. Appl.* **2005**, *323*, 513–521. [CrossRef]
7. Mihailescu, M.; Rădulescu, V.; Tersian, S. Homoclinic solutions of difference equations with variable exponents. *Topol. Methods Nonlinear Anal.* **2011**, *38*, 277–289.
8. Tahar, B.; Rafik, G. Existence and uniqueness of weak solution for a nonlocal problem involving the p-Laplacian. *IJPAM* **2015**, *98*, 11–21.
9. Ahmad, B.; Alruwaily, Y.; Alsaedi, A.; Ntouyas, S.K. Riemann-Stieltjes Integral boundary value problems involving mixed Riemann-Liouville and Caputo fractional derivatives. *J. Nonlinear Funct. Anal.* **2021**, *2021*, 11.
10. Timoumi, M. Infinitely many fast homoclinic solutions of damped vibration systems with superquadratic conditions or combined nonlinearities. *J. Nonlinear Var. Anal.* **2021**, *5*, 299–318.
11. Zuo, J.; An, T.; Li, M. Superlinear Kirchhoff-type problems of the fractional p-Laplacian without the (AR) condition. *Bound. Value Probl.* **2018**, *180*, 180. [CrossRef]
12. Fiscella, A.; Servadei, R.; Valdinoci, E. Density properties for fractional Sobolev spaces. *Ann. Acad. Sci. Fenn. Math.* **2015**, *40*, 235–253. [CrossRef]
13. Zuo, J.; An, T.; Ye, G.; Qiao, Z. Nonhomogeneous fractional p-Kirchhoff problems involving a critical nonlinearity. *Electron. J. Qual. Theory Differ. Eq.* **2019**, *180*, 1–15. [CrossRef]
14. Zuo, J.; An, T.; Yang, L.; Ren, X. The Nehari manifold for a fractional p-Kirchhoff system involving sign-changing weight function and concave-convex nonlinearities. *J. Funct. Spaces* **2019**, *2019*, 7624373. [CrossRef]
15. Applebaum, D. Lévy processesfrom probability to finance quantum groups. *Not. Am. Math. Soc.* **2004**, *51*, 1336–1347.
16. Caffarelli, L. Nonlocal diffusions, drifts and games. *Nonlinear Partial Differ. Eq.* **2012**, *7*, 37–52.

17. Laskin, N. Fractional quantum mechanics and Lévy path integrals. *Phys. Lett. A.* **2000**, *268*, 298–305. [CrossRef]
18. Autuori, G.; Pucci, P. Elliptic problems involving the fractional Laplacian in R^N. *J. Differ. Eq.* **2013**, *255*, 2340–2362. [CrossRef]
19. Mairi, B.; Guefaifia, R.; Boulaaras, S.; Bouali, T. Existence of positive solutions for a new class of nonlocal p(x)-Kirchhoff elliptic systems via sub-super solutions concept. *Appl. Sci. (APPS)* **2018**, *20*, 117–128.
20. Fiscella, A.; Pucci, P.; Zhang, B. $p-$fractional Hardy-Schrödinger-Kirhhoff systems with critical nonlinearities. *Adv. Nonlinear Anal.* **2019**, *8*, 1111–1131. [CrossRef]
21. Iannizzotto, A.; Tersian, S.A. Multiple homoclinic solutions for the discrete p-Laplacian via critical point theory. *J. Math. Anal. Appl.* **2013**, *403*, 173–182. [CrossRef]
22. Salah, B.; Rafik, G.; Tahar, B. Existence of Positive Solutions For a Class of Quasilinear Singular Elliptic Systems Involving Caffarelli-Kohn- Nirenberg Exponent With Sign-Changing Functions. *Indian J. Pure Appl. Math.* **2018**, *49*, 705–715.
23. Valdinoci, E. From the longjump random walkto the fractional Laplacian. *SeMA J.* **2009**, *49*, 33–44.
24. Xiang, M.; Zhang, B.; Yang, D. Multiplicity results for variable-order fractional Laplacian equations with variable growth. *Nonlinear. Anal.* **2019**, *178*, 190–204. [CrossRef]
25. Xiang, M.; Rădulescu, V.; Zhang, B. A critical fractional Choquard Kirchhoff problem with magnetic field. *Commun. Contemp. Math.* **2019**, *21*, 185004.
26. Ju, X.; Hu, D.; Xiang, M. The Nehari manifold method for discrete fractional p-Laplacian equations. *Adv. Differ. Eq.* **2020**, *559*, 1–21. [CrossRef]
27. Linda, M.; Tahar, B.; Rafik, G.; Mohamad, B. Existence of weak solutions for p-Laplacian problem with impulsive effects. *Appl. Sci. (APPS)* **2020**, *22*, 128–145.
28. Molica Bisci, G.; Rădulescu, V.; Servadei, R. *Variational Methods for Nonlocal Fractional Problems*; Cambridge University Press: Cambridge, UK , 2015.
29. Ciaurri, Ó.; Roncal, L.; Stinga, P.R.; Torrea, J.L.; Varona, J.L. Nonlocal discrete diffusion equation and the fractional discrete Laplacian, regularity and applications. *Adv. Math.* **2018**, *330*, 688–738. [CrossRef]
30. Xiang, M.; Zhang, B. Homoclinic solutions for fractional discrete Laplacian equations. *Nonlinear Anal.* **2020**, *198*, 111886. [CrossRef]
31. Huang, Y.; Oberman, A. Numerical methods for the fractional Laplacian: afinite difference-quadrature approach. *SIAM J. Numer. Anal.* **2014**, *52*, 3056–3084. [CrossRef]
32. Fabian, M.; Habala, P.; Hájek, P.; Montesinos, V.; Zizler, V. *Banach Space Theory*; Springer: Berlin/Heidelberg, Germany, 2011.
33. Servadei, R.; Valdinoci, E. Mountain Pass solutions for non-local elliptic operators. *J. Math. Anal. Appl.* **2012**, *389*, 887–898. [CrossRef]
34. Zuo, J.; An, T.; Fiscella, A. A critical Kirchhoff-type problem driven by a p(.)-fractional Laplace operator with variable s(.)-order. *Math. Methods Appl. Sci.* **2021**, *12*, 1071–1085. [CrossRef]
35. Zuo, J.; Fiscella, A.; Bahrouni, A. Existence and multiplicity results for p(.)&q(.) fractional Choquard problems with variable order. *Complex Var. Elliptic Eq.* **2022**, *67*, 500–516.
36. Choudhuri, D.; Zuo, J. Critical Kirchhoff p(.)&q(.)-fractional variable-order systems with variable exponent growth. *Anal. Math. Phys.* **2022**, *12*, 30.
37. Xiang, M.; Hu, D.; Yang, D. Least energy solutions for fractional Kirchhoff problems with logarithmic nonlinearity. *Nonlinear Anal.* **2020**, *198*, 111899. [CrossRef]