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Approximation by Tensor-Product Kind Bivariate Operator of a New Generalization of Bernstein-Type Rational Functions and Its GBS Operator

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Abstract: We introduce a tensor-product kind bivariate operator of a new generalization of Bernstein-type rational functions and its GBS (generalized Boolean sum) operator, and we investigate their approximation properties by obtaining their rates of convergence. Moreover, we present some graphical comparisons visualizing the convergence of tensor-product kind bivariate operator and its GBS operator.

Keywords: linear positive operator; rate of convergence; Bernstein-type rational function

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1. Introduction

Bernstein-type rational functions were defined by Balázs in [1] as follows:

$$R_n(f;u) = \frac{1}{(1+a_n u)^n} \sum_{k=0}^{n} f\left(\frac{k}{b_n}\right) \binom{n}{k} (a_n u)^k, u \geq 0, n \in \mathbb{N}, \tag{1}$$

where a_n and b_n are suitably chosen non-negative real sequences such that $b_n = na_n$ for each $n \in \mathbb{N}$, and f is a real-valued function on $[0,\infty)$.

In [2], Atakut and İspir introduced the bivariate operator of the Bernstein-type rational functions defined by (1) as follows:

$$R_{n,m}(f;u,v) = \sum_{j=0}^{n} \sum_{k=0}^{m} f\left(\frac{j}{b_n}, \frac{k}{b_m}\right) \binom{n}{j} \binom{m}{k} \frac{(a_n u)^j (a_m v)^k}{(1+a_n u)^n (1+a_m v)^m}, u,v \geq 0, n,m \in \mathbb{N}, \tag{2}$$

where a_n, a_m, b_n and b_m are suitably chosen non-negative sequences such that $b_h = ha_h$ for $h = n,m \in \mathbb{N}$, and f is a real-valued function on $[0,\infty) \times [0,\infty)$. They obtained an estimate by means of the usual first modulus of continuity and proved an asymptotic approximation theorem with the classical methods. Moreover, Atakut [3] presented some convergence results associated with the derivatives of the operator $R_{n,m}$ defined by (2).

Recently, a new generalization of Bernstein-type rational functions has been defined in [4] by:

$$R_n^G(f;u) = \sum_{k=0}^{n} f\left(\frac{k}{\gamma_n}\right) \binom{n}{k} \frac{(\alpha_n u)^k (\beta_n)^{n-k}}{(\beta_n + \alpha_n u)^n}, u \geq 0, n \in \mathbb{N}, \tag{3}$$

where f is a real-valued continuous function on $[0,\infty)$, and (α_n), (β_n) and (γ_n) are non-negative real sequences such that $\gamma_n = n\alpha_n$ satisfying the following properties:

$$\lim_{n\to\infty} \alpha_n = 0, \ \lim_{n\to\infty} \beta_n = 1 \text{ and } \lim_{n\to\infty} \gamma_n = \infty. \tag{4}$$

The operator R_n^G is a linear and positive operator. When $\beta_n = 1$, $\alpha_n = a_n$ and $\gamma_n = b_n$, it is reduced to the Bernstein-type rational functions given by (1). Therefore, it is a generalization of the Bernstein-type rational functions. Its Korovkin-type approximation results have been investigated in [4].

Recently, the approximation properties of lots of bivariate operators have been investigated. Readers can see the following references for details [5–15].

In this study, we introduce a tensor-product kind bivariate operator and its associated GBS (generalized Boolean sum) operator of the generalized Bernstein-type rational function R_n^G defined by (3), which is a generalization of the bivariate operator $R_{n,m}$ defined by (2). Moreover, we investigate their approximation properties on rectangular region $[0, r_1] \times [0, r_2]$ such that $r_1, r_2 > 0$. Lastly, we present an application including illustrative graphics visualizing the convergence of the tensor-product kind bivariate operator and its GBS operator, which also compare their convergence with the bivariate operator $R_{n,m}$ defined by (2).

2. Construction of Tensor-Product Kind Bivariate Operator

In this part, we introduce a tensor-product kind bivariate operator of the generalized Bernstein-type rational function R_n^G defined by (3) and investigate its approximation properties.

Let (α_1^n), (α_2^m), (β_1^n), (β_2^m), (γ_1^n) and (γ_2^m) be non-negative real sequences such that $\gamma_\phi^h = h\alpha_\phi^h$ for $(\phi, h) = (1, n), (2, m)$, fulfilling the following conditions:

$$\lim_{n,m\to\infty} \alpha_\phi^h = 0, \lim_{n,m\to\infty} \beta_\phi^h = 1 \text{ and } \lim_{n,m\to\infty} \gamma_\phi^h = \infty. \tag{5}$$

Let f be a real-valued continuous function on $[0, \infty) \times [0, \infty)$. We define the following tensor-product kind bivariate operator:

$$R_{n,m}^G(f; u, v) = \sum_{j=0}^{n} \sum_{k=0}^{m} f\left(\frac{j}{\gamma_1^n}, \frac{k}{\gamma_2^m}\right) s_{n,j}(u) s_{m,k}(v), u, v \geq 0, n, m \in \mathbb{N}, \tag{6}$$

where $s_{n,j}(u) = \binom{n}{j} \frac{(\alpha_1^n u)^j (\beta_1^n)^{n-j}}{(\beta_1^n + \alpha_1^n u)^n}$, $s_{m,k}(v) = \binom{m}{k} \frac{(\alpha_2^m v)^k (\beta_2^m)^{m-k}}{(\beta_2^m + \alpha_2^m v)^m}$. For any $\phi, \varphi \in \mathbb{R}$ and any real-valued continuous functions f, h on $[0, \infty) \times [0, \infty)$, we have the following relation:

$$R_{n,m}^G(\phi f + \varphi h; u, v) = \phi R_{n,m}^G(f; u, v) + \varphi R_{n,m}^G(h; u, v),$$

and if f is non-negative, then $R_{n,m}^G(f; .)$ is non-negative. Therefore, the bivariate operator $R_{n,m}^G$ is linear and positive. By denoting:

$${}_xR_n^G(f(\tau, \varsigma); u, \varsigma) := \sum_{j=0}^{n} f\left(\frac{j}{\gamma_1^n}, \varsigma\right) s_{n,j}(u),$$

$${}_yR_m^G(f(\tau, \varsigma); \tau, v) := \sum_{k=0}^{m} f\left(\tau, \frac{k}{\gamma_2^m}\right) s_{m,k}(v),$$

the bivariate operator $R_{n,m}^G$ is the tensorial product of ${}_xR_n^G$ and ${}_yR_m^G$ such that:

$$R_{n,m}^G = {}_x R_n^G \circ {}_y R_m^G = {}_y R_m^G \circ {}_x R_n^G.$$

Indeed, by denoting $g(\tau, v) := {}_y R_m^G(f(\tau, \varsigma); \tau, v)$, we obtain:

$$
\begin{aligned}
{}_xR_n^G\left({}_yR_m^G(f(\tau,\varsigma);\tau,v);u,v\right) &= {}_xR_n^G(g(\tau,v);u,v) \\
&= \sum_{j=0}^{n} g\left(\frac{j}{\gamma_1^n},v\right)s_{n,j}(u) \\
&= \sum_{j=0}^{n}\left(\sum_{k=0}^{m} f\left(\frac{j}{\gamma_1^n},\frac{k}{\gamma_2^m}\right)s_{m,k}(v)\right)s_{n,j}(u) \\
&= \sum_{j=0}^{n}\sum_{k=0}^{m} f\left(\frac{j}{\gamma_1^n},\frac{k}{\gamma_2^m}\right)s_{n,j}(u)s_{m,k}(v) \\
&= R_{n,m}^G(f(\tau,\varsigma);u,v).
\end{aligned}
$$

Similarly, we have the following relation:

$$
{}_yR_m^G\left({}_xR_n^G(f(\tau,\varsigma);u,\varsigma);u,v\right) = R_{n,m}^G(f(\tau,\varsigma);u,v).
$$

If $\alpha_\phi^h = a_h$, $\gamma_\phi^h = b_h$ and $\beta_\phi^h = 1$ for $(\phi,h) = (1,n),(2,m)$, then the tensor-product kind operator $R_{n,m}^G$ is reduced to the bivariate operator $R_{n,m}$ defined by (2). Therefore, the tensor-product kind operator $R_{n,m}^G$ is a generalization of the bivariate operator $R_{n,m}$ defined by (2)

Now, we give some auxilary results:

Lemma 1. *Let $R_{n,m}^G$ be the operator defined by (6) and $\psi_{i,j}(\tau,\varsigma) = \tau^i\varsigma^j$, $i,j = 0,1,2$, be the bivariate test functions. Then, we have the following equalities:*

$$
\begin{aligned}
R_{n,m}^G(\psi_{0,0};u,v) &= 1, \\
R_{n,m}^G(\psi_{1,0};u,v) &= \frac{u}{\beta_1^n+\alpha_1^n u}, \\
R_{n,m}^G(\psi_{0,1};u,v) &= \frac{v}{\beta_2^m+\alpha_2^m v}, \\
R_{n,m}^G(\psi_{2,0};u,v) &= \frac{\left(1-\frac{1}{n}\right)u^2}{(\beta_1^n+\alpha_1^n u)^2} + \frac{u}{\gamma_1^n(\beta_1^n+\alpha_1^n u)}, \\
R_{n,m}^G(\psi_{0,2};u,v) &= \frac{\left(1-\frac{1}{m}\right)v^2}{(\beta_2^m+\alpha_2^m v)^2} + \frac{v}{\gamma_2^m(\beta_2^m+\alpha_2^m v)}.
\end{aligned}
$$

Proof. By the proof of Lemma 1 of [4], we can write:

$$
R_{n,m}^G(\psi_{0,0};u,v) = \sum_{j=0}^{n}\sum_{k=0}^{m} 1.s_{n,j}(u)s_{m,k}(v) = 1
$$

$$
\begin{aligned}
R_{n,m}^G(\psi_{1,0};u,v) &= \sum_{j=1}^{n}\frac{j}{\gamma_1^n}s_{n,j}(u)\sum_{k=0}^{m}s_{m,k}(v) \\
&= \frac{u}{\beta_1^n+\alpha_1^n u}\sum_{j=0}^{n-1}s_{n-1,j}(u)\sum_{k=0}^{m}s_{m,k}(v) \\
&= \frac{u}{\beta_1^n+\alpha_1^n u},
\end{aligned}
$$

$$
\begin{aligned}
R_{n,m}^{G}(\psi_{2,0};u,v) &= \sum_{j=0}^{n}\sum_{k=0}^{m}\left(\frac{j}{\gamma_1^n}\right)^2 s_{n,j}(u)s_{m,k}(v)\\
&= \sum_{j=0}^{n}\left(\frac{j}{\gamma_1^n}\right)^2 s_{n,j}(u)\sum_{k=0}^{m}s_{m,k}(v)\\
&= \frac{\left(1-\frac{1}{n}\right)u^2}{\left(\beta_1^n+\alpha_1^n u\right)^2}\sum_{j=0}^{n-2}s_{n-2,j}(u)\sum_{k=0}^{m}s_{m,k}(v)\\
&\quad+\frac{u}{\gamma_1^n\left(\beta_1^n+\alpha_1^n u\right)}\sum_{j=0}^{n-1}s_{n-1,j}(u)\sum_{k=0}^{m}s_{m,k}(v)\\
&= \frac{\left(1-\frac{1}{n}\right)u^2}{\left(\beta_1^n+\alpha_1^n u\right)^2}+\frac{u}{\gamma_1^n\left(\beta_1^n+\alpha_1^n u\right)}.
\end{aligned}
$$

Similarly, $R_{n,m}^{G}(\psi_{0,1};u,v)$ and $R_{n,m}^{G}(\psi_{0,2};u,v)$ can be easily calculated by interchanging the roles of the components j, n and u of $s_{n,j}(u)$ with k, m and v and the components k, m and v of $s_{m,k}(v)$ with j, n and u, respectively. □

Remark 1. *From Lemma 1, we obtain:*

$$
R_{n,m}^{G}(\tau-u;u,v)=\frac{(1-\beta_1^n)u}{\beta_1^n+\alpha_1^n u}-\frac{\alpha_1^n u^2}{\beta_1^n+\alpha_1^n u},
$$

$$
R_{n,m}^{G}(\varsigma-v;u,v)=\frac{(1-\beta_2^m)v}{\beta_2^m+\alpha_2^m v}-\frac{\alpha_2^m v^2}{\beta_2^m+\alpha_2^m v},
$$

$$
\begin{aligned}
R_{n,m}^{G}\left((\tau-u)^2;u,v\right) &= \frac{\beta_1^n u}{\gamma_1^n\left(\beta_1^n+\alpha_1^n u\right)^2}+\frac{\left(\alpha_1^n+\left(\beta_1^n-1\right)^2-\frac{1}{n}\right)u^2}{\left(\beta_1^n+\alpha_1^n u\right)^2}\\
&\quad+\frac{2\alpha_1^n\left(\beta_1^n-1\right)u^3}{\left(\beta_1^n+\alpha_1^n u\right)^2}+\frac{\alpha_1^n u^4}{\left(\beta_1^n+\alpha_1^n u\right)^2},
\end{aligned}
$$

$$
\begin{aligned}
R_{n,m}^{G}\left((\varsigma-y)^2;u,v\right) &= \frac{\beta_2^m v}{\gamma_2^m\left(\beta_2^m+\alpha_2^m v\right)^2}+\frac{\left(\alpha_2^m+\left(\beta_2^m-1\right)^2-\frac{1}{m}\right)v^2}{\left(\beta_2^m+\alpha_2^m v\right)^2}\\
&\quad+\frac{2\alpha_2^m\left(\beta_2^m-1\right)v^3}{\left(\beta_2^m+\alpha_2^m v\right)^2}+\frac{\alpha_2^m v^4}{\left(\beta_2^m+\alpha_2^m v\right)^2}.
\end{aligned}
$$

3. Approximation Results

In this part, we firstly present a Volkov-type result for the tensor-product kind bivariate operator $R_{n,m}^{G}$.

Let $A\subset[0,\infty)\times[0,\infty)$ be a compact set of $\mathbb{R}^2$, and $C(A)$ be the space of all real-valued continuous functions f on A with the supremum norm $\|f\|=\sup\{|f(u,v)|:(u,v)\in A\}$.

Theorem 1. *Let $R_{n,m}^{G}$, $n,m\in\mathbb{N}$, be the tensor-product kind bivariate operator defined by (6) and (α_1^n), (α_2^m), (β_1^n), (β_2^m), (γ_1^n) and (γ_2^m) be real sequences fulfilling the condition (5). Then, for all $f\in C([0,r_1]\times[0,r_2])$, $r_1,r_2>0$, $R_{n,m}^{G}(f)$ converges uniformly to f on $[0,r_1]\times[0,r_2]$.*

Proof. By Lemma 1, the theorem can be proved by considering Volkov's theorem in [16] with similar methods to the proof of Theorem 1 of [4]; therefore, we omit its proof. □

Now, we obtain inequalities estimating the error of the approximation by the tensor-product kind bivariate operator $R^{G}_{n,m}$ defined by (6).

The complete modulus of continuity for bivariate functions $f \in C(A)$ is defined as follows:

$$\omega(f;\mu_1,\mu_2) = \sup\{|f(\tau,\varsigma) - f(u,v)| : |\tau - u| \le \mu_1, |\varsigma - v| \le \mu_2, (\tau,\varsigma),(u,v) \in A,\}$$

where $\mu_1, \mu_2 > 0$.

Moreover, the partial modulus of continuity according to x and y are defined by:

$$\begin{aligned}\omega^{(1)}(f;\mu_1) &= \sup\{|f(\tau,v) - f(u,v)| : |\tau - u| \le \mu_1, (\tau,v),(u,v) \in A\},\\ \omega^{(2)}(f;\mu_2) &= \sup\{|f(u,\varsigma) - f(u,v)| : |\varsigma - v| \le \mu_2, (u,\varsigma),(u,v) \in A\},\end{aligned}$$

which fulfill the properties of the classical modulus of continuity. The details of the modulus of continuity for the bivariate functions can be found in [17].

Secondly, we estimate the rate of convergence of the tensor-product kind bivariate operator $R^{G}_{n,m}$ defined in (6) by using the complete modulus of continuity.

Theorem 2. *Let $f \in C([0,r_1] \times [0,r_2])$, $r_1, r_2 > 0$. Then, the following inequality holds:*

$$\left|R^{G}_{n,m}(f;u,v) - f(u,v)\right| \le 4\omega(f;\mu^{u}_{n},\mu^{v}_{m}),$$

where $\mu^{u}_{n} := \left(R^{G}_{n,m}((\tau-u)^2;u,v)\right)^{1/2}$ and $\mu^{v}_{m} := \left(R^{G}_{n,m}((\varsigma-v)^2;u,v)\right)^{1/2}$.

Proof. Using the linearity and the positivity of the operator $R^{G}_{n,m}$ and taking properties of the complete modulus of continuity into account, we can write:

$$\begin{aligned}\left|R^{G}_{n,m}(f;u,v) - f(u,v)\right| &\le R^{G}_{n,m}(|f(\tau,\varsigma) - f(u,v)|;u,v)\\ &\le \omega(f;\mu_1,\mu_2)\left[1 + \frac{1}{\mu_1}R^{G}_{n,m}(|\tau-u|;u,v)\right.\\ &\quad + \frac{1}{\mu_2}R^{G}_{n,m}(|\varsigma-v|;u,v)\\ &\quad \left.+ \frac{1}{\mu_1\mu_2}R^{G}_{n,m}(|\tau-x||\varsigma-v|;u,v)\right].\end{aligned} \tag{7}$$

Applying the Cauchy–Schwarz inequality to (7), we obtain:

$$\begin{aligned}\left|R^{G}_{n,m}(f;u,v) - f(u,v)\right| &\le \omega(f;\mu_1,\mu_2)\left[1 + \frac{1}{\mu_1}\left(R^{G}_{n,m}\left((\tau-u)^2;u,v\right)\right)^{1/2}\right.\\ &\quad + \frac{1}{\mu_2}\left(R^{G}_{n,m}\left((\varsigma-v)^2;u,v\right)\right)^{1/2}\\ &\quad \left.+ \frac{1}{\mu_1\mu_2}\left(R^{G}_{n,m}\left((\tau-u)^2(\varsigma-v)^2;u,v\right)\right)^{1/2}\right].\end{aligned} \tag{8}$$

In (8), by considering Lemma 1 and choosing $\mu_1 =: \mu^{u}_{n} := \left(R^{G}_{n,m}((\tau-u)^2;u,v)\right)^{1/2}$ and $\mu_2 =: \mu^{v}_{m} := \left(R^{G}_{n,m}((\varsigma-v)^2;u,v)\right)^{1/2}$, we complete the proof of the theorem. □

We present in the following theorem the estimation of the rate of the convergence by the tensor-product kind bivariate operator $R^{G}_{n,m}$defined in (6) by means of the partial modulus of continuities.

Theorem 3. *Let $f \in C([0,r_1] \times [0,r_2])$, $r_1, r_2 > 0$. Then, the following inequality is valid:*

$$\left| R^G_{n,m}(f;u,v) - f(u,v) \right| \leq 2\left[\omega^{(1)}(f;\mu^u_n) + \omega^{(2)}(f;\mu^v_m)\right].$$

Proof. Considering the definition of the partial modulus of continuity and using the Cauchy–Schwarz inequality, we can write:

$$\begin{aligned}
\left| R^G_{n,m}(f;u,v) - f(u,v) \right| &\leq R^G_{n,m}(|f(\tau,\varsigma) - f(u,v)|;u,v) \\
&\leq R^G_{n,m}(|f(\tau,\varsigma) - f(u,\varsigma)|;u,v) + R^G_{n,m}(|f(u,\varsigma) - f(u,v)|;u,v) \\
&\leq \omega^{(1)}(f;\mu_1)\left[1 + \frac{1}{\mu_1}R^G_{n,m}(|\tau - u|;u,v)\right] \\
&\quad +\omega^{(2)}(f;\mu_2)\left[1 + \frac{1}{\mu_2}R^G_{n,m}(|\varsigma - v|;u,v)\right] \\
&\leq \omega^{(1)}(f;\mu_1)\left[1 + \frac{1}{\mu_1}\Big(R^G_{n,m}((\tau - u)^2;u,v)\Big)^{1/2}\right] \\
&\quad +\omega^{(2)}(f;\mu_2)\left[1 + \frac{1}{\mu_2}\Big(R^G_{n,m}((\varsigma - v)^2;u,v)\Big)^{1/2}\right].
\end{aligned}$$

Choosing $\mu_1 =: \mu^u_n := \Big(R^G_{n,m}((\tau - u)^2;u,v)\Big)^{1/2}$ and $\mu_2 =: \mu^v_m := \Big(R^G_{n,m}((\varsigma - v)^2;u,v)\Big)^{1/2}$, we complete the proof. □

Now, we investigate the rate of convergence of the operator $R^G_{n,m}$defined in (6) with the help of functions of the Lipschitz type.

Any function $f \in C(A)$ is called a function of Lipschitz type and denoted by $f \in Lip_M(a,b)$ if there exists an $M > 0$ such that:

$$|f(\tau,\varsigma) - f(u,v)| \leq M|\tau - u|^a|\varsigma - v|^b,$$

where $(\tau,\varsigma), (u,v) \in A$ are arbitrary and $0 < a, b \leq 1$.

Theorem 4. *Let $f \in Lip_M(a,b)$. Then, there exists an $M > 0$ such that:*

$$\left| R^G_{n,m}(f;u,v) - f(u,v) \right| \leq M(\mu^u_n)^a(\mu^v_m)^b,$$

for all $(x,y) \in [0,r_1] \times [0,r_2]$, where $\mu^u_n := \Big(R^G_{n,m}((\tau - u)^2;u,v)\Big)^{1/2}$ and $\mu^v_m := \Big(R^G_{n,m}((\varsigma - v)^2;u,v)\Big)^{1/2}$.

Proof. By the hypothesis of the theorem, we can write:

$$\begin{aligned}
\left| R^G_{n,m}(f;u,v) - f(u,v) \right| &\leq R^G_{n,m}(|f(\tau,\varsigma) - f(u,v)|;u,v) \\
&\leq MR^G_{n,m}(|\tau - u|^a|\varsigma - v|^b;u,v) \\
&= MR^G_{n,m}(|\tau - u|^a;x,y) \times R^G_{n,m}(|\varsigma - v|^b;u,v).
\end{aligned}$$

Respectively, applying the Hölder's inequality to the last inequality for $p_1 = \frac{2}{a}$, $q_1 = \frac{2}{2-a}$, $p_2 = \frac{2}{b}$ and $q_2 = \frac{2}{2-b}$ such that $\frac{1}{p_i} + \frac{1}{q_i} = 1, i = 1,2$, we obtain:

$$\begin{aligned}
\left| R^G_{n,m}(f;u,v) - f(u,v) \right| &\leq M\Big(R^G_{n,m}((\tau - u)^2;u,v)\Big)^{a/2}\Big(R^G_{n,m}(1;u,v)\Big)^{(2-a)/2} \\
&\quad \times\Big(R^G_{n,m}((\varsigma - v)^2;u,v)\Big)^{b/2}\Big(R^G_{n,m}(1;u,v)\Big)^{(2-b)/2} \\
&= M(\mu^u_n)^a(\mu^v_m)^b,
\end{aligned}$$

which completes the proof of the theorem. □

4. GBS Operator

In this part, we construct the GBS operator associated with the tensor-product kind bivariate operator $R^G_{n,m}$.

We recall some basic notations given by Bögel. The details of which can be found in references [18–20].

Let A be a compact subset of $\mathbb{R}^2$. A real-valued function on A is called Bögel-continuous function at $(\tau, \varsigma) \in A$ if:

$$\Delta_{(u,v)} f[\tau, \varsigma; u, v] = 0,$$

where $\Delta_{(u,v)} f[\tau, \varsigma; u, v]$ denotes the mixed difference defined by:

$$\Delta_{(u,v)} f[\tau, \varsigma; u, v] = f(u, v) - f(u, \varsigma) - f(\tau, v) + f(\tau, \varsigma).$$

Let A be a subset of $\mathbb{R}^2$. A real-valued function on A is Bögel-bounded function if there exists an $M > 0$ such that:

$$\left|\Delta_{(u,v)} f[\tau, \varsigma; u, v]\right| \leq M,$$

for all $(\tau, \varsigma), (u, v) \in A$.

Let A be a compact subset of $\mathbb{R}^2$. Then, each Bögel-continuous function is a Bögel-bounded function. Let $C_{\mathcal{B}}(A)$ denote the space of all the real-valued Bögel-continuous functions defined on A endowed with the following norm:

$$\|f\|_{\mathcal{B}} = sup\left\{\left|\Delta_{(u,v)} f[\tau, \varsigma; u, v]\right| : (u, v), (\tau, \varsigma) \in A\right\}.$$

It is obvious that $C(A) \subset C_{\mathcal{B}}(A)$.

For $f \in C([0, r_1] \times [0, r_2])$, $r_1, r_2 > 0$, we introduce the GBS (generalized Boolean sum) operator associated with the operator $R^G_{n,m}$ by:

$$B^G_{n,m}(f(\tau, \varsigma); u, v) = R^G_{n,m}(f(u, \varsigma) + f(\tau, v) - f(\tau, \varsigma); u, v), \tag{9}$$

for all $(\tau, \varsigma), (u, v) \in [0, r_1] \times [0, r_2]$ and $n, m \in \mathbb{N}$. We can definitely write:

$$\begin{aligned} B^G_{n,m}(f(\tau, \varsigma); u, v) &= \sum_{j=0}^{n} \sum_{k=0}^{m} s_{n,j}(u) s_{m,k}(v) \\ &\quad \times \left[f\left(u, \frac{k}{\gamma_2^m}\right) + f\left(\frac{j}{\gamma_1^n}, v\right) - f\left(\frac{j}{\gamma_1^n}, \frac{k}{\gamma_2^m}\right)\right] \end{aligned} \tag{10}$$

where $s_{n,j}(u) = \binom{n}{j} \frac{(\alpha_1^n u)^j (\beta_1^n)^{n-j}}{(\beta_1^n + \alpha_1^n u)^n}$, $s_{m,k}(v) = \binom{m}{k} \frac{(\alpha_2^m v)^k (\beta_2^m)^{m-k}}{(\beta_2^m + \alpha_2^m v)^m}$ and (α_1^n),(α_2^m),(β_1^n), (β_2^m), (γ_1^n) and (γ_2^m) are non-negative real sequences such that $\gamma_\phi^h = h\alpha_\phi^h$ for $(\phi, h) = (1, n), (2, m)$ fulfilling the condition of (5).

It is clear that $B^G_{n,m}$ maps $C_{\mathcal{B}}([0, r_1] \times [0, r_2])$ into itself, and it is linear and positive.

The mixed modulus of smoothness of $f \in C_{\mathcal{B}}(A)$ is defined in [21] by:

$$\omega_{mixed}(f; u, v) = \sup\{\left|\Delta_{(u,v)} f[\tau, \varsigma; u, v]\right| : |\tau - u| < \mu_1, |\varsigma - v| < \mu_2, (\tau, \varsigma), (u, v) \in A\}. \tag{11}$$

Theorem 5. *For any $f \in C_{\mathcal{B}}([0, r_1] \times [0, r_2])$, the following inequality is valid:*

$$\left|B^G_{n,m}(f; u, v) - f(u, v)\right| \leq 4\omega_{mixed}(f; \mu_n^u, \mu_m^v)$$

where $\mu_n^u := \left(R^G_{n,m}((\tau - u)^2; u, v)\right)^{1/2}$ *and* $\mu_m^v := \left(R^G_{n,m}((\varsigma - v)^2; u, v)\right)^{1/2}$.

Proof. By (11) and for $\lambda_1, \lambda_2 > 0$, the mixed modulus of smoothness ω_{mixed} possesses the following property:

$$\omega_{mixed}(f; \lambda_1\mu_1, \lambda_2\mu_2) \leq (1+\lambda_1)(1+\lambda_2)\omega_{mixed}(f; \mu_1, \mu_2),$$

by which for all $(\tau, \varsigma), (u, v) \in [0, r_1] \times [0, r_2]$, we obtain:

$$\begin{aligned} \left|\Delta_{(u,v)} f[\tau, \varsigma; u, v]\right| &\leq |f(u,v) - f(u,\varsigma) - f(\tau, v) + f(\tau, \varsigma)| \\ &\leq \omega_{mixed}(f; |\tau - u|, |\varsigma - v|) \\ &\leq \left(1 + \frac{|\tau - u|}{\mu_1}\right)\left(1 + \frac{|\varsigma - v|}{\mu_2}\right)\omega_{mixed}(f; \mu_1, \mu_2). \end{aligned} \tag{12}$$

By (9), we can write:

$$f(u, \varsigma) + f(\tau, v) - f(\tau, \varsigma) = f(u, v) - \Delta_{(u,v)} f[\tau, \varsigma; u, v].$$

Considering the definition of $R^G_{n,m}$ and $B^G_{n,m}$, we obtain:

$$\begin{aligned} B^G_{n,m}(f(\tau, \varsigma); u, v) &= R^G_{n,m}(f(u, \varsigma) + f(\tau, v) - f(\tau, \varsigma); u, v) \\ &= f(u, v) R^G_{n,m}(\psi_{0,0}; u, v) - R^G_{n,m}\left(\Delta_{(u,v)} f[\tau, \varsigma; u, v]; u, v\right). \end{aligned}$$

By (12) and taking the Cauchy–Schwarz inequality into account, we obtain:

$$\begin{aligned} \left|B^G_{n,m}(f; u, v) - f(u, v)\right| \leq\ & [R^G_{n,m}(1; u, v) + \frac{1}{\mu_1}(R^G_{n,m}((\tau - u)^2; u, v))^{1/2} \\ & + \frac{1}{\mu_2}(R^G_{n,m}((\varsigma - v)^2; u, v))^{1/2} \\ & + \frac{1}{\mu_1\mu_2}(R^G_{n,m}((\tau - u)^2; u, v) R^G_{n,m}((\varsigma - v)^2; u, v))^{1/2}] \\ & \times \omega_{mixed}(f; \mu_1, \mu_2). \end{aligned}$$

Choosing $\mu^u_n := \left(R^G_{n,m}((\tau - u)^2; u, v)\right)^{1/2}$ and $\mu^v_m := \left(R^G_{n,m}((\varsigma - v)^2; u, v)\right)^{1/2}$, we obtain the desired result. □

Now, we recall the Bögel-continuous functions of Lipschitz type. For $f \in C_{\mathcal{B}}(A)$, $(\tau, \varsigma), (u, v) \in A$ and $0 < a, b \leq 1$, if there exists an $M > 0$ such that:

$$\left|\Delta_{(u,v)} f[\tau, \varsigma; u, v]\right| \leq M|\tau - u|^a |\varsigma - v|^b,$$

then f is called a Bögel-continuous function of Lipschitz type and denoted by $Lip^{\mathcal{B}}_M(a, b)$.

Theorem 6. *Let $f \in Lip^{\mathcal{B}}_M(a, b)$. Then, for all $(u, v) \in [0, r_1] \times [0, r_2]$, we have the following inequality:*

$$\left|B^G_{n,m}(f; u, v) - f(u, v)\right| \leq M(\mu^u_n)^a (\mu^v_m)^b, M > 0,$$

where $\mu^u_n := \left(R^G_{n,m}((\tau - u)^2; u, v)\right)^{1/2}$ and $\mu^v_m := \left(R^G_{n,m}((\varsigma - v)^2; u, v)\right)^{1/2}$.

Proof. Since:

$$\begin{aligned}
B^G_{n,m}(f;u,v) &= R^G_{n,m}(f(u,\varsigma)+f(\tau,v)-f(\tau,\varsigma);u,v)\\
&= R^G_{n,m}\Big(f(u,v)-\Delta_{(u,v)}f[\tau,\varsigma;u,v];u,v\Big)\\
&= f(u,v)R^G_{n,m}(\psi_{0,0};u,v)-R^G_{n,m}\Big(\Delta_{(u,v)}f[\tau,\varsigma;u,v];u,v\Big),
\end{aligned}$$

we can write:

$$\begin{aligned}
\Big|B^G_{n,m}(f;u,v)-f(u,v)\Big| &\le R^G_{n,m}\Big(\Big|\Delta_{(u,v)}f[\tau,\varsigma;u,v]\Big|;u,v\Big)\\
&\le MR^G_{n,m}(|\tau-u|^a|\varsigma-v|^b;u,v)\\
&= MR^G_{n,m}(|\tau-u|^a;u,v)\times R^G_{n,m}(|\varsigma-v|^b;u,v).
\end{aligned}$$

Applying the Hölder's inequality to the last inequality by choosing $p_1=\frac{2}{a}$, $q_1=\frac{2}{2-a}$, $p_2=\frac{2}{b}$ and $q_2=\frac{2}{2-b}$ such that $\frac{1}{p_i}+\frac{1}{q_i}=1, i=1,2$, we obtain:

$$\begin{aligned}
\Big|B^G_{n,m}(f;u,v)-f(u,v)\Big| &\le M\Big(R^G_{n,m}((\tau-u)^2;u,v)\Big)^{a/2}\Big(R^G_{n,m}(1;u,v)\Big)^{(2-a)/2}\\
&\quad\times\Big(R^G_{n,m}((\varsigma-v)^2;u,v)\Big)^{b/2}\Big(R^G_{n,m}(1;u,v)\Big)^{(2-b)/2}\\
&= M(\mu^u_n)^a(\mu^v_m)^b,
\end{aligned}$$

which is the desired result. □

5. Graphical Comparisons

Let $\varphi(u,v)=(u-v)^2$ such that $(u,v)\in[0,3]\times[0,3]$.

1. Let us choose $n=m$, $\alpha^h_\phi=n^{-1/2}$, $\gamma^h_\phi=n^{1/2}$ and $\beta^h_\phi=1-5n^{-1}$ such that $(\phi,h)=(1,n),(2,m),n,m=1,2,...$

Figure 1 compares the approximation of $R^G_{5,5}(\varphi;u,v)$ (red), $R^G_{25,25}(\varphi;u,v)$ (yellow) and $R^G_{75,75}(\varphi;u,v)$ (green) to $\varphi(u,v)$ (blue) on $[0,3]\times[0,3]$. For increasing value of n, the approximation of $R^G_{n,n}(\varphi;u,v)$ to $\varphi(u,v)$ becomes better.

Figure 2 compares the approximation of $B^G_{5,5}(\varphi;u,v)$ (red), $B^G_{25,25}(\varphi;u,v)$ (yellow) and $B^G_{75,75}(\varphi;u,v)$ (green) to $\varphi(u,v)$ (blue) on $[0,3]\times[0,3]$. Similarly, for increasing value of n, the approximation of $B^G_{n,n}(\varphi;u,v)$ to $\varphi(u,v)$ becomes better.

2. Let us choose $n=m=75$, $\alpha^h_\phi=n^{-1/2}$, $\gamma^h_\phi=n^{1/2}$ and $\beta_1=\beta^h_\phi=1-5n^{-1}$, $\beta_2=\beta^h_\phi=1-10n^{-1}$ and $\beta_2=\beta^h_\phi=1-15n^{-1}$ such that $(\phi,h)=(1,n),(2,m),n,m=1,2,\ldots$

Figure 3 compares the approximation of $R^G_{75,75}(\varphi;u,v;\beta_1)$ (red), $R^G_{75,75}(\varphi;u,v;\beta_2)$ (yellow) and $R^G_{75,75}(\varphi;u,v;\beta_3)$ (green) to $\varphi(u,v)$ (blue) on $[0,3]\times[0,3]$. One can see that the approximation of $R^G_{75,75}(\varphi;u,v;\beta_3)$ (green) to $\varphi(u,v)$ (blue) is better than others.

Figure 4 compares the approximation of $B^G_{75,75}(\varphi;u,v;\beta_1)$ (red), $B^G_{75,75}(\varphi;u,v;\beta_2)$ (yellow) and $B^G_{75,75}(\varphi;u,v;\beta_3)$ (green) to $\varphi(u,v)$ (blue) on $[0,3]\times[0,3]$. One can see that the approximations of $B^G_{75,75}(\varphi;u,v;\beta_1)$ (red) and $B^G_{75,75}(\varphi;u,v;\beta_3)$ (green) are better in places of the sub-region of the region than $B^G_{75,75}(\varphi;u,v;\beta_2)$ (yellow).

3. Let us choose $n=m$, $\alpha^h_\phi=n^{-1/2}$, $\gamma^h_\phi=n^{1/2}$ and $\beta^h_\phi=1-5n^{-1}$ such that $(\phi,h)=(1,n),(2,m),n,m=1,2,...$

Figure 5, compares the approximation of the operators $R_{75,75}(\varphi;u,v)$ (green), $R^G_{75,75}(\varphi;u,v)$ (red) and $B^G_{75,75}(\varphi;u,v)$ (yellow) to $\varphi(u,v)$ (blue) on $[0,3]\times[0,3]$. One can see that the approximation of $B^G_{75,75}(\varphi;u,v;\beta_2)$ (yellow) to $\varphi(u,v)$ (blue) is the best. $B^G_{75,75}(\varphi;u,v;\beta_2)$ (yellow) is so close to $\varphi(u,v)$ (blue) that it almost coincides.

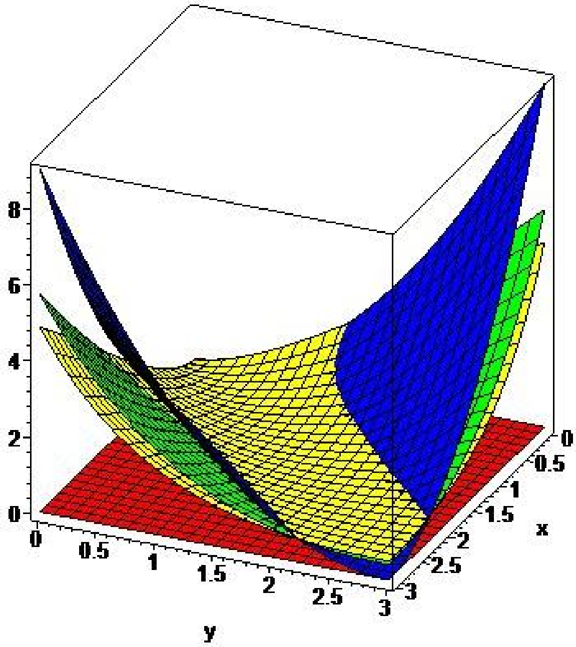

Figure 1. Approximation of $R^G_{n,n}(\varphi)$ to φ (blue) on $[0,3] \times [0,3]$ for $n = 5$ (red), $n = 25$ (yellow), $n = 75$ (green).

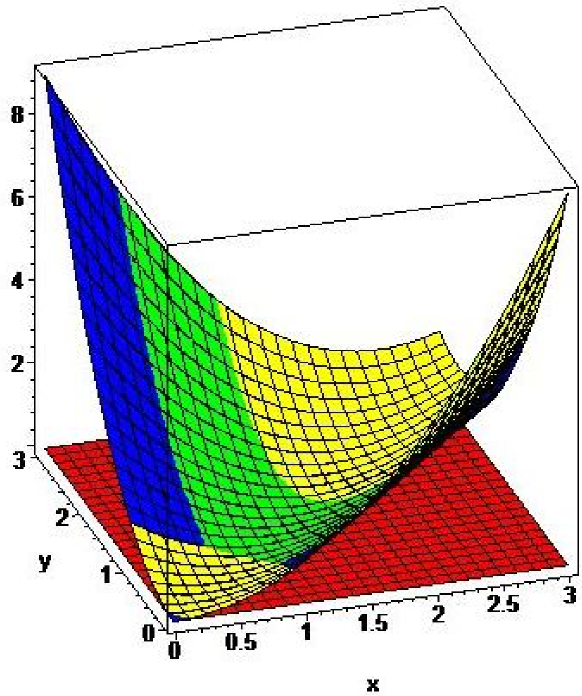

Figure 2. Approximation of $B^G_{n,n}(\varphi)$ to φ (blue) on $[0,3] \times [0,3]$ for $n = 5$ (red), $n = 25$ (yellow), $n = 75$ (green).

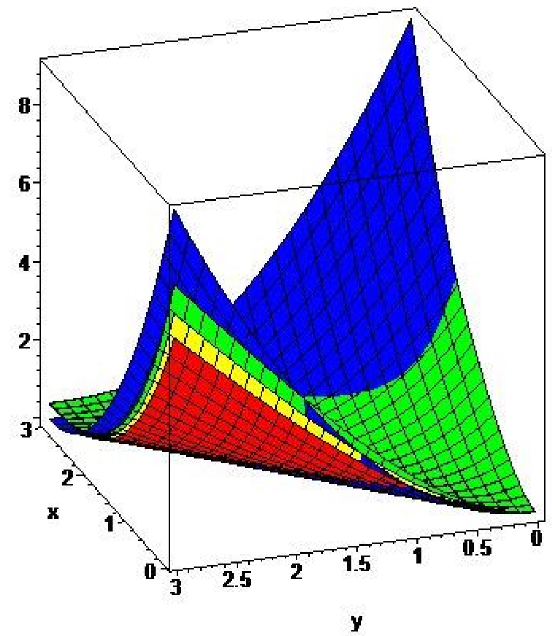

Figure 3. Comparison by Approximation of $R^{C}_{75,75}(\varphi)$ to φ (blue) on $[0,3] \times [0,3]$ for $\beta_1 = 1 - 5n^{-1}$ (red), $\beta_2 = 1 - 10n^{-1}$ (yellow) and $\beta_3 = 1 - 15n^{-1}$ (green).

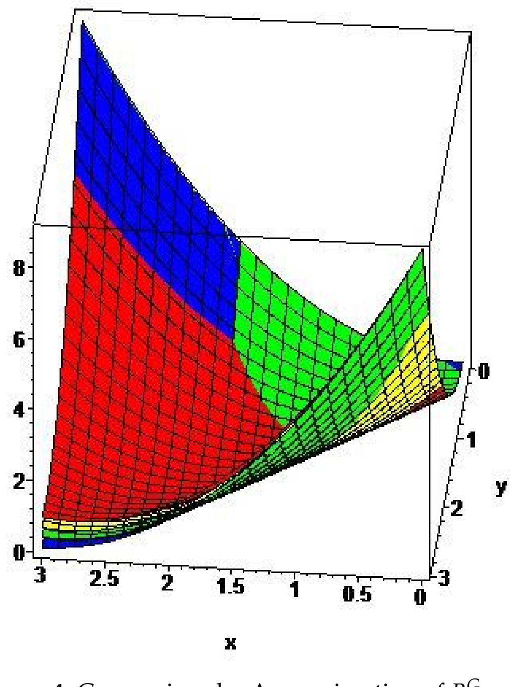

Figure 4. Comparison by Approximation of $B^{C}_{75,75}(\varphi)$ to φ (blue) on $[0,3] \times [0,3]$ for $\beta_1 = 1 - 5n^{-1}$ (red), $\beta_2 = 1 - 10n^{-1}$ (yellow) and $\beta_3 = 1 - 15n^{-1}$ (green).

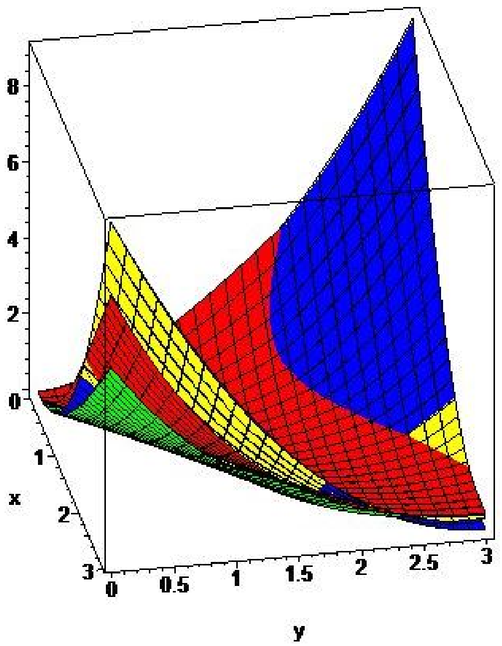

Figure 5. Comparison by Approximation of $R_{75,75}(\varphi)$ (green), $R^G_{75,75}(\varphi)$ (red) and $B^G_{75,75}(\varphi)$ (yellow) to φ (blue) on $[0,3] \times [0,3]$.

6. Conclusions

In this paper, we have introduced the tensor-product kind bivariate operator $R^G_{n,m}$ of the generalized Bernstein-type rational function R^G_n defined in [4] and its GBS (generalized Boolean sum) operator $B^G_{n,m}$, and we have investigated their approximation properties on rectangular region $[0,r_1] \times [0,r_2]$ such that $r_1, r_2 > 0$. Moreover, we have given some graphical comparisons visualizing the convergence of the tensor-product kind bivariate operator and its GBS operator, which also compare their convergence with the bivariate operator $R_{n,m}$ defined in [2].

The results of this paper demonstrate that the GBS operator $B^G_{n,m}$ possesses at least a better approximation than the tensor-product kind bivariate operator $R^G_{n,m}$, while the tensor-product kind bivariate operator $R^G_{n,m}$ has at least a better approximation than the bivariate operator $R_{n,m}$ defined in [2].

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