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On the Best Ulam Constant of the Linear Differential Operator with Constant Coefficients †

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Abstract: The linear differential operator with constant coefficients $D(y) = y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y, \quad y \in \mathcal{C}^n(\mathbb{R}, X)$ acting in a Banach space X is Ulam stable if and only if its characteristic equation has no roots on the imaginary axis. We prove that if the characteristic equation of D has distinct roots r_k satisfying $\operatorname{Re} r_k > 0$, $1 \leq k \leq n$, then the best Ulam constant of D is $K_D = \frac{1}{|V|}\int_0^\infty |\sum_{k=1}^{n} (-1)^k V_k e^{-r_k x}| dx$, where $V = V(r_1, r_2, \ldots, r_n)$ and $V_k = V(r_1, \ldots, r_{k-1}, r_{k+1}, \ldots, r_n)$, $1 \leq k \leq n$, are Vandermonde determinants.

Keywords: linear differential operator; Ulam stability; best constant; Banach space

MSC: 34D20; 39B82

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1. Introduction

In this paper, we denote by $\mathbb{K}$ the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. Let M and N be two linear spaces over the field $\mathbb{K}$.

Definition 1. *A function $\rho_M : M \to [0, \infty]$ is called a* gauge *on M if the following properties hold:*

(i) $\rho_M(x) = 0$ *if and only if* $x = 0$;

(ii) $\rho_M(\lambda x) = |\lambda| \rho_M(x)$ *for all* $x \in M, \lambda \in \mathbb{K}, \lambda \neq 0$.

Throughout this paper, we denote by $(X, \|\cdot\|)$ a Banach space over the field $\mathbb{C}$ and by $\mathcal{C}^n(\mathbb{R}, X)$ the linear space of all n times differentiable functions with continuous n-th derivatives, defined on $\mathbb{R}$ with values in X. $\mathcal{C}^0(\mathbb{R}, X)$ will be denoted as usual by $\mathcal{C}(\mathbb{R}, X)$. For $f \in \mathcal{C}^n(\mathbb{R}, X)$ define

$$\|f\|_\infty = \sup\{\|f(t)\| : t \in \mathbb{R}\}. \quad (1)$$

Then, $\|f\|_\infty$ is a gauge on $\mathcal{C}^n(\mathbb{R}, X)$. We suppose that $\mathcal{C}^n(\mathbb{R}, X)$ and $\mathcal{C}(\mathbb{R}, X)$ are endowed with the same gauge $\|\cdot\|_\infty$.

Let ρ_M and ρ_N be two gauges on the linear spaces M and N, respectively, and let $L : M \to N$ be a linear operator.

We denote by $\ker L = \{x \in M | Lx = 0\}$ and $R(L) = \{Lx | x \in M\}$ the kernel and the range of the operator L, respectively.

Definition 2. *We say that the operator L is Ulam stable if there exists $K \geq 0$ such that for every $\varepsilon > 0$ with $\rho_N(Lx) \leq \varepsilon$ there exists $z \in \ker L$ with the property $\rho_M(x - z) \leq K\varepsilon$.*

The Ulam stability of the operator L is equivalent to the stability of the associated equation $Lx = y$, $y \in R(L)$. An element $x \in M$ satisfying $\rho_N(Lx) \leq \varepsilon$ for some positive ε is called an approximate solution of the equation $Lx = y$, $y \in R(L)$. Consequently, Definition 2 can be reformulated as follows: The operator L is Ulam stable if for every approximate solution of $Lx = y, y \in R(L)$ there exists an exact solution of the equation near it. The problem of Ulam stability is due to Ulam [1]. Ulam formulated this problem during a conference at Madison University, Wisconsin, for the equation of the homomorphisms of a metric group. The first answer to Ulam's question was given by D.H. Hyers for the Cauchy functional equation in Banach spaces in [2]. In fact, a problem of this type was formulated in the famous book by Polya and Szegö for the Cauchy functional equation on the set of integers; see [3]. Since then, this research area received a lot of attention and was extended to the contexts of operators, functional, differential, or difference equations. For a broad overview on the topic, we refer the reader to [4,5].

The number K from Definition 2 is called an *Ulam constant* of L. In what follows, the infimum of all Ulam constants of L is denoted by K_L. Generally, the infimum of all Ulam constants of the operator L is not a Ulam constant of L (see [6,7]), but if it is, it will be called *the best Ulam constant* of L or, simply, *the Ulam constant* of the operator L. Finding the best Ulam constant of an equation or operator is a challenging problem because it offers the best measure of the error between the approximate and the exact solution. In [6,8], for linear and bounded operators acting on normed spaces their Ulam stability is characterized and representation results are given for their best Ulam constant. Using this result, D. Popa and I. Raşa obtained the best Ulam constant for the Bernstein, Kantorovich, and Stancu operators; see [9–12]. For more information on Ulam stability with respect to gauges and on the best Ulam constant of linear operators, we refer the reader to [4,13].

To the best of our knowledge, the first result on Ulam stability of differential equations was obtained by M. Obłoza [14]. Thereafter, the topic was deeply investigated by T. Miura, S. Miyajima, and S.E. Takahasi in [15–17] and by S. M Jung in [18], who gave some results for various differential equations and partial differential equations. For further details on Ulam stability, we refer the reader to [1,4,5].

Let $a_1, \ldots, a_n \in \mathbb{C}$ and consider the linear differential operator $D : \mathcal{C}^n(\mathbb{R}, X) \to \mathcal{C}(\mathbb{R}, X)$ defined by

$$D(y) = y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y, \quad y \in \mathcal{C}^n(\mathbb{R}, X). \tag{2}$$

Denote by $P(z) = z^n + a_1 z^{n-1} + \ldots + a_n$ the characteristic polynomial of the operator D, and let $r_1, \ldots, r_n$ be the complex roots of the characteristic equation $P(z) = 0$.

The problem of finding the best Ulam constant was first posed by Th. Rassias in [19]. Since then, various papers on this topic appeared, but there are only a few results on the best Ulam constant of differential equations and differential operators. In the sequel, we will provide a short overview of some important results concerning the Ulam stability and best Ulam constant of the differential operator D. In [16] the operator D is proven to be Ulam stable with the Ulam constant $\frac{1}{\prod_{k=1}^{n} |\operatorname{Re} r_k|}$ if and only if its characteristic equation has no roots on the imaginary axis. In [9], D. Popa and I. Raşa obtained sharp estimates for the Ulam constant of the first-order linear differential operator and the higher-order linear differential operator with constant coefficients. The best Ulam constant of the first-order linear differential operator with constant coefficients is obtained in [15]. Later, A.R. Baias and D. Popa obtained the best Ulam constant for the second-order linear differential operator with constant coefficients [20]. Recent results on Ulam stability for linear differential equations with periodic coefficients and on the best constant for Hill's differential equation were obtained by R. Fukutaka and M. Onitsuka in [21,22]. Important steps in finding the best Ulam constant were made also for higher-order difference equations with constant coefficients. For more details, we refer the reader to [23] and the references therein.

The aim of this paper is to determine the best Ulam constant for the n-order linear differential operator with constant coefficients acting in Banach spaces, for the case of distinct roots of the characteristic equation. Through this result, we improve and complement some extant results in the field.

2. Main Results

Let $a_1, \ldots, a_n \in \mathbb{C}$, and consider the linear differential operator $D : \mathcal{C}^n(\mathbb{R}, X) \to \mathcal{C}(\mathbb{R}, X)$ defined by

$$D(y) = y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y, \quad y \in \mathcal{C}^n(\mathbb{R}, X). \tag{3}$$

If $r_1, r_2, \ldots, r_n$ are distinct roots of the characteristic equation $P(z) = 0$, then the general solution of the homogeneous equation $D(y) = 0$ is given by

$$y_H(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \cdots + C_n e^{r_n x}, \tag{4}$$

where $\mathcal{C}_1, \ldots, \mathcal{C}_n \in X$ are arbitrary constants. Consequently,

$$\ker D = \left\{ \sum_{k=1}^{n} C_k e^{r_k x} | C_1, C_2, \ldots, C_n \in X \right\}. \tag{5}$$

The operator D is surjective, so according to the variation of constants method, for every $f \in \mathcal{C}(\mathbb{R}, X)$ there exists a particular solution of the equation $D(y) = f$ of the form

$$y_P(x) = \sum_{k=1}^{n} C_k(x) e^{r_k x}, \quad x \in \mathbb{R},$$

where $\mathcal{C}_1, \ldots, \mathcal{C}_n$ are functions of class $\mathcal{C}^1(\mathbb{R}, X)$ that satisfy

$$\begin{pmatrix} e^{r_1 x} & e^{r_2 x} & \ldots & e^{r_n x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} & \ldots & r_n e^{r_n x} \\ \ldots & \ldots & \ldots & \ldots \\ r_1^{n-1} e^{r_1 x} & r_2^{n-1} e^{r_2 x} & \ldots & r_n^{n-1} e^{r_n x} \end{pmatrix} \begin{pmatrix} C_1'(x) \\ C_2'(x) \\ \vdots \\ C_n'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f(x) \end{pmatrix}, \quad x \in \mathbb{R}. \tag{6}$$

In what follows, we denote for simplicity the Vandermonde determinants by $V := V(r_1, r_2, \ldots, r_n)$ and $V_k := V(r_1, r_2, \ldots, r_{k-1}, r_{k+1}, \ldots, r_n)$, $1 \le k \le n$. Consequently, we obtain

$$C_k'(x) = (-1)^{n+k} \frac{V_k}{V} e^{-r_k x} f(x), \quad k = 1, \ldots, n.$$

Hence, a particular solution of the equation $D(y) = f$ is given by

$$y_P(x) = \frac{1}{V} \sum_{k=1}^{n} (-1)^{n+k} V_k e^{r_k x} \int_0^x f(t) e^{-r_k t} dt, \quad x \in \mathbb{R}. \tag{7}$$

The main result concerning the Ulam stability of the operator D for the case of distinct roots of the characteristic equation is given in the next theorem.

Theorem 1. *Suppose that r_k, $1 \le k \le n$, are distinct roots of the characteristic equation with* $\operatorname{Re} r_k \ne 0$*, and let $\varepsilon > 0$. Then, for every $y \in \mathcal{C}^n(\mathbb{R}, X)$ satisfying*

$$\|D(y)\|_\infty \le \varepsilon \tag{8}$$

there exists a unique $y_H \in \ker D$ such that

$$\|y - y_H\|_\infty \le K\varepsilon, \tag{9}$$

where

$$K = \begin{cases} \dfrac{1}{|V|}\displaystyle\int_0^\infty \left|\sum_{k=1}^{n}(-1)^k V_k e^{-r_k x}\right| dx, & \text{if } \operatorname*{Re}_{1\le k\le n} r_k > 0; \\ \dfrac{1}{|V|}\displaystyle\int_0^\infty \left|\sum_{k=1}^{n}(-1)^k V_k e^{r_k x}\right| dx, & \text{if } \operatorname*{Re}_{1\le k\le n} r_k < 0; \\ \dfrac{1}{|V|}\displaystyle\int_0^\infty \left(\left|\sum_{k=1}^{p}(-1)^k V_k e^{-r_k x}\right| + \left|\sum_{k=p+1}^{n}(-1)^k V_k e^{r_k x}\right|\right) dx, & \text{if } \begin{matrix}\operatorname*{Re}_{1\le k\le p} r_k > 0; \\ \operatorname*{Re}_{p+1\le k\le n} r_k < 0.\end{matrix} \end{cases} \tag{10}$$

Proof. Existence. Suppose that $y \in \mathcal{C}^n(\mathbb{R}, X)$ satisfies (8), and let $D(y) = f$. Then, $\|f\|_\infty \le \varepsilon$ and

$$y(x) = \sum_{k=1}^{n} C_k e^{r_k x} + \frac{1}{V}\sum_{k=1}^{n}(-1)^{n+k} V_k e^{r_k x}\int_0^x f(t)e^{-r_k t}dt,\ x \in \mathbb{R},$$

for some $C_k \in X$, $1 \le k \le n$.

(i) First, let $\operatorname{Re} r_k > 0$, $1 \le k \le n$. Define $y_H \in KerD$ by the relation

$$y_H(x) = \sum_{k=1}^{n} \widetilde{C_k} e^{r_k x}, x \in \mathbb{R}, \quad \widetilde{C_k} \in X,$$

where

$$\widetilde{C_k} = C_k + (-1)^{n+k}\frac{V_k}{V}\int_0^\infty f(t)e^{-r_k t}dt, \quad 1 \le k \le n.$$

Since $\|f(t)e^{-r_k t}\| \le \varepsilon|e^{-r_k t}| = \varepsilon e^{-t \operatorname{Re} r_k}$, $t \ge 0$ and $\int_0^\infty e^{-t\operatorname{Re} r_k}dt$ is convergent, it follows that $\int_0^\infty f(t)e^{-r_k t}dt$ is absolutely convergent, so the constants $\widetilde{C_k}$, $1 \le k \le n$ are well defined. Then,

$$\begin{aligned} y(x) - y_H(x) &= \sum_{k=1}^{n} C_k e^{r_k x} + \frac{1}{V}\sum_{k=1}^{n}(-1)^{n+k}V_k e^{r_k x}\int_0^x f(t)e^{-r_k t}dt - \sum_{k=1}^{n}\widetilde{C_k}e^{r_k x} \\ &= \sum_{k=1}^{n} C_k e^{r_k x} + \frac{1}{V}\sum_{k=1}^{n}(-1)^{n+k}V_k e^{r_k x}\int_0^x f(t)e^{-r_k t}dt \\ &- \sum_{k=1}^{n} C_k e^{r_k x} - \frac{1}{V}\sum_{k=1}^{n}(-1)^{n+k}V_k e^{r_k x}\int_0^\infty f(t)e^{-r_k t}dt \\ &= -\frac{1}{V}\sum_{k=1}^{n}(-1)^{n+k}V_k e^{r_k x}\int_x^\infty f(t)e^{-r_k t}dt \\ &= -\frac{1}{V}\sum_{k=1}^{n}(-1)^{n+k}V_k\int_x^\infty f(t)e^{r_k(x-t)}dt, \quad x \in \mathbb{R}. \end{aligned}$$

Now, letting $t - x = u$ in the above integral we obtain

$$\begin{aligned} y(x) - y_H(x) &= -\frac{1}{V}\sum_{k=1}^{n}(-1)^{n+k}V_k\int_0^\infty f(u+x)e^{-r_k u}du, \\ &= \frac{(-1)^{n+1}}{V}\int_0^\infty\left(\sum_{k=1}^{n}(-1)^k V_k e^{-r_k u}\right)f(u+x)du, \quad x \in \mathbb{R}. \end{aligned}$$

Hence

$$\begin{aligned}\|y(x)-y_H(x)\| &\leq \int_0^\infty \left|\frac{1}{V}\sum_{k=1}^n (-1)^k V_k e^{-r_k u}\right| \cdot \big\|f(u+x)\big\| du, \quad x\in\mathbb{R} \\ &\leq \frac{\varepsilon}{|V|}\int_0^\infty \left|\sum_{k=1}^n (-1)^k V_k e^{-r_k u}\right| du, \quad x\in\mathbb{R},\end{aligned}$$

therefore

$$\|y-y_0\|_\infty \leq K\varepsilon.$$

(ii) Let $\operatorname{Re} r_k < 0$, $1 \leq k \leq n$. The proof follows analogously, defining

$$y_H(x) = \sum_{k=1}^n \widetilde{C_k} e^{r_k x}, x\in\mathbb{R}, \quad \widetilde{C_k}\in X,$$

with

$$\widetilde{C_k} = C_k - (-1)^{n+k}\frac{V_k}{V}\int_{-\infty}^0 f(t)e^{-r_k t}dt \quad 1\leq k\leq n.$$

Then,

$$\begin{aligned}y(x)-y_H(x) &= \sum_{k=1}^n C_k e^{r_k x} + \frac{1}{V}\sum_{k=1}^n (-1)^{n+k} V_k e^{r_k x}\int_0^x f(t)e^{-r_k t}dt \\ &- \sum_{k=1}^n C_k e^{r_k x} + \frac{1}{V}\sum_{k=1}^n (-1)^{n+k} V_k e^{r_k x}\int_{-\infty}^0 f(t)e^{-r_k t}dt \\ &= \frac{1}{V}\sum_{k=1}^n (-1)^{n+k} V_k e^{r_k x}\int_{-\infty}^x f(t)e^{-r_k t}dt, \\ &= \frac{1}{V}\sum_{k=1}^n (-1)^{n+k} V_k \int_{-\infty}^x f(t)e^{r_k(x-t)}dt, \\ &= \frac{(-1)^n}{V}\int_0^\infty \left(\sum_{k=1}^n (-1)^k V_k e^{r_k u}\right) f(x-u)du, \quad x\in\mathbb{R},\end{aligned}$$

where $u = x - t$. Hence,

$$\|y(x)-y_H(x)\| \leq \frac{1}{|V|}\int_0^\infty \left|\sum_{k=1}^n (-1)^k V_k e^{r_k u}\right| \cdot \big\|f(x-u)\big\| du \leq K\varepsilon, x\in\mathbb{R},$$

which entails

$$\|y-y_H\|_\infty \leq K\varepsilon.$$

(iii) Let $\operatorname{Re} r_k > 0$, $1 \leq k \leq p$, and $\operatorname{Re} r_k < 0$, $p+1 \leq k \leq n$. Define y_H by the relation

$$y_H(x) = \sum_{k=1}^n \widetilde{C_k} e^{r_k x}, x\in\mathbb{R}, \quad \widetilde{C_k}\in X,$$

with

$$\begin{aligned}\widetilde{C_k} &= C_k + (-1)^{n+k}\frac{V_k}{V}\int_0^\infty f(t)e^{-r_k t}dt, \quad 1\leq k\leq p, \\ \widetilde{C_k} &= C_k - (-1)^{n+k}\frac{V_k}{V}\int_{-\infty}^0 f(t)e^{-r_k t}dt, \quad p+1\leq k\leq n.\end{aligned}$$

Then,

$$y(x)-y_H(x) = \frac{1}{V}\sum_{k=1}^n (-1)^{n+k} V_k e^{r_k x}\int_0^x f(t)e^{-r_k t}dt$$

$$-\frac{1}{V}\sum_{k=1}^{p}(-1)^{n+k}V_k e^{r_k x}\int_0^{\infty}f(t)e^{-r_k t}dt+\frac{1}{V}\sum_{k=p+1}^{n}(-1)^{n+k}V_k e^{r_k x}\int_{-\infty}^{0}f(t)e^{-r_k t}dt,$$

$$=-\frac{1}{V}\sum_{k=1}^{p}(-1)^{n+k}V_k e^{r_k x}\int_x^{\infty}f(t)e^{-r_k t}dt+\frac{1}{V}\sum_{k=p+1}^{n}(-1)^{n+k}V_k e^{r_k x}\int_{-\infty}^{x}f(t)e^{-r_k t}dt$$

$$=-\frac{1}{V}\sum_{k=1}^{p}(-1)^{n+k}V_k\int_x^{\infty}f(t)e^{r_k(x-t)}dt+\frac{1}{V}\sum_{k=p+1}^{n}(-1)^{n+k}V_k\int_{-\infty}^{x}f(t)e^{r_k(x-t)}dt.$$

Letting $x-t=-u$, and $x-t=u$ correspondingly, in the previous integrals, it follows that

$$y(x)-y_H(x)=-\frac{1}{V}\sum_{k=1}^{p}(-1)^{n+k}V_k\int_0^{\infty}f(x+u)e^{-r_k u}du$$
$$+\frac{1}{V}\sum_{k=p+1}^{n}(-1)^{n+k}V_k\int_0^{\infty}f(x-u)e^{r_k u}du,\quad x\in\mathbb{R}$$

and

$$\|y(x)-y_H(x)\|\leq\int_0^{\infty}\left(\left|\frac{1}{V}\sum_{k=1}^{p}(-1)^{n+k}V_k e^{-r_k u}\right|\|f(x+u)\|\right)du$$
$$+\int_0^{\infty}\left(\left|\frac{1}{V}\sum_{k=p+1}^{n}(-1)^{n+k}V_k e^{r_k u}\right|\|f(x-u)\|\right)du$$
$$\leq\frac{\varepsilon}{|V|}\int_0^{\infty}\left(\left|\sum_{k=1}^{p}(-1)^k V_k e^{-r_k u}\right|+\left|\sum_{k=p+1}^{n}(-1)^k V_k e^{r_k u}\right|\right)du,\quad x\in\mathbb{R}.$$

Therefore, we have

$$\|y-y_0\|_{\infty}\leq K\varepsilon.$$

Its existence is proved. Uniqueness. Suppose that for some $y\in\mathcal{C}^n(\mathbb{R},X)$ satisfying (8), there exist $y_1,y_2\in\ker D$ such that

$$\|y-y_j\|_{\infty}\leq K\varepsilon,\quad j=1,2.$$

Then,

$$\|y_1-y_2\|_{\infty}\leq\|y_1-y\|_{\infty}+\|y-y_2\|_{\infty}\leq 2K\varepsilon.$$

However, $y_1-y_2\in\ker D$; hence, there exist $C_k\in X$, $1\leq k\leq n$ such that

$$y_1(x)-y_2(x)=\sum_{k=1}^{n}C_k e^{r_k x},\quad x\in\mathbb{R}. \tag{11}$$

If $(C_1,C_2,\ldots,C_n)\neq(0,0,\ldots,0)$, then

$$\|y_1-y_2\|_{\infty}=\sup_{x\in\mathbb{R}}\|y_1(x)-y_2(x)\|=+\infty,$$

which contradicts the boundedness of y_1-y_2. We conclude that $C_k=0$, $1\leq k\leq n$; therefore, $y_1=y_2$. The theorem is proven.

□

Theorem 2. *If r_k are distinct roots of the characteristic equation with* $\operatorname{Re} r_k\neq 0$, $1\leq k\leq n$, *then the best Ulam constant of D is given by*

$$K_D = \begin{cases} \dfrac{1}{|V|}\displaystyle\int_0^\infty \left|\sum_{k=1}^{n}(-1)^k V_k e^{-r_k x}\right| dx, & \textit{if} \underset{1\le k\le n}{\operatorname{Re}} r_k > 0; \\ \dfrac{1}{|V|}\displaystyle\int_0^\infty \left|\sum_{k=1}^{n}(-1)^k V_k e^{r_k x}\right| dx, & \textit{if} \underset{1\le k\le n}{\operatorname{Re}} r_k < 0; \\ \dfrac{1}{|V|}\displaystyle\int_0^\infty \left(\left|\sum_{k=1}^{p}(-1)^k V_k e^{-r_k x}\right| + \left|\sum_{k=p+1}^{n}(-1)^k V_k e^{r_k x}\right|\right) dx, & \textit{if} \begin{array}{l}\underset{1\le k\le p}{\operatorname{Re}} r_k > 0; \\ \underset{p+1\le k\le n}{\operatorname{Re}} r_k < 0.\end{array} \end{cases} \tag{12}$$

Proof. Suppose that D admits a Ulam constant $K < K_D$.

(i) First, let $\operatorname{Re} r_k > 0$, $1 \le k \le n$. Then,

$$K_D = \frac{1}{|V|}\int_0^\infty \left|\sum_{k=1}^{n}(-1)^k V_k e^{-r_k x}\right| dx.$$

Let $h(x) = \sum\limits_{k=1}^{n}(-1)^k V_k e^{-r_k x},\ x \in \mathbb{R}$. Take $s \in X$, $\|s\| = 1$, $\theta > 0$ is arbitrary chosen, and consider $f : \mathbb{R} \to X$ given by

$$f(x) = \frac{\overline{h(x)}}{|h(x)| + \theta e^{-x}} s, \quad x \in \mathbb{R}.$$

Obviously, the function f is continuous on $\mathbb{R}$ and $\|f(x)\| \le 1$ for all $x \in \mathbb{R}$. Let $\widetilde{y}$ be the solution of $D(y) = f$, given by

$$\widetilde{y}(x) = \sum_{k=1}^{n} C_k e^{r_k x} + \frac{1}{V}\sum_{k=1}^{n}(-1)^{n+k} V_k e^{r_k x}\int_0^x f(t) e^{-r_k t} dt \tag{13}$$

with the constants

$$C_k = -(-1)^{n+k}\frac{V_k}{V}\int_0^\infty f(t) e^{-r_k t} dt, \quad 1 \le k \le n.$$

The improper integrals in the definition of C_k $1 \le k \le n$ are obviously absolutely convergent since $\|f(x)\| \le 1$, $x \in \mathbb{R}$, and $\operatorname{Re} r_k > 0$, $1 \le k \le n$. Then,

$$\begin{aligned} \widetilde{y}(x) &= -\frac{1}{V}\sum_{k=1}^{n}\left((-1)^{n+k} V_k \int_0^\infty f(t) e^{-r_k t} dt\right) e^{r_k x} \\ &+ \frac{1}{V}\sum_{k=1}^{n}(-1)^{n+k} V_k e^{r_k x}\int_0^x f(t) e^{-r_k t} dt \\ &= -\frac{1}{V}\sum_{k=1}^{n}(-1)^{n+k} V_k e^{r_k x}\int_x^\infty f(t) e^{-r_k t} dt \\ &= \frac{(-1)^{n+1}}{V}\sum_{k=1}^{n}(-1)^k V_k \int_x^\infty f(t) e^{r_k (x-t)} dt. \end{aligned}$$

Using the substitution $x - t = -u$, $\widetilde{y}(x)$ becomes

$$\widetilde{y}(x) = \frac{(-1)^{n+1}}{V}\sum_{k=1}^{n}(-1)^k V_k \int_0^\infty f(x+u) e^{-r_k u} du, \quad x \in \mathbb{R}. \tag{14}$$

Since f is bounded and $\operatorname{Re} r_k > 0$, $1 \le k \le n$, it follows that $\widetilde{y}(x)$ is bounded on $\mathbb{R}$. Furthermore, $\|D(\widetilde{y})\|_\infty \le 1$, and the Ulam stability of D for $\varepsilon = 1$ with the constant K leads to the existence of $y_H \in \ker D$, given by

$$y_H(x) = \sum_{k=1}^{n} C_k e^{r_k x}, \quad x \in \mathbb{R},$$

$C_k \in X$, $1 \le k \le n$, with the property

$$\|\widetilde{y} - y_H\|_\infty \le K. \tag{15}$$

If $(C_1, C_2, \dots, C_n) \ne (0, 0, \dots, 0)$ we have, in view of the boundedness of $\widetilde{y}$,

$$\lim_{x \to \infty} \|\widetilde{y}(x) - y_H(x)\| = +\infty, \tag{16}$$

a contradiction with the existence of K satisfying (15). Therefore, $C_1 = C_2 = \dots = C_n = 0$, and the relation (15) becomes

$$\|\widetilde{y}(x)\| \le K, \text{ for all } x \in \mathbb{R}. \tag{17}$$

Now let $x = 0$ in (17). We obtain, in view of (14),

$$\frac{1}{|V|} \left\| \int_0^\infty \left(\sum_{k=1}^{n} (-1)^k V_k e^{-r_k u} \right) f(u) du \right\| \le K,$$

or equivalently

$$\frac{1}{|V|} \left\| \int_0^\infty h(u) f(u) du \right\| = \frac{1}{|V|} \int_0^\infty \frac{|h(u)|^2}{|h(u)| + \theta e^{-u}} du \le K, \quad \forall \theta > 0. \tag{18}$$

Let $I(\theta) = \int_0^\infty \frac{|h(u)|^2}{|h(u)| + \theta e^{-u}} du$ and $I_0 = \int_0^\infty |h(u)| du$. We show that $\lim_{\theta \to 0} I(\theta) = I_0$. Indeed,

$$\begin{aligned} |I(\theta) - I_0| &\le \int_0^\infty \left| \frac{|h(u)|^2}{|h(u)| + \theta e^{-u}} - |h(u)| \right| du \\ &= \theta \int_0^\infty \frac{|h(u)| e^{-u}}{|h(u)| + \theta e^{-u}} du \\ &\le \theta \int_0^\infty e^{-u} du = \theta, \quad \theta > 0. \end{aligned}$$

Consequently, $\lim_{\theta \to 0} I(\theta) = I_0$. Letting $\theta \to 0$ in (18), we have $K_D \le K$, which is a contradiction to the supposition $K < K_D$.

(ii) The case $\operatorname{Re} r_k < 0$, $1 \le k \le n$, follows analogously. Let $h(x) = \sum_{k=1}^{n} (-1)^k V_k e^{r_k x}$, $x \in \mathbb{R}$, and f be given by

$$f(x) = \frac{\overline{h(-x)}}{|h(-x)| + \theta e^{x}} s,$$

for $s \in X$, $\|s\| = 1$, $x \in \mathbb{R}$ and $\theta > 0$ be arbitrary chosen. Obviously, the function f is continuous on $\mathbb{R}$ and $\|f(x)\| \le 1$ for all $x \in \mathbb{R}$. Let $\widetilde{y}$ be the solution of $D(y) = f$, given by

$$\widetilde{y}(x) = \sum_{k=1}^{n} C_k e^{r_k x} + \sum_{k=1}^{n} (-1)^{n+k} \frac{V_k}{V} e^{r_k x} \int_0^x f(t) e^{-r_k t} dt \tag{19}$$

with the constants

$$C_k = (-1)^{n+k}\frac{V_k}{V}\int_{-\infty}^{0} f(t)e^{-r_k t}dt, \quad 1 \le k \le n.$$

Using a similar reasoning as in the previous case, we obtain

$$\widetilde{y}(x) = \frac{(-1)^n}{V}\int_0^\infty \left(\sum_{k=1}^{n}(-1)^k V_k e^{r_k u}\right) f(x-u)du, \ x \in \mathbb{R}.$$

Since f is bounded and $\operatorname{Re} r_k < 0$, $1 \le k \le n$, it follows that $\widetilde{y}(x)$ is bounded on $\mathbb{R}$. Furthermore, $\|D(\widetilde{y})\|_\infty \le 1$ and the Ulam stability of D for $\varepsilon = 1$ with the constant K leads to the existence of $y_H \in \ker D$, given by

$$y_H(x) = \sum_{k=1}^{n} \widetilde{C_k} e^{r_k x}, \quad x \in \mathbb{R},$$

$\widetilde{C_k} \in X$, $1 \le k \le n$, such that

$$\|\widetilde{y} - y_H\|_\infty \le K. \tag{20}$$

If $(\widetilde{C_1}, \widetilde{C_2}, \ldots, \widetilde{C_n}) \neq (0, 0, \ldots, 0)$ it follows that $\widetilde{y} - y_H$ is unbounded, a contradiction to the existence of K satisfying (20).
Therefore, $\widetilde{C_1} = \widetilde{C_2} = \cdots = \widetilde{C_n} = 0$, and the relation (20) becomes

$$\|\widetilde{y}(x)\| \le K, \text{ for all } x \in \mathbb{R}. \tag{21}$$

Now, let $x = 0$ in (21). We have

$$\frac{1}{|V|}\left\|\int_0^\infty \left(\sum_{k=1}^{n}(-1)^k V_k e^{r_k u}\right) f(-u)du\right\| \le K,$$

or equivalently

$$\frac{1}{|V|}\left\|\int_0^\infty h(u)f(-u)du\right\| = \frac{1}{|V|}\int_0^\infty \frac{|h(u)|^2}{|h(u)| + \theta e^{-u}}du \le K, \quad \forall \theta > 0. \tag{22}$$

Let $I(\theta) = \int_0^\infty \frac{|h(u)|^2}{|h(u)|+\theta e^{-u}}du$ and $I_0 = \int_0^\infty |h(u)|du$. The arguments used in the proof of the previous case lead to $\lim\limits_{\theta \to 0} I(\theta) = I_0$. Letting $\theta \to 0$ in (22), we have $K_D \le K$, a contradiction to the supposition $K < K_D$.

(iii) Consider $\operatorname{Re} r_k > 0$, $1 \le k \le p$ and $\operatorname{Re} r_k < 0$, $p+1 \le k \le n$. Let

$$h_1(x) = \sum_{k=p+1}^{n} (-1)^k V_k e^{r_k x}, \quad h_2(x) = \sum_{k=1}^{p}(-1)^k V_k e^{-r_k x}, \quad x \in \mathbb{R}.$$

Take an arbitrary $\theta > 0$, $s \in X$, $\|s\| = 1$ and define

$$f(x) = \begin{cases} \frac{\overline{h_1(-x)}}{|h_1(-x)|+\theta e^{x}}s, & \text{if } x \in (-\infty, -\theta] \\ \frac{-\overline{h_2(x)}}{|h_2(x)|+\theta e^{-x}}s, & \text{if } x \in [\theta, +\infty) \\ \varphi(x), & \text{if } x \in (-\theta, \theta). \end{cases} \tag{23}$$

where $\varphi : (-\theta, \theta) \to X$ is an affine function chosen such that f is continuous on $\mathbb{R}$.
Remark that $\|f\|_\infty \le 1$.

$$\widetilde{y}(x) = \sum_{k=1}^{n} C_k e^{r_k x} + \frac{1}{V} \sum_{k=1}^{n} (-1)^{n+k} V_k e^{r_k x} \int_0^x f(t) e^{-r_k t} dt \tag{24}$$

with the constants

$$\begin{aligned} C_k &= -(-1)^{n+k} \frac{V_k}{V} \int_0^{\infty} f(t) e^{-r_k t} dt, \quad 1 \le k \le p. \\ C_k &= (-1)^{n+k} \frac{V_k}{V} \int_{-\infty}^{0} f(t) e^{-r_k t} dt, \quad p+1 \le k \le n. \end{aligned}$$

Consequently,

$$\widetilde{y}(x) = \frac{(-1)^n}{V} \int_0^{\infty} \left(\left(\sum_{k=p+1}^{n} (-1)^k V_k e^{r_k u} \right) f(x-u) - \left(\sum_{k=1}^{p} (-1)^k V_k e^{-r_k u} \right) f(x+u) \right) du.$$

Since f is bounded, taking account of the sign of $\operatorname{Re} r_k$, $1 \le k \le n$, it follows that $\widetilde{y}(x)$ is bounded. The relation $\|D(y)\|_{\infty} = \|f\|_{\infty} \le 1$ and the stability of D for $\varepsilon = 1$ with the Ulam constant K leads to the existence of an exact solution $y_H \in \ker D$ given by

$$y_H(x) = \sum_{k=1}^{n} \widetilde{C}_k e^{r_k x}, \quad x \in \mathbb{R},$$

such that

$$\|\widetilde{y} - y_H\|_{\infty} \le K. \tag{25}$$

For $(\widetilde{C_1}, \widetilde{C_2}, \dots, \widetilde{C_n}) \ne (0, 0, \dots, 0)$, the solution y_H is unbounded; therefore, the relation (25) is true only for $y_H(x) = 0$, $x \in \mathbb{R}$. Consequently, relation (25) becomes

$$\|\widetilde{y}(x)\| \le K, \quad x \in \mathbb{R}. \tag{26}$$

For $x = 0$, we have $\|\widetilde{y}(0)\| \le K$. However,

$$\begin{aligned} \widetilde{y}(0) &= \frac{(-1)^n}{V} \int_0^{\infty} \Big(h_1(u) f(-u) - h_2(u) f(u) \Big) du \\ &= \frac{(-1)^n}{V} \left\{ \int_{\theta}^{\infty} h_1(u) f(-u) du - \int_{\theta}^{\infty} h_2(u) f(u) du \right\} \\ &+ \frac{(-1)^n}{V} \int_0^{\theta} \Big(h_1(u) f(-u) - h_2(u) f(u) \Big) du \\ &= \frac{(-1)^n}{V} \left\{ \int_{\theta}^{\infty} \frac{|h_1(u)|^2}{|h_1(u)| + \theta e^{-u}} du + \int_{\theta}^{\infty} \frac{|h_2(u)|^2}{|h_2(u)| + \theta e^{-u}} du \right\} \\ &+ \frac{(-1)^n}{V} \int_0^{\theta} \Big(h_1(u) f(-u) - h_2(u) f(u) \Big) du. \end{aligned}$$

Analogous to the previous cases, it can be proven that if $\theta \to 0$, then

$$\begin{aligned} \int_{\theta}^{\infty} \frac{|h_1(u)|^2}{|h_1(u)| + \theta e^{-u}} du &\longmapsto \int_0^{\infty} |h_1(u)| du \\ \int_{\theta}^{\infty} \frac{|h_2(u)|^2}{|h_2(u)| + \theta e^{-u}} du &\longmapsto \int_0^{\infty} |h_2(u)| du, \end{aligned}$$

and

$$\int_0^{\theta} \Big(h_1(u) f(-u) - h_2(u) f(u) \Big) du \longmapsto 0,$$

in view of the relation

$$\|f(u)\| = \|\varphi(u)\| \leq 1, \quad u \in [-\theta, \theta].$$

Hence, letting $\theta \to 0$ in (26), we have $K_D \leq K$, which is a contradiction.
□

Theorem 3. *If r_k, $1 \leq k \leq n$, are real and distinct roots of the characteristic equation and $a_n \neq 0$, then the best Ulam constant of the operator D is*

$$K_D = \frac{1}{|\prod_{k=1}^{n} r_k|} = \frac{1}{|a_n|}. \tag{27}$$

Proof. In [16], it is proven that D is Ulam stable with the Ulam constant $K_D = \frac{1}{|\prod_{k=1}^{n} r_k|}$. We show further that this is also the best Ulam constant of the operator D. Suppose that D admits a Ulam constant $K < K_D$. Let $\varepsilon > 0$ and

$$\widetilde{y}(x) = \frac{\varepsilon}{a_n}, \quad x \in \mathbb{R}.$$

Then, $\|D(\widetilde{y})\|_\infty = \varepsilon$ and since D is Ulam stable with the constant K, it follows that there exists $y_H \in \ker D$ such that

$$\|\widetilde{y} - y_H\|_\infty \leq K\varepsilon. \tag{28}$$

Clearly, if y_H is not identically $0 \in X$, then it is unbounded so relation (28) cannot hold. Therefore, $y_H(x) = 0$ for all $x \in \mathbb{R}$ and relation (28) becomes $\|\widetilde{y}\|_\infty \leq K\varepsilon$, or $K_D \leq K$, which is a contradiction. □

The previous results lead to the following identity.

Proposition 1. *If r_k, $1 \leq k \leq n$, are real distinct, nonzero numbers then*

$$\frac{1}{|r_1 r_2 \cdots r_n|} = K_D, \tag{29}$$

where K_D is given by (12).

Proof. For real and distinct roots r_k, $1 \leq k \leq n$, of the characteristic equation, the best Ulam constant is given on one hand by relation (12), Theorem 2, and on the other hand by relation (27) in Theorem 3. □

Next, we obtain as well an explicit representation of the best Ulam constant for the case of complex and distinct roots of the characteristic equation having the same imaginary part.

Theorem 4. *If the characteristic equation of D admits outside of the imaginary axis distinct roots having the same imaginary part, then the best Ulam constant of D is given by*

$$K_D = \frac{1}{\prod_{k=1}^{n} |\operatorname{Re} r_k|}. \tag{30}$$

Proof. Suppose that $r_k = \rho_k + i\alpha$, $\rho_k \in \mathbb{R} \setminus \{0\}$, $1 \leq k \leq n$, $\alpha \in \mathbb{R}$. Then, the Vandermonde determinants become

$$V(r_1, r_2, \ldots, r_n) = \prod_{1 \leq k < j \leq n} (r_j - r_k) = \prod_{1 \leq k < j \leq n} (\rho_j + i\alpha - \rho_k - i\alpha) = V(\rho_1, \rho_2, \ldots, \rho_n)$$

and analogously

$$V_k(r_1, r_2, \dots, r_{k-1}, r_{k+1}, \dots, r_n) = V_k(\rho_1, \rho_2, \dots, \rho_{k-1}, \rho_{k+1}, \dots, \rho_n), \quad 1 \le k \le n.$$

On the other hand, for $\operatorname{Re} r_k > 0, 1 \le k \le n$, we have

$$\begin{aligned} |\sum_{k=1}^{n}(-1)^k V_k e^{-r_k x}| &= |\sum_{k=1}^{n}(-1)^k V_k e^{-\rho_k x} e^{-i\alpha}| \\ &= |e^{-i\alpha x}||\sum_{k=1}^{n}(-1)^k V_k e^{-\rho_k x}| = |\sum_{k=1}^{n}(-1)^k V_k e^{-\rho_k x}| \end{aligned}$$

and analogously for the other expressions in (10). Consequently, in view Theorem 2 and Proposition 1, the best Ulam constant of D becomes

$$K_D = \frac{1}{\prod_{k=1}^{n} |\rho_k|}.$$

□

Theorem 2 is an extension of the result given in [20] for distinct roots of the characteristic equation. Indeed, the particular case $n = 2$ corresponds to the second-order linear differential operator given by

$$D(y) = y'' + a_1 y' + a_2 y, \quad a_1, a_2 \in \mathbb{C}, \tag{31}$$

and the best Ulam constant in this case is

$$K_D = \begin{cases} \frac{1}{|r_1 - r_2|} \int_0^\infty |e^{-r_1 x} - e^{-r_2 x}| dx, & \text{if } \operatorname{Re} r_1 > 0, \operatorname{Re} r_2 > 0, \\ \frac{1}{|r_1 - r_2|} \int_0^\infty |e^{r_1 x} - e^{r_2 x}| dx, & \text{if } \operatorname{Re} r_1 < 0, \operatorname{Re} r_2 < 0, \\ \frac{1}{|r_1 - r_2|} \left| \frac{1}{\operatorname{Re} r_1} - \frac{1}{\operatorname{Re} r_2} \right|, & \text{if } \operatorname{Re} r_1 \cdot \operatorname{Re} r_2 < 0 \end{cases} \tag{32}$$

An explicit representation of K_D for the second-order linear differential operator with real coefficients is given in the next theorem.

Theorem 5. *If $D(y) = y'' + a_1 y' + a_2 y$, $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, then the best Ulam constant of the operator is*

$$K_D = \begin{cases} \frac{1}{|a_2|}, & \text{if } a_1^2 - 4a_2 \ge 0, \\ \frac{1}{a_2} \coth \frac{|a_1|\pi}{2\sqrt{4a_2 - a_1^2}}, & \text{if } a_1^2 - 4a_2 < 0. \end{cases} \tag{33}$$

Proof. Let $\delta = a_1^2 - 4a_2$.

(i) If $\delta \ge 0$, then $r_1, r_2 \in \mathbb{R}$ and, in view of [20] (Theorem 3) and Vieta's formulas,

$$K_D = \frac{1}{|r_1 r_2|} = \frac{1}{|a_2|}.$$

(ii) If $\delta < 0$, then $r_{1,2} = \alpha \pm i\beta$, $\alpha, \beta \in \mathbb{R}$, $\beta \ne 0$. Suppose first $\alpha > 0$. Then,

$$\begin{aligned} K_D &= \frac{1}{2|\beta|} \int_0^\infty e^{-\alpha x} |e^{-i\beta x} - e^{i\beta x}| dx \\ &= \frac{1}{2|\beta|} \int_0^\infty e^{-\alpha x} | - 2i \sin \beta x| dx = \frac{1}{|\beta|} \int_0^\infty e^{-\alpha x} |\sin(|\beta| x)| dx. \end{aligned}$$

Now, letting $|\beta|x = t$ in the above integral, taking account

$$\int_0^\infty e^{-px}|\sin x|dx = \frac{1}{1+p^2}\coth\frac{p\pi}{2}, \quad p > 0,$$

we obtain

$$K_D = \frac{1}{\beta^2}\int_0^\infty e^{-\frac{\alpha}{|\beta|}t}|\sin t|dt = \frac{1}{\alpha^2+\beta^2}\coth\frac{\alpha}{2|\beta|}\pi = \frac{1}{a_2}\coth\frac{|a_1|\pi}{2\sqrt{4a_2 - a_1^2}}.$$

We can prove this analogously for $\alpha < 0$.

□

3. Conclusions

In this paper, we obtain the best Ulam constant for an n-order linear differential operator with constant coefficients acting in a Banach space for the case of distinct roots of the characteristic equation. This result gives an optimal evaluation of the difference between an approximate solution and an exact solution of the equation associated to the differential operator. Consequently, these results can be applied in the study of perturbations of a dynamical systems governed by differential equations and in some branches of science as engineering, mechanics, and economy.

It will be interesting to obtain a closed-form (if possible) for the best Ulam constant of the n-order differential operator for the case of multiple roots of the characteristic equation.

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References

1. Ulam, S.M. *A Collection of Mathematical Problems*; Interscience: New York, NY, USA, 1960.
2. Hyers, D.H. On the stability of the linear functional equation. *Proc. Natl. Acad. Sci. USA* **1941**, *27*, 222. [CrossRef] [PubMed]
3. Pölya, G.; Szegö, G. *Problems and Theorems in Analysis*; Springer: Berlin/Heidelberg, Germany, 1972.
4. Brzdek, J.; Popa, D.; Rasa, I.; Xu, B. *Ulam Stability of Operators*; Academic Press: Cambridge, MA, USA, 2018.
5. Hyers, D.H.; Isac, G.; Rassias, T. *Stability of Functional Equations in Several Variables*; J. Birkhäuser Boston, Inc.: Boston, MA, USA, 1998.
6. Hatori, O.; Kobayashi, K.; Miura, T.; Takagi, H.; Takahasi, S.E. On the best constant o Hyers-Ulam stability. *J. Nonlinear Convex Anal.* **2004**, *5*, 387–393.
7. Popa, D.; Rasa, I. Hyers-Ulam stability of the linear differential operator with nonconstant coeffcients. *Appl. Math. Comput.* **2012**, *219*, 1562–1568.
8. Takahasi, S.E.; Takagi, H.; Miura, T.; Miyajima, S. The Hyers-Ulam stability constants of first order linear differential operators. *J. Math. Anal. Appl.* **2004**, *296*, 403–409. [CrossRef]
9. Popa, D.; Rasa, I. On the Hyers-Ulam stability of the linear differential equation. *J. Math. Anal. Appl.* **2011**, *381*, 530–537. [CrossRef]
10. Popa, D.; Rasa, I. On the stability of some classical operators from approximation theory. *Expo. Math.* **2013**, *31*, 205–214. [CrossRef]
11. Popa, D.; Rasa, I. Best constant in stability of some positive linear operators. *Aequ. Math.* **2016**, *90*, 719–726. [CrossRef]
12. Popa, D.; Rasa, I. On the best constant in Hyers-Ulam stability of some positive linear operators. *J. Math. Anal. Appl.* **2014**, *412*, 103–108. [CrossRef]
13. Brzdek, J.; Popa, D.; Rasa, I. Hyers-Ulam stability with respect to gauges. *J. Math. Anal. Appl.* **2017**, *453*, 620–628. [CrossRef]
14. Obloza, M. Hyers stability of the linear differential equation. *Rocznik Nauk.-Dydakt. Prace Mat.* **1993**, *13*, 259–270.

15. Miura, T.; Miyajima, S.; Takahasi, S.E. A characterization of Hyers-Ulam stability of first order linear differential operators. *J. Math. Anal. Appl.* **2003**, **286**, 136–146. [CrossRef]
16. Miura, T.; Miyajima, S.; Takahasi, S.E. Hyers-Ulam stability of linear differential operator with constant coeffcients. *Math. Nachrichten* **2003**, *258*, 90–96. [CrossRef]
17. Takagi, H.; Miura, T.; Takahasi, S.E. Essential norms and stability constants of weighted composition operators on C(X). *Bull. Korean Math. Soc.* **2003**, *40*, 583–591. [CrossRef]
18. Jung, S.M. Ulam stability of a system of first order linear differential equations with constant coefficients. *J. Math. Anal. Appl.* **2006**, *320*, 549–561. [CrossRef]
19. Rassias, T.M.; Tabor, J. What is left of Hyers-Ulam stability? *J. Geom.* **1992**, *1*, 65–69.
20. Baias, A.R.; Popa, D. On the best Ulam constant of the second order linear differential operator. *Rev. De La Real Acad. De Cienc. Exactas Físicas Y Naturales. Ser. A. Matemáticas* **2020**, *114*, 23. [CrossRef]
21. Fukutaka, R.; Onitsuka, M. Best constant in Hyers–Ulam stability of first-order homogeneous linear differential equations with a periodic coefficient. *J. Math. Anal. Appl.* **2019**, *473*, 1432–1446. [CrossRef]
22. Fukutaka, R.; Onitsuka, M. Best constant for Ulam stability of Hill's equations. *Bull. Des Sci. Math.* **2020**, *163*, 102888. [CrossRef]
23. Baias, A.R.; Popa, D. On the best Ulam constant of a higher order linear difference equation. *Bull. Des Sci. Math.* **2021**, *166*, 102928. [CrossRef]