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The Extendability of Cayley Graphs Generated by Transposition Trees

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Abstract: A connected graph Γ is k-extendable for a positive integer k if every matching M of size k can be extended to a perfect matching. The extendability number of Γ is the maximum k such that Γ is k-extendable. In this paper, we prove that Cayley graphs generated by transposition trees on $\{1, 2, \ldots, n\}$ are $(n-2)$-extendable and determine that the extendability number is $n-2$ for an integer $n \geq 3$.

Keywords: extendability; cayley graphs; transposition trees; bubble-sort graphs; star graphs

MSC: 05C25; 05C70

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1. Introduction

Cayley graphs on a group and a generating set have been an important class of graphs in the study of interconnection networks for parallel and distributed computing [1–6]. Some recent results about topological properties and routing problems on the networks based on Cayley graphs on the symmetric groups with the set of transpositions as the generating sets, including two special classes, the star graphs [5] and bubble-sort graphs [1], can be found in [6–9].

Throughout this paper, we consider finite, simple connected graph. Let Γ be a graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. A graph H is a subgraph of Γ if $V(H) \subseteq V(\Gamma)$ and $E(H) \subseteq E(\Gamma)$. The induced subgraph $\Gamma[C]$ is the subgraph of Γ with vertex set C and edge set $\{uv | u, v \in C, uv \in E(\Gamma)\}$. Let G be a group, S a subset of G such that the identity element does not belong to S and $S = S^{-1}$, where $S^{-1} = \{\tau^{-1} | \tau \in S\}$. The *Cayley graph* Γ, denoted by $\Gamma = \mathrm{Cay}(G, S)$, is the graph whose vertex set $V(\Gamma) = G$ and u, v are adjacent if and only if $u^{-1}v \in S$. It's known that Γ is connected if and only if S is a generating set of G. Furthermore, obviously, all Cayley graphs are vertex-transitive (see [10]).

We denote $\mathfrak{S}_n$ as the symmetric group on n letters (set of all permutations on $\{1, 2, \ldots, n\}$). Now let us restrict S to be a subset of transpositions on $\{1, 2, \ldots, n\}$. Clearly all Cayley graphs $\mathrm{Cay}(\mathfrak{S}_n, S)$ are $|S|$-regular bipartite graphs. The *transposition generating graph* of S, denoted by $T(S)$, is the graph with vertex set $\{1, 2, \ldots, n\}$ and two vertices s and t are adjacent if and only if the transposition (st) is in S. If $T(S)$ is a tree, it is called *transposition trees*.

An edge set $M \subseteq E(\Gamma)$ is called a *matching* of Γ if no two of them share an end-vertex. Moreover, a matching of Γ is said to be *perfect* if it covers all vertices of Γ. A connected graph Γ having at least $2k+2$ vertices is said to be *k-extendable*, introduced by Plummer [11], if each matching M of k edges is contained in a perfect matching of Γ. Any k-extendable graph is $(k-1)$-extendable, but the converse is not true [11]. The *extendability number* of Γ, denoted by $ext(\Gamma)$, is the maximum k such that Γ is k-extendable. Plummer [11,12] studied the relationship between n-extendability and other graph properties. For more research results related to matching extendability, one can refer to [13–17]. Yu et al. [18]

classified the 2-extendable Cayley graphs of finite abelian groups. Chen et al. [19] classified the 2-extendable Cayley graphs of dihedral groups. Recently, Gao et al. [20] characterize the 2-extendable quasi-abelian Cayley graphs. Their research is focused on 2-extendability of some Cayley graphs; in this paper, we focus on the general extendability, i.e., $(n-2)$-extendability of Cayley graphs generated by transposition trees.

We proceed as follows. In Section 2, we provide preliminaries and previous related results on Cayley graphs. In Section 3, we give our main results: show that all Cayley graphs generated by transposition trees are $(n-2)$-extendable and then determine their extendability numbers are $n-2$.

2. Preliminaries

In this section, we shall give some definitions and known results which will be used in this paper.

Denote by $\mathfrak{S}_n$ the group of all permutations on $[n] = \{1, 2, \ldots, n\}$. Obviously, $|\mathfrak{S}_n| = n!$. For convenience, we use $\mathbf{x} = x_1x_2 \ldots x_n$ to denote the permutation $\binom{1\ 2\ \ldots n}{x_1\ x_2 \ldots x_n}$ (see [21]); (st) to denote the permutation $\binom{1 \ldots s \ldots t \ldots n}{1 \ldots t \ldots s \ldots n}$, which is called a *transposition*. Obviously, $x_1 \ldots x_s \ldots x_t \ldots x_n(st) = x_1 \ldots x_t \ldots x_s \ldots x_n$. The identity permutation $12 \ldots n$ is denoted by $\mathbf{1}$. A permutation of $\mathfrak{S}_n$ is said to be *even* (resp. *odd*) if it can be written as a product of an even (resp. odd) number of transpositions. Let S be a subset of transpositions. Clearly, the Cayley graph $\mathrm{Cay}(\mathfrak{S}_n, S)$ is a bipartite graph with one partite set containing the vertices corresponding to odd permutations and the other partite set containing the vertices corresponding to even permutations.

To better describe a transposition set S as the generating set, we use a simple way to depict S via a graph. The *transposition generating graph* $T(S)$ is the graph with vertex set $[n]$ and two vertices s and t are adjacent if and only if $(st) \in S$. If $T(S)$ is a tree, it is called *transposition trees*, we denote by $\mathbb{T}_n$ the set of Cayley graphs $\mathrm{Cay}(\mathfrak{S}_n, S)$ generated by transposition trees. For any graph $\mathcal{T}_n(S) = \mathrm{Cay}(\mathfrak{S}_n, S) \in \mathbb{T}_n$, $\mathbf{x} = x_1x_2 \ldots x_n$ is adjacent to $\mathbf{y} = y_1y_2 \ldots y_n$ if and only if for $(st) \in S$, $x_s = y_t$, $x_t = y_s$ and $x_k = y_k$ for $k \neq s, t$, that is $\mathbf{y} = \mathbf{x}(st)$. In this case, we say that the edge $e = \mathbf{xy}$ is an (st)-edge and denote $g(e) = (st)$, which is the edge e corresponding to transposition. Let $E_{st} = \{e \in E(\mathcal{T}_n(S)) | e \text{ is an } (st)\text{-edge}\}$. Obviously, for every transposition $(st) \in S$, E_{st} is a perfect matching of $\mathcal{T}_n(S)$. We have the following propositions about Cayley graphs generated by transpositions:

Proposition 1 ([10], p. 52)**.** *Let $\Gamma = \mathrm{Cay}(\mathfrak{S}_n, S)$ be a Cayley graph generated by transpositions. Then, Γ is connected if and only if $T(S)$ is connected.*

Proposition 2 ([22])**.** *Let S and S' be two sets of transpositions on $[n]$. Then, $\mathrm{Cay}(\mathfrak{S}_n, S)$ and $\mathrm{Cay}(\mathfrak{S}_n, S')$ are isomorphic if and only if $T(S)$ and $T(S')$ are isomorphic.*

In all Cayley graphs $\mathbb{T}_n$, there are two classes which are most important, when $T(S)$ is isomorphic to the star $K_{1,n-1}$ and the path P_n. If $T(S) \cong K_{1,n-1}$, $\mathrm{Cay}(\mathfrak{S}_n, S)$ is called *n-dimensional star graph* and denoted by ST_n. If $T(S) \cong P_n$, $\mathrm{Cay}(\mathfrak{S}_n, S)$ is called *n-dimensional bubble-sort graph* and denoted by BS_n. The star graph and the bubble-sort graph are illustrated in Figures 1 and 2 for the case n = 4. Both ST_n and BS_n are connected bipartite $(n-1)$-regular graph of order $n!$. When $n = 3$, $\mathcal{T}_3(S) \cong ST_3 \cong BS_3 \cong C_6$; $n = 4$, up to isomorphism, there are exactly two different graphs ST_4 and BS_4 (see [23]).

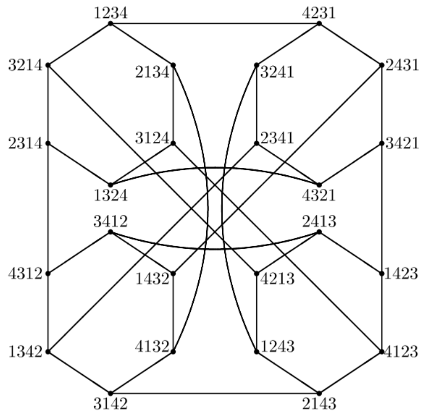

Figure 1. The star graph $ST_4 = \mathrm{Cay}(\mathfrak{S}_4, \{(12),(13),(14)\})$.

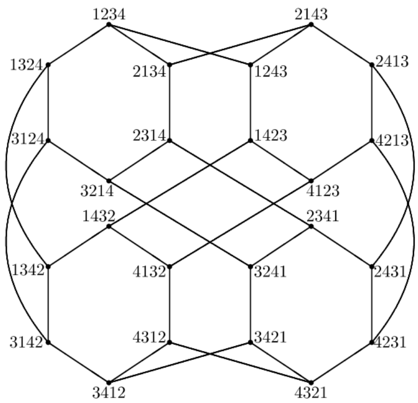

Figure 2. The Bubble-sort graph $BS_4 = \mathrm{Cay}(\mathfrak{S}_4, \{(12),(23),(34)\})$.

Let $\mathbf{x} = x_1x_2\ldots x_n$ be a vertex of $\mathcal{T}_n(S)$. We say that x_i is the *i-th coordinate* of $\mathbf{x}$, denoted by $(\mathbf{x})_i$. It is easy to see that the Cayley graph $\mathcal{T}_n(S)$ has the following proposition:

Proposition 3 ([23,24]). *Let $T(S)$ be a transposition tree of order n, j one of its leaf and $\mathcal{T}_n^{\{i\}}(S)$ $(1 \le i \le n)$ the subgraph of $\mathcal{T}_n(S)$ induced by those vertices $\mathbf{x}$ with $(\mathbf{x})_j = i$. Then, $\mathcal{T}_n(S)$ consists of n vertex-disjoint subgraphs: $\mathcal{T}_n^{\{1\}}(S), \mathcal{T}_n^{\{2\}}(S), \ldots, \mathcal{T}_n^{\{n\}}(S)$; each isomorphic to another Cayley graph $\mathcal{T}_{n-1}(S') = \mathrm{Cay}(\mathfrak{S}_{n-1}, S')$ with $S' = S \backslash \tau$, where τ is the transposition corresponding to the edge incident to the leaf j.*

Readers can refer to [10,21] for the terminology and notation not defined in this paper.

3. Main Results

First, we will give some useful lemmas.

The *Cartesian product* $\Gamma_1 \square \Gamma_2$ of graphs Γ_1 and Γ_2 is a graph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$. Two vertices (u,v) and (u',v') are adjacent in $\Gamma_1 \square \Gamma_2$ if either $u = u'$ and $vv' \in E(\Gamma_2)$ or $uu' \in E(\Gamma_1)$ and $v = v'$. Clearly $\Gamma_1 \square \Gamma_2 = \Gamma_2 \square \Gamma_1$.

Lemma 1. *Let T be a labeled tree of order n, e any edge of T, and T_1, T_2 two components of $T - e$, where $|V(T_1)| = r$. Furthermore, let S (S^-, S_1, S_2, respectively) be the transposition set on $\{1,2,\ldots,n\}$ satisfying $T(S) = T$ ($T(S^-) = T - e$, $T(S_1) = T_1$, $T(S_2) = T_2$. Then, $\mathrm{Cay}(\mathfrak{S}_n, S^-)$ has $\binom{n}{r}$ components and each component is isomorphic to $\mathrm{Cay}(\mathfrak{S}_r, S_1) \square \mathrm{Cay}(\mathfrak{S}_{n-r}, S_2)$.*

Proof. Without loss of generality, we can assume $r \le \lfloor \frac{n}{2} \rfloor$.

When $r = 1$, T_1 is an isolated vertex, e is a pendant edge and $S_1 = \emptyset$. Then, $\mathrm{Cay}(\mathfrak{S}_1, S_1)\square\mathrm{Cay}(\mathfrak{S}_{n-1}, S_2) = \mathrm{Cay}(\mathfrak{S}_{n-1}, S_2)$. The lemma is true, following from Proposition 3.

When $r \geq 2$, we relabel T as follows: Relabel the vertices of T_1 as $\{1, 2, \ldots, r\}$ and the vertices of T_2 as $\{r+1, r+2, \ldots, n\}$. Let S', S'^{-}, S'_1, S'_2 be the corresponding transposition sets. Obviously, $S'^{-} = S'_1 \cup S'_2$. By Proposition 2, we know that $\mathrm{Cay}(\mathfrak{S}_n, S) \cong \mathrm{Cay}(\mathfrak{S}_n, S')$, $\mathrm{Cay}(\mathfrak{S}_n, S^{-}) \cong \mathrm{Cay}(\mathfrak{S}_n, S'^{-})$, and so on. Thus, we only need to prove the corresponding result on S', S'^{-}, S'_1 and S'_2. Since $T - e$ is disconnected, $\mathrm{Cay}(\mathfrak{S}_n, S'^{-})$ is also disconnected by Proposition 1. Let Γ_1 be the component of $\mathrm{Cay}(\mathfrak{S}_n, S'^{-})$ containing the identity element $\mathbf{1}$. Since T_1 and T_2 are connected, S'_1 generates $\mathfrak{S}_r$ and S'_2 generates $\mathfrak{S}_{n-r}$ (let $\mathfrak{S}_{n-r}$ be symmetric group on $\{r+1, r+2, \ldots, n\}$). Then, the vertices in Γ_1 can be represented as $\mathbf{v} = x_1x_2\ldots x_rx_{r+1}\ldots x_n$, where $x_1x_2\ldots x_r$ is a permutation on $\{1, 2, \ldots, r\}$ and $x_{r+1}\ldots x_n$ is a permutation on $\{r+1, r+2, \ldots, n\}$. Furthermore, let $\mathbf{v} = x_1x_2\ldots x_rx_{r+1}\ldots x_n$ and $\mathbf{v}' = x'_1x'_2\ldots x'_rx'_{r+1}\ldots x'_n$ be two vertices in Γ_1. Then, $\mathbf{v}$ and $\mathbf{v}'$ are adjacent if and only if for $j, k \leq r$ and $(jk) \in S'_1$, $x_k = x'_j$, $x_j = x'_k$ and $x_l = x'_l$ for other digits, or, for $j, k \geq r+1$ and $(jk) \in S'_2$, $x_k = x'_j$, $x_j = x'_k$ and $x_l = x'_l$ for other digits. Thus, $\Gamma_1 \cong \mathrm{Cay}(\mathfrak{S}_r, S'_1)\square\mathrm{Cay}(\mathfrak{S}_{n-r}, S'_2)$ and $|V(\Gamma_1)| = r!(n-r)!$. Since $\mathrm{Cay}(\mathfrak{S}_n, S'^{-})$ is vertex-transitive, all components of $\mathrm{Cay}(\mathfrak{S}_n, S'^{-})$ are isomorphic and there exist $\frac{n!}{r!(n-r)!} = \binom{n}{r}$ components in it. $\square$

We need to consider the extendability of the Cartesian product when we investigate the extendability of $\mathcal{T}_n(S)$. The following lemmas are used several times in the proof of our theorem.

Lemma 2 ([25,26]). *If Γ is a k-extendable graph, then $\Gamma\square K_2$ is $(k+1)$-extendable.*

Lemma 3 ([25]). *If Γ_1 and Γ_2 are k-extendable and l-extendable graphs, respectively, then their Cartesian product $\Gamma_1\square\Gamma_2$ is $(k+l+1)$-extendable.*

Lemma 4 ([27]). *A bipartite Cayley graph is 2-extendable if and only if it is not a cycle.*

In order to prove the main result, we need other definitions and notations. The *symmetric difference* of two sets A and B is defined as the set $A\triangle B = (A-B)\cup(B-A)$. Let Γ be a connected graph. If $e = uv \in E(\Gamma)$, denote $V(e) = \{u, v\}$ and $E(v) = \{e | V(e) \cap \{v\} \neq \emptyset\}$.

Let $\mathbf{x}$ be a permutation of $[n]$. The smallest positive integer k for which $\mathbf{x}^k$ is the identity permutation, this number k is called the *order* of $\mathbf{x}$, denoted by $o(\mathbf{x}) = k$. $fix(\mathbf{x})$ denotes the set of points in $[n]$ fixed by $\mathbf{x}$ (see [10]). Let $\overline{fix(\mathbf{x})} = [n] - fix(\mathbf{x})$. As we know, there is another way of writing the permutation as products of disjoint cycles which are commutative (see [21]). For example, if $\mathbf{x} \in \mathfrak{S}_9$, $\mathbf{x} = 324, 158, 967$, then $\mathbf{x} = (134)(68)(79) = (68)(134)(79)$, and further $fix(\mathbf{x}) = \{2, 5\}$, $\overline{fix(\mathbf{x})} = \{1, 3, 4, 6, 7, 8, 9\}$, $|\overline{fix(\mathbf{x})}| = 7$. We say that $\mathbf{x}$ is a type of $(m_1m_2m_3)(m_4m_5)(m_6m_7)$ permutation. Clearly $\mathbf{x}^6 = \mathbf{1}$ and $o(\mathbf{x}) = 6$.

Theorem 1. *Any Cayley graph $\mathcal{T}_n(S) \in \mathbb{T}_n$ is $(n-2)$-extendable for any integer $n \geq 3$.*

Proof. We prove the theorem by induction on n. For $n = 3$, the $\mathcal{T}_3(S)$ is 6-cycle, which is 1-extendable. For $n = 4$, the $\mathcal{T}_4(S)$ is a 3-regular bipartite Cayley graph, which is not a cycle. $\mathcal{T}_4(S)$ is 2-extendable by Lemma 4.

Now we assume the statement is true for all integers smaller than n $(n \geq 5)$. Let S be a subset of transpositions on $[n]$. The transposition generating graph $T(S)$ is a tree. We will show that any matching M of size $(n-2)$ can be extended to a perfect matching of $\mathcal{T}_n(S)$.

Let M be a matching with $(n-2)$ edges. There are $(n-1)$ classes of edges in $\mathcal{T}_n(S)$ because of $|S| = n-1$. We may suppose that $E_{s_4t_4} \cap M = \emptyset$. Let $S^{-} = S\backslash(s_4t_4)$. By Lemma 1, $\mathrm{Cay}(\mathfrak{S}_n, S^{-})$ has $\binom{n}{r}$ connected components and each component is isomorphic to $\mathrm{Cay}(\mathfrak{S}_r, S_1)\Box\mathrm{Cay}(\mathfrak{S}_{n-r}, S_2)$. We may assume $1 \le r \le \lfloor \frac{n}{2} \rfloor$ by the symmetry of Cartesian product. For the convenience, we denote the components of $\mathcal{T}_n(S)\backslash E_{s_4t_4} = \mathrm{Cay}(\mathfrak{S}_n, S^-)$ by $\mathcal{C}_i(i = 1, 2, \ldots, l)$, where $l = \binom{n}{r}$.

Claim 1. $\mathcal{C}_i$ is $(n-3)$-extendable.

If $r = 1$, the transposition (s_4t_4) corresponding to the edge is a leaf of $T(S)$, $\mathcal{C}_i \cong \mathcal{T}_{n-1}(S')$ by Proposition 3, where $S' = S^- = S\backslash(s_4t_4)$, $\mathcal{C}_i$ is $(n-3)$-extendable by the inductive hypothesis.

If $r = 2$, $\mathcal{T}_2(S) = K_2$, $\mathcal{C}_i \cong K_2\Box\mathrm{Cay}(\mathfrak{S}_{n-2}, S_2) = K_2\Box\mathcal{T}_{n-2}(S_2)$. $\mathcal{T}_{n-2}(S_2)$ is $(n-4)$-extendable by the inductive hypothesis. $\mathcal{C}_i$ is $(n-3)$-extendable by Lemma 2.

If $r \ge 3$, by the inductive hypothesis $\mathrm{Cay}(\mathfrak{S}_r, S_1) \cong \mathcal{T}_r(S_1)$ is $(r-2)$-extendable and $\mathrm{Cay}(\mathfrak{S}_{n-r}, S_2) \cong \mathcal{T}_{n-r}(S_2)$ is $(n-r-2)$-extendable. Hence, $\mathrm{Cay}(\mathfrak{S}_r, S_1)\Box\mathrm{Cay}(\mathfrak{S}_{n-r}, S_2)$ is $(n-3)$-extendable by Lemma 3. We get the Claim.

Let $J = \{i | E(\mathcal{C}_i) \cap M \neq \emptyset\}$. If $|J| \ge 2$, then $|E(\mathcal{C}_i) \cap M| \le n-3$. When $i \in J$, each edge set $E(\mathcal{C}_i) \cap M$ can be extended to a perfect matching of $\mathcal{C}_i$, which is defined by $M(\mathcal{C}_i)$. Clearly, $M \subset \bigcup_{i \in J} M(\mathcal{C}_i)$. When $i \notin J$, let $M(\mathcal{C}_i)$ be an arbitrary perfect matching of $\mathcal{C}_i$. Then, $\bigcup_{i=1}^{l} M(\mathcal{C}_i) = \left(\bigcup_{i \in J} M(\mathcal{C}_i)\right) \cup \left(\bigcup_{i \notin J} M(\mathcal{C}_i)\right)$ is a perfect matching of $\mathrm{Cay}(\mathfrak{S}_n, S^-)$, which is also a perfect matching of $\mathcal{T}_n(S)$.

When $|J| = 1$, without loss of generality, we assume that $M \subset E(\mathcal{C}_1)$ and $\mathcal{C}_1$ contains the identity permutation $\mathbf{1}$. If M can be extended to a perfect matching of $\mathcal{C}_1$, we are done. Suppose that M cannot be extended to a perfect matching of $\mathcal{C}_1$. Let $e_2 = v_1v_2$ be an edge in M. $M\backslash e_2$ can be extended to a perfect matching of $\mathcal{C}_1$ (since $|M\backslash e_2| = n-3$), which is denoted by $M'(\mathcal{C}_1)$. Let $E(v_1) \cap M'(\mathcal{C}_1) = e_1$, $E(v_2) \cap M'(\mathcal{C}_1) = e_3$, $V(e_1) = \{v_0, v_1\}$, $V(e_3) = \{v_2, v_3\}$ and $e_4 = E(v_3) \cap E_{s_4t_4}$. By the transitivity of $\mathcal{C}_1$ and without loss of generality, we can assume that $v_0 = \mathbf{1}$. Let $o(g(e_1)g(e_2)g(e_3)g(e_4)) = a$, $v_i = \prod_{j=1}^{i} g(e_j)$, and $e_{4b+1} \in E_{s_1t_1}$, $e_{4b+2} \in E_{s_2t_2}$, $e_{4b+3} \in E_{s_3t_3}, e_{4b+4} \in E_{s_4t_4}$ $(b = 0, \ldots, a-1)$, where $\{(s_1t_1), (s_2t_2), (s_3t_3), (s_4t_4)\} \subset S$. It is easy to see $g(e_2) \neq g(e_i)$ $(i = 1, 3)$, $g(e_4) \neq g(e_i)$ $(i = 1, 2, 3)$, $g(e_1)g(e_2)g(e_3) \neq g(e_4)$, $v_3 = g(e_1)g(e_2)g(e_3)$ is an odd permutation and $v_4 = g(e_1)g(e_2)g(e_3)g(e_4)$ is an even permutation. The cardinality of $\overline{fix(v_3)}$ can only be 2, 4, 5 and 6. We discuss these four cases one by one in order to prove that M can be extended to a perfect matching of $\mathcal{T}_n(S)$.

Case 1. $|\overline{fix(v_3)}| = 2$.

In this case, v_3 is a transposition and $o(v_3) = 2$. There are two subcases for the order of v_4.

Subcase 1.1. v_4 is a type of $(m_1m_2)(m_3m_4)$ permutation.

We have $o(v_4) = 2$, $(v_4)^2 = \mathbf{1}$. Note that $v_i = \prod_{j=1}^{i} g(e_j)$, where $i \in [8]$. Hence, there is an 8-cycle $C_8 = v_0e_1v_1e_2 \ldots v_7e_8v_8$ $(v_8 = v_0)$. The vertex $v_{4b+i} \in V(\mathcal{C}_{b+1})$ $(i = 0, 1, 2, 3;$ $b = 0, 1)$. We may take a perfect matching $M'(\mathcal{C}_2)$ of $\mathcal{C}_2$ such that $e_5 \in M'(\mathcal{C}_2), e_7 \in M'(\mathcal{C}_2)$ and $e_6 \notin M'(\mathcal{C}_2)$ because of $\mathcal{C}_2 \cong \mathcal{C}_1$. Now we take $M'' = (M'(\mathcal{C}_1) \bigcup M'(\mathcal{C}_2)) \triangle E(C_8)$. Clearly $M \subset M''$, M'' is a perfect matching of subgraph $\mathcal{T}_n(S)[V(\mathcal{C}_1) \bigcup V(\mathcal{C}_2)]$. Let $M(\mathcal{C}_i)$ be a perfect matching of $\mathcal{C}_i$ $(i = 3, \ldots, l)$. Hence, $\bigcup_{i=3}^{l} M(\mathcal{C}_i) \bigcup M''$ is a perfect matching of $\mathcal{T}_n(S)$.

Subcase 1.2. v_4 is a type of $(m_1m_2m_3)$ permutation.

We have $o(v_4) = 3$, $(v_4)^3 = \mathbf{1}$. Note that $v_i = \prod_{j=1}^{i} g(e_j)$, where $i \in [12]$. Hence, there is a 12-cycle $C_{12} = v_0e_1v_1e_2 \ldots v_{11}e_{12}v_{12}$ $(v_{12} = v_0)$. The vertex $v_{4b+i} \in V(\mathcal{C}_{b+1})$ $(i = 0, 1, 2, 3; b = 0, 1, 2)$. We may take a perfect matching $M'(\mathcal{C}_{b+1})$ of $\mathcal{C}_{b+1}$ such that $e_{4b+1} \in M'(\mathcal{C}_{b+1}), e_{4b+3} \in M'(\mathcal{C}_{b+1})$ and $e_{4b+2} \notin M'(\mathcal{C}_{b+1})$ $(b = 1, 2)$ because of $\mathcal{C}_{b+1} \cong \mathcal{C}_1$.

Now we take $M'' = \left(\bigcup_{i=1}^{3} M'(\mathcal{C}_i)\right) \triangle E(C_{12})$. Clearly $M \subset M''$, M'' is a perfect matching of subgraph $\mathcal{T}_n(S)[\bigcup_{i=1}^{3} V(\mathcal{C}_i)]$. Let $M(\mathcal{C}_i)$ be a perfect matching of $\mathcal{C}_i$ $(i = 4, \dots, l)$. Hence, $\bigcup_{i=4}^{l} M(\mathcal{C}_i) \bigcup M''$ is a perfect matching of $\mathcal{T}_n(S)$.

Case 2. $|\overline{fix(v_3)}| = 4$.

In this case, v_3 is a type of $(m_1m_2m_3m_4)$ permutation and $o(v_3) = 4$. There are two subcases.

Subcase 2.1. v_4 is a type of $(m_1m_2m_3m_4)(m_5m_6)$ permutation.

We have $o(v_4) = 4$, $(v_4)^4 = \mathbf{1}$. Note that $v_i = \prod_{j=1}^{i} g(e_j)$, where $i \in [16]$. Hence, there is a 16-cycle $C_{16} = v_0e_1v_1e_2 \dots v_{15}e_{16}v_{16}$ $(v_{16} = v_0)$. The vertex $v_{4b+i} \in V(\mathcal{C}_{b+1})$ $(i = 0,1,2,3; b = 0,1,2,3)$. We may take a perfect matching $M'(\mathcal{C}_{b+1})$ of $\mathcal{C}_{b+1}$ such that $e_{4b+1} \in M'(\mathcal{C}_{b+1}), e_{4b+3} \in M'(\mathcal{C}_{b+1})$ and $e_{4b+2} \notin M'(\mathcal{C}_{b+1})$ $(b = 1,2,3)$ because of $\mathcal{C}_{b+1} \cong \mathcal{C}_1$. Now we take $M'' = \left(\bigcup_{i=1}^{4} M'(\mathcal{C}_i)\right) \triangle E(C_{16})$. Clearly $M \subset M''$, M'' is a perfect matching of subgraph $\mathcal{T}_n(S)[\bigcup_{i=1}^{4} V(\mathcal{C}_i)]$. Let $M(\mathcal{C}_i)$ be a perfect matching of $\mathcal{C}_i$ $(i = 5, \dots, l)$. Hence, $\bigcup_{i=5}^{l} M(\mathcal{C}_i) \bigcup M''$ is a perfect matching of $\mathcal{T}_n(S)$.

Subcase 2.2. v_4 is a type of $(m_1m_2m_3m_4m_5)$ permutation.

We have $o(v_4) = 5$, $(v_4)^5 = \mathbf{1}$. Note that $v_i = \prod_{j=1}^{i} g(e_j)$, where $i \in [20]$. Hence, there is a 20-cycle $C_{20} = v_0e_1v_1e_2 \dots v_{19}e_{20}v_{20}$ $(v_{20} = v_0)$. The vertex $v_{4b+i} \in V(\mathcal{C}_{b+1})$ $(i = 0,1,2,3; b = 0,1,2,3,4)$. We may take a perfect matching $M'(\mathcal{C}_{b+1})$ of $\mathcal{C}_{b+1}$ such that $e_{4b+1} \in M'(\mathcal{C}_{b+1}), e_{4b+3} \in M'(\mathcal{C}_{b+1})$ and $e_{4b+2} \notin M'(\mathcal{C}_{b+1})$ $(b = 1,2,3,4)$ because of $\mathcal{C}_{b+1} \cong \mathcal{C}_1$. Now we take $M'' = \left(\bigcup_{i=1}^{5} M'(\mathcal{C}_i)\right) \triangle E(C_{20})$. Clearly $M \subset M''$, M'' is a perfect matching of subgraph $\mathcal{T}_n(S)[\bigcup_{i=1}^{5} V(\mathcal{C}_i)]$. Let $M(\mathcal{C}_i)$ be a perfect matching of $\mathcal{C}_i$ $(i = 6, \dots, l)$. Hence, $\bigcup_{i=6}^{l} M(\mathcal{C}_i) \bigcup M''$ is a perfect matching of $\mathcal{T}_n(S)$.

Case 3. $|\overline{fix(v_3)}| = 5$.

In this case, v_3 is a type of $(m_1m_2m_3)(m_4m_5)$ permutation and $o(v_3) = 6$. There are four subcases.

Subcase 3.1. v_4 is a type of $(m_1m_2m_3)(m_4m_5m_6)$ permutation.

We have $o(v_4) = 3$, $(v_4)^3 = \mathbf{1}$. There is a 12-cycle $C_{12} = v_0e_1v_1e_2 \dots v_{11}e_{12}v_{12}$ $(v_{12} = v_0)$ in subgraph $\mathcal{T}_n(S)[\bigcup_{i=1}^{3} V(\mathcal{C}_i)]$, where $v_{4b+i} \in V(\mathcal{C}_{b+1})$ for $i = 0,1,2,3; b = 0,1,2$. The rest of the proof is similar to Subcase 1.2.

Subcase 3.2. v_4 is a type of $(m_1m_2m_3m_4)(m_5m_6)$ permutation.

We have $o(v_4) = 4$, $(v_4)^4 = \mathbf{1}$. There is a 16-cycle $C_{16} = v_0e_1v_1e_2 \dots v_{15}e_{16}v_{16}$ $(v_{16} = v_0)$ in subgraph $\mathcal{T}_n(S)[\bigcup_{i=1}^{4} V(\mathcal{C}_i)]$, where $v_{4b+i} \in V(\mathcal{C}_{b+1})$ for $i = 0,1,2,3$; $b = 0,1,2,3$. The rest of the proof is similar to Subcase 2.1.

Subcase 3.3. v_4 is a type of $(m_1m_2m_3m_4m_5)$ permutation.

We have $o(v_4) = 5$, $(v_4)^5 = \mathbf{1}$. There is a 20-cycle $C_{20} = v_0e_1v_1e_2 \dots v_{19}e_{20}v_{20}$ $(v_{20} = v_0)$ in subgraph $\mathcal{T}_n(S)[\bigcup_{i=1}^{5} V(\mathcal{C}_i)]$, where $v_{4b+i} \in V(\mathcal{C}_{b+1})$ for $i = 0,1,2,3$; $b = 0,1,2,3,4$. The rest of the proof is similar to Subcase 2.2.

Subcase 3.4. v_4 is a type of $(m_1m_2m_3)(m_4m_5)(m_6m_7)$ permutation.

We have $o(v_4) = 6$, $|\overline{fix(v_4)}| = 7$ and $n \geq 7, l = \binom{n}{r} \geq 7$, $(v_4)^6 = \mathbf{1}$. $v_i = \prod_{j=1}^{i} g(e_j)$, where $i \in [24]$. Hence, there is a 24-cycle $C_{24} = v_0 e_1 v_1 e_2 \ldots v_{23} e_{24} v_{24}\,(v_{24} = v_0)$. The vertex $v_{4b+i} \in V(\mathcal{C}_{b+1})$ $(i = 0,1,2,3; b = 0,1,2,3,4,5)$. We may take a perfect matching $M'(\mathcal{C}_{b+1})$ of $\mathcal{C}_{b+1}$ such that $e_{4b+1} \in M'(\mathcal{C}_{b+1}), e_{4b+3} \in M'(\mathcal{C}_{b+1})$ and $e_{4b+2} \notin M'(\mathcal{C}_{b+1})$ $(b = 1,2,3,4,5)$ because of $\mathcal{C}_{b+1} \cong \mathcal{C}_1$. Now we take $M'' = \left(\bigcup_{i=1}^{6} M'(\mathcal{C}_i)\right) \triangle E(C_{24})$. Clearly, $M \subset M''$, M'' is a perfect matching of subgraph $\mathcal{T}_n(S)[\bigcup_{i=1}^{6} V(\mathcal{C}_i)]$. Let $M(\mathcal{C}_i)$ be a perfect matching of $\mathcal{C}_i$ $(i = 7, \ldots, l)$. Hence, $\bigcup_{i=7}^{l} M(\mathcal{C}_i) \bigcup M''$ is a perfect matching of $\mathcal{T}_n(S)$.

Case 4. $|\overline{fix(v_3)}| = 6$.

In this case, v_3 is a type of $(m_1m_2)(m_3m_4)(m_5m_6)$ permutation and $o(v_3) = 2$. There are three subcases.

Subcase 4.1. v_4 is a type of $(m_1m_2)(m_3m_4)(m_5m_6)(m_7m_8)$ permutation.

We have $o(v_4) = 2$. There is an 8-cycle $C_8 = v_0 e_1 v_1 e_2 \ldots v_7 e_8 v_8\,(v_8 = v_0)$ in subgraph $\mathcal{T}_n(S)[\bigcup_{i=1}^{2} V(\mathcal{C}_i)]$, where $v_{4b+i} \in V(\mathcal{C}_{b+1})$ for $i = 0,1,2,3; b = 0,1$. The rest of the proof is similar to Subcase 1.1.

Subcase 4.2. v_4 is a type of $(m_1m_2m_3m_4)(m_5m_6)$ permutation.

We have $o(v_4) = 4$. There is a 16-cycle $C_{16} = v_0 e_1 v_1 e_2 \ldots v_{15} e_{16} v_{16}\,(v_{16} = v_0)$ in subgraph $\mathcal{T}_n(S)[\bigcup_{i=1}^{4} V(\mathcal{C}_i)]$, where $v_{4b+i} \in V(\mathcal{C}_{b+1})$ for $i = 0,1,2,3; b = 0,1,2,3$. The rest of the proof is similar to Subcase 2.1.

Subcase 4.3. v_4 is a type of $(m_1m_2m_3)(m_4m_5)(m_6m_7)$ permutation.

We have $o(v_4) = 6$. There is a 24-cycle $C_{24} = v_0 e_1 v_1 e_2 \ldots v_{23} e_{24} v_{24}\,(v_{24} = v_0)$ in subgraph $\mathcal{T}_n(S)[\bigcup_{i=1}^{6} V(\mathcal{C}_i)]$, where $v_{4b+i} \in V(\mathcal{C}_{b+1})$ for $i = 0,1,2,3; b = 0,1,2,3,4,5$. The rest of the proof is similar to Subcase 3.4.

In conclusion, any matching M of size $n-2$ can be extended to a perfect matching of $\mathcal{T}_n(S)$. The proof is complete. □

The extendability number of Γ, denoted by $ext(\Gamma)$, is the maximum k such that Γ is k-extendable. As we know that $\mathcal{T}_n(S) \in \mathbb{T}_n$ is an $(n-1)$-regular bipartite Cayley graph and not $(n-1)$-extendable. We can obtain the extendability number of $\mathcal{T}_n(S)$ by Theorem 1.

Corollary 1. *$ext(\mathcal{T}_n(S)) = n - 2$ for $n \geq 3$.*

4. Concluding Remarks

In this paper, we prove that Cayley graph $\mathcal{T}_n(S)$ generated by transposition trees on $\{1, 2, \ldots, n\}$ is $(n-2)$-extendable and determine that the extendability number is $n-2$, which enriches the results on the extendability of Cayley graphs. Here, the transposition generating graph of S is a tree. A natural problem is whether we can generalize transposition trees to general connected graphs which is worth of further investigation. We present a conjecture.

Conjecture 1. *Let S be a transposition generating set of the symmetric group $\mathfrak{S}_n$. Then, the Cayley graph* $\mathrm{Cay}(\mathfrak{S}_n, S)$ *is $(|S| - 1)$-extendable.*

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