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Classical and Bayesian Inference of a Progressive-Stress Model for the Nadarajah–Haghighi Distribution with Type II Progressive Censoring and Different Loss Functions

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Abstract: Accelerated life testing (ALT) is a time-saving technology used in a variety of fields to obtain failure time data for test units in a fraction of the time required to test them under normal operating conditions. This study investigated progressive-stress ALT with progressive type II filtering with the lifetime of test units following a Nadarajah–Haghighi (NH) distribution. It is assumed that the scale parameter of the distribution obeys the inverse power law. The maximum likelihood estimates and estimated confidence intervals for the model parameters were obtained first. The Metropolis–Hastings (MH) algorithm was then used to build Bayes estimators for various squared error loss functions. We also computed the highest posterior density (HPD) credible ranges for the model parameters. Monte Carlo simulations were used to compare the outcomes of the various estimation methods proposed. Finally, one data set was analyzed for validation purposes.

Keywords: Bayesian approach; non-Bayesian approach; progressive-stress model; inverse power low; Nadarajah–Haghighi distribution; type II progressive censoring; balanced squared loss function; balanced LINEX loss function

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1. Introduction

Nadarajah and Haghighi [1] proposed a generalization of exponential distribution as an alternative to the regularly used Weibull and gamma distributions in lifetime models. The mode of the Nadarajah and Haghighi (NH) distribution is zero. Recent studies have shown that the NH distribution is extremely adaptable and can be used to model survival data, reliability issues, fatigue life studies, and hydrological data. It can have hazard rate functions that are constant, decreasing, increasing, upside-down bathtub (unimodal), bathtub-shaped, and decreasing–increasing–decreasing; see, for example, [2–5] and the references given therein.

The NH distribution is said to apply to a random variable T if its probability density function (PDF) is provided by

$$f(t;\theta,\lambda) = \theta\lambda(1+\lambda t)^{\theta-1}e^{(1-(1+\lambda t)^{\theta})},\ \theta,\lambda,t>0, \tag{1}$$

and the related cumulative density function (CDF) is given by

$$F(t;\theta,\lambda) = 1 - e^{(1-(1+\lambda t)^{\theta})},\ \theta,\lambda,t > 0, \tag{2}$$

where θ and λ are the shape and scale parameters, respectively.

Some individuals or observations may be lost and/or eliminated from the experiment prior to failure in practical applications for reliability and survival analyses. This results in censored data, with no access to the whole data information on failure times for some observations. Progressive type II censoring, first proposed by [6], is one of the most common censoring systems in lifetime scenarios and has received significant attention in the literature. The authors of [7] showed, for example, that in many instances, type II progressive-censoring schemes, which include ordinary type II one-stage right censoring as a particular case, could greatly outperform standard type II censoring. Reference [8] provides a brief overview of the progressive type II censoring technique. Furthermore, we suggest interested readers see [9] for a full examination of the many facets of progressive censoring, ranging from methodology to applications. Mimicking the seminal work of [10], the progressive type II censoring strategy may be represented as follows: Before conducting a life experiment, the experimenter fixes a sample size n, the number of complete observations to be observed, m and progressive-censoring-scheme $R_1,\ R_2,\ \cdots, R_m$, with $n = m + \sum_{i=1}^{m-1} R_i$. The n units are placed on the life test at the same time. Soon after the initial failure, R_1 surviving units are picked at random and removed from the experiment. Following that, R_2 items are eliminated immediately after the second failure, and so on. This procedure is repeated until all $R_\imath$ remaining units are deleted following the m-th failure. Note that the R_i values are fixed before the experiment begins. Furthermore, if $R_1 = R_2 = \cdots = R_m = 0$, then $n = m$ which corresponds to the entire sample. Furthermore, if $R_1 = R_2 = \cdots = R_{m-1} = 0$, then $R_m = n - m$, which corresponds to the traditional type II right-censoring technique. See [9] for more information on progressive censoring and important references. The authors of [11] propose a new censoring scheme, the type II progressive hybrid censoring scheme, which combines progressive type II and hybrid censoring schemes. A hybrid censoring system is a combination of type I and type II censoring schemes, as defined by [12]. We suggest readers see the book by [9] for a thorough account and results of progressive censoring.

However, in many industrial applications, one of the major requirements for the majority of the production process is that the system is required to operate for a longer period of time. In addition, it is very important to maintain the desired level of system reliability regarding the lifetime of products. In such scenarios, however, life testing under normal stress can lead to a lengthy procedure with expensive cost. Therefore, standard procedures related to progressive censoring may not be advisable. To remedy this situation, the study of accelerated life testing (ALT) has been developed (see [13] for details and the references cited therein). The test procedure makes it possible to quickly obtain information on the life distribution of products by inducing early failure with a much stronger stress than normal. An important added feature in ALT is the step-stress accelerated life test (SSALT). There are two main types of SSALTs: the simple SSALT and the multiple-step SSALT. In the simple SSALT there is a single change of stress during the test. For example, the authors of [14] showed optimum simple SSALT plans in an exponential cumulative exposure (CE) model, the author of [15] studied an exponential CE model with a threshold parameter in the simple SSALT, and the authors of [16] demonstrated optimum modified simple SSALT plans in an exponential CE model, with the consideration that it is desirable to increase the stress at some finite rate. The authors of [17] dealt with a Weibull step-stress model in the simple SSALT. The ALT for modified Kies exponential lifetime distribution based on a progressive-censoring scheme was introduced by [18].

Of particular note is that even though there are some works on complete and censored NH data from both Bayesian and frequentist perspectives, little attention has been paid to the progressive-stress model for the Nadarajah–Haghighi distribution with type II progressive censoring. Accordingly, in this study, we considered classical and Bayesian inference of the NH progressive step-stress model with the SSALT. In addition, we carried out simulation studies to compare the finite sample performance of the estimation methods based on different loss functions.

The rest of the paper is organized as follows. In Section 2, we propose the cumulative exposure model of NH distribution. We discuss the maximum likelihood estimation shape and scale parameters of the NH step-stress model with the SSALT in Section 3. In Section 4, we discuss Bayesian estimation using BSEL and BLINEX loss functions via Markov chain Monte Carlo (MCMC) algorithms. The interval estimation such as asymptotic confidence interval, HPD interval of credibility and Bootstrap confidence intervals is outlined in Section 5. Section 6 describes the simulation studies. An illustrative example is presented in Section 7. Some final remarks are presented in Section 8.

2. A Cumulative Exposure Model of NH Distribution

The basic assumptions under progressive-stress accelerated life testing of the NH distribution are given as follows:

- Under usual conditions, the lifetime of a unit follows NH($\theta,\ \lambda$).
- The progressive-stress φ(t) is directly proportional to the time t with constant rate β, i.e., $\varphi(t) = \beta t,\ \beta > 0$.
- The scale parameter λ of the CDF in (2) satisfies the inverse power law, as follows

$$\lambda(t) = \frac{1}{a[\varphi(t)]^b}$$

- It is assumed that a and b are unknown physical positive parameters and need to be estimated.
- Assume n is the total number of units tested, $\varphi_0 < \varphi_1(t) < \ldots < \varphi_k(t)$ are the stress levels in the test, and φ_0 is the use stress. Under each progressive-stress level, identical units $\varphi_i(t) = \beta_i t, i = 1,2,\ldots,k,\ n_i$ are tested, and the progressive type II censoring is performed as follows: When the first failure $t_{i1:m_i:n_i}$ occurs, R_{i1} units are picked at random from the remaining $n_1 - 1$ surviving units. When the second failure $t_{i2:m_i:n_i}$ occurs, R_{i2} items from the remaining $n_i - 2 - R_{i1}$ units are withdrawn at random. When the $m_i - th$ failure occurs, $t_{im_i:m_i:n_i}$, the test is terminated, and all remaining $R_{im_i:m_i:n_i} = n_i - m_i - \sum\limits_{j=1}^{m_i-1} R_{im_i}$ items are removed.
- The complete samples and type II censored samples are clearly specific examples of this technique. Under the progressive-stress $\varphi_i(t)$, the observed progressive-censoring data are $t_{i1:m_i:n_i} < t_{i2:m_i:n_i} < \cdots < t_{im_i:m_i:n_i}$, $i = 1,2,\ldots,k$.
- The linear cumulative exposure model (CEM) accounts for the effect of changing stress; for more details, see [13].
- The CDF in progressive stress, $\varphi_i(t)$, and the linear cumulative exposure model is given as follows

$$G_i(t) = F_i(\Delta t),\ i = 1,2,\ldots,k,$$

where $\Delta t = \int_0^t \frac{1}{\lambda_i(u)} du = \frac{a\beta_i^b t^{b+1}}{b+1}$, F $(\cdot)$ denotes the CDF of the NH distribution under progressive-stress $\varphi_i(t)$ with scale parameter λ. Thus,

$$G_i(t) = 1 - e^{\left(1-\left(1+\frac{a\beta_i^b t^{b+1}}{b+1}\right)^{\theta}\right)},\ t > 0,\ a,\ b,\ \theta > 0,\ i = 1,2,\ldots,k. \tag{3}$$

The corresponding PDF is given by

$$g_i(t) = a\theta\beta_i^b t^b \left(1 + \frac{a\beta_i^b t^{b+1}}{b+1}\right)^{\theta-1} e^{\left[1-\left(1+\frac{a\beta_i^b t^{b+1}}{b+1}\right)^{\theta}\right]}, \tag{4}$$

where $t > 0,\ a,\ b,\ \theta > 0,$

3. Maximum Likelihood Estimation

In this section, we discuss the maximum likelihood estimates (MLEs) for the parameters a, b, and θ under the progressive-stress ALT based on progressively type II censored data. We assume that the observed data under the stress levels $\varphi_i(t),\ i = 1, \ldots, k$ and $j = 1, \ldots, m_{i,}$. The likelihood function is obtained as

$$L(a,b,\theta) = \prod_{i=1}^{k} c_i \prod_{j=1}^{m_{i,}} g_i(t_{ij})\left[1 - G_i(t_{ij})\right]^{R_{ij}},\ t_{ij} > 0, \tag{5}$$

where $c_i = n_i(n_i - 1 - R_{i1})(n_i - 2 - R_{i2})\ldots\left(n_i - m_i + 1 - \sum_{j=1}^{m_i-1} R_{ij}\right)$. By substituting (3) and (4) in (5), we obtain

$$L(a,b,\theta) = \prod_{i=1}^{k} c_i \prod_{j=1}^{m_{i,}} a\theta\beta_i^b t_{ij}^b \left(1 + \frac{a\beta_i^b t_{ij}^{b+1}}{b+1}\right)^{\theta-1} e^{\left[1-\left(1+\frac{a\beta_i^b t_{ij}^{b+1}}{b+1}\right)^{\theta}\right]} \left\{ e^{\left[1-\left(1+\frac{a\beta_i^b t_{ij}^{b+1}}{b+1}\right)^{\theta}\right]} \right\}^{R_{ij}}.$$

Then,

$$L(a,b,\theta) = \prod_{i=1}^{k} c_i \prod_{j=1}^{m_{i,}} a\theta\beta_i^b t_{ij}^b \left(1 + \frac{a\beta_i^b t_{ij}^{b+1}}{b+1}\right)^{\theta-1} \left\{ e^{\left[1-\left(1+\frac{a\beta_i^b t_{ij}^{b+1}}{b+1}\right)^{\theta}\right]} \right\}^{R_{ij}+1} \tag{6}$$

The logarithm of (6) is given by

$$\begin{aligned} log\ L(a,b,\theta) = & \sum_{i=1}^{k} log\ c_i \\ & + \sum_{i=1}^{k} m_i(log a + log\theta) \\ & + b\sum_{i=1}^{k} m_i log\ \beta_i \\ & + b\sum_{i=1}^{k}\sum_{j=1}^{m_i} log(t_{ij}) \\ & + (\theta-1)\sum_{i=1}^{k}\sum_{j=1}^{m_i} log\left(1 + \frac{a\beta_i^b t_{ij}^{b+1}}{b+1}\right) + \sum_{i=1}^{k}\sum_{j=1}^{m_i}(R_{ij}+1)\left[1 - \left(1 + \frac{a\beta_i^b t_{ij}^{b+1}}{b+1}\right)^{\theta}\right]. \end{aligned} \tag{7}$$

By taking the first partial derivatives of (7) with respect to a, b, and θ and then equating each to zero, we obtain

$$\frac{\partial log\ L}{\partial\ a} = \frac{\sum_{i=1}^{k} m_i}{a} + (\theta-1)\sum_{i=1}^{k}\sum_{j=1}^{m_i} \frac{\beta_i^b t_{ij}^{b+1}}{(b+1)\left(1 + \frac{a\beta_i^b t_{ij}^{b+1}}{b+1}\right)} + \sum_{i=1}^{k}\sum_{j=1}^{m_i} \frac{\theta\beta_i^b t_{ij}^{b+1}}{(b+1)}(R_{ij}+1)\left(1 + \frac{a\beta_i^b t_{ij}^{b+1}}{b+1}\right)^{\theta-1} = 0, \tag{8}$$

$$\begin{aligned}\frac{\partial log\ L}{\partial\ b} = & \sum_{i=1}^{k} m_i log\beta_i \\ & + \sum_{i=1}^{k}\sum_{j=1}^{m_i} t_{ij} \\ & + (\theta-1)\sum_{i=1}^{k}\sum_{j=1}^{m_i} \frac{a\beta_i^b t_{ij}^{b+1}\left[\frac{log\beta_i}{b+1}+\frac{logt_{ij}}{b+1}-1\right]}{\left(1+\frac{a\beta_i^b t_{ij}^{b+1}}{b+1}\right)} \\ & + \sum_{i=1}^{k}\sum_{j=1}^{m_i}\theta\left(R_{ij}+1\right)\left[1-\frac{a\beta_i^b t_{ij}^{b+1}\left[\frac{log\beta_i}{b+1}+\frac{logt_{ij}}{b+1}-1\right]}{(b+1)^2}\right] = 0,\end{aligned} \tag{9}$$

$$\begin{aligned}\frac{\partial log\ L}{\partial\ \theta} = & \frac{\sum_{i=1}^{k} m_i}{\theta} \\ & + \theta\sum_{i=1}^{k}\sum_{j=1}^{m_i} log\left(1+\frac{a\beta_i^b t_{ij}^{b+1}}{b+1}\right) \\ & - \sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{\theta\beta_i^b t_{ij}^{b+1}}{(b+1)}\left(R_{ij}+1\right)\left(1+\frac{a\beta_i^b t_{ij}^{b+1}}{b+1}\right)^{\theta} log\left(1+\frac{a\beta_i^b t_{ij}^{b+1}}{b+1}\right) = 0.\end{aligned} \tag{10}$$

Equating the first partial derivatives of (7) to zero and solving the resulting Equations (8)–(10) numerically, we could numerically obtain the MLEs $\hat{a}$, $\hat{b}$ and $\hat{\theta}$ for the three unknown parameters a, b, and θ. As a result, numerical solutions to the nonlinear system in (8)–(10) can be found using an iterative approach such as Newton–Raphson.

4. Bayesian Estimation

In this section, we study the Bayes estimates of the parameters a, b, and θ using the balanced squared error loss (BSEL) and balanced linear-exponential (BLINEX) error loss function. It is well known that Bayesian analysis begins with the prior specification for the unknown parameters, and in the absence of prior knowledge, noninformative priors are often preferred to minimize the impacts of the prior distributions. Thus, in this study, we assumed that the three parameters are independent, and the noninformative priors of a, b, and θ are given as

$$\pi(\Theta) = \pi(a,b,\theta) = \pi_1(a)\pi_2(b)\pi_3(\theta) \propto \frac{1}{ab\theta}, a,b,\theta > 0, \tag{11}$$

where Θ is the parameter model. The resulting joint posterior density function is obtained by

$$\pi^*(a,b,\theta|t) \propto \pi(a,b,\theta)L(a,b,\theta),\ \text{which shows that}\pi^*(a,b,\theta|t) \propto \frac{1}{ab\theta}\prod_{i=1}^{k} c_i \tag{12}$$

Bayes Estimation Using BSEL and BLINEX Loss Functions

We followed the work of [19–21] and determined that the Bayes parameter estimators of a function w using the BSEL function are given by

$$\hat{w}_{BSEL} = \omega\hat{w}_{ML} + (1-\omega)\int_{\Theta}\hat{w}_{ML}\ \pi^*(\Theta|\,t)\ d\Theta \tag{13}$$

where $\hat{w}_{ML}$ is the MLE of w, and $0 < \omega < 1$ and $\pi^*(.|t)$ is defined in (12). Similarly, the Bayes parameter estimator of a function w using the BLINEX loss function is obtained as

$$\hat{w}_{BLINEX} = -\frac{1}{\gamma} ln\left[\omega e^{-\gamma\hat{w}_{ML}} + (1-\omega)\int_{\Theta} e^{-\gamma\hat{w}_{ML}\pi^*(\Theta|t)}\ d\Theta\right] \tag{14}$$

where $\gamma \neq 0$ is the shape parameter of the BLINEX loss function. We used the MCMC algorithm to solve the integrals in (13) and (14). The conditional posterior distributions of the parameters a, b and θ are obtained as follows

$$\pi^*(a|b,\theta,t) \propto \frac{1}{a}\prod_{i=1}^{k}\prod_{j=1}^{m_i}\beta_i^b t_{ij}^b\left(1+\frac{a\beta_i^b t_{ij}^{b+1}}{b+1}\right)^{\theta-1}\left\{e^{[1-(1+\frac{a\beta_i^b t_{ij}^{b+1}}{b+1})^{\theta}]}\right\}^{R_{ij}+1}, \tag{15}$$

$$\pi^*(b|a,\theta,t) \propto \frac{1}{b}\prod_{i=1}^{k}\prod_{j=1}^{m_i}\beta_i^b t_{ij}^b\left(1+\frac{a\beta_i^b t_{ij}^{b+1}}{b+1}\right)^{\theta-1}\left\{e^{[1-(1+\frac{a\beta_i^b t_{ij}^{b+1}}{b+1})^{\theta}]}\right\}^{R_{ij}+1}, \tag{16}$$

and

$$\pi^*(\theta|b,a,t) \propto \frac{1}{\theta}\prod_{i=1}^{k}\prod_{j=1}^{m_i}\beta_i^b t_{ij}^b\left(1+\frac{a\beta_i^b t_{ij}^{b+1}}{b+1}\right)^{\theta-1}\left\{e^{[1-(1+\frac{a\beta_i^b t_{ij}^{b+1}}{b+1})^{\theta}]}\right\}^{R_{ij}+1}. \tag{17}$$

Because it is difficult to obtain the conditional posterior distributions of the parameters a, b, and θ in the form of some well-known distributions, we employed the Metropolis–Hastings (MH) algorithm for generating posterior samples for the above conditional posterior distributions to obtain the appropriate Bayes estimates of the unknown parameters. However, drawing samples directly from the conditional posterior distributions of model parameters may be problematic. As a result, we need to choose appropriate known distributions to obtain these conditional distributions. More information about this topic can be found in [22–27]. In this study, we followed the steps of the MH algorithm to create a random sample from these conditional distributions:

Step 1: For the parameters (a, b, θ), set the initial guess to (a^0, b^0, θ^0).
Step 2: Set j = 1.
Step 3: Create $a \sim N(a^j, \sigma_{11}), b \sim N(b^j, \sigma_{22})$ and $\theta \sim N(\theta^j, \sigma_{33})$, where σ is the variance–covariance matrix.
Step 4: Compute $p = \frac{\pi(a^j, b^j, \theta^j | \underline{x})}{\pi(a^{j-1}, b^{j-1}, \theta^{j-1} | \underline{x})}$.
Step 5: With probability $min(1, p)$, accept (a_j, b_j, θ_j),
Step 6: To obtain B number of samples for the parameters (a, b, θ), repeat steps (3) to (5) B times.

5. Interval Estimation

We created confidence intervals (CIs) utilizing the MLE's normality criterion using the asymptotic CI (ACI) as well as the highest posterior density (HPD) intervals for the parameters a, b, and θ.

5.1. Asymptotic Confidence Interval

The ACI was obtained using the normality property of MLEs with the parameters a, b, and θ. It is known that the MLEs of (a, b, θ) have an asymptotic distribution under some regularity conditions, such that

$$(\hat{a} - a) \sim N(0, \sigma_{11}),\ \left(\hat{b} - b\right) \sim N(0, \sigma_{22}),\ \left(\hat{\theta} - \theta\right) \sim N(0, \sigma_{33}),$$

where the variance–covariance matrix for the parameters (a, b, θ) is σ_{ii}. The $100(1-\omega)\%$ percent confidence intervals for the parameters $(\theta_1, \theta_2, \theta_3) = (a, b, \theta)$ are written as $\left(\hat{\theta}_i \mp Z_{1-\frac{\omega}{2}}\sqrt{\sigma_{ii}}\right)$, i = 1, 2, 3, where σ_{ii} indicates the $(i,i)th$ the variance–covariance matrix σ and $\left(\hat{\theta}_1 \equiv \hat{a},\ \hat{\theta}_2 \equiv \hat{b}, \hat{\theta}_3 \equiv \hat{\theta}\right)$.

5.2. HPD Interval of Credibility

The HPD credible interval $(L,\ U)$ was constructed for a random quantity θ^* that is derived by debating the following expression.

$$p(L \leq \theta^* \leq U) = \int_L^U \pi^*(\theta^*|t)d\theta^* = 1 - \gamma$$

Because finding the interval $(L,\ U)$ analytically is difficult, we used the posterior samples acquired in Section 4, to obtain the requisite HPD credible intervals using the technique described in [28].

5.3. Bootstrap Confidence Intervals

The CIs based on parametric bootstrap sampling utilizing the percentile intervals were produced; for additional information, see [29]. The bootstrap CIs were calculated by using the following algorithm:

(1) Calculate the MLE values of the parameters using the original data θ, a, and b.
(2) To make a bootstrap sample t^*, use the variables $\hat{\theta}$, $\hat{a}$, and $\hat{b}$.
(3) The bootstrap estimates $\hat{\theta}^*$, $\hat{a}^*$, and $\hat{b}^*$, respectively, are obtained based on t^*.
(4) To obtain the bootstrap samples, repeat steps 1–3 several times and organize each estimate in ascending order

$$\left\{\hat{\theta}^{*[1]},\ \hat{\theta}^{*[2]}, \ldots,\ \hat{\theta}^{*[I]}\right\}, \left\{\hat{a}^{*[1]},\ \hat{a}^{*[2]}, \ldots,\ \hat{a}^{*[I]}\right\} \text{and } \left\{\hat{b}^{*[1]},\ \hat{b}^{*[2]}, \ldots, b^{*[I]}\right\},$$

The $100(1-\delta)\%$ percentile bootstrap CIs for ω are then calculated as follows:

$$(\hat{\omega}_{iL},\ \hat{\omega}_{iU}) = \left(\hat{\omega}_i{}^{*[\frac{\delta}{2}I]},\ \hat{\omega}_i{}^{*[(1-\frac{\delta}{2})I]}\right), i = 1, 2, 3, \tag{18}$$

where $\hat{\omega}_1{}^* = \hat{\theta}^*$, $\hat{\omega}_2{}^* = \hat{a}^*$ and $\hat{\omega}_3{}^* = \hat{b}^*$.

6. Simulation Study

We carried out a simulation study according to the following algorithm to obtain the likelihood and Bayesian estimation of the NH parameters based on progressive stress and study their properties through the mean squared error (MSE), bias, and length of confidence intervals.

Step 1: Using the algorithm presented in [10], $k \geq 2$ progressively type II censored random samples are generated from the uniform (0,1) distribution $\left(U_{i1}, U_{i2}, \ldots, U_{im_{i,}}\right)$, for given values of $m_{i,}, i = 1, 2, \ldots, k$.

Step 2: To compare the performance of the estimation procedures developed in the study, we consider the following two schemes for each stress: Scheme 1: $R_{im_i} = n_i - m_i$, $R_{ij} = 0; j = 1, \ldots, m_i - 1$. Scheme 2: $R_{i1} = n_i - m_i$, $R_{ij} = 0; j = 1, \ldots, m_i$.

Step 3: Progressively type II censored random samples $(t_{i1}, \ldots, t_{im_i})$ are produced, and from inverse CDF (3), we specify the values of parameters as follows: In Table 1 ($\theta = 1.7,\ a = 1.3,\ b = 2$), $k = 2$, and $\beta_1 = 40,\ \beta_2 = 80$. In Table 2 ($\theta = 0.8,\ a = 0.5, b = 1.3$), $k = 2$, and $\beta\_1 = 40$, $\beta\_2 = 80$. In Table 3 ($\theta = 3,\ a = 2.5,\ b = 0.6$), $k = 2$, and $\beta_1 = 40,\ \beta_2 = 80$. In Table 4 ($\theta = 1.7$, a = 1.3, b = 2), $k = 4$, and $\beta_1 = 40,\ \beta_2 = 80,\ \beta_3 = 110,\ \beta_4 = 150$. In Table 5 ($\theta = 0.8$, a = 0.5, $b = 1.3$), $k = 4$, and $\beta_1 = 40,\ \beta_2 = 80,\ \beta_3 = 110,\ \beta_4 = 150$. In Table 6 ($\theta = 3,\ a = 2.5,\ b = 0.6$), $k = 4$, and $\beta_1 = 40,\ \beta_2 = 80,\ \beta_3 = 110,\ \beta_4 = 150$.

Step 4: The MLEs $\left(\hat{\theta},\ \hat{a},\ \hat{b}\right)$ are obtained numerically by solving the likelihood equations with respect to $(\theta,\ a,\ b)$ in (8)–(10) by using an iterative Newton–Raphson algorithm using the maxlik function of the "maxlik" package in the R program; for more information in this topic see [30].

Step 5: Based on (15)–(17), and the MH algorithm, the Bayesian estimations with the BSEL and BLINEX loss functions of the parameters (θ, a, b) are computed by (13) and (14), respectively.

Step 6: The above steps are repeated I times based on I different samples, and then the average of likelihood and Bayesian estimations are computed, with their MSE, bias, and length of confidence intervals (LCI) of the parameters (θ, a, b).

Step 7: In length of CI (LCI) of the MLE of each parameter, we compute the ACI for likelihood estimators and bootstrap CIs with the percentile algorithm and t algorithm, which can be denoted as LBP and LBT, respectively. In the LCI of Bayesian estimation, we compute the HPD for each loss function, denoted by the LCCI.

Simulation Results

Based on the two censoring schemes, considering two cases of levels of the stress (simple ramp when $k = 2$ and multi ramp when $k = 4$ $(k > 2)$), simulation results are presented in Tables 1–6. First, Tables 1–3 show the likelihood and Bayesian estimations of the parameters (θ, a, b) in the simple ramp with their bias, MSE, and LCI for different sample sizes (n_i), and number of failures (m_i) which can be obtained by rounding $r \times n_i$. Second, Tables 4–6 show the likelihood and Bayesian estimations of the parameters (θ, a, b) in the multi-ramp with their bias, MSE, and LCI for different sample sizes (n_i), and number of failures (m_i). Bayesian estimations of the parameters based on the BLINEX loss function, considering two values of c (c = −0.5, 0.5), are also included. The number of MH iterations of the algorithm is 10,000, and the number of simulations is 1000. The coverage probabilities of CI were set to 95%.

From the simulation results presented in Tables 1–6, the following points can be observed:

(1) For fixed values of the sample sizes n_i, by increasing the censored sample sizes, m_i, the bias, MSE, and LCI of the estimates decrease for the two different censored schemes.
(2) For fixed values of m_i, by increasing the sample sizes n_i, the bias, MSE, and LCI decrease for different censored schemes.
(3) For fixed values of m_i or n_i or scheme, by increasing the level of stress k, the bias, MSE, and LCI decrease.
(4) For fixed values of m_i or n_i or scheme, we note that Scheme 2 is better than Scheme 1 for some or all parameters.
(5) The bias and MSE reduce significantly, and the symmetric and asymmetric Bayesian estimations are better than the MLE in the considered scenarios.
(6) The LCI reduces significantly, the symmetric and asymmetric Bayesian estimations of the HPD are better than the ACI of MLE.
(7) We observe that the shortest lengths of the CI are the bootstrap CI.

Table 1. MLE and Bayesian estimation with different loss functions in the simple ramp when $\theta = 1.7$, $a = 1.3$, $b = 2$.

$k = 2$	$\theta = 1.7, a = 1.3, b = 2$			MLE		BSEL		BLINEX (c = −0.5)		BLINEX (c = 0.5)		MLE			BSEL	BLINEX (c = −0.5)	BLINEX (c = 0.5)
n_1, n_2	Scheme	r		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	LCI	LBP	LBP	LCCI	LCCI	LCCI
20, 15	1	65%	θ	−0.6751	1.5008	−0.1938	0.0886	−0.1172	0.0801	−0.2619	0.1099	4.0094	0.1331	0.1271	0.8592	0.9821	0.7748
			a	0.3824	1.5335	−0.0521	0.0788	0.0209	0.0994	−0.1174	0.0740	4.6193	0.1432	0.1453	1.0048	1.1145	0.8922
			b	0.0673	0.6760	−0.2972	0.1145	−0.2616	0.0964	−0.3314	0.1345	3.2139	0.0998	0.0993	0.6315	0.6551	0.6181
		85%	θ	−0.4879	0.7308	−0.0879	0.0657	−0.0097	0.0736	−0.1581	0.0724	2.7529	0.0899	0.0894	0.9552	1.0524	0.8647
			a	0.5008	0.9984	0.0504	0.0612	0.0195	0.0890	0.0426	0.0691	3.3911	0.1105	0.1101	0.9821	0.9834	0.9084
			b	0.0625	0.6469	−0.1622	0.0558	−0.1260	0.0471	−0.1967	0.0667	3.1026	0.0978	0.0965	0.6718	0.6934	0.6551
	2	65%	θ	0.0872	0.5633	0.0550	0.0583	0.1391	0.0909	−0.0206	0.0450	2.9235	0.0894	0.0875	0.9262	1.0470	0.8281
			a	−0.2725	0.5104	0.0517	0.1058	0.1262	0.1418	−0.0158	0.0866	2.5902	0.0805	0.0818	1.2014	1.3056	1.0823
			b	0.5035	0.8594	0.0607	0.0411	0.1011	0.0500	0.0218	0.0360	3.0529	0.0908	0.0915	0.7305	0.7490	0.7048
		85%	θ	0.0823	0.5078	0.0542	0.0580	0.1346	0.0879	−0.0171	0.0448	2.8345	0.0804	0.0803	0.9137	0.9103	0.8448
			a	−0.1801	0.4902	0.0863	0.1041	0.1611	0.1509	0.0185	0.0778	2.6534	0.0858	0.0860	1.0977	1.2237	1.0492
			b	0.3257	0.5593	0.0494	0.0363	0.0879	0.0437	0.0126	0.0323	2.6404	0.0854	0.0845	0.7149	0.7384	0.6916
40, 50	1	65%	θ	−0.9678	1.4864	−0.3139	0.1585	−0.2489	0.1345	−0.3726	0.1900	3.7765	0.1250	0.1174	0.9207	1.0006	0.8607
			a	0.3126	1.0050	0.1260	0.1244	0.1974	0.1754	0.0601	0.0913	4.4408	0.1433	0.1373	1.2116	1.3391	1.0992
			b	−0.0262	0.5537	−0.4534	0.2334	−0.4262	0.2114	−0.4791	0.2558	2.9167	0.0938	0.0929	0.6567	0.6774	0.6325
		85%	θ	−0.9531	1.0410	−0.2143	0.1193	−0.1488	0.1099	−0.2738	0.1380	1.4282	0.0451	0.0455	1.0419	1.1471	0.9624
			a	0.3033	0.9427	0.1230	0.1202	0.1374	0.1701	0.0582	0.0902	3.1609	0.1034	0.1027	1.2028	1.1908	1.1697
			b	0.2468	0.3993	−0.2597	0.1017	−0.2310	0.0904	−0.2869	0.1145	2.2816	0.0720	0.0720	0.7346	0.7532	0.7076
	2	65%	θ	0.0189	0.5195	0.0350	0.0660	0.1067	0.0902	−0.0307	0.0557	2.8259	0.0875	0.0878	0.9662	1.0601	0.9024
			a	−0.1043	0.3458	0.0534	0.1026	0.1154	0.1347	−0.0034	0.0834	2.2699	0.0749	0.0750	1.1326	1.2150	1.0415
			b	0.2988	0.3316	0.0806	0.0416	0.1104	0.0489	0.0519	0.0365	1.9309	0.0610	0.0610	0.7083	0.7181	0.6939
		85%	θ	0.0199	0.2902	0.0345	0.0627	0.1023	0.0901	−0.0155	0.0547	2.1112	0.0691	0.0689	0.9521	0.9602	0.9010
			a	−0.0041	0.2644	0.0499	0.1013	0.1106	0.1325	0.0046	0.0829	2.0168	0.0606	0.0617	1.1228	1.1344	1.1043
			b	0.1356	0.1506	0.0437	0.0332	0.0711	0.0381	0.0173	0.0302	1.4262	0.0475	0.0476	0.6648	0.6811	0.6482

Table 2. MLE and Bayesian estimation with different loss functions in the simple ramp when $\theta = 0.8$, $a = 0.5$, $b = 1.3$.

$k = 2$	$\theta = 0.8$, $a = 0.5$, $b = 1.3$			MLE		BSEL		BLINEX ($c = -0.5$)		BLINEX ($c = 0.5$)		MLE			BSEL	BLINEX ($c = -0.5$)	BLINEX ($c = 0.5$)
n_1, n_2	Scheme	r		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	LCI	LBP	LBP	LCCI	LCCI	LCCI
20, 15	1	65%	θ	−0.6718	2.8664	−0.1568	0.1203	−0.1274	0.1278	−0.1845	0.1151	6.0949	0.2060	0.1973	0.9950	1.0839	0.9307
			a	0.2440	1.6298	0.1466	0.1261	0.1736	0.1496	0.1202	0.1050	4.9146	0.1549	0.1554	1.0924	1.1438	1.0436
			b	0.0957	0.4839	−0.0738	0.0664	−0.0268	0.0708	−0.1174	0.0675	2.7025	0.0858	0.0848	0.9578	1.0205	0.8990
		85%	θ	−0.2515	0.1789	−0.0586	0.1017	−0.0262	0.1189	−0.0889	0.0893	1.3336	0.0416	0.0414	1.0767	1.1327	0.9832
			a	0.1999	0.2632	0.1647	0.1249	0.1903	0.1468	0.1398	0.1054	1.8532	0.0598	0.0607	1.1127	1.1571	1.0571
			b	0.2494	0.3102	0.0044	0.0617	0.0480	0.0721	−0.0363	0.0566	1.9532	0.0594	0.0579	0.9574	1.0115	0.8972
	2	65%	θ	0.1997	0.2701	0.1153	0.1081	0.1526	0.1338	0.0801	0.0876	1.8819	0.0604	0.0602	1.0837	1.1369	0.9657
			a	−0.1490	0.1301	0.0586	0.0855	0.0816	0.0993	0.0365	0.0737	1.2884	0.0409	0.0399	1.0231	1.0804	0.9543
			b	0.2680	0.2961	0.0933	0.0665	0.1342	0.0823	0.0551	0.0557	1.8574	0.0593	0.0599	0.9271	0.9837	0.8748
		85%	θ	0.1673	0.2162	0.0845	0.0495	0.1121	0.0621	0.0586	0.0400	1.7014	0.0522	0.0516	0.7796	0.8472	0.7347
			a	−0.0699	0.0976	0.0664	0.0581	0.0850	0.0669	0.0486	0.0508	1.1941	0.0379	0.0380	0.8233	0.8664	0.7746
			b	0.1521	0.2192	0.0372	0.0230	0.0627	0.0273	0.0130	0.0206	1.7367	0.0583	0.0588	0.5765	0.6031	0.5623
40, 50	1	65%	θ	−0.6809	3.2570	−0.3695	0.1771	−0.3528	0.1723	−0.3849	0.1822	6.5550	0.2254	0.2022	0.6022	0.6613	0.5605
			a	0.6260	1.1068	0.3023	0.1956	0.3278	0.2238	0.2769	0.1692	3.3163	0.1154	0.1066	1.1525	1.2097	1.0901
			b	0.2043	0.1752	0.0435	0.0965	0.0945	0.1144	−0.0056	0.0849	1.4326	0.0473	0.0469	1.1800	1.2411	1.1403
		85%	θ	−0.4415	0.2093	−0.2621	0.0908	−0.2500	0.0879	−0.2733	0.0942	0.4708	0.0153	0.0151	0.5155	0.5453	0.4948
			a	0.4369	0.3465	0.2701	0.1401	0.2903	0.1585	0.2502	0.1231	1.5470	0.0474	0.0479	0.9661	1.0058	0.9208
			b	0.2543	0.1467	0.0283	0.0340	0.0554	0.0390	0.0022	0.0311	1.1229	0.0349	0.0349	0.7075	0.7235	0.6965
	2	65%	θ	0.0872	0.1598	0.1037	0.0866	0.1337	0.1045	0.0753	0.0720	1.5300	0.0468	0.0472	0.9517	1.0225	0.9087
			a	−0.1012	0.0663	0.0297	0.0519	0.0452	0.0580	0.0149	0.0467	0.9282	0.0277	0.0276	0.8375	0.8677	0.7991
			b	0.1905	0.2101	0.0796	0.0639	0.1103	0.0757	0.0506	0.0552	1.6350	0.0535	0.0526	0.8925	0.9435	0.8638
		85%	θ	0.0839	0.1248	0.0757	0.0492	0.0973	0.0593	0.0550	0.0409	1.3462	0.0423	0.0426	0.7645	0.8162	0.7216
			a	−0.0484	0.0473	0.0579	0.0489	0.0717	0.0550	0.0446	0.0435	0.8311	0.0265	0.0264	0.7049	0.7358	0.6820
			b	0.0961	0.1294	0.0243	0.0259	0.0433	0.0286	0.0059	0.0241	1.3594	0.0432	0.0432	0.6307	0.6517	0.6262

Table 3. MLE and Bayesian estimation with different loss functions in the simple ramp when $\theta = 3$, $a = 2.5$, $b = 0.6$.

$k = 2$	$\theta = 3$, $a = 0.5$, $b = 0.6$			MLE		BSEL		BLINEX (c = −0.5)		BLINEX (c = 0.5)			MLE		BSEL	BLINEX (c = −0.5)	BLINEX (c = 0.5)
n_1, n_2	**Scheme**	**r**		**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**LCI**	**LBP**	**LBP**	**LCCI**	**LCCI**	**LCCI**
20, 15	1	65%	θ	−1.0081	1.4542	−0.3499	0.1451	−0.2693	0.0983	−0.4242	0.2008	2.5956	0.0851	0.0847	0.5826	0.6182	0.5601
			a	0.0679	1.2060	−0.3724	0.1673	−0.2938	0.1205	−0.4438	0.2221	4.2987	0.1386	0.1410	0.6487	0.7208	0.6095
			b	−0.1176	0.1318	−0.2613	0.0778	−0.2521	0.0737	−0.2702	0.0819	1.3473	0.0437	0.0433	0.3603	0.3730	0.3459
		85%	θ	−0.4454	0.4194	−0.1202	0.0252	−0.0733	0.0168	−0.1652	0.0375	1.8440	0.0591	0.0591	0.3991	0.4098	0.3920
			a	−0.0649	0.2608	−0.1168	0.0311	−0.0670	0.0235	−0.1639	0.0433	1.9865	0.0654	0.0637	0.5281	0.5430	0.5133
			b	−0.0581	0.0912	−0.1632	0.0352	−0.1555	0.0331	−0.1707	0.0374	1.1620	0.0367	0.0374	0.3568	0.3624	0.3499
	2	65%	θ	−0.1533	0.4265	−0.0128	0.0410	0.0808	0.0511	−0.0995	0.0484	2.4897	0.0734	0.0726	0.8016	0.8379	0.7820
			a	−0.2400	0.6128	−0.0100	0.0597	0.0877	0.0755	−0.0987	0.0639	2.9224	0.0901	0.0898	0.9570	1.0232	0.9205
			b	0.2240	0.1830	0.0290	0.0270	0.0438	0.0293	0.0145	0.0253	1.4291	0.0427	0.0426	0.6181	0.6302	0.6014
		85%	θ	−0.0076	0.2889	0.0133	0.0152	0.0632	0.0198	−0.0346	0.0158	2.1077	0.0662	0.0651	0.4780	0.4812	0.4698
			a	−0.1365	0.3228	0.0253	0.0269	0.0803	0.0347	−0.0268	0.0255	2.1629	0.0679	0.0684	0.6379	0.6699	0.6282
			b	0.1288	0.1135	0.0058	0.0144	0.0155	0.0150	−0.0039	0.0140	1.2206	0.0386	0.0384	0.4462	0.4534	0.4430
40, 50	1	65%	θ	−1.1973	2.0957	−0.4388	0.2123	−0.3633	0.1551	−0.5082	0.2761	3.1917	0.1021	0.0996	0.5524	0.5850	0.5390
			a	0.4242	1.8779	−0.4240	0.2012	−0.3566	0.1528	−0.4852	0.2543	5.1106	0.1639	0.1643	0.5646	0.6197	0.5296
			b	−0.3040	0.1460	−0.3420	0.1222	−0.3367	0.1188	−0.3471	0.1255	0.9077	0.0310	0.0305	0.2663	0.2715	0.2631
		85%	θ	−1.1130	1.6602	−0.1645	0.0369	−0.1219	0.0252	−0.2053	0.0517	2.5459	0.0791	0.0792	0.3912	0.3950	0.3890
			a	0.3356	0.3522	−0.1170	0.0290	−0.0745	0.0218	−0.1575	0.0393	4.9114	0.1509	0.1504	0.4752	0.4874	0.4623
			b	−0.1376	0.0633	−0.2416	0.0645	−0.2372	0.0625	−0.2459	0.0665	0.8264	0.0268	0.0265	0.2928	0.2964	0.2898
	2	65%	θ	−0.0736	0.3191	−0.0093	0.0421	0.0786	0.0525	−0.0899	0.0477	2.1964	0.0688	0.0681	0.8118	0.8396	0.7883
			a	−0.0731	0.2996	0.0195	0.0535	0.1048	0.0712	−0.0585	0.0517	2.1275	0.0674	0.0659	0.8891	0.9502	0.8511
			b	0.1074	0.0593	0.0351	0.0180	0.0438	0.0190	0.0264	0.0171	0.8576	0.0288	0.0285	0.4791	0.4815	0.4721
		85%	θ	−0.0128	0.2589	0.0097	0.0167	0.0557	0.0204	−0.0344	0.0173	1.3010	0.0590	0.0589	0.5014	0.5122	0.4937
			a	0.0694	0.2707	0.0260	0.0300	0.0731	0.0363	−0.0188	0.0284	1.3429	0.0612	0.0612	0.6600	0.6807	0.6407
			b	0.0647	0.0409	0.0031	0.0099	0.0090	0.0101	−0.0027	0.0097	0.7513	0.0240	0.0241	0.3942	0.3955	0.3921

Table 4. MLE and Bayesian estimation with different loss functions in multi ramp when $\theta = 1.7$, $a = 1.3$, $b = 2$.

k = 4	$\theta = 1.7, a = 1.3, b = 2$			MLE		BSEL		BLINEX (c = −0.5)		BLINEX (c = 0.5)		MLE			BSEL	BLINEX (c = −0.5)	BLINEX (c = 0.5)
n_1, n_2, n_3, n_4	Scheme	r		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	LCI	LBP	LBP	LCCI	LCCI	LCCI
20, 15, 18, 10	1	65%	θ	−0.8715	1.0614	−0.2242	0.1068	−0.1519	0.0943	−0.2893	0.1302	2.1561	0.0965	0.0965	0.8956	0.9632	0.8047
			a	0.8096	1.4846	0.0001	0.0882	0.0688	0.1142	−0.0625	0.0765	3.7822	0.1691	0.1694	1.1078	1.2537	0.9950
			b	0.1157	0.3767	−0.3089	0.1190	−0.2801	0.1037	−0.3365	0.1355	2.3651	0.1070	0.1066	0.6109	0.6291	0.5869
		85%	θ	−0.5590	0.7077	−0.1089	0.0307	−0.0677	0.0259	−0.1476	0.0388	2.4667	0.1144	0.1135	0.5240	0.5597	0.4972
			a	0.6354	0.9014	0.0470	0.0402	0.0905	0.0539	0.0062	0.0323	3.0655	0.1441	0.1385	0.6795	0.7253	0.6408
			b	0.0962	0.3299	−0.1872	0.0476	−0.1704	0.0419	−0.2036	0.0538	2.2221	0.1003	0.1017	0.4437	0.4436	0.4419
	2	65%	θ	0.1463	0.4847	0.0373	0.0756	0.1170	0.1094	−0.0345	0.0617	2.9452	0.1315	0.1295	0.9935	1.0840	0.8752
			a	−0.3137	0.3755	0.0228	0.1139	0.0899	0.1463	−0.0388	0.0959	2.0658	0.0946	0.0937	1.2244	1.3813	1.1134
			b	0.3591	0.4138	0.0913	0.0437	0.1225	0.0519	0.0613	0.0379	2.0944	0.0977	0.0973	0.7276	0.7407	0.7164
		85%	θ	0.1331	0.3552	0.0306	0.0226	0.0738	0.0306	−0.0100	0.0192	2.8690	0.1216	0.1226	0.5557	0.6039	0.5304
			a	−0.1476	0.3143	0.0460	0.0416	0.0882	0.0533	0.0067	0.0348	2.1222	0.1006	0.0992	0.7528	0.8065	0.7120
			b	0.2191	0.2586	0.0238	0.0139	0.0419	0.0154	0.0060	0.0132	1.8008	0.0761	0.0764	0.4427	0.4514	0.4406
25, 20, 20, 25	1	65%	θ	−1.0371	1.2431	−0.2741	0.1331	−0.2060	0.1158	−0.3356	0.1606	1.6059	0.0765	0.0765	0.9131	1.0396	0.8033
			a	0.9794	0.9451	0.0684	0.1174	0.1367	0.1581	0.0052	0.0917	2.7529	0.1255	0.1214	1.1129	1.2247	1.0286
			b	0.2007	0.4168	−0.3360	0.1360	−0.3096	0.1204	−0.3612	0.1525	2.4078	0.0999	0.1001	0.5456	0.5477	0.5355
		85%	θ	−0.8327	0.8583	−0.1666	0.0526	−0.1276	0.0443	−0.2031	0.0638	1.5933	0.0719	0.0724	0.6202	0.6524	0.5959
			a	0.8746	0.8324	0.0859	0.0508	0.1282	0.0662	0.0464	0.0406	2.9330	0.1322	0.1275	0.7932	0.8515	0.7411
			b	0.2137	0.3082	−0.1874	0.0491	−0.1719	0.0440	−0.2027	0.0547	2.0106	0.0873	0.0873	0.4630	0.4716	0.4563
	2	65%	θ	0.0105	0.4456	0.0441	0.0754	0.1172	0.1023	−0.0232	0.0628	2.6189	0.1169	0.1171	1.0020	1.0986	0.9428
			a	−0.1869	0.3433	−0.0015	0.1093	0.0559	0.1339	−0.0545	0.0955	2.1788	0.0930	0.0930	1.1695	1.2602	1.0874
			b	0.3025	0.2693	0.0981	0.0430	0.1247	0.0504	0.0725	0.0374	1.6543	0.0717	0.0713	0.6976	0.7207	0.6846
		85%	θ	0.0224	0.2149	0.0203	0.0238	0.0630	0.0310	−0.0196	0.0212	1.8168	0.0821	0.0822	0.6106	0.6641	0.5580
			a	−0.0716	0.2182	0.0536	0.0487	0.0929	0.0604	0.0168	0.0415	1.8115	0.0809	0.0810	0.7661	0.8193	0.7406
			b	0.1356	0.1320	0.0284	0.0175	0.0444	0.0191	0.0127	0.0164	1.3223	0.0592	0.0601	0.4837	0.4910	0.4805

Table 5. MLE and Bayesian estimation with different loss functions in multi ramp when $\theta = 0.8$, $a = 0.5$, $b = 1.3$.

k = 4	$\theta = 0.8, a = 0.5, b = 1.3$			MLE		BSEL		BLINEX (c = −0.5)		BLINEX (c = 0.5)		MLE			BSEL	BLINEX (c = −0.5)	BLINEX (c = 0.5)
n_1, n_2, n_3, n_4	Scheme	r		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	LCI	LBP	LBP	LCCI	LCCI	LCCI
20, 15, 18, 10	1	65%	θ	−0.4626	0.2492	−0.2234	0.1145	−0.1991	0.1159	−0.2461	0.1151	0.7363	0.0344	0.0328	0.8227	0.8569	0.7749
			a	0.3913	0.3915	0.2019	0.1463	0.2270	0.1703	0.1772	0.1240	1.9158	0.0833	0.0839	1.1213	1.1835	1.0524
			b	0.1880	0.1968	−0.0547	0.0723	−0.0107	0.0790	−0.0959	0.0703	1.5765	0.0739	0.0737	0.9501	1.0167	0.8885
		85%	θ	−0.2908	0.1596	−0.1193	0.0548	−0.1007	0.0563	−0.1370	0.0545	1.0747	0.0506	0.0498	0.7117	0.7482	0.6735
			a	0.2322	0.2134	0.1603	0.0924	0.1795	0.1058	0.1415	0.0804	1.5669	0.0688	0.0679	0.8484	0.8828	0.8237
			b	0.1623	0.1711	−0.0366	0.0257	−0.0133	0.0263	−0.0588	0.0263	1.4928	0.0685	0.0673	0.5739	0.5880	0.5635
	2	65%	θ	0.1299	0.1748	0.1374	0.1253	0.1716	0.1523	0.1044	0.1022	1.5594	0.0733	0.0726	1.0883	1.1717	1.0231
			a	−0.1473	0.0858	0.0164	0.0596	0.0339	0.0670	−0.0005	0.0536	0.9932	0.0429	0.0434	0.8881	0.9247	0.8439
			b	0.2240	0.2153	0.0817	0.0631	0.1126	0.0743	0.0525	0.0551	1.5945	0.0698	0.0698	0.9148	0.9542	0.8811
		85%	θ	0.1153	0.1522	0.0809	0.0505	0.1036	0.0606	0.0592	0.0421	1.4724	0.0720	0.0710	0.7057	0.7504	0.6762
			a	−0.0763	0.0775	0.0679	0.0594	0.0842	0.0672	0.0520	0.0526	1.0509	0.0466	0.0465	0.8191	0.8463	0.7834
			b	0.1143	0.1520	0.0234	0.0241	0.0429	0.0264	0.0046	0.0228	1.4627	0.0616	0.0614	0.5870	0.5948	0.5762
25, 20, 20, 25	1	65%	θ	−0.5191	0.2829	−0.2937	0.1456	−0.2745	0.1430	−0.3117	0.1490	0.4551	0.0212	0.0194	0.8007	0.8642	0.7501
			a	0.5151	0.4880	0.2251	0.1420	0.2473	0.1616	0.2032	0.1237	1.8517	0.0845	0.0843	1.0437	1.0777	0.9973
			b	0.1694	0.1096	−0.0074	0.0847	0.0334	0.0947	−0.0465	0.0789	1.1161	0.0488	0.0475	1.1251	1.1954	1.0986
		85%	θ	−0.3831	0.1711	−0.2176	0.0699	−0.2051	0.0671	−0.2295	0.0729	0.6114	0.0285	0.0277	0.5299	0.5513	0.5081
			a	0.3178	0.2274	0.2112	0.1011	0.2299	0.1151	0.1928	0.0883	1.3947	0.0643	0.0637	0.8576	0.8927	0.8125
			b	0.1490	0.1018	0.0134	0.0287	0.0367	0.0318	−0.0089	0.0269	1.0124	0.0450	0.0420	0.6588	0.6696	0.6415
	2	65%	θ	0.1017	0.1693	0.1241	0.0918	0.1547	0.1115	0.0951	0.0755	1.5643	0.0706	0.0690	1.0020	1.0791	0.9475
			a	−0.1549	0.0652	−0.0193	0.0454	−0.0055	0.0490	−0.0326	0.0426	0.7969	0.0354	0.0353	0.7545	0.7783	0.7351
			b	0.2107	0.1870	0.0883	0.0566	0.1150	0.0655	0.0629	0.0499	1.4821	0.0700	0.0698	0.8434	0.8594	0.8319
		85%	θ	0.0992	0.1270	0.0802	0.0429	0.0992	0.0500	0.0619	0.0369	1.3428	0.0595	0.0595	0.7124	0.7547	0.6790
			a	−0.0549	0.0520	0.0444	0.0417	0.0577	0.0465	0.0314	0.0374	0.8686	0.0394	0.0392	0.6876	0.7238	0.6680
			b	0.0887	0.1077	0.0285	0.0241	0.0455	0.0266	0.0120	0.0225	1.2399	0.0551	0.0552	0.5933	0.6070	0.5709

Table 6. MLE and Bayesian estimation with different loss functions in multi ramp when $\theta = 3$, $a = 2.5$, $b = 0.6$.

$k = 4$	$\theta = 3, a = 2.5, b = 0.6$			MLE		BSEL		BLINEX (c = −0.5)		BLINEX (c = 0.5)		MLE			BSEL	BLINEX (c = −0.5)	BLINEX (c = 0.5)
n_1, n_2, n_3, n_4	Scheme	r		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	LCI	LBP	LBP	LCCI	LCCI	LCCI
20, 15, 18, 10	1	65%	θ	−1.3150	2.3042	−0.3331	0.1341	−0.2518	0.0897	−0.4078	0.1876	2.9740	0.0931	0.0934	0.5858	0.6235	0.5653
			a	0.8989	3.0227	−0.3491	0.1514	−0.2737	0.1096	−0.4177	0.2009	5.8365	0.1831	0.1819	0.6546	0.7089	0.6228
			b	−0.0913	0.0606	−0.2622	0.0764	−0.2552	0.0731	−0.2691	0.0798	0.8971	0.0275	0.0271	0.3234	0.3309	0.3182
		85%	θ	−0.4708	0.4836	−0.1116	0.0243	−0.0654	0.0166	−0.1557	0.0358	2.0072	0.0672	0.0661	0.4192	0.4311	0.4174
			a	0.1134	0.3691	−0.0982	0.0274	−0.0510	0.0217	−0.1430	0.0372	2.3408	0.0769	0.0765	0.5218	0.5481	0.5058
			b	−0.1124	0.0513	−0.1800	0.0387	−0.1749	0.0370	−0.1852	0.0405	0.7709	0.0247	0.0247	0.2934	0.2941	0.2908
	2	65%	θ	−0.1024	0.4661	−0.0198	0.0413	0.0717	0.0510	−0.1043	0.0486	2.6473	0.0861	0.0860	0.8006	0.8313	0.7613
			a	−0.1680	0.7442	−0.0116	0.0668	0.0803	0.0824	−0.0956	0.0693	3.3186	0.1090	0.1096	0.9976	1.0673	0.9505
			b	0.1489	0.0759	0.0331	0.0167	0.0426	0.0178	0.0237	0.0159	0.9086	0.0284	0.0285	0.4847	0.4906	0.4771
		85%	θ	0.0948	0.4053	0.0163	0.0164	0.0650	0.0211	−0.0303	0.0166	2.4820	0.0808	0.0806	0.4926	0.5031	0.4903
			a	−0.1387	0.3914	0.0311	0.0291	0.0823	0.0368	−0.0174	0.0271	2.3925	0.0765	0.0772	0.6587	0.6836	0.6349
			b	0.0754	0.0501	0.0002	0.0083	0.0061	0.0085	−0.0057	0.0082	0.8269	0.0263	0.0273	0.3591	0.3616	0.3564
25, 20, 20, 25	1	65%	θ	−1.0913	1.4984	−0.3755	0.1641	−0.2985	0.1155	−0.4466	0.2206	2.1751	0.0671	0.0668	0.5890	0.6191	0.5681
			a	0.1753	0.6860	−0.3636	0.1586	−0.2929	0.1171	−0.4282	0.2069	3.1748	0.1003	0.1006	0.6294	0.6672	0.6001
			b	−0.1634	0.0712	−0.2865	0.0879	−0.2809	0.0849	−0.2920	0.0909	0.8272	0.0254	0.0253	0.2929	0.2975	0.2878
		85%	θ	−0.6934	0.7946	−0.1339	0.0307	−0.0891	0.0214	−0.1768	0.0435	2.1972	0.0689	0.0680	0.4372	0.4488	0.4279
			a	0.1442	0.6040	−0.0950	0.0278	−0.0492	0.0224	−0.1385	0.0372	3.0604	0.1013	0.0911	0.5472	0.5677	0.5313
			b	−0.1170	0.0424	−0.1894	0.0414	−0.1851	0.0398	−0.1936	0.0429	0.6642	0.0197	0.0199	0.2797	0.2791	0.2786
	2	65%	θ	−0.1862	0.6036	−0.0300	0.0419	0.0594	0.0488	−0.1123	0.0510	2.9582	0.0956	0.0961	0.7994	0.8209	0.7782
			a	−0.0428	0.6685	−0.0307	0.0599	0.0549	0.0696	−0.1088	0.0656	3.2022	0.1008	0.1016	0.9397	1.0007	0.9127
			b	0.1438	0.0634	0.0441	0.0147	0.0515	0.0157	0.0367	0.0139	0.8105	0.0261	0.0261	0.4317	0.4365	0.4286
		85%	θ	−0.0443	0.1186	0.0114	0.0177	0.0587	0.0219	−0.0340	0.0182	1.3394	0.0427	0.0432	0.5242	0.5270	0.5207
			a	−0.0603	0.1282	0.0288	0.0309	0.0779	0.0382	−0.0179	0.0290	1.3843	0.0435	0.0431	0.6507	0.6920	0.6365
			b	0.0485	0.0237	0.0034	0.0071	0.0082	0.0072	−0.0014	0.0070	0.5732	0.0183	0.0181	0.3187	0.3206	0.3177

7. An Illustrative Example

We used a practical experiment to test the MH distribution's superiority over its competitors. In this case, [13] employed a previous dependability experiment. This data set contains stress time from ramp-voltage studies with small light bulbs. In the ramp-voltage trials, 62 light bulbs were utilized in the first level of stress, and 61 light bulbs were used in the second level of stress, with ramp rates of 2.01 and 2.015 V/h, respectively, as well as a 2 V experimental design stress. The data are for a complete sample that has not been censored as follows:

The first level under progressive stress 2.01 V/h, t_1, and the second level under progressive stress 2.015 V/h, t_2, are shown in Figures 1 and 2, respectively.

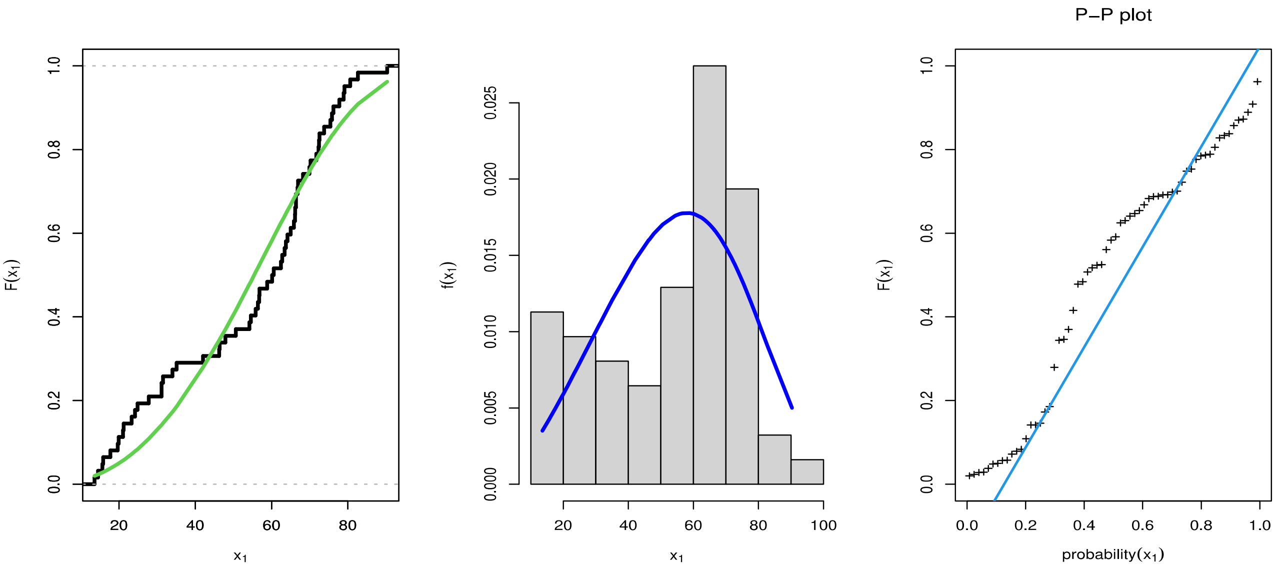

Figure 1. Fitted CDF with empirical CDF, estimated PDF, and P-P plots for first stress levels.

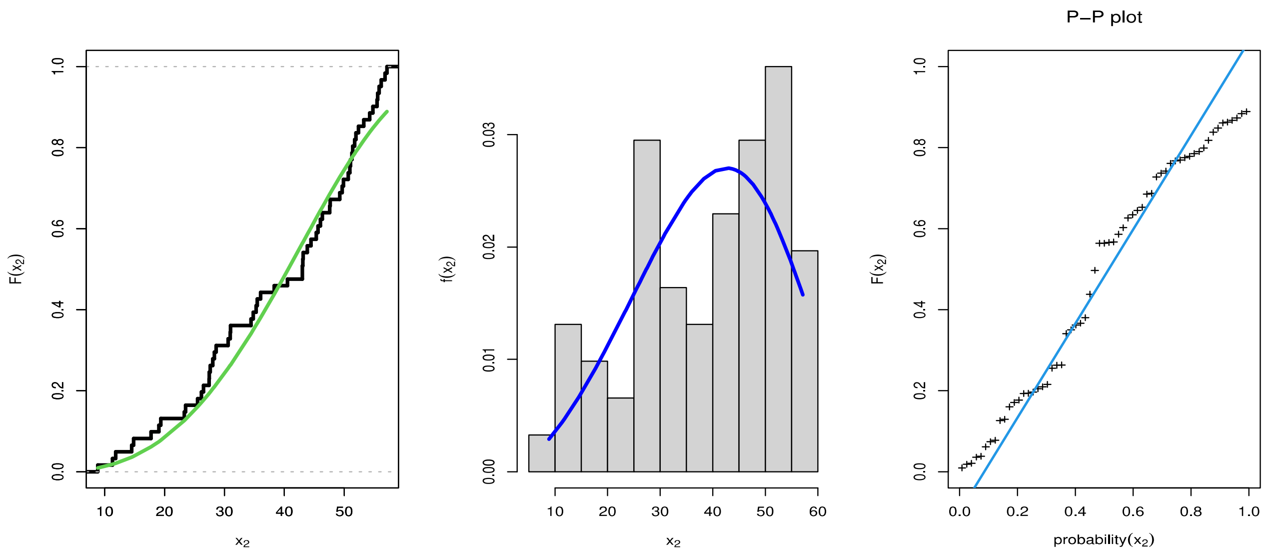

Figure 2. Fitted CDF with empirical CDF, estimated PDF, and P-P plots for second stress levels.

To see if the real data set follows the NH distribution, we utilized the Kolmogorov–Smirnov distance (KSD) test to obtain the p-value for making decisions. Table 7 contains the MLEs for the NH distribution with standard error (SE) and Akaike information criteria (AIC). Figures 1 and 2 show the fitted CDF, estimated PDF, and P-P plots for each stress level of real data. These figures confirmed that the NH distribution is a fit for each stress level of this real data. According to the results of KSD and P-values for the NH distribution in Table 7 and Figures 1 and 2, we conclude that the NH distribution is suitable for fitting each stress level of real data. If the P-value of the KS test is more than 0.05, then we may conclude that the distribution is a fit for this data. When the P-value of the KS test increases, the line of estimated cdf is near the empirical cdf, and the line of estimated pdf has drowned the histogram of data. Table 7 shows the KS test values and the P-value of the KS test < 0.5 as shown in Figures 1 and 2. Table 8 shows the MLEs and Bayesian estimates for the parameters of the progressive-stress model and the MLEs and Kolmogorov–Smirnov test for each stress level of real data. Table 8 results were obtained after those of Table 7.

Table 7. MLE and Kolmogorov–Smirnov test for each stress level of real data.

	First Level		**Second Level**	
	Estimates	**SE**	**Estimates**	**SE**
θ	3.3823	0.8429	2.6078	0.5590
a	1.10×10^{-5}	3.96×10^{-6}	8.33×10^{-6}	3.08×10^{-6}
b	1.3756	0.0320	1.6994	0.0507
AIC	554.7177		493.5818	
KSD	0.1124		0.1113	
p-value	0.3852		0.4362	

Table 8. MLEs and Bayesian estimates for parameters of progressive-stress model.

	MLE				**Bayesian**			
	Estimates	**SE**	**Lower**	**Upper**	**Estimates**	**SE**	**Lower**	**Upper**
θ	11.9739	0.5590	0.0000	66.3388	12.3834	0.4691	4.0420	21.6353
a	1.84×10^{-5}	3.08×10^{-6}	1.84×10^{-5}	1.84×10^{-5}	1.92×10^{-5}	2.75×10^{-6}	1.08×10^{-6}	1.71×10^{-5}
b	1.0260	0.0507	0.9979	1.0541	1.0322	0.0409	0.9997	1.0539
AIC	1074.528							

As illustrated in Figure 3, we see that Bayes estimators outperform MLEs in most cases. The estimated parameters maximize the log-likelihood function. The roots were computed, and they invariably pinpoint the global maximum rather than the local maximum. By displaying the log-likelihood function, we were able to confirm our findings. As seen by the blue dot, the estimate is at its maximum location along the curve.

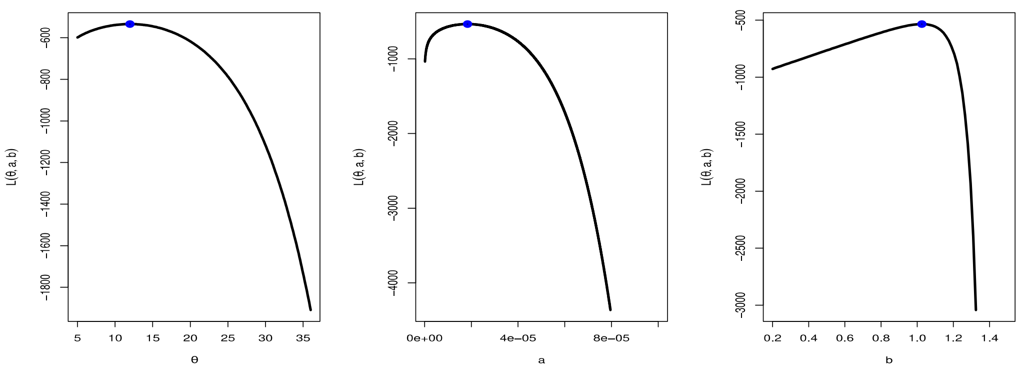

Figure 3. Profile–likelihood for the three parameters for the real data set. Where ge-03 is $g \times 10^{-3}$ and g is value of parameter g without virgule.

Figures 4 and 5 illustrate the trace plots and normal curves of the posterior distribution for MCMC estimation, respectively. These numbers also showed that the MCMC sequences converged quickly to become stationary. Furthermore, as shown in Figure 6, we used the random drawings to generate random draws for the sub-survivors and overall survivors as well as the Bayesian estimators and 95% credible intervals for those functions at various times. The normal distribution is often used to estimate the conditional posterior distributions of the parameters because they are generally symmetric and unimodal (see Figure 5). Figure 6 indicate that Bayes point and interval estimates of the sub-survivor functions and the overall survivor function.

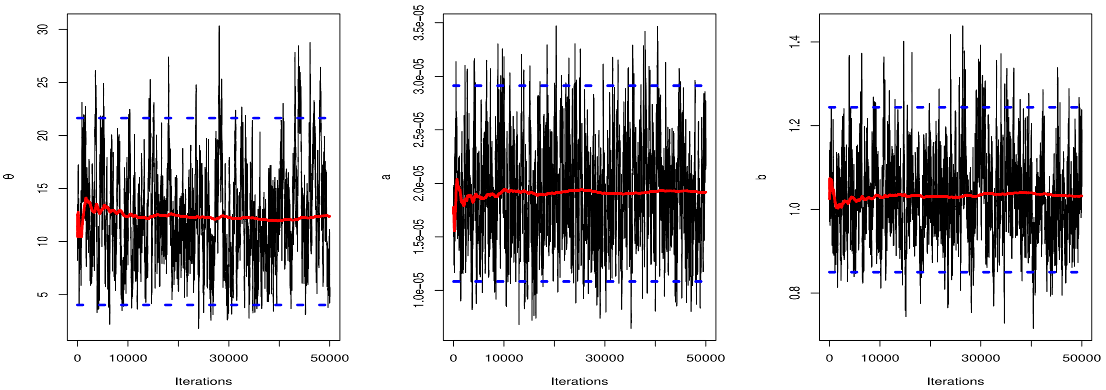

Figure 4. Iterations of MCMC results for progressive-stress model. Where *g*e-06 is $g \times 10^{-6}$ and g is value of parameter g without virgule.

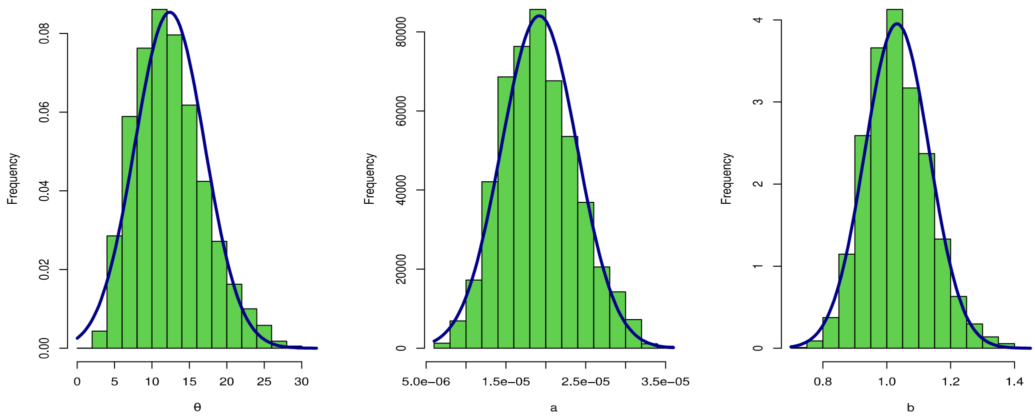

Figure 5. Convergence of MCMC results for progressive-stress model.

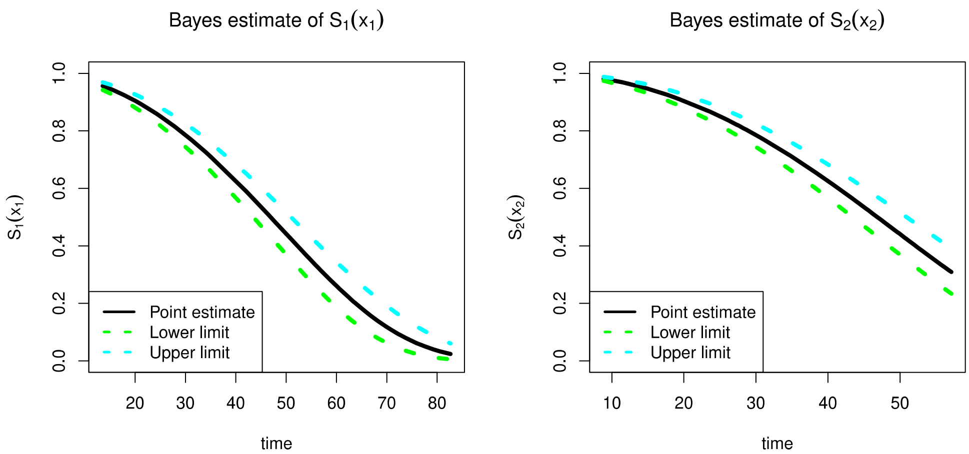

Figure 6. Bayes point and interval estimates of the sub-survivor functions and the overall survivor function.

8. Concluding Remarks

In this work, we studied classical and Bayesian estimation procedures for the progressive-stress model for the NH distribution with type II progressive-censoring data. Specifically, we first considered the MLEs of the NH parameters and adopted their asymptotic normal distribution to construct approximate confidence intervals. Then, we studied Bayesian estimators and the corresponding HPD intervals of the unknown parameters with suitable prior distributions and different loss functions. Numerical results from simulation studies and real data examples showed that the performance of the proposed methods is satisfactory for statistical inference of the step-stress accelerated model with the NH distribution based on progressive type II censored data.

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References

1. Nadarajah, S.; Haghighi, F. An extension of the exponential distribution. *Statistics* **2011**, *45*, 543–558. [CrossRef]
2. Khan, M.N.; Saboor, A.; Cordeiro, G.M.; Nazir, M. A weighted Nadarajah and Haghighi distribution. *UPB Sci. Bull. Ser. A Appl. Math. Phys.* **2018**, *80*, 133–140.
3. Tahir, M.H.; Cordeiro, G.M.; Ali, S.; Dey, S.; Manzoor, A. The inverted Nadarajah–Haghighi distribution: Estimation methods and applications. *J. Stat. Comput. Simul.* **2018**, *88*, 2775–2798. [CrossRef]
4. Chesneau, C.I.; Okorie, E.I.; Bakouch, H.S. A Skewed Nadarajah–Haghighi Distribution with Some Applications. *J. Indian Soc. Probab. Stat.* **2020**, *21*, 225–245. [CrossRef]
5. Mansoor, M.; Tahir, M.H.; Alzaatreh, A.; Cordeiro, G.M. The Poisson Nadarajah—Haghighi Distribution: Properties and Applications to Lifetime Data. *Int. J. Reliab. Qual. Saf. Eng.* **2020**, *27*, 2050005. [CrossRef]
6. Cohen, A.C. Progressively Censored Samples in Life Testing. *Technometrics* **1963**, *5*, 327. [CrossRef]
7. Hofmann, G.; Cramer, E.; Balakrishnan, N.; Kunert, G. An asymptotic approach to progressive censoring. *J. Stat. Plan. Inference* **2005**, *130*, 207–227. [CrossRef]
8. Lawless, J.F. *Statistical Models and Methods for Lifetime Data*; John Wiley and Sons: Hoboken, NJ, USA, 2003.
9. Balakrishnan, N.; Aggarwala, R. *Progressive Censoring: Theory, Methods, and Applications*; Springer: Berlin/Heidelberg, Germany, 2000.
10. Balakrishnan, N.; Sandhu, R.A. A simple simulation algorithm for generating progressive Type-II censored samples. *Am. Stat.* **1995**, *199549*, 229–230.
11. Kundu, D.; Joarder, A. Analysis of Type-II progressively hybrid censored data. *Comput. Stat. Data Anal.* **2006**, *50*, 2509–2528. [CrossRef]
12. Gupta, R.D.; Kundu, D. Hybrid censoring schemes with exponential failure distribution. *Commun. Stat.-Theory Methods* **1998**, *27*, 3065–3083. [CrossRef]
13. Nelson, W.B. *Accelerated Testing: Statistical Models, Test Plans, and Data Analysis*; Wiley: New York, NY, USA, 1990.
14. Miller, R.; Nelson, W. Optimum Simple Step-Stress Plans for Accelerated Life Testing. *IEEE Trans. Reliab.* **1983**, *R-32*, 59–65. [CrossRef]
15. Xiong, C. Step Stress Model with Threshold Parameter. *J. Stat. Comput. Simul.* **1999**, *63*, 349–360. [CrossRef]
16. Park, S.-J.; Yum, B.-J. Optimal design of accelerated life tests under modified stress loading methods. *J. Appl. Stat.* **1998**, *25*, 41–62. [CrossRef]
17. Lu, M.W.; Rudy, R.J. Step-stress accelerated test. *Int. J. Mater. Prod. Technol.* **2002**, *17*, 425–434. [CrossRef]
18. El-Raheem, A.M.A.; Almetwally, E.M.; Mohamed, M.S.; Hafez, E.H. Accelerated life tests for modified Kies exponential lifetime distribution: Binomial removal, transformers turn insulation application and numerical results. *AIMS Math.* **2021**, *6*, 5222–5255. [CrossRef]
19. Ahmadi, J.; Jozani, M.J.; Marchand, E.; Parsian, A. Bayes estimation based on k-record data from a general class of distributions under balanced type loss functions. *J. Stat. Plan Inference* **2009**, *139*, 1180–1189. [CrossRef]

20. Ahmadi, J.; Jozani, M.J.; Marchand, E.; Parsian, A. Prediction of k-records from a general class of distributions under balanced type loss functions. *Metrika* **2008**, *70*, 19–33. [CrossRef]
21. Ahmed, E.A. Bayesian estimation based on progressive Type-II censoring from two-parameter bathtub-shaped lifetime model: An Markov chain Monte Carlo approach. *J. Appl. Stat.* **2014**, *41*, 752–768. [CrossRef]
22. Contreras-Reyes, J.E. Lerch distribution based on maximum nonsymmetric entropy principle: Application to Conway's game of life cellular automaton. *Chaos Solitons Fractals* **2021**, *151*, 111272. [CrossRef]
23. Almongy, H.; Alshenawy, F.; Almetwally, E.; Abdo, D. Applying Transformer Insulation Using Weibull Extended Distribution Based on Progressive Censoring Scheme. *Axioms* **2021**, *10*, 100. [CrossRef]
24. Bantan, R.; Hassan, A.S.; Almetwally, E.; Elgarhy, M.; Jamal, F.; Chesneau, C.; Elsehetry, M. Bayesian Analysis in Partially Accelerated Life Tests for Weighted Lomax Distribution. *Comput. Mater. Contin.* **2021**, *68*, 2859–2875. [CrossRef]
25. Alotaibi, R.; Al Mutairi, A.; Almetwally, E.M.; Park, C.; Rezk, H. Optimal Design for a Bivariate Step-Stress Accelerated Life Test with Alpha Power Exponential Distribution Based on Type-I Progressive Censored Samples. *Symmetry* **2022**, *14*, 830. [CrossRef]
26. Alotaibi, R.; Khalifa, M.; Almetwally, E.M.; Ghosh, I.; Rezk, H. Classical and Bayesian Inference of a Mixture of Bivariate Exponentiated Exponential Model. *J. Math.* **2021**, *2021*, 1–20. [CrossRef]
27. Tolba, A.H.; Almetwally, E.M.; Ramadan, D.A. Bayesian Estimation of A one Parameter Akshaya Distribution with Progressively Type II Censord Data. *J. Stat. Appl. Probab.* **2022**, *11*, 565–579.
28. Chen, M.H.; Shao, Q.M. Monte Carlo estimation of Bayesian credible and HPD intervals. *J. Comput. Graph. Stat.* **1999**, *8*, 69–92.
29. Tibshirani, R.; Efron, B. *An Introduction to the Bootstrap*; Chapman & Hall, Inc.: New York, NY, USA, 1993.
30. R Core Team. *A Language and Environment for Statistical Computing*; R Foundation for Statistical Computing: Vienna, Austria, 2021.