mathematics

Article

Non-Markovian Inverse Hawkes Processes

Youngsoo Seol

Department of Mathematics, Dong-A University, Busan 49315, Korea; prosul76@dau.ac.kr

Abstract: Hawkes processes are a class of self-exciting point processes with a clustering effect whose jump rate is determined by its past history. They are generally regarded as continuous-time processes and have been widely applied in a number of fields, such as insurance, finance, queueing, and statistics. The Hawkes model is generally non-Markovian because its future development depends on the timing of past events. However, it can be Markovian under certain circumstances. If the exciting function is an exponential function or a sum of exponential functions, the model can be Markovian with a generator of the model. In contrast to the general Hawkes processes, the inverse Hawkes process has some specific features and self-excitation indicates severity. Inverse Markovian Hawkes processes were introduced by Seol, who studied some asymptotic behaviors. An extended version of inverse Markovian Hawkes processes was also studied by Seol. With this paper, we propose a non-Markovian inverse Hawkes process, which is a more general inverse Hawkes process that features several existing models of self-exciting processes. In particular, we established both the law of large numbers (LLN) and Central limit theorems (CLT) for a newly considered non-Markovian inverse Hawkes process.

Keywords: Hawkes process; non-Markovian inverse Hawkes process; self-exciting point processes; central limit theorems; law of large numbers

MSC: 60G55; 60F05; 60F10

check for updates

Citation: Seol, Y. Non-Markovian Inverse Hawkes Processes. *Mathematics* **2022**, *10*, 1413. https://doi.org/10.3390/math10091413

Academic Editor: Vasile Preda

Received: 28 March 2022
Accepted: 21 April 2022
Published: 22 April 2022

1. Introduction

Hawkes processes [1] are widely used and useful models of simple point processes. They are self-exciting and exhibit clustering effects. The intensity process for a point process is composed of the summation of the baseline intensity plus other terms that depend upon the history of the whole past of the point process in comparison with a standard Poisson process. In applications, the Hawkes process is typically used as an expressive model for temporal phenomena of the stochastic process which evolve in continuous time, such as in modeling high-frequency trading. The Hawkes process is a natural generalization of the Poisson process and captures both the self-exciting property and clustering effect. This process is a very variable model which is amenable to statistical analysis. Therefore, it has wide applications in insurance [2], neuroscience [3], criminology [4], seismology [5], DNA modeling [6], and finance [7]. In general, the self-exciting and clustering characteristics of the Hawkes process make it highly desirable for computations in financial applications [8], such as in modeling the associated defaults and evaluating derivatives of the credit in finance [7,9]. There are many situations which require time-dependent frameworks when it comes to model adjustment. The Hawkes process generally can be categorized by the linear and nonlinear cases of Hawkes processes based on the intensity. Hawkes [1] introduced the linear process which can be studied via an immigration-birth representation [10]. Most applications for the Hawkes process consider exclusively the linear case. The stability [11], law of large numbers (LLN) [12], Bartlett spectrum [13], central limit theorem (CLT) [14], and large deviation principles (LDP) [15] have all been studied and are understood very well. The nonlinear Hawkes process is much less studied, mainly due to the deficiency of the immigration-birth representation and computational tractability, although some efforts in this direction have been made. The first nonlinear case was studied by Brémaud and Massoulié [16]. Recently, Zhu [2,17–20] investigated several results for both the linear and nonlinear models. The central limit theorem for the nonlinear model was investigated by Zhu [19] and the large deviation principles were obtained by Zhu [18]. Jaisson and Rosenbaum [21,22] studied some limit theorems and rough fractional diffusions as scaling limits of nearly unstable Hawkes processes. Zhu [2] also studied the applications of the Hawkes process in financial mathematics. Some variations and extensions of the Hawkes process were studied by Dassios and Zhao [9], Ferro, Leiva, and Moller [23], Karabash and Zhu [24], Mehrdad and Zhu [25], and Zhu [26]. Seol [27] considered the arrival time τ_n and the inverse of the Hawkes process and studied the limit theorems for τ_n. Recently, data-driven models are gaining attention due to the development of storage technology. In contrast to the continuous-time scheme, in the real world, events are often recorded in a discrete-time scheme. It is more important that the data are collected in a fixed phase or the data only show the aggregate results. For example, continuous-time Hawkes models can be spaced unevenly in time, whereas a discrete-time Hawkes model can be spaced evenly in time, and thus, a discrete-time Hawkes process has wide application in many fields. Seol [28] proposed a 0–1 discrete Hawkes process starting from empty history and proved some limit behaviors, such as the law of large numbers (LLN), invariance principles, and central limit theorem (CLT). Recently, Wang [29,30] studied limit behaviors of a discrete-time Hawkes process with random marks, and furthermore, proved the large and moderate deviations for a discrete-time Hawkes process with marks. Seol [31,32] has studied the moderate deviation principle of marked Hawkes processes and also studied asymptotic behaviors for the compensator processes of Hawkes models. Furthermore, Gao and Zhu [33–36] made some progress in the direction of limit behaviors other than the large time scale limits. Studies have also been reported on modifying and extending the classical Hawkes process. Firstly, the intensity of the baseline was given by time-inhomogeneous data (see [37]). Secondly, the immigrant values can be obtained by a Cox process with shot noise intensity, which was known for the dynamic contagion model (see [9]). Thirdly, the immigrants can be conditioned on the renewal process instead of the Poisson process, which generalizes the classical Hawkes process. This is known as the renewal Hawkes process (see [38]). Recently, Seol [39] showed that the inverse case of Markovian Hawkes processes can be represented by several existing models of self-exciting processes, and studied the asymptotic behaviors of the inverse Markovian Hawkes processes. Seol [40] further studied the extended version of the inverse case of the Markovian Hawkes model. In contrast to the general Hawkes processes, the inverse Hawkes process has some specific features and important applications to several financial models, such as a shot-noise process and a jump-diffusion process with no diffusions. In the current paper, we considered a non-Markovian inverse Hawkes process which comprised the remarkable properties of several self-exciting point processes. We also studied the limit theorems of a non-Markovian inverse Hawkes process. This paper has been organized into two parts. The general review of the Hawkes process and the statement of the main theorems are reported in Section 1. The proofs of the main theorems with some auxiliary results have been provided in Section 2.

1.1. The General Hawkes Process

In this section, we formally introduce the general Hawkes process, which was introduced by Brémaud and Massoulié [16].

Let $\mathcal{B}(\mathbb{R})$ be a Borel $\sigma-$algebra and $\Upsilon_t^{-\infty} := \sigma(\mathbb{N}(C), C \subset (-\infty, t], C \in \mathcal{B}(\mathbb{R}))$ be an increasing function of the family of $\sigma-$algebras with $\mathbb{N}$ is a simple point process on $\mathbb{R}$. Any nonnegative $\Upsilon_t^{-\infty}$-progressively measurable process λ_t with:

$$E\left[\mathbb{N}(a,b]|\Upsilon_a^{-\infty}\right] = E\left[\int_a^b \lambda_s ds|\Upsilon_a^{-\infty}\right]$$

as for all intervals, $(a,b]$ is called an $\Upsilon_t^{-\infty}$-intensity of $\mathbb{N}$. We use the notation $\mathbb{N}_t := \mathbb{N}(0,t]$ to present the number of points in the interval $(0,t]$. The general definition of Hawkes process is a simple point process $\mathbb{N}$ admitting an $\Upsilon_t^{-\infty}$-intensity:

$$\lambda_t := \lambda\left(\int_{-\infty}^{t} h(t-s)\mathbb{N}(ds)\right),$$

where $\lambda(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is left continuous and locally integrable, $h(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ with the condition $\|h\|_{L^1} = \int_0^\infty h(t)dt < \infty$. In the literature, $\lambda(\cdot)$ and $h(\cdot)$ are usually referred to as the rate function and exciting function, respectively. The assumption of the local integrability of $\lambda(\cdot)$ makes sure that the process is non-explosive, while the left continuity assumption makes sure that λ_t is Υ_t-predictable. The Hawkes process is generally non-Markovian because the future development of a self-exciting point process is determined by timing of the past events, whereas it is Markovian as a special case. If the exciting function h is an exponential function or a sum of function of exponentials, the process is Markovian with a generator of the process. However, the difficulty arises when h is neither an exponential function nor a sum of function of exponentials, in which case the process becomes non-Markovian. When $h(t) = pe^{-qt}$, the structure of the Hawkes process is Markovian in the manner that $Z_t = \int_{-\infty}^{t-} pe^{-q(t-s)} d\mathbb{N}_s$ is Markovian, satisfying the dynamics:

$$dZ_t = -qZ_t dt + pd\mathbb{N}_t,$$

where $\mathbb{N}_t$ has the intensity $\nu + Z_{t-}$ at time t and the Z_t process has the infinitesimal generator:

$$\Gamma f(z) = -qzf'(z) + (\nu + z)[f(z+p) - f(z)].$$

It is well known (see [36]) that:

$$\frac{1}{t}\int_0^t Z_s ds \to \frac{\nu}{q-p},$$

and:

$$\frac{1}{\sqrt{t}}\left[\int_0^t Z_s ds - \frac{\nu}{q-p}\cdot t\right] \to N\left(0, \frac{p^2\nu q}{(q-p)^3}\right),$$

in distribution as $t \to \infty$.

The Hawkes processes can generally be classified as linear and nonlinear case models based on the intensity $\lambda(\cdot)$. When $\lambda(\cdot)$ is linear, we call the process a linear Hawkes process, and furthermore, for $\lambda(l) = \nu + l$ for some $\nu > 0$, and $\|h\|_{L^1} < 1$, we can use a useful method, i.e., the immigration-birth representation, also known as the Galton-Watson theory. The limit results are well understood and more explicitly represented. The limit behaviors of the linear Hawkes processes with marks were reported by Zhu [25]. Daley and Vere-Jones [12] investigated the law of large numbers (LLN) of the linear case model as shown in the following equation:

$$\frac{N_t}{t} \to \frac{\nu}{1-\|h\|_{L^1}} \text{ as } t \to \infty.$$

The functional central limit theorem (FCLT) of a linear multivariate Hawkes model under certain assumptions has been investigated by Bacry et al. [14] and the results are given by:

$$\frac{N_{\cdot t} - \cdot\mu t}{\sqrt{t}} \to \sigma B(\cdot), \text{ as } t \to \infty,$$

where $B(\cdot)$ is the standard Brownian motion and the convergence used in [14] is weak convergence on $D[0,1]$, and the space of càdlàg function on $[0,1]$ is equipped with Skorokhod topology. Here:

$$\mu = \frac{\nu}{1 - \|h\|_{L^1}} \text{ and } \sigma^2 = \frac{\nu}{(1 - \|h\|_{L^1})^3}.$$

Bordenave and Torrisi [15] showed that under the conditions $0 < \|h\|_{L^1} < 1$ and $\int_0^\infty th(t)dt < \infty$, $\mathbb{P}(\frac{N_t}{t} \in \cdot)$ satisfies the large deviation principle(LDP) with a good rate function $I(\cdot)$, where $\theta = \theta_x$ is the unique solution in $(-\infty, \|h\|_{L^1} - 1 - \log \|h\|_{L^1})$, of:

$$\mathbb{E}(e^{\theta D}) = \frac{x}{\nu + x\|h\|_{L^1}}, \quad x > 0$$

where D in the equation denotes the total number of descendants of an immigrant, including the immigrant themself. Zhu [20] showed that under the conditions $\|h\|_{L^1} < 1$ and $\sup_{t>0} t^{3/2} h(t) \leq C < \infty$, for any Borel set β and time sequence $\sqrt{n} \ll \kappa(n) \ll n$, there exist a moderate deviation principle:

$$-\inf_{x \in \beta^\circ} L(x) \leq \liminf_{t \to \infty} \frac{t}{\kappa(t)^2} \log \mathbb{P}\left(\frac{1}{\kappa(t)}(N_t - \mu t) \in \beta\right)$$
$$\leq \limsup_{t \to \infty} \frac{t}{\kappa(t)^2} \log \mathbb{P}\left(\frac{1}{\kappa(t)}(N_t - \mu t) \in \beta\right) \leq -\inf_{x \in \bar{\beta}} L(x)$$

where $L(x) = \frac{x^2(1-\|h\|_{L^1})^3}{2\nu}$.

When $\lambda(\cdot)$ is nonlinear, we call the process a nonlinear Hawkes process, and the general Galton-Watson theory cannot be applied. The nonlinear model is much more challenging to study because of the lack of immigration-birth representation with computational tractability. For nonlinear Hawkes processes, Brémaud and Massoulié [16] provided a stationary solution with convergence to equilibrium in a nonstationary version under certain conditions. Furthermore, Massoulié [41] showed that the stability of nonlinear Hawkes processes with random marks can also be extended to the Markovian case. The author also proved stability without the Lipschitz condition for $\lambda(\cdot)$. Furthermore, Brémaud [16] discussed the nonlinear case of the Hawkes process in terms of the rate of extinction. In [19], Zhu showed a functional central limit theorem (FCLT) for the nonlinear case of the Hawkes process. The large deviation principle has also been proved by Zhu [17] with a special case of the Hawkes process when $h(\cdot)$ is an exponential function or a sum of exponential functions. For a general $h(\cdot)$, Zhu [18] provided a large deviation principle based on a level-3 nonlinear Hawkes process.

1.2. Inverse Markovian Hawkes Process

In a recent paper by Seol [39], an inverse version of the Markovianvian Hawkes process was developed and studied. In contrast to the general Hawkes processes, the new model has some specific features. Hawkes processes predict that if someone has made more jumps in the past, they will make more jumps in the future. The inverse Hawkes process reveals that the greater the jumps in the past, the larger the jumps in the future. Unlike the general Hawkes process, the inverse Hawkes process is self-excited by the jump size, but the self-excitation is generated by the intensity of the general Hawkes process. Specifically, self-excitation indicates frequency in the general Hawkes process, but severity in the inverse version of the Markovian Hawkes process. Inverse Markovian Hawkes process can be represented by a combination of existing models of the self-exciting process, which means that if $p = 0$, then Z_t can be expressed as a shot-noise process, such as $Z_t = Z_0 e^{-qt} + \int_0^t \nu e^{-q(t-s)} dN_s$, and if $\nu = 0$, then it can be represented by a jump-diffusion process with no diffusions, such as the following model $Z_t = Z_0 \exp(-qt + \log(1+p)N_t)$.

Seol [39] first proposed an inverse version of the Markovian Hawkes process, which was defined as:

$$dZ_t = -qZ_t dt + (\nu + pZ_{t-})dN_t,$$

where N_t is Poisson with intensity 1 and $p > 0$, $q > 0$, and $\nu > 0$ and it follows that:

$$d(e^{qt}Z_t) = (pZ_{t-} + \nu)e^{qt}dN_t,$$

and since we assumed $Z_0 = 0$, we get:

$$Z_t = \int_0^t (pZ_{s-} + \nu)e^{-q(t-s)}dN_s.$$

The Z_t process has the infinitesimal generator:

$$\Gamma f(z) = -qzf'(z) + f(z + pz + \nu) - f(z).$$

Under certain assumptions, Seol [39] investigated:

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t Z_s ds = \frac{\nu}{q-p},$$

and:

$$\frac{1}{\sqrt{t}}\left[\int_0^t Z_s ds - \frac{\nu}{q-p}\cdot t\right] \to N\left(0, \frac{\nu^2 + 2\nu p\frac{\nu}{q-p} + p^2\frac{\nu^2(p+q+2)}{(q-p)(2q-2p-p^2)}}{(q-p)^2}\right),$$

in distribution as $t \to \infty$. Furthermore, Seol [40] introduced a model combining the Hawkes process and inverse Hawkes process, which is an extended version of the inverse Markovian Hawkes process. The extended model can be defined as

$$dZ_t = -qZ_t dt + p_1 dN_t^{(1)} + (\nu_2 + p_2 Z_{t-})dN_t^{(2)},$$

where $N_t^{(1)}$ is a simple point process with intensity $\nu_1 + Z_{t-}$ at time t and $N_t^{(2)}$ is a Poisson process with intensity 1, where q, p_1, p_2, ν_1, and ν_2 are all positive constants. The infinitesimal generator of Z_t process is given by:

$$\Gamma f(z) = -qzf'(z) + (\nu_1 + z)[f(z + p_1) - f(z)] + f(z + \nu_2 + p_2 z) - f(z).$$

Under certain assumptions, Seol [40] investigated:

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t Z_s ds = \frac{\nu_1 p_1 + \nu_2}{q - p_1 + p_2},$$

as $t \to \infty$, and:

$$\frac{1}{\sqrt{t}}\left[\int_0^t Z_s ds - \frac{\nu_1 p_1 + \nu_2}{q - p_1 + p_2}\cdot t\right] \to N\left(0, \sigma^2\right),$$

in distribution as $t \to \infty$, where:

$$\begin{aligned}\sigma^2 :&= \frac{1}{(q-p_1-p_2)^2}\left[p_1^2(\nu_1 + \mathbb{E}[Z_\infty]) + \mathbb{E}[(\nu_1 + p_2 Z_\infty)^2]\right] \\ &= \frac{(p_1^2\nu_1 + \nu_1^2)K_1K_2 + (p_1^2\nu_1 + 2\nu_1 p_2)K_2K_3 + p_2^2(K_3K_5 + K_1K_4)}{K_1^3K_2},\end{aligned}$$

and $K_i, i \in 1,2,3,4,5$ are constants and:

$$\begin{aligned}
K_1 &= q - p_1 + p_2, \\
K_2 &= 2q - 2p_1 - p_2 - p_2^2, \\
K_3 &= \nu_1 p_1 + \nu_2, \\
K_4 &= \nu_1 p_1^2 + \nu_2^2, \\
K_5 &= p_1^2 + 2\nu_1 p_1 + 2\nu_2 p_2 + 2\nu_2.
\end{aligned}$$

1.3. Main Results of This Paper

We now give the statement of the main parts. We will investigate asymptotic results of the Hawkes process with a non-Markovian inverse structure that extends the Markovian inverse Hawkes process. Our results mainly consist of both the central limit theorems (CLT) and the law of large numbers (LLN).

We developed a new model which is a non-Markovian inverse Hawkes process. It is very natural to define the general inverse Hawkes process as follows:

$$Z_t = \int_0^t (pZ_{s-} + \nu) h(t-s) dN_s, \tag{1}$$

where $h(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is integrable, i.e., $\|h\|_{L^1} = \int_0^\infty h(t)dt < \infty$. We used the assumptions stated below throughout the paper.

Assumption 1. *(1)* $N(-\infty, 0] = 0$, *which means that the Hawkes model has empty history,*
(2) $\frac{1}{\sqrt{t}} \int_0^t \int_t^\infty h(s-u) ds du \to 0$ *as* $t \to \infty$,
(3) $\int_0^\infty h(t) t^{1/2} dt < \infty$.

The first asymptotic result is a law of large number for our considered model.

Theorem 1. *Let* Z_t *be defined as* (1). *Under Assumption 1, we have:*

$$\frac{1}{t} \int_0^t Z_s ds \to \frac{\nu \|h\|_{L^1}}{1 - p\|h\|_{L^1}}, \tag{2}$$

in probability as $t \to \infty$.

when $h(t) = e^{-qt}$, it recovers the Markovian case, and:

$$\frac{1}{t} \int_0^t Z_s ds \to \frac{\nu \|h\|_{L^1}}{1 - p\|h\|_{L^1}} = \frac{\nu}{q - p}, \tag{3}$$

in probability as $t \to \infty$. The second asymptotic result is the central limit theorem.

Theorem 2. *Let* Z_t *be defined as* (1). *Under the Assumption 1, we have:*

$$\frac{1}{\sqrt{t}} \left[\int_0^t Z_s ds - \frac{\nu \|h\|_{L^1}}{1 - p\|h\|_{L^1}} \cdot t \right] \to N\left(0, \sigma^2\right), \tag{4}$$

in distribution as $t \to \infty$, where:

$$\begin{aligned}\sigma^2 :&= \frac{\|h\|_{L^1}^2}{(1-p\|h\|_{L^1})^2}\mathbb{E}[(pZ_\infty+\nu)^2] \\ &= \frac{\|h\|_{L^1}^2}{(1-p\|h\|_{L^1})^2}\left[p^2\mathbb{E}[Z_\infty^2]+2p\nu\mathbb{E}[Z_\infty]+\nu^2\right] \\ &= \frac{\nu^2\|h\|_{L^1}^2}{(1-p\|h\|_{L^1})^2}\left[p^2\mathbb{E}\left[\left(\sum_{k=0}^{\infty}p^k h^{*(k+1),N}(\infty)\right)^2\right]+2p\frac{\|h\|_{L^1}}{1-p\|h\|_{L^1}}+1\right],\end{aligned}$$

where:

$$\begin{aligned}&h^{*(k+1,N)}(t) \\ &:= \int_{0<s_{k+1}<s_k<\cdots<s_1\le t} h(t-s_1)h(s_1-s_2)\cdots h(s_k-s_{k+1})dN_{s_{k+1}}dN_{s_k}\cdots dN_{s_1}.\end{aligned}$$

when $h(t)=e^{-qt}$, it recovers the Markovian case, and:

$$\frac{1}{\sqrt{t}}\left[\int_0^t Z_s ds - \frac{\nu}{q-p}\cdot t\right] \to N\left(0, \frac{\nu^2+2\nu p\frac{\nu}{q-p}+p^2\frac{\nu^2(p+q+2)}{(q-p)(2q-2p-p^2)}}{(q-p)^2}\right), \tag{5}$$

in distribution as $t \to \infty$.

2. Proofs of the Main Results

In the current section, we will give the proofs of our main theorems and the related auxiliary results. The following are the key results to prove the main results. The key result is devoted to the distributional properties of the non-Markovian inverse Hawkes processes. Both the first and second moments of Z_t have been computed in Section 2.1. The main theorems of the paper are validated in Sections 2.2 and 2.3.

2.1. Some Auxiliary Results

In this section, we will obtain closed formulae for the moments of Z_t. In particular, the first and second moments will be discussed.

Proposition 1. *Let Z_t be defined as (1). Under the Assumption 1, we have:*

(i)

$$\mathbb{E}[Z_\infty] = \frac{\nu\|h\|_{L^1}}{1-p\|h\|_{L^1}}. \tag{6}$$

(ii)

$$\mathbb{E}[Z_\infty^2] = \nu^2\mathbb{E}\left[\left(\sum_{k=0}^{\infty}p^k h^{*(k+1),N}(\infty)\right)^2\right], \tag{7}$$

provided it is finite, where:

$$\begin{aligned}&h^{*(k+1,N)}(t) \\ &:= \int_{0<s_{k+1}<s_k<\cdots<s_1\le t} h(t-s_1)h(s_1-s_2)\cdots h(s_k-s_{k+1})dN_{s_{k+1}}dN_{s_k}\cdots dN_{s_1}.\end{aligned}$$

Proof. (i) Note that:

$$Z_t = \int_0^t (pZ_{s-}+\nu)h(t-s)dN_s. \tag{8}$$

By taking expectations:

$$\mathbb{E}[Z_t] = \int_0^t (p\mathbb{E}[Z_s] + \nu)h(t-s)ds.$$

By taking $t \to \infty$ on both sides of the above renewal equation, we get:

$$\mathbb{E}[Z_\infty] = \frac{\nu\|h\|_{L^1}}{1 - p\|h\|_{L^1}}.$$

(ii) We recall again that:

$$Z_t = \int_0^t (pZ_{s-} + \nu)h(t-s)dN_s.$$

By solving this renewal type equation, we get:

$$Z_t = \nu \sum_{k=0}^{\infty} p^k h^{*(k+1),N}(t), \tag{9}$$

where:

$$h^{*(k+1,N)}(t) := \int_{0<s_{k+1}<s_k<\cdots<s_1\leq t} h(t-s_1)h(s_1-s_2)\cdots h(s_k-s_{k+1})dN_{s_{k+1}}dN_{s_k}\cdots dN_{s_1}.$$

Therefore, we have:

$$\mathbb{E}[Z_\infty^2] = \nu^2 \mathbb{E}\left[\left(\sum_{k=0}^{\infty} p^k h^{*(k+1),N}(\infty)\right)^2\right].$$

provided it is finite. □

2.2. Proof of the Law of Lagre Number

The followings are the proofs of the first main theorems.
From the definition of Z_t, we get:

$$\int_0^t Z_s ds = \int_0^t \int_0^s (pZ_{u-} + \nu)h(s-u)dN_u ds, \tag{10}$$

and from Fubini's theorem:

$$\begin{aligned}\int_0^t Z_s ds &= \int_0^t (pZ_{u-} + \nu)\int_0^{t-u} h(s)dsdN_u \\ &= \|h\|_{L^1}\int_0^t (pZ_{u-} + \nu)du + \|h\|_{L^1}\int_0^t (pZ_{u-} + \nu)dM_u + \mathcal{E}_t,\end{aligned}$$

where the error term $\mathcal{E}_t$ is defined as:

$$\begin{aligned}\mathcal{E}_t :&= \int_0^t (pZ_{u-} + \nu)\int_{t-u}^{\infty} h(s)dsdN_u \\ &= \int_0^t (pZ_{u-} + \nu)\int_u^{\infty} h(s)dsdN_u\end{aligned}$$

and $M_t = N_t - t$ is a martingale. We can compute that:

$$\mathbb{E}[\mathcal{E}_t] = \int_0^t (p\mathbb{E}[Z_{u-}] + \nu)\int_u^{\infty} h(s)dsdu.$$

Since Z_u is uniform integrable and h is integrable:

$$\frac{1}{t}\mathbb{E}[\mathcal{E}_t] = \frac{1}{t}\int_0^t (p\mathbb{E}[Z_{u-}] + \nu)\int_u^\infty h(s)dsdu \to 0$$

as $t \to \infty$. Thus:

$$\frac{1}{t}\mathcal{E}_t \to 0,$$

in probability. Let:

$$G_t := \frac{1}{t}\int_0^t (pZ_{u-} + \nu)dM_u.$$

Then:

$$\mathbb{E}[G_t^2] = \frac{1}{t^2}\int_0^t \mathbb{E}[(pZ_{u-} + \nu)^2]du \to 0$$

as $t \to \infty$, since Z_u^2 is uniform integrable. Thus:

$$G_t := \frac{1}{t}\int_0^t (pZ_{u-} + \nu)dM_u \to 0,$$

in probability. Therefore, we get:

$$\frac{1}{t}\int_0^t Z_s ds \to \frac{\nu\|h\|_{L^1}}{1 - p\|h\|_{L^1}},$$

in probability as $t \to \infty$.

This completes the proof of Theorem 1.

2.3. Proof of Central Limit Theorem

In the current section, we will give the proofs of the second main theorem. Let us recall that the error term $\mathcal{E}_t$ is defined as:

$$\begin{aligned}\mathcal{E}_t : &= \int_0^t (pZ_{u-} + \nu)\int_{t-u}^\infty h(s)dsdN_u \\ &= \int_0^t (pZ_{u-} + \nu)\int_u^\infty h(s)dsdN_u\end{aligned}$$

and $M_t = N_t - t$ is a martingale. We can compute that:

$$\mathbb{E}[\mathcal{E}_t] = \int_0^t (p\mathbb{E}[Z_{u-}] + \nu)\int_u^\infty h(s)dsdu.$$

From Assumption 1, we get:

$$\frac{\mathbb{E}(\mathcal{E}_t)}{\sqrt{t}} \to 0 \text{ as } t \to \infty. \tag{11}$$

Thus:

$$\frac{1}{\sqrt{t}}\mathcal{E}_t \to 0,$$

in probability. Applying the central limit theorem for the martingales properties (see Theorem VIII-3.11 of [42] for details):

$$\frac{1}{\sqrt{t}}\int_0^t (pZ_{u-}+\nu)dM_u \to N\Big(0,\mathbb{E}[(pZ_\infty+\nu)^2]\Big), \tag{12}$$

in distribution as $t\to\infty$.

Hence, using the results for $\mathbb{E}[Z_\infty]$ and $\mathbb{E}[Z_\infty^2]$ in Proposition 1, we get:

$$\frac{1}{\sqrt{t}}\left[\int_0^t Z_s ds - \frac{\nu\|h\|_{L^1}}{1-p\|h\|_{L^1}}\cdot t\right] \to N\Big(0,\sigma^2\Big),$$

in distribution as $t\to\infty$, where:

$$\begin{aligned}\sigma^2 :&= \frac{\|h\|_{L^1}^2}{(1-p\|h\|_{L^1})^2}\mathbb{E}[(pZ_\infty+\nu)^2] \\ &= \frac{\|h\|_{L^1}^2}{(1-p\|h\|_{L^1})^2}\Big[p^2\mathbb{E}[Z_\infty^2]+2p\nu\mathbb{E}[Z_\infty]+\nu^2\Big] \\ &= \frac{\nu^2\|h\|_{L^1}^2}{(1-p\|h\|_{L^1})^2}\left[p^2\mathbb{E}\left[\left(\sum_{k=0}^\infty p^k h^{*(k+1),N}(\infty)\right)^2\right]+2p\frac{\|h\|_{L^1}}{1-p\|h\|_{L^1}}+1\right],\end{aligned}$$

where:

$$\begin{aligned}&h^{*(k+1,N)}(t) \\ &:= \int_{0<s_{k+1}<s_k<\cdots<s_1\le t} h(t-s_1)h(s_1-s_2)\cdots h(s_k-s_{k+1})dN_{s_{k+1}}dN_{s_k}\cdots dN_{s_1}.\end{aligned}$$

This completes the proof of Theorem 2.

3. Discussion

Statistical tools for modeling and analyzing temporal and spatial data are based on point processes, which are well-understood objects in probability theory. These models can be used to describe complex systems in sociology, biology, criminology, seismology, finance, and many other fields. The Poisson process is the most standard point process, which has independent time increments. However, the real-world data do not often show independent time increments. The Hawkes process exhibits characteristics such as self-excitation, clustering, and contagion. However, many key theoretical results in this field are still unknown, despite their importance for applications. This article contributed to a better understanding of the large-time behavior of self-exciting point processes. In particular, the author investigated the theoretical properties of Hawkes processes. The author studied various asymptotics of the Hawkes model, including the law of large numbers and central limit theorems for non-Markovian inverse Hawkes processes which were newly introduced in this paper.

4. Conclusions

Among simple point processes, Hawkes processes [1] are the most popular and useful model. They exhibit self-exciting clustering effects. The intensity process for a point process is composed of the summation of the baseline intensity plus other terms that depend upon the history of the whole past of the point process in comparison with a standard Poisson process. In applications, the Hawkes process is typically used as an expressive model for temporal phenomena of stochastic processes which evolve in continuous time, such as in modeling high-frequency trading. The Hawkes process is the natural generalization of the Poisson process and captures both the self-exciting property and clustering effect. This process is a very variable model which is amenable to statistical analysis. Therefore, it has wide applications in insurance [2], neuroscience [3], criminology [4], seismology [5], DNA modeling [6], and finance [7]. The self-exciting and clustering properties of the Hawkes process make it highly suitable for computations in financial applications, including modeling defaults associated with credit and evaluating derivatives of credit in finance. There are many situations that require time-dependent frameworks when it comes to model adjustment. On the basis of the intensity, the Hawkes process can be categorized into linear and nonlinear cases. In the current paper, we considered a non-Markovian inverse Hawkes process which incorporated the remarkable properties of several self-exciting point processes. We also studied the limit theorems of a non-Markovian inverse Hawkes process. Besides the examples already mentioned, the results of this paper impact society in a broader manner. This paper can clarify how complex systems self-excite and cluster for researchers and the general public. By using these techniques, we can improve, refine, and better understand current applications of self-exciting point processes. Both academics and the general public will be inspired to use them in the future.

Funding: This research is supported by the Dong-A University research grant.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Conflicts of Interest: The author declares no competing interest.

References

1. Hawkes, A.G. Spectra of some self-exciting and mutually exciting point process. *Biometrika* **1971**, *58*, 83–90. [CrossRef]
2. Zhu, L. Ruin probabilities for risk processes with non-stationary arrivals and subexponential claims. *Insur. Math. Econom.* **2013**, *53*, 544–550. [CrossRef]
3. Johnson, D.H. Point process models of single-neuron discharges. *J. Comput. Neurosci.* **1996**, *3*, 275–299. [CrossRef] [PubMed]
4. Mohler, G.O.; Short, M.B.; Brantingham, P.J.; Schoenberg F.P.; Tita, G.E. Self-exciting point process modelling of crime. *J. Am. Stat. Assoc.* **2011**, *106*, 100–108. [CrossRef]
5. Hawkes, A.G.; Adamopoulos, L. Cluster models for earthquakesregional comparisons. *Bull. Int. Statist. Inst.* **1973**, *45*, 454–461.
6. Reynaud-Bouret, P.; Schbath, S. Adaptive estimation for Hawkes processes; application to genome analysis. *Ann. Statist.* **2010** *38*, 2781–2822. [CrossRef]
7. Errais, E.; Giesecke, K.; Goldberg, L. Affine Point Processes and Portfolio Credit Risk. *SIAM J. Financ. Math.* **2010**, *1*, 642–665. [CrossRef]
8. Duffie, D.; Filipović, D.; Schachermayer, W. Affine processes and applications in finance. *Ann. Appl. Probab.* **2003** *13*, 984–1053. [CrossRef]
9. Dassios, A.; Zhao, H. A Dynamic Contagion Process. *Adv. Appl. Probab.* **2011**, *43*, 814–846. [CrossRef]
10. Hawkes, A.G.; Oakes, D. A cluster process representation of self-exciting process. *J. Appl. Prob.* **1974**, *11*, 493–503. [CrossRef]
11. Karabash, D. On Stability of Hawkes process. *arXiv* **2013**, arXiv: 1201.1573.
12. Daley, D.J.; Vere-Jones, D. *An Introduction to the Theory of Point Processes*, 2nd ed.; Springer: Berlin/Heidelberg, Germany, 2003. Volumes I and II.
13. Kelbert, M.; Leonenko, N.; Belitsky, V. On the Bartlett spectrum of randomized Hawkes processes. *Math. Commun.* **2013**, *18*, 393–407.
14. Bacry, E.; Delattre, S.; Hoffmann, M.; Muzy, J.F. Scaling limits for Hawkes processes and application to financial statistics. *Stoch. Process. Appl.* **2012**, *123*, 2475–2499. [CrossRef]
15. Bordenave, C.; Torrisi, G.L. Large deviations of Poisson cluster processes. *Stoch. Models.* **2007**, *23*, 593–625. [CrossRef]
16. Brémaud, P.; Massoulié, L. Stability of nonlinear Hawkes processes. *Ann. Probab.* **1996**, *24*, 1563–1588. [CrossRef]
17. Zhu, L. Large deviations for Markovian nonlinear Hawkes processes. *Ann. Appl. Probab.* **2015**, *25*, 548–581. [CrossRef]
18. Zhu, L. Process-level large deviations for nonlinear Hawkes point processes. *Ann. Inst. H. Poincaré Probab. Statist.* **2014**, *50*, 845–871. [CrossRef]
19. Zhu, L. Central limit theorem for nonlinear Hawkes processes. *J. Appl. Prob.* **2013**, *50*, 760–771. [CrossRef]
20. Zhu, L. Moderate deviations for Hawkes processes. *Statist. Probab. Lett.* **2013**, *83*, 885–890. [CrossRef]
21. Jaisson, T.; Rosenbaum, M. Limit theorems for nearly unstable Hawkes processes. *Ann. Appl. Probab.* **2015**, *25*, 600–631. [CrossRef]
22. Jaisson, T.; Rosenbaum, M. Rough fractional diffusions as scaling limits of nearly unstable heavy tailed Hawkes processes. *Ann. Appl. Probab.* **2016**, *26*, 2860–2882. [CrossRef]
23. Fierro, R.; Leiva, V.; Mller, J. The Hawkes process with different exciting functions and its asymptotic behavior. *J. Appl. Prob.* **2015**, *52*, 37–54.

24. Karabash, D.; Zhu, L. Limit theorems for marked Hawkes processes with application to a risk model. *Stoch. Model.* **2015**, *31*, 433–451. [CrossRef]
25. Mehrdad, B.; Zhu, L. On the Hawkes Process with Different Exciting Functions. *arXiv* **2015**, arXiv: 1403.0994.
26. Zhu, L. Limit theorems for a Cox-Ingersoll-Ross process with Hawkes jumps. *J. Appl. Prob.* **2014**, *51*, 699–712. [CrossRef]
27. Seol, Y. Limit theorem for inverse process T_n of linear Hawkes process. *Acta Math. Sin. Engl. Ser.* **2017**, *33*, 51–60. [CrossRef]
28. Seol, Y. Limit theorems of discrete Hawkes Processes. *Statist. Probab. Lett.* **2015**, *99*, 223–229. [CrossRef]
29. Wang, H. Limit theorems for a discrete-time marked Hawkes process. *arXiv* **2020**, arXiv: 2003.02810.
30. Wang, H. Large and moderate deviations for a discrete-time marked Hawkes process. *arXiv* **2020**, arXiv: 2003.05772.
31. Seol, Y. Moderate deviations for Marked Hawkes Processes. *Acta Math. Sin. Engl. Ser.* **2017**, *33*, 1297–1304. [CrossRef]
32. Seol, Y. Limit theorems for the compensator of Hawkes Processes. *Statist. Probab. Lett.* **2017**, *127*, 165–172. [CrossRef]
33. Gao, F.; Zhu, L. Some asymptotic results for nonlinear Hawkes processes. *Stoch. Process. Appl.* **2018**, *128*, 4051–4077. [CrossRef]
34. Gao, X.; Zhu, L. Limit theorems for Markovian Hawkes processes with a large initial intensity. *Stoch. Process. Appl.* **2018**, *128*, 3807–3839. [CrossRef]
35. Gao, X.; Zhu, L. Large deviations and applications for Markovian Hawkes processes with a large initial intensity. *Bernoulli* **2018**, *24*, 2875–2905. [CrossRef]
36. Gao, X.; Zhu, L. Functional central limit theorem for stationary Hawkes processes and its application to infinite-serve queues. *Queueing Syst.* **2018**, *90*, 161–206. [CrossRef]
37. Gao, X.; Zhou, X.; Zhu, L. Tranform Analysis for Hawkes processes with applications. *Quant. Financ.* **2018**, *18*, 265–282. [CrossRef]
38. Wheatley, S.; Filimonov, V.; Sorrette, D. The Hawkes process with renewal immigration & its estimation with an EM algorithm. *Comput. Stat. Data Anal.* **2016**, *94*, 120–135.
39. Seol, Y. Limit theorems for an inverse Markovian Hawkes Processes. *Statist. Probab. Lett.* **2019**, *155*, 108580. [CrossRef]
40. Seol, Y. Asymptotics for an extended inverse Markovian Hawkes Process. *J. Korean Math. Soc.* **2021**, *58*, 819–833.
41. Massoulié, L. Stability results for a general class of interacting point processes dynamics., applications. *Stoch. Process. Appl.* **1998**, *75*, 1–30. [CrossRef]
42. Jacod, J.; Shiryaev, A.N. *Limit Theorems for Stochastic Processes*; Springer: Berlin/Heidelberg, Germany, 1987.