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Stability of Solutions to Systems of Nonlinear Differential Equations with Discontinuous Right-Hand Sides: Applications to Hopfield Artificial Neural Networks

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Abstract: In this paper, we study the stability of solutions to systems of differential equations with discontinuous right-hand sides. We have investigated nonlinear and linear equations. Stability sufficient conditions for linear equations are expressed as a logarithmic norm for coefficients of systems of equations. Stability sufficient conditions for nonlinear equations are expressed as the logarithmic norm of the Jacobian of the right-hand side of the system of equations. Sufficient conditions for the stability of solutions of systems of differential equations expressed in terms of logarithmic norms of the right-hand sides of equations (for systems of linear equations) and the Jacobian of right-hand sides (for nonlinear equations) have the following advantages: (1) in investigating stability in different metrics from the same standpoints, we have obtained a set of sufficient conditions; (2) sufficient conditions are easily expressed; (3) robustness areas of systems are easily determined with respect to the variation of their parameters; (4) in case of impulse action, information on moments of impact distribution is not required; (5) a method to obtain sufficient conditions of stability is extended to other definitions of stability (in particular, to p-moment stability). The obtained sufficient conditions are used to study Hopfield neural networks with discontinuous synapses and discontinuous activation functions.

Keywords: differential equations with discontinuous right-hand sides; Hopfield artificial neural networks; stability

MSC: 34D20; 34A36

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1. Introduction

Hopfield, Cohen–Grossberg and similar neural networks have been actively studied recently due to their applications in physics and engineering [1–4]. Hopfield neural networks (HNNs) have found many applications in associative memory, repetitive learning, classification of patterns, optimization problems and many others.

Today, two basic mathematical models are employed for neural network research: either local field neural network models or static neural models. The basic model of local field neural network is described as

$$\frac{dx_i(t)}{dt} = -x_i(t) + \sum_{j=1}^{n} w_{ij} g_j(x_j(t)) + I_i, i = 1, 2, \ldots, n, \tag{1}$$

where g_i is a function of the ith neuron activation, x_i is the state of the ith neuron, I_i is the external input imposed on the ith neuron, w_{ij} stands for the synaptic connectivity value between the jth neuron and the ith neuron, and n is the number of neurons in the network.

A static neural network is defined by the system of equations

$$\frac{dx_i(t)}{dt} = -x_i(t) + g_i\left(\sum_{j=1}^{n} w_{ij}x_j(t) + I_i\right), i = 1, 2, \ldots, n, \tag{2}$$

where we used the same notation as above.

The local fields neural network (1) was introduced by Hopfield and it is the Hopfield neural network that is usually referred to in the literature. The neural network (1) models bidirectional associative memory networks [5] and cellular neural networks [6].

Static neural networks (2) are often referred to as Cohen–Grossberg networks. They are widely used in optimization problems and in modeling brain processes, so-called brain-state-in-a-box neural networks [7]. The research on stability for models such as (2) was opened with the classical work [8].

The stability results for the basic model (2) as well as the results for a more general model,

$$\frac{dx_i(t)}{dt} = a_i(x_i)[b_i(x_i) - \sum_{i=1}^{n} c_{ij}\varphi_j(x_j)]$$

were obtained in [8,9].

Below, we present a brief review of the papers devoted to the stability of solutions for systems of ordinary differential equations with discontinuous right-hand sides. Here, we examined the stability of dynamic neural networks and obtained sufficient conditions for their absolute and local asymptotic stability. The fixed points of the neural network are associated with local minima of the network energy function. Interest in seeking for local minima is due to the study of the memory problem for neural networks. Clearly, the more local minima a neural network has, the greater potential memory it holds.

When solving computational mathematics problems on neural networks, asymptotically stable networks appear to be more preferable in general.

The derivation of sufficient conditions for stable neural networks in general is a rather complicated problem. Its solution is known only in a few special cases.

In [10], sufficient conditions for stability for neural networks (1) have been obtained based on Gershgorin circles. In [11], sufficient conditions for the global stability of solutions of systems of Equation (1) have been obtained by a mapping method. In [12], the results of works [10,11] have been generalized and new Lyapunov functions have been constructed.

In [13], it was proven that the diagonal stability of the interconnection matrix implies the existence and uniqueness of an equilibrium and global stability of the equilibrium.

In [14], it was shown that the negative semidefiniteness of matrices ensures the stability of Hopfield neural networks described by the Equation (1). In [15], the number of sufficient conditions for the local exponential stability of HNNs was presented. In [16], the algorithm of matrix norms was applied to the study of nonlinear dynamical systems.

The stability of recurrent systems that model identification problems was investigated in [17].

Extensive literature is devoted to researching the stability of neural networks with various time delays [18–22].

Constructing a mathematical model, we have to abstract from many phenomena—for example, from the uncertainty. In [21], the stability of fuzzy cellular neural networks based on the union of cellular neural networks and fuzzy logic methods has been studied. The cellular neural networks are modeled by systems of differential equations with discontinuous right-hand sides.

Along with continuous activation functions, there are a great number of applications that are modeled by neural networks with discontinuous activation functions. Similar models have been studied in [23].

The theory of differential equations with discontinuous right-hand sides is given in [24].

Recall, following [24], the definitions of solutions for differential equations with discontinuous right-hand sides and their stability.

Consider an equation or a system of equations in vector form

$$\frac{dx}{dt} = f(t, x), \tag{3}$$

where $f(t, x)$ is a piece-wise continuous function or a vector function in domain $\Omega : \{x \in R_n, t \in [0, \infty)\}$; M is a measure zero set of the function $f(t, x)$ discontinuity points.

Each point (t, x) we associate with a set $F(t, x)$ in n dimensional space. This set is constructed as follows. If the function $f(t, x)$ is continuous at the point (t, x), the set $F(t, x)$ contains just one point which matches with $f(t, x)$. If f has a discontinuity point at (t, x), the set $F(t, x)$ is defined according to the related physical problem. One such method is described in Section 1.5 in [25].

Definition 1 ([24])**.** *A solution of the Equation (3) is called a solution of differential inclusion*

$$\frac{dx}{dt} \in F(t, x), \tag{4}$$

i.e., absolutely continuous vector function $x(t)$ defined in interval or segment I and for which the inclusion $\frac{dx}{dt} \in F(t, x)$ is satisfied almost everywhere in I.

Definition 2 ([24])**.** *The solution $x = \varphi(t), t_0 \leq t < \infty$ of differential inclusion (4) is called stable if, for any $\epsilon > 0$, there exists $\delta > 0$ so that, for every $\tilde{x}_0$, $|\tilde{x}_0(t_0) - \varphi(t_0)| < \delta$ each solution $\tilde{x}(t)$ with the initial condition $\tilde{x}(t_0) = \tilde{x}_0$, $t_0 \leq t < \infty$ exists and satisfies the inequality*

$$|\tilde{x}(t) - \varphi(t)| < \epsilon \quad \text{for} \quad t_0 \leq t < \infty.$$

Definition 3 ([24])**.** *The solution $x = \varphi(t), t_0 \leq t < \infty$ of differential inclusion (4) is called asymptotically stable if it is stable and, in addition, $\lim\limits_{t \to \infty} |\tilde{x}(t) - \varphi(t)| = 0$.*

Definition 4 ([24])**.** *The solution $x = \varphi(t)$, $t_0 \leq t < \infty$, of differential inclusion (4) is called stable in general if it is asymptotically stable for any initial $x_0 \in R_n$, where R_n is n-dimensional vector space.*

Intense research on the stability of systems of ordinary differential equations with discontinuous right-hand sides began in the middle of the last century in connection with increasing interest in automatic control problems. In addition to the issues of automatic control, automatic regulation [26] and the theory of relay systems, differential equations with discontinuous right-hand sides are widely used to model various problems in physics and engineering—in particular, the classical problem of dry friction [27]. Differential equations for automatic control with variable structures and discontinuous right-hand sides are obtained from differential equations with continuous right-hand sides when passing to the limit along a parameter [24].

Today, the stability of solutions of systems of ordinary differential equations with discontinuous right-hand sides is an active and growing field.

This is because there are numerous applications of systems of differential equations with discontinuous right-hand sides (Filippov systems) for various problems in physics, techniques, biology and medicine. A detailed bibliography is given in [28].

Recently, stability theory with discontinuous coefficients has been extended to numerical mathematics. There are widely used various methods to determine solutions for systems of linear and nonlinear algebraic equations.

In [29], the authors have developed a continuous method for solving nonlinear operator equations. Each nonlinear operator equation is assigned with the Cauchy problem. Convergence of the method is based on Lyapunov stability theory.

Collocation methods for solving initial and boundary problems for differential equations and the theory of B, D, G, P stability of their solutions has been developed and presented in [30–32]. The latter also contains an extensive bibliography.

In [33], the second Lyapunov's method was used to investigate semistability finite-time stability differential inclusions for systems of differential equations with discontinuity of the first kind on various manifolds. Semistability has a wider range of applications than the stability condition.

Research is performed in several directions: (1) systems of differential equations with one [34] and two [35] relays have been studied.

Stability of solutions of differential equations with one relay

$$\frac{dx_i(t)}{dt} = p_i \mathrm{sgn}\, x_i + \sum_{j=1}^{n} c_{ij} x_j, i = 1, 2, \dots, n. \tag{5}$$

has been studied in [34].

In [35], the author investigates the stability of solutions of systems of differential equations

$$\begin{aligned} \tfrac{dx_1(t)}{dt} &= a_1 sgn x_1(t) + b_1 sgn x_2(t) + \textstyle\sum_{j=2}^{n} c_{1,j} x_j(t), \\ \tfrac{dx_2(t)}{dt} &= a_2 sgn x_1(t) + b_2 sgn x_2(t) + \textstyle\sum_{j=2}^{n} c_{2,j} x_j(t), \\ \tfrac{dx_i(t)}{dt} &= \textstyle\sum_{j=1}^{n} c_{i,j} x_j(t), i = 3, .., n, \end{aligned} \tag{6}$$

with constant coefficients.

Stability of solutions of systems of differential equations with relay [34,35] is based on the study of transfer functions. There have been obtained necessary and sufficient conditions for the stability of solutions for the systems (5), (6) expressed in terms of coefficients of equations.

Numerous works have been devoted to the stability of systems of nonlinear switching differential equations. For a bibliography, see [36].

Another class of problems is related to the study of sliding modes in automatic regulation and control systems [37]. It is interesting to note that sliding modes are present in ecology models [28].

When studying the stability of systems of nonlinear differential equations with discontinuous right-hand sides, Lyapunov's functions method [38–40] has been used.

Stability of neural networks described by the equations

$$\frac{dx_i}{dt} = -c_i x_i + \sum_{j=1}^{n} a_{ij} \varphi_j(x_j), \tag{7}$$

$i = 1, 2, \dots, n$ and more general equations

$$\frac{dx_i}{dt} = -c_i x_i - \sum_{j=1}^{n} a_{ij} \sum_{k=1}^{n} a_{jk} g_k(x_k), \tag{8}$$

$i = 1, 2, \dots, n$ by the second Lyapunov method has been investigated in [41] assuming that the functions $g_k(x)$, $k = 1, 2, \dots, n$ are continuous. The following conditions are imposed on the functions $\varphi_j(x_j)$

A1. Each function $\varphi_j(x_j)$ is defined everywhere for $-\infty < x_j < \infty$, continuous and one-valued;

A2. Each function $\varphi_j(x_j)$ lies in the first and the third quadrant; $x_j \neq 0$, moreover, the inequalities are fulfilled $x_j \varphi_j(x_j) > 0$, $j = 1, 2, \dots, n$;

A3. $\lim_{|x_j| \to \infty} \int_0^{x_j} \varphi_j(\rho) d\rho = +\infty$, $j = 1, 2, \dots, n$.

Stability of neural networks with discontinuous coefficients $g_i(x)$, $i = 1, 2, \ldots, n$ has been studied in [42,43].

It was assumed in [42] that

$$g_i(x) = g(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases}$$

$i = 1, 2, \ldots, n$. To ensure the stability of a neural network, the method based on the majorization of the nonlinear part of Equation (7) by a constant and further solving the differential has been proposed.

Stability of the solutions of Equation (7) with discontinuous nonlinear functions was investigated with the second Lyapunov method in [43]. The study of sliding mode stability is also reported in [43].

The detailed research of neural networks including Hopfield networks is given in [44]. The stability of neural networks with various activation functions in general has been studied, as well as the stability at separated stationary points. The basic technique of neural network stability in [44] appears to be the use of Lyapunov's and energy functions. In [44], one can find an extensive bibliography on the stability of neural networks described with differential and difference equations.

The exponential stability of a Hopfield neural network on the timeline has been investigated in [45]. Stability of the neural networks described by differential equations

$$\frac{dx_i(t)}{dt} = -e_i x_i + \sum_{j=1}^{n} g_{ij}(x_j), i = 1, 2, \ldots, n, \tag{9}$$

with functions $g_{ij}(x_j), i, j = 1, 2, \ldots, n$ having discontinuities of the first kind at separate points has been studied in [25,46].

In this paper, we investigate the stability of solutions of systems of linear and nonlinear equations with discontinuous right-hand sides. We obtained sufficient conditions for the asymptotic stability for systems of differential equations used when studying HNNs' stability with discontinuous synapses and activation functions.

We study the stability of solutions for systems of differential equations regardless of how an inclusion equation is defined. With this approach, it is essential to use the first Lyapunov method.

It is possible to suggest that in applying the second Lyapunov method, one has to construct separate Lyapunov–Krasovski functionals for each area where the right-hand side of the differential equation system is continuous.

The paper is divided into the Introduction, three sections and the Conclusions. Section 2 introduces the definitions and the notation used throughout the paper. Section 3 examines the stability of solutions of differential equations with discontinuous right-hand sides. In Section 4, we analyze the stability of Hopfield neural networks. The obtained results are drawn in the final section.

2. Definitions and Notations

We now introduce a few definitions used in this paper.

Here, $D^k g(t, u_1, \ldots, u_n)$ stands for a partial derivative $D^k g(t, u_1, \ldots, u_n) = \partial g(t, u_1, \ldots, u_n)/\partial u_k$, $k = 1, 2, \ldots, n$.

Moreover, we employ the following notation: $B(a, r) = \{z \in B : \|z - a\| \leq r\}$, $S(a, r) = \{z \in B : \|z - a\| = r\}$, $Re(K) = \Re(K) = (K + K^*)/2$, $\Lambda(K) = \lim\limits_{h \downarrow 0}(\|I + hK\| - 1)h^{-1}$. Here, B is a Banach space, $a \in B$, K is a linear and bounded operator on B, $\Lambda(K)$ is the logarithmic norm [47] of the operator K, K^* is the conjugate operator to K, and I stands for the identity operator.

The main properties of the logarithmic norm are given in [47].

If A is an $n \times n$ matrix, then $\Lambda(A)$ can readily be computed for the corresponding norms of linear vector spaces.

The logarithmic norm is known for operators in the most frequently used spaces.

Let $A = \{a_{ij}\}$, $i, j = 1, 2, \dots, n$ be a real matrix.

In the n-dimensional space R_n of vectors $x = (x_1, \dots, x_n)$, the following norms are often used:

- $\|x\|_1 = \sum\limits_{i=1}^{n} |x_i|$;
- $\|x\|_2 = \max\limits_{1 \le i \le n} |x_i|$;
- $\|x\|_3 = (\sum\limits_{i=1}^{n} x_i^2)^{1/2}$.

Below are some expressions of the logarithmic norm of a matrix $A = (a_{ij})$ corresponding to the norms of the vectors given above:

$$\Lambda_1(A) = \max_{1 \le j \le n} \left(a_{jj} + \sum_{i \ne j} |a_{ij}| \right);$$

$$\Lambda_2(A) = \max_{1 \le i \le n} \left(a_{ii} + \sum_{j \ne i} |a_{ij}| \right);$$

$$\Lambda_3(A) = \lambda_{\max} \left(\frac{A + A^*}{2} \right),$$

where A^* is the conjugate matrix for A.

3. Stability of Solutions to Equations Systems with Discontinuous Right-Hand Sides

3.1. Stability of Solutions to Linear Equations Systems with Discontinuous Coefficients

Consider the Cauchy problem

$$\frac{dx_i(t)}{dt} = \sum_{j=1}^{n} a_{ij}(t) x_j(t), t \ge 0, \tag{10}$$

$$x_i(0) = x_i, i = 1, 2, \dots, n, \tag{11}$$

with discontinuous coefficients $a_{ij}(t), i, j = 1, 2, \dots, n$.

We assume that the functions $a_{ij}(t)$ are continuous everywhere except a countable set of points $\zeta_1, \zeta_2, \dots$, where the functions have discontinuities.

The following statement is valid.

Theorem 1. *Let the following conditions be satisfied:*

(1) Functions $a_{ij}(t)$ are continuous everywhere except a countable set of points $\zeta_1, \zeta_2, ..$, where the functions have discontinuities. There is at most a finite number of discontinuities on each $[0, A], 0 < A < \infty$. The coefficients $\{a_{ij}(t)\}$ at $\zeta_1, \zeta_2, \dots$, have discontinuities of the first kind or discontinuities of the second kind integrable in L-metric:

$$\int_{\zeta_l^1}^{\zeta_{l+1}^1} |a_{ij}(t)| dt \le c_{lij} < \infty, i, j = 1, 2, \dots, n, l = 1, 2, \dots$$

Here, ζ_l^1 are the points that satisfy $\zeta_i < \zeta_i^1 < \zeta_{i+1}, i = 0, 1, \dots, \zeta_0 = 0$;

(2) The Cauchy problem (10)–(11) has a solution for $t \ge 0$ and any initial conditions;

(3) The inequality $\Lambda(A(t)) \le -\kappa, \kappa > 0$ is valid everywhere except a set of points $\zeta_1, \zeta_2, \dots$ Here, $A(t) = \{a_{ij}(t)\}_{i,j=1}^{n}$.

Then, zero solution of the system (10) is asymptotically stable in general.

Set $\Lambda(A(t))$ equal to zero at discontinuity points ζ_i. By the theorem, we have a finite number of discontinuities in each time interval. Thus, it does not change the value $\int_0^t \Lambda(A(\tau))d\tau$.

Proof of Theorem 1. Consider a time interval $[0, \zeta_1)$. The Wintner estimate is valid [47] within this interval

$$\|x(t)\| \leq \|x(0)\| \exp\left\{\int_0^t \Lambda(A(\tau))d\tau\right\}, t \in [0, \zeta_1). \tag{12}$$

The function $\|x(t)\|$ is continuous for $t \geq 0$, and then the Inequality (12) is correct for $t \in [0, \zeta_1]$. Therefore,

$$\|x(\zeta_1)\| \leq \|x(0)\| \exp\left\{\int_0^{\zeta_1} \Lambda(A(\tau))d\tau\right\}.$$

Consider an interval $[\zeta_1, \zeta_2]$. First, assume that functions $a_{ij}(t)$ have discontinuities of the first kind at ζ_2. Let $A^+(t) = \{a^+_{ij}(t)\}, i, j = 1, 2, \dots, n$ be a matrix with elements defined by

$$a^+_{ij}(t) = \begin{cases} a_{ij}(t), t \neq \zeta_2, \\ \lim\limits_{t \to \zeta_2 + 0} a_{ij}(t), t = \zeta_2. \end{cases}$$

We take $x_i(\zeta_1), i = 1, 2, \dots, n$, as initial values. Repeat the arguments above, for $t \in [\zeta_1, \zeta_2]$, and we have

$$\begin{aligned} &\|x(t)\| \leq \|x(\zeta_1)\| \exp\left\{\textstyle\int_{\zeta_1}^t \Lambda(A^+(\tau))d\tau\right\} = \\ &= \|x(\zeta_1)\| \exp\left\{\textstyle\int_{\zeta_1}^t \Lambda(A(\tau))d\tau\right\} \leq \|x(0)\| \exp\left\{\textstyle\int_0^t \Lambda(A(\tau))d\tau\right\}. \end{aligned}$$

Next, we consider the case where functions $a_{ij}(t)$ have discontinuities of the second kind and integrals $\int_0^\infty |a_{ij}(t)|dt$ exist. Obviously, for $t \in [0, \zeta_1)$, the Inequality (12) is valid. Since the function $x(t)$ is continuous, the inequality is valid on the interval $[0, \zeta_1]$. From the continuity of $\|x(t)\|$, it follows that a point $\zeta_1^1 (\zeta_1 < \zeta_1^1 < \zeta_2)$ such that

$$\|x(t)\| \leq \|x_0\| + \|x_0\| \frac{\exp\left\{\int_0^t \Lambda(A(\tau))d\tau\right\} - 1}{2}$$

exists for $t \in [\zeta_1, \zeta_1^1]$.

By taking $x_i(\zeta_1^1), i = 1, 2, \dots, n$ as initial conditions and repeating the arguments above, we verify immediately the validity of the inequality

$$\|x(t)\| \leq \|x_0\| \exp\left\{\int_0^t \Lambda(A(\tau))d\tau\right\},$$

for $0 \leq t \leq \zeta_2$. Repeating the process in each interval $[\zeta_l, \zeta_{l+1}]$, we can observe that the inequality is correct for $0 \leq t \leq \infty$. □

Let us consider the case when coefficients $a_{ij}(t)$ have a countable number of discontinuity points.

Let the function $a_{11}(t)$ have a countable number of discontinuity points located in interval $[b_1, b_2]$ with measure $\Delta = |b_2 - b_1|, b_1 > 0$.

The following assertion is true.

Theorem 2. *Let the following conditions be satisfied:*

(1) The functions a_{ij} are continuous everywhere except a countable number of points located in the interval $[b_1, b_2], b_1 > 0, \Delta = |b_2 - b_1|$.

(2) The Cauchy problem (10), (11) has a solution for all $t \geq 0$ and for any initial conditions.

(3) The inequality $\Lambda(A(t)) \leq -\kappa, \kappa > 0$ holds everywhere except the interval $[b_1, b_2]$. Here, $A(t) = \{a_{ij}(t)\}, i, j = 1, 2, ..., n$;

(4) The inequality is valid

$$\int_0^{b_1} \Lambda(A(\tau))d\tau + A\Delta < 0,$$

where $A = \sup_{\tau \in [b_1, b_2]} \|A(\tau)\|$.

Then, a trivial solution of the system of Equation (10) is asymptotically stable.

Proof of Theorem 2. Consider $[0, b_1]$. From Wintner inequality [47], it follows that for $t \in [0, b_1]$

$$\|x(t)\| \leq \|x(0)\| \exp\left\{ \int_0^t \Lambda(A(\tau))d\tau \right\}.$$

Take $[b_1, b_2]$. Here, we have

$$x(t) = x(b_1) + \int_{b_1}^t A(\tau)x(\tau)d\tau,$$

and the integral is understood in the sense of Lebesgue.

Thus,

$$\|x(t)\| \leq \|x(b_1)\| + \int_{b_1}^t \|A(\tau)\| \|x(\tau)\| d\tau \leq \|x(b_1)\| + A \int_{b_1}^t \|x(\tau)\| d\tau.$$

where $A = \max_{b_1 \leq t \leq b_2} \|A(t)\|$.

From Gronwall–Bellman inequality, it follows that

$$\|x(t)\| \leq \|x(b_1)\| \exp\{A(t - b_1)\}.$$

Thus, for $t \in [b_1, b_2]$,

$$\begin{aligned} &\|x(t)\| \leq \|x(0)\| \exp\left\{ \int_0^{b_1} \Lambda(A(\tau))d\tau + A(t - b_1) \right\} \\ &\leq \exp\left\{ \int_0^{b_1} \Lambda(A(\tau))d\tau + A(b_2 - b_1) \right\} \|x(0)\|. \end{aligned} \tag{13}$$

Therefore, if $\int_0^{b_1} \Lambda(A(\tau))d\tau + A\Delta < 0$, then $\|x(b_2)\| \leq \|x(0)\|$ and a trivial solution of the system of Equation (10) is stable. □

It is easy to see that the obtained results can be extended to systems of switching differential equations. At the same time, the stability condition is extended to systems of differential equations with an arbitrary number of relays. Moreover, the suggested method allows one to obtain sufficient conditions for the stability of solutions of systems of nonlinear equations with relay. Similarly, based on the results presented in Sections 2 and 3, one can formulate sufficient conditions for the stability of switching systems.

Example. We consider a system of differential equations with relay

$$\frac{dx_k(t)}{dt} = \sum_{l=1}^m a_{kl}(t) sgn\, x_l(t) + \sum_{l=1}^n b_{kl}(t)x_l(t), \;\; k = 1, 2, \ldots, n.$$

Theorem 1 implies that for the asymptotic stability of the trivial solution of this system, it is sufficient to fulfill the following conditions: for each $t \in [0, \infty)$

$$b_{kk}(t) + \sum_{l=1}^{m} |a_{kl}(t)| + \sum_{l=1, l\neq k}^{n} |b_{kl}(t)| \leq -\xi, \xi > 0.$$

3.2. Stability of Solutions for Systems of Nonlinear Non-Autonomous Differential Equations with Discontinuous Right-Hand Sides

First, let us recall the sufficient stability conditions for systems of nonlinear differential equations with continuous right-hand sides that we gave previously [48], and which we extensively use below.

Consider the system of equations

$$\frac{dx_i(t)}{dt} = a_i(t, x_1(t), \ldots, x_n(t)),\ i = 1, 2, \ldots, n, \tag{14}$$

with the initial conditions

$$x_i(0) = x_i,\ i = 1, 2, \ldots, n. \tag{15}$$

Let $x^*(t) = (x_1^*(t), \ldots, x_n^*(t))$ be a steady-state solution of the Cauchy problems (14) and (15).

Let the functions $a_i(t, u_1, ..., u_n)$ be continuous with respect to the first variable and have partial derivatives with respect to other variables satisfying the Lipschitz condition with a coefficient A:

$$|D^j a_i(t, x_1^*, \ldots, x_n^*) - D^j a_i(t, y_1^*, \ldots, y_n^*)| \leq A(|x_1^* - y_1^*| + \ldots + |x_n^* - y_n^*|),\ i, j = 1, \ldots, n. \tag{16}$$

Let $\chi = \text{const} > 0$. Let, for $t \in [0, \infty)$, the following conditions be satisfied

$$D^i a_i(t, x_1^*(t), \ldots, x_n^*(t)) + \sum_{j=1, j\neq i}^{n} \left| D^j a_i(t, x_1^*(t), \ldots, x_n^*(t)) \right| < -\chi < 0,\ i = 1, \ldots, n. \tag{17}$$

Theorem 3 ([48])**.** *Let the system (14) have a steady-state solution $x^*(t) = (x_1^*(t), \ldots, x_n^*(t))$. Let the functions $a_i(t, x_1, \ldots, x_n), i = 1, 2, \ldots, n$ be continuous with respect to the first variable, continuously differentiate to other variables and partial derivatives satisfy the Lipschitz condition (16). Let, for all $t \geq 0$, the conditions (17) be satisfied. Then, the steady-state solution $x^*(t)$ of the system of Equation (14) is asymptotically stable in the R_n^3 space metric of n-dimensional vectors $v = (v_1, \ldots, v_n)$ with norm $\|v\| = \max_{1\leq j\leq n} |v_j|$.*

Consider a system of nonlinear equations

$$\frac{du_i(t)}{dt} = a_i(t, u_1(t), \ldots, u_n(t)),\ i = 1, 2, \ldots, n,\ t \geq 0, \tag{18}$$

with initial conditions

$$u_i(0) = u_i,\ i = 1, 2, \ldots, n. \tag{19}$$

Their right-hand sides are continuous everywhere except a countable set of values $(\zeta_i, u_1^i, \ldots, u_n^i), i = 1, 2, \ldots,$ in which they have discontinuities.

For the sake of simplicity, we consider two cases here:

(1) there are discontinuities with respect to variable t;

(2) there are discontinuities with respect to variable u_1.

Consider the first case. Assume that the functions $a_j(t; u_1, \ldots, u_n), j = 1, 2, \ldots, n$ have discontinuities with respect to t at points $\zeta_i, i = 1, 2, \ldots; 0 < \zeta_1 < \zeta_2 < \ldots$. For convenience, let $j = 1$. The functions $a_i(t; u_1, \ldots, u_n), i = 2, 3, \ldots, n$ are assumed to be continuous.

We impose the following constraints on $a_i(t, u_1, \ldots, u_n), i = 1, 2, \ldots, n$:

(1) functions $a_i(t, u_1, \dots, u_n)$, $i = 2, \dots, n$ are continuous with respect to $t(t \in [0, \infty))$ and have partial derivatives that satisfy the Lipschitz condition with coefficient q with respect to other variables

$$|D^k a_i(t, u_1, \dots, u_n) - D^k a_i(t, v_1, \dots, v_n)| \leq q \sum_{i=1}^{n} |u_i - v_i|, k = 1, 2, \dots, n,\ t \in [0, \infty); \quad (20)$$

(2) the function $a_1(t, u_1, \dots, u_n)$ is continuous for $t \in [0, \zeta_1) \cup_{i=1}^{\infty} (\zeta_i, \zeta_{i+1})$. For t, it has partial derivatives with respect to other variables satisfying the Lipschitz condition with coefficient q

$$|D^j a_1(t, u_1, \dots, u_n) - D^j a_1(t, v_1, \dots, v_n)| \leq q \sum_{i=1}^{n} |u_i - v_i|, , j = 1, 2, \dots, n; \quad (21)$$

(3) for $t \in [0, \infty)$

$$D^i a_i(t, x_1^*(t), \dots, x_n^*(t)) + \sum_{j=1, j \neq i}^{n} \left| D^j a_i(t, x_1^*(t), \dots, x_n^*(t)) \right| < -\chi < 0,\ i = 2, 3, \dots, n; \quad (22)$$

(4) for $t \in [0, \zeta_1) \cup_{i=1}^{\infty} (\zeta_i, \zeta_{i+1})$

$$D^1 a_1(t, x_1^*(t), \dots, x_n^*(t)) + \sum_{j=2}^{n} \left| D^j a_1(t, x_1^*(t), \dots, x_n^*(t)) \right| < -\chi < 0, \quad (23)$$

where $\chi = \text{const} > 0$.

Now, consider the time interval $t \in [0, \zeta_1]$.

Let $\|u(0)\| \leq \delta$, where $\delta \leq \chi/(4qn^2)$. In [48], it was shown that in order to fulfill the conditions (20)–(23) in time interval $t \in [0, \zeta_1]$, the trajectory of the solution of the Cauchy problems (18) and (19) does not leave a ball $B(0, \delta)$. It was also shown that for $t \in [0, \zeta_1]$, it holds that

$$\|u(t)\| \leq e^{-\chi t/4} \|u(0)\| \leq e^{-\chi t/4} \delta. \quad (24)$$

Since the function $\|u(t)\|$ is continuous for $t \in [0, \infty)$, we can find $\zeta_1', \zeta_1 < \zeta_1' < \zeta_2$ such that $|\zeta_1' - \zeta_1| < |\zeta_2 - \zeta_1|/10$ and $\|u(t)\| \leq e^{-\chi \zeta_1/8} \|u(0)\|$ for $t \in [\zeta_1, \zeta_1']$. Obviously, for $t \in [\zeta_1', \zeta_2]$, $\|u(t)\| \leq e^{-\chi(t - \zeta_1')/4} \|u(\zeta_1')\| \leq e^{-\chi(t - \zeta_1')/4} e^{-\chi \zeta_1/8} \|u(0)\|$.

For $t = \zeta_2$, we have $\|u(\zeta_2)\| \leq e^{-\chi(\zeta_2 - \zeta_1')/4} e^{-\chi \zeta_1/8} \|u(0)\| < e^{-\chi \Delta_1/8} e^{-\chi \Delta_0/8} \|u(0)\|$. Here and below, $\Delta_k = |\zeta_{k+1} - \zeta_k|$.

Consider the interval $[\zeta_2, \zeta_3]$. From function $\|u(t)\|, t \in [0, \infty)$ continuity, it follows that there is an interval $[\zeta_2, \zeta_2']$ such that $|\zeta_2' - \zeta_2| < \Delta_3/10$ and $\|u(t)\| \leq e^{-\chi(\Delta_0 + \Delta_1)/8} \|u_0\|$ for $t \in [\zeta_2, \zeta_2']$. Then, if $t \in [\zeta_2', \zeta_3]$ $\|u(t)\| \leq e^{-\chi(t - \zeta_2')/4} \|u(\zeta_2')\| \leq e^{-\chi(t - \zeta_2')/4} e^{-\chi(\Delta_0 + \Delta_1)/8} \|u_0\|$, $\|u(\zeta_3)\| < e^{-\chi(\zeta_3 - \zeta_2)/8} e^{-\chi(\Delta_0 + \Delta_1)/8} \|u_0\| = e^{-\chi(\Delta_0 + \Delta_1 + \Delta_2)/8} \|u_0\|$.

Repeating the process, we have for $t \in [\zeta_k, \zeta_{k+1}] : \|u(\zeta_{k+1})\| < e^{-\chi(\sum_{l=0}^{k} \Delta_l)/8}$. Therefore, $t \to \infty$ $\|u(t)\| \to 0$.

Asymptotic stability is proven.

Theorem 4. *Let the following conditions be fulfilled:*
(1) the Cauchy problems (14) and (15) has a steady-state solution $x^*(t)$, $x^*(t) = (x_1^*(t), \dots, x_n^*(t))$, $t \geq 0$;
(2) functions $a_{ij}(t, x_1, x_2, \dots, x_n)$ are continuous with respect to variables $x_1, \dots, x_n$ and have a set of countable discontinuities $\zeta_1, \dots, \zeta_n, ...$ with respect to t. Moreover, in each finite time interval $[0, T)$, there is at most a finite number of discontinuities;
(3) at every point of continuity with respect to t, functions $a_{ij}(t, x_1, \dots, x_n)$ have partial derivatives with respect to $x_1, \dots, x_n$ and satisfy the Inequalities (20), (21);
(4) the conditions (22), (23) are fulfilled.

Then, a steady-state solution of the Cauchy problems (14) and (15) is asymptotically stable.

Now, we move on to the case where functions $a_i(t, u_1, \ldots, u_n)$, $i = 1, 2, \ldots, n$ have discontinuities with respect to u_i, $i = 1, 2, \ldots, n$.

For simplicity, we restrict the discussion to the case where the function $a_1(t, u_1, \ldots, u_n)$ has a discontinuity at u_1 for $u_1 = u_1^*$.

Assume that a gap occurs at time $t = \eta_1 > 0$ and $u_1(\eta_1) = u_1^*$.

Let, for $-\infty < u_1 < u_1^*$, $-\infty < u_i < \infty, i = 2, \ldots, n$ and for $u_1^* < u_1 < \infty$, $-\infty < u_i < \infty, i = 2, \ldots, n$, functions $a_i(t, u_1, \ldots, u_n), i = 1, \ldots, n$ have partial derivatives that satisfy the Lipschitz condition $|D^j a_i(t, u_1, \ldots, u_n) - D^j a_i(t, v_1, \ldots, v_n)| \leq q \sum\limits_{i=1}^{n} |u_i - v_i|, j = 1, 2, \ldots, n$.

Consider a time interval $[0, \eta_1)$. The conditions of Theorem 3 are verified in each $[0, b] \subset [0, \eta_1)$. Therefore, for $t \in [0, \eta_1)$, the inequality occurs $\|u(t)\| \leq e^{-\chi t/4}\delta < \delta$.

Although the function $\|u(t)\|$ is continuous for $t \in [0, \infty)$, the inequality $\|u(t)\| \leq e^{-\chi t/4}\delta$ is valid for $t \in [0, \eta_1]$.

Consider special features of the transition of the Cauchy problem solution trajectories (18) and (19) through $t = \eta_1$.

There are two possible cases:

(1) there is an interval $(\eta_1, \eta_1 + \Delta_1]$, in which $u_1(t) \neq u_1(\eta_1)$;

(2) there is a time interval $[\eta_1, \eta_1 + \Delta_2]$, in which $u_1(t) = u_1(\eta_1)$.

We will study each case separately.

First, from continuity of the function $\|u(t)\|$, it follows that there is $h, h < \eta_1/10$ so that $t \in [\eta_1, \eta_1 + h]$, $\|u(t)\| < e^{-\chi t/8}\delta$.

Therefore, for $t \in [0, \eta_1 + h]$, the inequality $\|u(t)\| \leq e^{-\chi t/8}\delta$ is valid. Clearly, $\|u(\eta_1 + h)\| \leq e^{-\chi(\eta_1+h)/8}\delta$.

For $t \in [\eta_1 + h, \infty)$, the inequality $\|u(t)\| \leq e^{-\chi(t-\eta_1-h)/4}\|u(\eta_1 + h)\| \leq e^{-\chi(t-\eta_1-h)/4}e^{-\chi(\eta_1+h)/8}\delta \leq e^{-\chi t/8}\delta$ is valid.

Therefore, $\|u(t)\| \leq e^{-\chi t/8}\delta$, for $t \in [0, \infty)$.

Thus, for the first case, the stability of the steady-state solution for the system of Equation (18) is proven.

Now, we move on to the second case. Since $u_1(t) = u_1(\eta_1)$ for $t \in [\eta_1, \eta_1 + \Delta_1]$, in this time interval instead of (18), one should observe the following system of equations

$$0 = a_1(t, u_1(\eta_1), u_2(t), \ldots, u_n(t)). \tag{25}$$

$$\frac{du_i(t)}{dt} = a_i(t, u_1(\eta_1), u_2(t), \ldots, u_n(t)), \ i = 2, \ldots, n. \tag{26}$$

The system of Equation (26) is considered under initial condition $u_i(\eta_1) = u_i$, $i = 2, 3, \ldots, n$.

We assume that functions $u_2(t), \ldots, u_n(t)$ satisfy the condition (25) for $t \in [\eta_1, \eta_1 + \Delta_1]$.

The system of Equation (26) is studied similarly to the system (18) in the space of lower dimension. Sufficient conditions of stability for the solution of the system (26) for $t \in [\eta_1, \eta_1 + \Delta_1]$ are constructed similarly to the sufficient conditions of stability for the solution of the system (18) on the time interval $t \in [0, \zeta_1]$. We omit the details. Finally, we investigate the time interval $t \in [\eta_1 + \Delta_1, \infty]$ and employ the arguments given in [48].

Now, we must consider the case of a countable set of discontinuities. It suffices to observe the case where there are discontinuities with respect to u_1 for $\bar{u}_1^i$, $i = 1, 2, \ldots$. We suggest that the discontinuities occur at the time moments t_i^*: $a_1(t_i^*, \bar{u}_1^i(t_i^*), \ldots, u_n(t_i^*))$, $i = 1, 2, \ldots$. As above, we will assume that at each finite time interval $[0, T]$, there is a finite number of discontinuities.

Here, we also must consider two cases:

(1) $u_1^i(t) \neq \bar{u}_1^i(t_i^*)$ in an interval $t \in (t_i^*, t_i^* + h_i^1)$;

(2) there is an interval $[t_i^*, t_i^* + h_i]$, where $u_1^i(t) = \bar{u}_i^i(t_i^*)$.

For convenience, we observe the first case. The second one leads to the case of a system of a lower dimension.

To each time moment t_i^*, $i = 1, 2, \ldots$ associated with a function discontinuity $a_1(t, u_1(t), \ldots, u_n(t))$ we assign a number ζ_i' so that $t_i^* < \zeta_i' < t_{i+1}^*$, $|\zeta_i' - t_i^*| \leq |t_{i+1}^* - t_i^*|/4$, $i = 1, 2, \ldots$.

Repeating the above arguments for each time interval $[0, \zeta_1']$, $[\zeta_i', \zeta_{i+1}']$, $i = 1, 2, \cdots$, we verify the validity of the following statement.

Theorem 5. *Let the following conditions be fulfilled:*
(1) the Cauchy problems (14)–(15) has a steady-state solution $x^*(t)$, $x^*(t) = (x_1^*(t), \ldots, x_n^*(t))$;
(2) functions $a_{ij}(t, x_1, \ldots, x_n)$ are continuous with respect to variables $(t, x_1, \ldots, x_n)$ everywhere except a countable set of discontinuities with respect to variables $x_1, \ldots, x_n$ that occur at time moments t_i^*, $i = 1, 2, \ldots$. Moreover, in each finite time interval, there is a finite number of discontinuities;
(3) at continuity points, functions $a_{ij}(t, x_1, \ldots, x_n)$ have partial derivatives with respect to variables $x_1, \ldots, x_n$ that satisfy the Lipschitz condition;
(4) the conditions (22), (23) are fulfilled.

Then, a steady-state solution of the Cauchy problems (14)–(15) is asymptotically stable.

4. Stability of Hopfield Neural Networks

In this section, we investigate the stability of Hopfield neural networks, which are modeled by a system of nonlinear differential equations

$$\frac{dx_i(t)}{dt} = -a_i(t)x_i(t) + \sum_{j=1}^{n} w_{i,j}(t) g_j(x_j(t)), \tag{27}$$

with discontinuous coefficients $a_i(t)$ and activation functions $g_i(x)$, $i = 1, 2, \ldots, n$.

We will perform our study of the stability of neural networks (27) in two stages. The first stage includes a case with discontinuous coefficients $a_i(t)$. The second one considers discontinuity of activation function $g_i(x)$, $i = 1, 2, \ldots, n$.

First, let functions $a_i(t)$, $i = 1, 2, \ldots, n$ have discontinuities of the first kind. It is enough to restrict ourselves to the case of one point of discontinuity. Assume that the function $a_{11}(t)$ is discontinuous at the point b_1, $0 < b_1 < \infty$. Without loss of generality, we suggest $g_j(0) = 0$, $j = 1, 2, \ldots, n$, $|g_i(x)| \leq \alpha_i|x|$, $i = 1, 2, \ldots, n$.

Now, we investigate the stability of the zero solution of the system of Equation (27).

In the interval $(0, b_1]$, the norm of the solution of Equation (27) for initial value

$$x(0) = x_0, x(0) = (x_1(0), \ldots, x_n(0)), \tag{28}$$

is estimated by the inequality

$$\begin{aligned} \|x(t)\| &\leq \exp\Big\{\textstyle\int_0^t \Lambda(A(\tau))d\tau\Big\}\|x(0)\| \\ &+ \textstyle\int_0^t \exp\Big\{\int_s^t \Lambda(A(\tau))d\tau\Big\}\|F(t, x(s))\|ds, \end{aligned} \tag{29}$$

where $A(t) = \{a_{ij}(t)\}$, $i, j = 1, 2, \ldots, n$,

$$F(t, x(t)) = \left(\sum_{j=1}^{n} w_{1j}(t) g_j(x(t)), \ldots, \sum_{j=1}^{n} w_{nj}(t) g_j(x(t))\right)^T.$$

Proceeding with the Inequality (29), we have

$$\|x(t)\| \leq \exp\left\{\int_0^t \Lambda(A(\tau))d\tau\right\}\|x(0)\| + \gamma \int_0^t \exp\left\{\int_s^t \Lambda(A(\tau))d\tau\right\}\|x(s)\|ds, \tag{30}$$

where γ is defined from the inequality $\|F(t, x(t))\| \leq \gamma\|x(t)\|$.

From the Inequality (30), using well-known methods, we have the estimate

$$\|x(t)\| \leq \exp\left\{\int_0^t \Lambda(A(\tau))d\tau + \gamma t\right\}\|x(0)\|, t \in [0, b_1].$$

Taking $x(b_1) = (x_1(b_1), \ldots, x_n(b_1))$ as the initial value and repeating the arguments given in the proof of Theorem 1, we obtain the inequality

$$\|x(t)\| \leq \exp\left\{\int_0^t \Lambda(A(\tau))d\tau \ + \gamma t\right\}\|x(0)\|,$$

which is valid for $t \in [0, \infty)$.

From this inequality, it follows that when the condition

$$\left\{\int_0^t \Lambda(A(\tau))d\tau + \gamma t\right\} < 0,$$

is satisfied, the system of Equation (27) is asymptotically stable in general.

Thus, the following statement has been proven.

Theorem 6. *Let the following conditions be fulfilled:*

(1) functions $a_i, i = 1, 2, \ldots, n$ are continuous everywhere in $[0, \infty)$ except a finite number of points where they have discontinuities of the first kind;

(2) functions $g_i(t)$ are continuous;

(3) $|g_j(x(t))| \leq \alpha_j |x(t)|$;

(4) in $t \in [0, \infty)$, the following condition is satisfied

$$\int_0^t \Lambda(A(\tau))d\tau + \gamma t < 0,$$

where gamma is defined from the inequality $\|F(t, x(t))\| \leq \gamma \|x(t)\|$.

Then, the Hopfield neural network is stable in general.

Consider the case of discontinuity in synapses $w_{ij}(t), i, j = 1, 2, \ldots, n$. For convenience, we restrict ourselves to the discontinuity of the function $w_{11}(t)$ at the time moment $b_1, 0 < b_1 < \infty$.

Let us represent the system of Equation (27) as

$$\begin{aligned}\frac{dx_1(t)}{dt} &= -a_1(t)x_1(t) + w_{11}(t)g_1'(0)x_1(t) \\ &\quad + w_{11}(t)u_1(x_1(t)) + \sum_{j=2}^{n} w_{1j}(t)g_j(x_j(t)),\end{aligned}$$

$$\frac{dx_i(t)}{dt} = -a_i(t)x_i(t) + \sum_{j=1}^{n} w_{ij}(t)g_j(x_j(t)), i = 2, 3, \ldots, n. \tag{31}$$

Here, $u_1(x_1(t)) = g_1(x_1(t)) - g_1'(0)x_1(t)$.

It is essential that $|u_1(x_1(t))| = o(|x_1(t)|)$, since we examine the trivial solution of the system (27). Therefore,

$$|u_1(x_1(t))| = |g(x_1(t)) - g_1(0) - g_1'(0)x_1(t)| \leq B|x_1(t)|^2,$$

where $B = \max\limits_{0<\theta(x_1(t))<1} |g''(\theta(x_1(t)))|$.

Obviously, the system of Equation (31) has a structure similar to that of the system of Equation (27). The difference is that the coefficient for $x_1(t)$ now is equal to $-a_1(t) + w_{11}(t)g_1'(0)$, and the vector function $F(x(t))$ contains $w_{11}(t)u_1(x(t))$ instead of $w_{11}(t)g_1(x(t))$.

Taking this remark into account, the assertion of Theorem 7 extends to the system (31).

Finally, we consider the case which involves discontinuous activation functions. Clearly, the system of Equation (18) is a special case of the system of Equation (18). Theorem 5's statements are readily extended to this.

5. Conclusions

In this paper, we obtain sufficient conditions of asymptotic stability for solutions of linear and nonlinear systems of ordinary differential equations with discontinuous right-hand sides. We have derived conditions for local asymptotic stability and stability in general and the obtained sufficient conditions have been used to investigate the stability of Hopfield neural networks with discontinuous synapses and activation functions. The proposed method for studying Hopfield neural networks can also be applied to other types of artificial neural networks.

The authors hope to continue their study in the following directions:

- stability of solutions of systems of differential equations with discontinuous right-hand sides and delays;
- stability of solutions of systems of parabolic equations with discontinuous right-hand sides;
- stability of solutions of systems of hyperbolic equations with discontinuous right-hand sides.

We intend to use the obtained results in the following fields:

- Ecology. There are a lot of regions with dramatic climate change. Models with discontinuities describe the dynamics of populations very well.
- Problems of automatic regulation and control.
- Mathematical models of immunology during therapy.

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References

1. Hopfield, J.J. Neurons with graded response have collective computational properties like those of two-state neurons (associative memory/neural network/stability/action potentials). *Proc. Natl. Acad. Sci. USA* **1984**, *81*, 3088–3092. [CrossRef] [PubMed]
2. Hopfield, J.J.; David, W.T. Computing with Neural Circuits: A Model Source. *Sci. New Ser.* **1986**, *233*, 625–633.
3. Michel, A.N.; Gray, D.L. Analysis and synthesis of neural networks with lower block triangular interconnected structure. *IEEE Trans. Circuits Syst.* **1990**, *37*, 1267–1283. [CrossRef]
4. Li, J.H.; Michel, A.N.; Porod, W. Qualitative analysis and synthesis of a class of neural networks. *IEEE Trans. Circuits Syst.* **1988**, *35*, 976–985. [CrossRef]
5. Kosko, B. Bidirectional associative memories. *IEEE Trans. Syst. Man Cybern.* **1988**, *18*, 49–60. [CrossRef]
6. Chua, L.O.; Yang, L. Cellular neural networks: Theory. *IEEE Trans. Circuits Syst.* **1988**, *35*, 1257–1272. [CrossRef]
7. Varga, I.; Elek, G.; Zak, S.H. On the brain-state-in-a-convex-domain neural models. *Neural Netw.* **1996**, *9*, 1173–1184. [CrossRef]
8. Grossberg, S. Content-addresable memory storage by neural networks: A general model and global Liapunov method. In *Computational Neuroscience*; ISchwartz, E.L., Ed.; MIT Press: Cambridge, MA, USA, 1990; pp. 56–65.
9. Cohen, M.A.; Grossberg, S. Absolute stability of global pattern formation and parallel memory storage by competitive neural networks. *IEEE Trans. Syst. Man Cybern.* **1983**, *SMC-13*, 815–826. [CrossRef]
10. Hizsch, M.M. Convergent activation dynamics in continuous time networks. *Neural Netw.* **1989**, *2* , 331–349.
11. Kelly, D.G. Stability of contractive nonlinear neural networks. *IEEE Trans. Biomed. Eng.* **1990**, *3*, 231–242. [CrossRef]

12. Matsuoka, K. Stability conditions for nonlinear continuous neural networks with asymmetric connection weights. *Neural Netw.* **1992**, *5*, 495–500. [CrossRef]
13. Kaszkurewicz, E.; Bhaya, A. On a class of globally stable neural circuits. *IEEE Trans. Circuits Syst. I Fundam. Theory Appl.* **1994**, *41*, 171–174. [CrossRef]
14. Forti, M.; Manetti, S.; Marini, M. Necessary and sufficient conditions for absolute stability of neural networks. *IEEE Trans. Circuits Syst. I Fundam. Theory Appl.* **1994**, *41*, 491–494. [CrossRef]
15. Yang, H.; Dillon, T.S. Exponential stability and oscillation of Hopfield graded response neural networks. *IEEE Trans. Neural Netw.* **1994**, *5*, 719–729. [CrossRef]
16. Fang, Y.; Kincaid, T.G. Stability Analysis of Dynamical Neural Networks. *IEEE Trans. Neural Netw.* **1996**, *7*, 996–1006. [CrossRef] [PubMed]
17. Korkobi, T.; Chtourou, M.; Djemel, M. Stability Analysis of Neural Networks-Band System Identification. *Model. Simul. Eng.* **2009**, *2008*, 343940.
18. Yin, L.; Chen, Y.; Zhao, Y. Global Exponential Stability for a Class Neural Networks with Continuously Distributed Delays. *Adv. Dyn. Syst. Appl.* **2009**, *4*, 221–229.
19. Mou, S.; Gao, H.; Lam, J.; Qiang, W. A new criterion of delay-dependent asymptotic stability for hopfield neural networks with time delay. *IEEE Trans. Neural Netw.* **2008**, *19*, 532–535.
20. Sun, C.; Song, S.; Feng, C. On Global Robust Exponential Stability of Interval Neural Networks with delays. In Proceedings of the 2002 International Joint Conference on Neural Networks, Honoluu, HI, USA, 12–17 May 2002.
21. Zhao, K.; Li, Y. Robust Stability Analysis of Fuzzy Neural Network with Delays. *Math. Probl. Eng.* **2009**, *2009*, 1–13. [CrossRef]
22. Zong, G.-D.; Liu, J. New Delay-dependent Global Asymptotic Stability Condition for Hopfield Neural Networks with Time-varying Delays. *Int. J. Autom. Comput.* **2009**, *6*, 415–419. [CrossRef]
23. McCulloch, W.S.; Pitts, W. A logical calculus of the ideas immanent in nervous activity. *Bull. Math. Biophys.* **1943**, *5*, 115–133. [CrossRef]
24. Filippov, A.F. *Differential Equations with Discontinuous Right-Hand Side*; Nauka: Moscow, Russia, 1985; p. 224.
25. Boikov, I.V. *Stability of Solutions of Differential Equations*; Publishing House of Penza State University: Penza, Russia, 2008; p. 244.
26. Emel'yanov, S.V. Theory of automatic control systems with variable structure: Origin and initial stage of development. In *Nonlinar Dynamics and Control*; Emel'yanov, S.V., Korovin, S.V., Eds.; Fizmatlit: Moscow, Russia, 2004; pp. 5–16.
27. Painleve, P. *Lecons Sur le Frottement*; Hermann: Paris, France, 1895; p. 111.
28. Tan, X.; Qin, W.; Liu, X.; Yang, J.; Jiang, S. Sliding bifurcation analysis and global dynamics for a Filippov predator-prey system. *J. Nonlinear Sci. Appl.* **2016**, *9*, 3948–3961. [CrossRef]
29. Boikov, I.V. On a continuous method for solving nonlinear operator equations. *Differ. Equ.* **2012**, *48*, 1308–1314. [CrossRef]
30. Veldhuizen, V.M. On D-stability and B-stability. *Numer. Math.* **1983**, *43*, 349–357. [CrossRef]
31. Solak, M.K. A note on robust G-stability. In Proceedings of the 1990 American Control Conference, San Diego, CA, USA, 23–25 May 1990.
32. Shokri, A.; Vigo-Aguiar, J.; Mehdizadeh, K.M.; Garcia-Rubio, R. A new implicit six-step P-stable method for the numerical solution of Schrodinger equation. *Int. J. Comput. Math.* **2020**, *97*, 802–817. [CrossRef]
33. Hui, Q.; Haddad, W.M.; Bhat, S.P. Semistability, Finite-Time Stability, Differential Inclusions, and Discontinuous Dynamical Systems Having a Continuum of Equilibria. *IEEE Trans. Autom. Control* **2009**, *54*, 24–65.
34. Anosov, D.V. On stability of relay system equilibrium. *Avtomat. Telemekh.* **1959**, *20*, 135–149.
35. Losev, A.A. Stability analysis of the zero solution of a relay system of ordinary differential equations with two relays. *Differ. Equ.* **2017**, *53*, 1005–1020. [CrossRef]
36. Aleksandrov, A.Y.; Mason, O. On Diagonal Stability of Positive Systems with Switches and Delays. *Autom. Remote Control* **2018**, *79*, 2114–2127. [CrossRef]
37. Utkin, V.I.; Guldner, J.; Shi, J.X. *Sliding Mode Control in Electro-Mechanical Systems*; CRC Press: Boca Raton, FL, USA, 2009; p. 503.
38. Bainov, D.D.; Simeonov, P.S. *Systems with Impulse Effect: Stability, Theory and Applications*; Ellis Horwood Limited: Chichester, UK; John Wiley and Sons: New York, NY, USA, 1989; 255p.
39. Perestyuk, N.A.; Plotnikov, V.A.; Samoilenko, A.M.; Skripnik, N.V. *Differential Equations with Impulse Effects: Multivalued Right-Hand Sides with Discontinuities (De Gruyter Studies in Mathematics: 40)*; Walter De Gruyter GmbH Co.: Berlin, Germany; Boston, MA, USA, 2011; p. 307.
40. Anashkin, O.V.; Dovzhik, N.V.; Mit'ko, O.V. Stability of solutions of differential equations with impulse effect. *Din. Sist.* **2010**, *28*, 3–10.
41. Dudnikov, E.E.; Rybashov, M.V. Absolute Stability of a Class of Feedback Neural Networks. *Autom. Remote Control* **1999**, *12*, 33–40.
42. Borisov, J.I.; Kuz'michev, J.A.; Smolitzkii, H.P. On equilibrium states in a continuous model of neural Hopfield networks. *J. Instrum. Eng.* **1994**, *3–4*, 5–12.
43. Petryakova, E.A. Hopfield neural network with asymmetric matrix of communication coefficients between neurons. *J. Instrum. Eng.* **1994**, *37*, 24–32.
44. Gupta, M.M.; Jin, L.; Hamma, N. *Static and Dynamic Neural Networks. From Fundamentals to Advanced Theory*; Wiley Interscience: New York, NY, USA, 2005; p. 722.

45. Lukyanova, T.A. On exponential stability of Hopfield neural network on time scale. *Rep. Natl. Acad. Sci. Ukr.* **2011**, *10*, 13–17.
46. Boikov, I.V. Stability of Hopfield neural networks. *Autom. Remote Control* **2003**, *64*, 1474–1487. [CrossRef]
47. Daletskii, Y.L.; Krein, M.G. *Stability of Solutions of Differential Equations in Banach Space*; Nauka: Moscow, Russia, 1970; p. 536.
48. Boikov, I.V. Stability of Steady-State Solutions of Systems of Nonlinear Nonautonomous Delay Differential equations. *Differ. Equ.* **2018**, *54*, 1–23. [CrossRef]