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Hopf Differential Graded Galois Extensions

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Abstract: We introduce the concept of Hopf dg Galois extensions. For a finite dimensional semisimple Hopf algebra H and an H-module dg algebra R, we show that $\mathcal{D}(R\#H) \cong \mathcal{D}(R^H)$ is equivalent to that R/R^H is a Hopf differential graded Galois extension. We present a weaker version of Hopf differential graded Galois extensions and show the relationships between Hopf differential graded Galois extensions and Hopf Galois extensions.

Keywords: differential graded algebras; derived categories; Galois extensions

MSC: 16T05; 18E30; 18G10

1. Introduction

The Hopf Galois extension was introduced in [1]. It was shown that for a finite dimensional semisimple Hopf algebra H and a left H-module algebra R, the smash product $R\#H$ is Morita equivalent to R^H if and only if R/R^H is an H^*-Galois extension. Now suppose R is a differential graded (dg) algebra and the differential is compatible with the H-module action. The Hopf Galois extension on dg algebra R and the equivalence between dg module categories gr-$(R\#H)$ and gr-R^H follows easily from [1]. However, if we consider the derived categories $\mathcal{D}(R\#H)$ and $\mathcal{D}(R^H)$, then the problem is subtle.

In the present paper, we focus our attention on the relationship between the derived categories $\mathcal{D}(R\#H)$ and $\mathcal{D}(R^H)$. We introduce the concept of Hopf dg Galois extensions and show that $R\#H$ and R^H is derived equivalent to each other if and only if R/R^H is a Hopf dg Galois extension. In some situations, for example, when R is a positive graded algebra, the concept of Hopf dg Galois extensions is precisely equal to the concept of Hopf Galois extensions. Thus, we can consider the Hopf dg Galois extension as a generality of the Hopf Galois extension.

For this purpose, we proceed as follows. We first review the basic facts on derived categories and derived functors. In Section 4, we define the Hopf dg Galois extensions. We show that $R\#H$ and R^H is derived equivalent to each other if and only if R/R^H is a Hopf dg Galois extension in Theorem 2. Finally, we give some conditions for the quotient categories of derived categories $\mathcal{D}(R\#H)$ and $\mathcal{D}(R^H)$ to be equivalent.

2. Preliminaries

Throughout this paper, k is a field of characteristic 0 and all algebras are k-algebras; unadorned $\otimes$ means $\otimes_k$ and Hom means Hom_k. Recall that a differential graded (dg) algebra is a $\mathbb{Z}$-graded algebra $A = \bigoplus_{n\in\mathbb{Z}} A^n$ equipped with a differential d of degree 1 such that $d(ab) = d(a)b + (-1)^{|a|}ad(b)$, where $a, b \in A$ are homogeneous elements and $|a|$ is the degree of a.

Suppose A is an algebra without gradings. We may view A as a dg algebra $\bigoplus_{n\in\mathbb{Z}} A_n$ concentrated in degree zero, where

(1) $A_0 = A$,
(2) $A_n = 0$, for every $n \neq 0$,
(3) the differential $d = 0$.

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Unless otherwise stated, all modules in this paper are right modules. Let A and B be dg algebras. A (right) dg A-module M is a (right) A-module M, which has a grading $M = \bigoplus_{n\in\mathbb{Z}} M^n$ and a differential d such that $M^n A^m \in M^{n+m}$ and $d(ma) = d(m)a + (-1)^n md(a)$, for $m \in M^n$ and $a \in A^m$. We call M a dg (A, B)-bimodule if M, which comes with one grading and one differential, is both a left dg A-module and a right dg B-module.

Let A and B be dg algebras. Let M be a dg (A, B)-module and N be a right dg B-bimodule. Let

$$\mathrm{Hom}^\bullet_B(M, N) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}^n_B(M, N),$$

where $\mathrm{Hom}^n_B(M, N)$ is the set of all graded B-module maps of degree n. Then, $\mathrm{Hom}^\bullet_B(M, N)$ is a right dg A-module with a differential defined by $d(f) = d_N \circ f - (-1)^n f \circ d_M \in \mathrm{Hom}^{n+1}_B(M, N)$, for $f \in \mathrm{Hom}^n_B(M, N)$ and $n \in \mathbb{Z}$. Let T be a right dg A-module. Then, the tensor product $T \otimes_A M$ is a right dg B-module with differential $d(t \otimes m) = d(t) \otimes m + (-1)^n t \otimes d(m)$ for $t \in T^n$ and $m \in M$.

Let A and B be dg algebras. $\mathcal{M}(A)$ will denote the dg module category of dg A-modules. $\mathcal{D}(A)$ will denote the derived category of dg A-modules. For a dg (A, B)-module M, we have two functors:

$$\mathrm{Hom}^\bullet_B(M, -)\colon \mathcal{M}(B) \to \mathcal{M}(A),$$

and

$$- \otimes_A M\colon \mathcal{M}(A) \to \mathcal{M}(B).$$

These two functors compose an adjoint pair $(- \otimes_A M, \mathrm{Hom}^\bullet_B(M, -))$, see ([2], Lemma 19.11).

Let

$$\mathrm{RHom}^\bullet_B(M, -)\colon \mathcal{D}(B) \to \mathcal{D}(A)$$

denote the right derived functor of $\mathrm{Hom}^\bullet_B(M, -)$ and

$$- \otimes^L_A M\colon \mathcal{D}(A) \to \mathcal{D}(B)$$

denote the left derived functor of $- \otimes_A M$. Due to the adjoint above, $(- \otimes^L_A M, R\,\mathrm{Hom}^\bullet_B(M, -))$ is an adjoint pair, see ([3], Section 5.8).

Let H be a finite dimensional semisimple Hopf algebra with counit ε. We say that R is a left H-module algebra, if there is a left H-module action on R such that

(1) $h \cdot a \in R^n$,
(2) $h \cdot (ab) = \Sigma(h_{(1)} \cdot a)(h_{(2)} \cdot b)$,
(3) $h \cdot 1 = \varepsilon(h) \cdot 1$,

for every $a, b \in R$ and $h \in H$.

Let R be a left H-module algebra. For a left H-module M, we write $M^H = \{m \in M \mid h \cdot m = \varepsilon(h)m, \text{ for all } h \in H\}$. Let $\bar{S}$ denote the inverse of the antipode S. It is well known that R^H is a subalgebra of R and R has an $(R^H, R\#H)$-bimodule structure defined by

$$r_1.r.(r_2\#h) = (\bar{S}h) \cdot (r_1 r r_2),$$

and R has a $(R\#H, R^H)$-bimodule structure defined by

$$(r_1\#h).r.r_2 = r_1(h \cdot r)r_2,$$

where the notation "." denotes the multiplication on the module R and the notation "·" denotes the H-module action on the algebra R, see ([4], Sections 1.7 and 4.1).

The Hopf Galois extension is defined in [1]. R/R^H is said to be right H^*-Galois if the map

$$\gamma\colon R \otimes_{R^H} R \to R \otimes H^*,\ r_1 \otimes r_2 \mapsto (r_1 \otimes 1)\rho(r_2)$$

is surjective, where R is considered as a right H^*-comodule and ρ is the comodule structure map. By ([1], Theorem 1.2), R/R^H is right H^*-Galois if and only if the map $R \otimes_{R^H} R \to R\#H, r_1 \otimes r_2 \mapsto (r_1\#t)(r_2\#1)$ is surjective.

Let $\mathcal{C}$ be a triangulated category and $\mathcal{B}$ be a full triangulated subcategory of $\mathcal{C}$. We call $\mathcal{B}$ a thick subcategory if the following condition is satisfied:

If $f\colon X \to Y$ is a map in $\mathcal{C}$ which is contained in a distinguished triangle

$$X \to Y \to Z \to X[1]$$

where Z is in $\mathcal{B}$, and if the map f also factors through an object W of $\mathcal{B}$, then X and Y are objects of $\mathcal{B}$.

If $\mathcal{B}$ is a thick subcategory of $\mathcal{C}$, then the quotient category $\frac{\mathcal{C}}{\mathcal{B}}$ is a triangulated category. For the thick subcategory, we have the following proposition.

Proposition 1 ([5], Proposition 1.3)**.** *A full triangulated subcategory $\mathcal{B}$ of a triangulated category $\mathcal{C}$ is thick if and only if every object of $\mathcal{C}$ that is a direct summand of an object of $\mathcal{B}$ is itself an object of $\mathcal{B}$.*

3. The Equivalences of Triangulated Categories

Let B be a dg algebra and e be an idempotent in B^0 such that $d(e) = 0$. Then, $A = eBe$ is a dg algebra, Be is a dg (B, A)-bimodule and eB is a dg (A, B)-bimodule. For the dg (B, A)-bimodule Be, we may find a dg (B, A)-bimodule P and a dg (B, A)-bimodule morphism $p\colon P \to Be$ such that p is a quasi-isomorphism and P is K-projective both as a left dg B-module and as a right dg A-module. Similarly, we may find a dg (A, B)-bimodule Q and a dg (A, B)-bimodule morphism $q\colon Q \to eB$ such that q is a quasi-isomorphism and Q is K-projective both as a left dg A-module and as a right dg B-module. Then, the functor $- \otimes^L_A Be\colon \mathcal{D}(B) \to \mathcal{D}(A)$ is isomorphic to the functor $- \otimes_A P\colon \mathcal{D}(B) \to \mathcal{D}(A)$ and the functor $- \otimes^L_A eB\colon \mathcal{D}(A) \to \mathcal{D}(B)$ is isomorphic to the functor $- \otimes_A Q\colon \mathcal{D}(A) \to \mathcal{D}(B)$.

Since $(- \otimes^L_A eB,\ \mathrm{RHom}^\bullet_B(eB, -))$ is an adjoint pair between $\mathcal{D}(B)$ and $\mathcal{D}(A)$, we have a bijection

$$\Psi\colon\ \mathrm{Hom}_{\mathcal{D}(B)}(Be \otimes^L_A eB, B) \to \mathrm{Hom}_{\mathcal{D}(A)}(Be, Be),$$

since $\mathrm{RHom}^\bullet_B(eB, B) \cong Be$ in $\mathcal{D}(A)$. Below, we set

$$\psi = \Psi^{-1}(\mathrm{Id}_{Be}), \tag{1}$$

where Id_{Be} is the identity morphism in $\mathrm{Hom}_{\mathcal{D}(A)}(Be, Be)$.

Similarly, $(- \otimes_A Q,\ \mathrm{Hom}^\bullet_B(Q, -))$ is an adjoint pair between $\mathcal{D}(B)$ and $\mathcal{D}(A)$. For every $i \in \mathbb{Z}$, there exists an isomorphism of dg (B, B)-bimodules

$$\alpha_i\colon\ \mathrm{Hom}^\bullet_B(P \otimes_A Q, B[i]) \to \mathrm{Hom}^\bullet_A(P, \mathrm{Hom}^\bullet_B(Q, B[i])),$$

such that for $f \in \mathrm{Hom}^n_B(P \otimes_A Q, B[i]), x \in P, y \in Q$, we have

$$\alpha_i(f)(x)\colon y \mapsto f(x \otimes y).$$

Note that both Q and eB are K-projective as right dg B-modules. It follows that the quasi-isomorphism $q\colon Q \to eB$, when viewed as a right dg B-module morphism, is indeed a homotopic equivalence. Hence the dg (B, B)-bimodule morphism

$$\mathrm{Hom}^\bullet_A(P, \mathrm{Hom}^\bullet_B(q, B[i]))\colon\ \mathrm{Hom}^\bullet_A(P, \mathrm{Hom}^\bullet_B(eB, B[i])) \to \mathrm{Hom}^\bullet_A(P, \mathrm{Hom}^\bullet_B(Q, B[i]))$$

is a quasi-isomorphism. Since $\mathrm{Hom}^\bullet_B(eB, B[i]) \cong Be[i]$ as dg (B, A)-bimodules, let β_i denote the quasi-isomorphism from $\mathrm{Hom}^\bullet_A(P, Be[i])$ to $\mathrm{Hom}^\bullet_A(P, \mathrm{Hom}^\bullet_B(Q, B[i]))$. Thus, we have the following isomorphism

$$\Phi_i = (H^0(\beta_i))^{-1} \circ H^0(\alpha_i)\colon\ \mathrm{Hom}_{\mathcal{D}(B)}(P \otimes_A Q, B[i]) \to \mathrm{Hom}_{\mathcal{D}(A)}(P, Be[i]).$$

Let $\phi := m \circ (p \otimes_A q)$ be the composition

$$P \otimes_A Q \xrightarrow{p \otimes q} Be \otimes_A eB \xrightarrow{m} B, \tag{2}$$

where m is the multiplication map in B, that is, $m(b_1 \otimes b_2) = b_1 b_2$. Then, ϕ is a dg (B, B)-bimodule morphism. For $b \in B^i$ such that $d(b) = 0$, let l_b denote the map $l_b \colon B \to B[i]$, $a \mapsto ba$ for $a \in B$, and let l'_b denote the map $l'_b : Be \to Be[i]$, $ae \mapsto bae$, for $a \in B$. Then, we have the following lemma.

Lemma 1. *Retain the notation above,* $\Phi_i(l_b \circ \phi) = l'_b \circ p$.

Proof. By the definitions, for $x \in P$, $y \in Q$, $\alpha_i(l_b \circ \phi)(x) \colon y \mapsto bp(x)q(y)$ and $\beta_i(l'_b \circ p)(x) \colon y \mapsto bp(x)q(y)$. Thus, $\Phi_i(l_b \circ \phi) = l'_b \circ p$. □

Since $P \cong Be$ in $\mathcal{D}(A)$ and $Be \otimes^L_A eB \cong P \otimes_A Q$ in $\mathcal{D}(B)$, we have the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}(B)}(Be \otimes^L_A eB, B) & \xrightarrow{\Psi} & \mathrm{Hom}_{\mathcal{D}(A)}(Be, Be) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{Hom}_{\mathcal{D}(B)}(P \otimes_A Q, B) & \xrightarrow{\Phi_0} & \mathrm{Hom}_{\mathcal{D}(A)}(P, Be). \end{array}$$

Hence the morphism ψ may be represented by ϕ as defined in (2). That is, we have $\mathrm{cone}(\phi) \cong \mathrm{cone}(\psi)$ in $\mathcal{D}(B)$. Moreover, we may use Φ_0 to conduct calculations instead of using Ψ.

Let A, B be dg algebras. Let N be a dg (A, B)-bimodule. The bimodule structure implies a natural map $l_A \colon A \to \mathrm{RHom}^\bullet_B(N, N)$, sending $a \in A$ to the left module action on N. In [5], Rickard characterized the Morita equivalence of derived categories. For dg algebras, we have the following lemma.

Lemma 2 ([5], Theorem 6.4)**.** *Let A, B be dg algebras. Let N be a dg (A, B)-bimodule. Then, the functor $- \otimes^L_A N \colon \mathcal{D}(A) \to \mathcal{D}(B)$ gives an equivalence of triangulated categories if and only if*

(1) *N is a compact object of $\mathcal{D}(B)$.*
(2) *N is a weak generator in $\mathcal{D}(B)$.*
(3) *The map $l_A \colon A \to \mathrm{RHom}^\bullet_B(N, N)$ is a quasi-isomorphism.*

Now we can get the following theorem.

Theorem 1. *Let B be a dg algebra and e be an idempotent in B^0 such that $d(e) = 0$. Set $A = eBe$. The following conditions are equivalent.*

(1) *$F = - \otimes^L_A eB \colon \mathcal{D}(A) \to \mathcal{D}(B)$ is an equivalence of triangulated categories.*
(2) *$G = - \otimes^L_B Be \colon \mathcal{D}(B) \to \mathcal{D}(A)$ is an equivalence of triangulated categories.*
(3) *The morphism $\psi \colon Be \otimes^L_A eB \to B$ is an isomorphism in $\mathcal{D}(B)$.*

Proof. (1)⇔(2) F is left adjoint to $G' = \mathrm{RHom}^\bullet_B(eB, -) \colon \mathcal{D}(B) \to \mathcal{D}(A)$. The functors G and G' are naturally isomorphic to each other since eB is a compact K-projective dg module in $\mathcal{D}(B)$ and $\mathrm{Hom}^\bullet_B(eB, B) \cong Be$, see ([6], Section 2.1). Then, (F, G) is an adjoint pair. Therefore F is an equivalence of triangulated categories if and only if G is an equivalence of triangulated categories.

(1)⇒(3) Since F and G are equivalences, the functors $-\otimes_A P\colon \mathcal{D}(B)\to\mathcal{D}(A)$ and $-\otimes_B Q\colon \mathcal{D}(A)\to\mathcal{D}(B)$ are equivalences. For every $n\in\mathbb{Z}$, we have the following morphisms of groups.

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}(A)}(P,P[n]) &\xrightarrow{-\otimes_A Q} \mathrm{Hom}_{\mathcal{D}(B)}(P\otimes_A Q, P\otimes_A Q[n])\\ &\xrightarrow{\mathrm{Hom}_{\mathcal{D}(B)}(P\otimes_A Q,\phi[n])} \mathrm{Hom}_{\mathcal{D}(B)}(P\otimes_A Q, B[n])\\ &\xrightarrow{\Phi_n} \mathrm{Hom}_{\mathcal{D}(A)}(P, Be[n])\\ &\xrightarrow{\mathrm{Hom}_{\mathcal{D}(A)}(P,p[n])^{-1}} \mathrm{Hom}_{\mathcal{D}(A)}(P,P[n]).\end{aligned}$$

By Lemma 1, the composition above is the identity morphism. Since the morphisms $-\otimes_A Q$, Φ_n and $\mathrm{Hom}_{\mathcal{D}(A)}(P,p[n])^{-1}$ are isomorphisms, $\mathrm{Hom}_{\mathcal{D}(B)}(P\otimes_A Q,\phi[n])$ is an isomorphism.

By (1) and (2), (F,G) is an adjoint pair and then $(-\otimes_A P, -\otimes_B Q)$ is an adjoint pair. So, we have $P\otimes_A Q\cong B\otimes_B P\otimes_A Q\cong B$ in $\mathcal{D}(B)$. Thus, $\mathrm{Hom}_{\mathcal{D}(B)}(B,\phi[n])$ is an isomorphism for every $n\in\mathbb{Z}$. Hence, ϕ is an isomorphism in $\mathcal{D}(B)$ and ϕ is a quasi-isomorphism of dg modules. Then, ψ is an isomorphism.

(3)⇒(1) The morphism of dg modules $\phi\colon P\otimes_A Q\to B$ is a quasi-isomorphism since ψ is an isomorphism. Then, $P\otimes_A Q\cong B$ in $\mathcal{D}(B)$. By Lemma 2, the functor $-\otimes_B P\otimes_A Q\colon \mathcal{D}(B)\to\mathcal{D}(B)$ is an equivalence. Since we have isomorphisms $Q\otimes_B P\cong eB\otimes_B Be\cong A$ in $\mathcal{D}(A)$, the functor $-\otimes_A Q\otimes_B P\colon \mathcal{D}(A)\to\mathcal{D}(A)$ is an equivalence. Thus, the functor $-\otimes_A Q\colon \mathcal{D}(A)\to\mathcal{D}(B)$ is an equivalence. Hence, $F=-\otimes_A^L eB\colon \mathcal{D}(A)\to\mathcal{D}(B)$ is an equivalence of triangulated categories. □

4. Hopf DG Galois Extensions

Let H be a finite dimensional semisimple Hopf algebra with integral t such that $\varepsilon(t)=1$. Suppose that R is a dg algebra with the differential d. We call R a left dg H-module algebra if R is a left graded H-module algebra and the differential of R is compatible with the H-module action, that is,

$$d(h\cdot r)=h\cdot d(r)$$

for $h\in H$ and $r\in R$. Since R is a dg algebra, the smash product $R\#H$ is a dg algebra with the differential $\delta=d\#\mathrm{Id}$ and R^H is a dg subalgebra of R. Let $e=1_R\#t\in R\#H$. Then, e is an idempotent in R^0 and $\delta(e)=0$. Thus, $e(R\#H)e$ is a dg algebra with differential δ. By direct calculation, we have the following isomorphisms ([7], Lemma 3.1).

(1) The map $R^H\to e(R\#H)e$, $r\mapsto e(r\#1)e$, is an isomorphism of dg algebras.
(2) The map $R\to(R\#H)e$, $r\mapsto(r\#1)e$, is an isomorphism of dg $(R\#H,R^H)$-bimodules.
(3) The map $R\to e(R\#H)$, $r\mapsto e(r\#1)$, is an isomorphism of dg $(R^H,R\#H)$-bimodules.

Let $B=R\#H$ and $A=eBe\cong R^H$. Let $p\colon P\to R$ be the dg $(R\#H,R^H)$-bimodule quasi-isomorphism such that P is K-projective on both sides. Let $q\colon Q\to R$ be the dg $(R^H,R\#H)$-bimodule quasi-isomorphism such that Q is K-projective on both sides. Recall the dg $(R\#H,R\#H)$-bimodule morphism

$$\phi\colon P\otimes_{R^H}Q\to R\#H,\, x\otimes y\mapsto p(x)q(y)$$

defined above. Now we can define the Hopf dg Galois extension.

Definition 1. *For a dg left H-module algebra R, R/R^H is called dg H^*-Galois if the morphism $\phi\colon P\otimes_{R^H}Q\to R\#H$ is a quasi-isomorphism.*

Now we have the following theorem for dg H^*-Galois extensions.

Theorem 2. *Let H be a finite dimensional semisimple Hopf algebra with integral t such that $\varepsilon(t) = 1$. Let R be a left dg H-module algebra. The following conditions are equivalent.*

(1) *R/R^H is dg H^*-Galois.*
(2) (a) *The map $l_{R\#H}\colon R\#H \to \mathrm{RHom}^{\bullet}_{R^H}(R,R)$ is a quasi-isomorphism,*
    (b) *R is a compact object in $\mathcal{D}(R^H)$.*
(3) *R is a weak generator in $\mathcal{D}(R\#H)$.*

Proof. Let $B = R\#H$, $e = 1_R\#t$, then $A = eBe = R^H$. Thus, the condition (1) is equivalent to Theorem 1 (3). By Lemma 2, the condition (2) is equivalent to Theorem 1 (2) and the condition (3) is equivalent to Theorem 1 (1). Then, by Theorem 1, (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). □

The following results will show the relation between Hopf Galois extensions and Hopf dg Galois extensions.

Lemma 3. *Let H be a finite dimensional semisimple Hopf algebra and R be a dg left H-module algebra. Then, $R_{R\#H}$ is a weak generator in $\mathcal{D}(R\#H)$ if and only if for every dg $R\#H$-module M, $H^n(M^H) = 0$ for every $n \in \mathbb{Z}$ implies $H^n(M) = 0$ for every $n \in \mathbb{Z}$.*

Proof. Given a dg $R\#H$-module (M, d_M), by ([7], Lemma 2.2), for every $n \in \mathbb{Z}$,

$$\mathrm{Hom}_{R\#H}(R, M[n]) \cong \mathrm{Hom}_R(R, M[n])^H \cong (\mathrm{Ker}\ d^n_M)^H.$$

Then, for every $n \in \mathbb{Z}$, we have

$$\mathrm{Hom}_{K(R\#H)}(R, M[n]) \cong (\mathrm{Ker}\ d^n_M/\mathrm{Im}\ d^{n-1}_M)^H \cong (H^n(M))^H.$$

Since H is semisimple, $(-)^H \cong \mathrm{Hom}_H(k,-)$ is an exact functor. Therefore, $(H^n(M))^H \cong H^n(M^H)$ for every $n \in \mathbb{Z}$. By [7] Proposition 2.5, R is a K-projective dg $R\#H$-module. Thus, for every $n \in \mathbb{Z}$,

$$\mathrm{Hom}_{\mathcal{D}(R\#H)}(R, M[n]) \cong \mathrm{Hom}_{K(R\#H)}(R, M[n]) \cong (H^n(M))^H \cong H^n(M^H).$$

Hence, $R_{R\#H}$ is a weak generator in $\mathcal{D}(R\#H)$ if and only if for every dg $R\#H$-module M, $\mathrm{Hom}_{\mathcal{D}(R\#H)}(R, M[n]) = 0$ for every $n \in \mathbb{Z}$ implies $M \cong 0$ in $\mathcal{D}(R\#H)$, if and only if for every dg $R\#H$-module M, $H^n(M^H) = 0$ for every $n \in \mathbb{Z}$ implies $H^n(M) = 0$ for every $n \in \mathbb{Z}$. □

Corollary 1. *Let H be a finite dimensional semisimple Hopf algebra with integral t such that $\varepsilon(t) = 1$. Let R be a left H-module dg algebra. If R/R^H is right dg H^*-Galois, then the map $\varphi\colon R \otimes_{R^H} R \to R\#H, r_1 \otimes r_2 \mapsto (r_1\#t)(r_2\#1)$ is a quasi-isomorphism.*

Proof. Consider the short exact sequence of dg $R\#H$-modules

$$0 \to \mathrm{Ker}\ \varphi \to R \otimes_{R^H} R \to \mathrm{Im}\ \varphi \to 0.$$

Since $(-)^H$ is an exact functor, we have the short exact sequence

$$0 \to (\mathrm{Ker}\ \varphi)^H \to (R \otimes_{R^H} R)^H \to (\mathrm{Im}\ \varphi)^H \to 0.$$

Since $(R \otimes_{R^H} R)^H = t(R \otimes_{R^H} R) = (tR) \otimes_{R^H} R = R^H \otimes_{R^H} R \cong R$, for every $\alpha \in (\mathrm{Ker}\ \varphi)^H$, there exists $r \in R$ such that $\alpha = 1 \otimes r \in (R \otimes_{R^H} R)^H$. Then, $\varphi(\alpha) = \varphi(1 \otimes r) = (1\#t)(r\#1) = 0$. However, by [1] [Lemma 0.5], $(1\#H)(R\#1) \cong H \otimes R$ as vector spaces by

$$\eta\colon (1\#H)(R\#1) \to H \otimes R,\ (1\#h)(r\#1) \mapsto h_{(2)} \otimes (h_{(1)} \cdot r),$$

and

$$\eta^{-1}\colon H\otimes R\to(1\#H)(R\#1),\ h\otimes r\mapsto(1\#h_{(2)})((S^{-1}h_{(1)})\cdot r\#1).$$

Thus, $(1\#t)(r\#1)=0$ if and only if $r=0$, which means $(\text{Ker }\varphi)^H=0$. Therefore, $H^n((\text{Ker }\varphi)^H)=0$ for every $n\in\mathbb{Z}$. By Lemma 3, $H^n(\text{Ker }\varphi)=0$ for every $n\in\mathbb{Z}$. Then, $H^n(R\otimes_{R^H}R)\cong H^n(\text{Im }\varphi)$ for every $n\in\mathbb{Z}$.

Consider another short exact sequence of dg $R\#H$-modules

$$0\to\text{Im }\varphi\to R\#H\to\text{Coker }\varphi\to 0.$$

Since $(-)^H$ is an exact functor, we have the short exact sequence

$$0\to(\text{Im }\varphi)^H\to(R\#H)^H\to(\text{Coker }\varphi)^H\to 0.$$

By ([1], Lemma 0.5), $(R\#H)^H=(1\#t)(R\#1)$. However,

$$\begin{aligned}(\text{Im }\phi)^H &= ((R\#t)(R\#1))^H\\ &= (1\#t)(R\#t)(R\#1)\\ &= (R^H\#t)(R\#1)\\ &= (1\#t)(R^H\#1)(R\#1)\\ &= (1\#t)(R\#1).\end{aligned}$$

Thus, the map $(\text{Im }\varphi)^H\to(R\#H)^H$ is surjective. Then, $(\text{Coker }\varphi)^H=0$. Therefore, $H^n((\text{Coker }\varphi)^H)=0$ for every $n\in\mathbb{Z}$. By Lemma 3, $H^n(\text{Coker }\varphi)=0$ for every $n\in\mathbb{Z}$. So, we have $H^n(\text{Im }\varphi)\cong H^n(R\#H)$ for every $n\in\mathbb{Z}$. Thus, $H^n(R\otimes_{R^H}R)\cong H^n(R\#H)$ for every $n\in\mathbb{Z}$. Hence, φ is a quasi-isomorphism. □

Corollary 2. *Let H be a finite dimensional semisimple Hopf algebra with integral t such that $\varepsilon(t)=1$. Let $R=\bigoplus_{n\geq 0}R^n$ be a left H-module dg algebra. Then, R/R^H is dg H^*-Galois if and only if R/R^H, forgetting the differentials, is right H^*-Galois.*

Proof. Suppose that R/R^H is dg H^*-Galois. Then, by Corollary 1, the map $\varphi\colon R\otimes_{R^H}R\to R\#H, r_1\otimes r_2\mapsto(r_1\#t)(r_2\#1)$ is a quasi-isomorphism. Since $H^0(R)=\text{Ker }d_R^0$ and $1_R\in\text{Ker }d_R^0$, the map φ is surjective. Thus, R/R^H is right H^*-Galois.

Suppose that R/R^H is right H^*-Galois. Then, by [1] Theorem 1.2, R is a dg finitely generated projective left R^H-module, and for every dg $R\#H$-module M, $M^H\otimes_{R^H}R\cong M$ as dg $R\#H$-modules. Thus, $H^n(M^H)=0$ for every $n\in\mathbb{Z}$ implies $H^n(M)=0$ for every $n\in\mathbb{Z}$. By Lemma 3, $R_{R\#H}$ is a weak generator in $\mathcal{D}(R\#H)$. Thus, R/R^H is dg H^*-Galois. □

If R is a dg algebra concentrated in degree 0, then Corollary 2 shows that R/R^H is dg H^*-Galois if and only if R/R^H, forgetting the differentials, is H^*-Galois. Thus, the definition of dg H^*-Galois is an extension of the definition of H^*-Galois.

5. The Equivalences of Quotient Categories

Suppose that B is a dg algebra and e is an idempotent in B^0 such that $d(e)=0$. Then, eBe is a dg algebra. Let $A=eBe$. Let

$$\mathcal{D}_0(B)=\{M\in\mathcal{D}(B)\mid\text{Hom}_{\mathcal{D}(B)}(M,B[n])=0,n\in\mathbb{Z}\}$$

and

$$\mathcal{D}_0(A)=\{N\in\mathcal{D}(A)\mid\text{Hom}_{\mathcal{D}(A)}(N,Be[n])=0,n\in\mathbb{Z}\}.$$

By Proposition 1, it is clear that $\mathcal{D}_0(B)$ (resp. $\mathcal{D}_0(A)$) is a thick triangulated subcategory of $\mathcal{D}(B)$ (resp. $\mathcal{D}(A)$). Let $\mathcal{D}_q(B)$ denote the quotient category $\frac{\mathcal{D}(B)}{\mathcal{D}_0(B)}$ and $\mathcal{D}_q(A)$ denote the quotient category $\frac{\mathcal{D}(A)}{\mathcal{D}_0(A)}$. Let π denote the natural quotient functor. Theorem 1 shows that the map $\phi\colon Be\otimes_A^L eB\to B$ is an isomorphism in $\mathcal{D}(B)$ if and only if $\mathcal{D}(A)\cong\mathcal{D}(B)$. In this section, we will give some equivalent conditions for the quotient categories $\mathcal{D}_q(B)$ and $\mathcal{D}_q(A)$ being equivalent.

Theorem 3. *Let B be a dg algebra and e be an idempotent in B^0 such that $d(e) = 0$. Let $A = eBe$ be a dg algebra. The following conditions are equivalent.*

(1) *The map $l_B\colon B \to \mathrm{RHom}^{\bullet}_A(Be, Be)$ is a quasi-isomorphism.*
(2) $\mathrm{Hom}_{\mathcal{D}(B)}(cone(\psi), B[n]) = 0$ *for all $n \in \mathbb{Z}$.*
(3) *The functor $- \otimes^L_B Be\colon \mathcal{D}(B) \to \mathcal{D}(A)$ implies an equivalence of triangulated categories from $\mathcal{D}_q(B)$ to $\mathcal{D}_q(A)$.*

Proof. (1)⇔(2) Consider the composition $\Phi_n \circ \mathrm{Hom}_{\mathcal{D}(B)}(\phi, B[n])$,

$$\mathrm{Hom}_{\mathcal{D}(B)}(B, B[n]) \xrightarrow{\mathrm{Hom}_{\mathcal{D}(B)}(\phi, B[n])} \mathrm{Hom}_{\mathcal{D}(A)}(P, P[n]) \xrightarrow{\Phi_n} \mathrm{Hom}_{\mathcal{D}(A)}(P, Be[n]).$$

Since Φ_n is an isomorphism for every n, by Lemma 1, the condition (1) is equivalent to $\mathrm{Hom}_{\mathcal{D}(B)}(\phi, B[n])$ being an isomorphism for every n. Consider the distinguished triangle in $\mathcal{D}(B)$,

$$P \otimes_A Q \xrightarrow{\phi} B \longrightarrow \mathrm{cone}(\phi) \longrightarrow P \otimes_A Q[1].$$

In the following proving process, we write $(-)^*_n$ for the functor $\mathrm{Hom}_{\mathcal{D}(B)}(-, B[n])$ temporarily to simplify the notation. Then, we have the long exact sequence

$$\cdots \to (B)^*_n \xrightarrow{(\phi)^*_n} (P \otimes_A Q)^*_n \to (\mathrm{cone}(\phi))^*_{n+1} \to (B)^*_{n+1} \xrightarrow{(\phi)^*_{n+1}} (P \otimes_A Q)^*_{n+1} \to \cdots.$$

Thus, we have that the functor $(\phi)^*_n$ is an isomorphism for every n if and only if $(\mathrm{cone}(\psi))^*_n \cong (\mathrm{cone}(\phi))^*_n = 0$ for every n.

(2)⇔(3) Consider the distinguished triangle in $\mathcal{D}(B)$

$$Be \otimes^L_A eB \xrightarrow{\varphi} B \longrightarrow \mathrm{cone}(\varphi) \longrightarrow Be \otimes^L_A eB[1].$$

Then, we have a distinguished triangle in $\mathcal{D}_q(B)$

$$\pi(Be \otimes^L_A eB) \xrightarrow{\pi(\varphi)} \pi(B) \longrightarrow \pi(\mathrm{cone}(\varphi)) \longrightarrow \pi(Be \otimes^L_A eB[1]).$$

Suppose that $\mathrm{Hom}_{\mathcal{D}(B)}(\mathrm{cone}(\psi), B[n]) = 0$ for all $n \in \mathbb{Z}$, then $\pi(\mathrm{cone}(\varphi)) = 0$ in $\mathcal{D}_q(B)$. Thus, $\pi(Be \otimes^L_A eB) \cong \pi(B)$ in $\mathcal{D}_q(B)$. Since $eB \otimes^L_B Be \cong A$ in $\mathcal{D}(A)$, we have $\pi(eB \otimes^L_B Be) \cong \pi(A)$ in $\mathcal{D}_q(A)$. Therefore, the functor $- \otimes^L_B Be\colon \mathcal{D}(B) \to \mathcal{D}(A)$ implies an equivalence of triangulated categories from $\mathcal{D}_q(B)$ to $\mathcal{D}_q(A)$.

Suppose that the functor $- \otimes^L_B Be\colon \mathcal{D}(B) \to \mathcal{D}(A)$ implies an equivalence of triangulated categories from $\mathcal{D}_q(B)$ to $\mathcal{D}_q(A)$; then, $\pi(Be \otimes^L_A eB) \cong \pi(B)$ in $\mathcal{D}_q(B)$. Thus, $\pi(\mathrm{cone}(\varphi)) = 0$ in $\mathcal{D}_q(B)$, that is, $\mathrm{Hom}_{\mathcal{D}(B)}(\mathrm{cone}(\psi), B[n]) = 0$ for all $n \in \mathbb{Z}$. □

Let H be a finite dimensional semisimple Hopf algebra with integral t such that $\varepsilon(t) = 1$. Let R be a left H-module algebra. Let $B = R\#H$ and $e = 1\#t$ in Theorem 3; then, $A \cong eBe \cong R^H$ as dg algebras. Thus, Theorem 3 shows some equivalent conditions of the quasi-isomorphism $R\#H \to \mathrm{RHom}^{\bullet}_{R^H}(R, R)$.

Corollary 3. *Let B be a dg algebra and e be an idempotent in B^0 such that $d(e) = 0$. Let $A = eBe$ be a dg algebra. If $\mathrm{Hom}_{\mathcal{D}(B)}(B, B[n]) = 0, \mathrm{Hom}_{\mathcal{D}(A)}(Be, Be[n]) = 0$, for $n \leq \alpha$ or $n \geq \beta$, then the following conditions are equivalent.*

(1) *The map $l_B\colon B \to \mathrm{RHom}^{\bullet}_A(Be, Be)$ is a quasi-isomorphism.*

(2) $\mathrm{Hom}_{\mathcal{D}(B)}(cone(\psi), B[n]) = 0$ *for* $\alpha + 1 \leq n \leq \beta$.
(3) $\mathrm{Hom}_{\mathcal{D}(B)}(cone(\psi), B[n]) = 0$ *for all* $n \in \mathbb{Z}$.
(4) *The functor* $- \otimes^L_B Be \colon \mathcal{D}(B) \to \mathcal{D}(A)$ *implies an equivalence of triangulated categories from* $\mathcal{D}_q(B)$ *to* $\mathcal{D}_q(A)$.

Proof. By Theorem 3, it is clear that (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) and (3) $\Rightarrow$ (2). It suffices to show (2) $\Rightarrow$ (3).

(2) $\Rightarrow$ (3) By the proof of Theorem 3, if we have $\mathrm{Hom}_{\mathcal{D}(B)}(B, B[n]) = 0$ and $\mathrm{Hom}_{\mathcal{D}(A)}(Be, Be[n]) = 0$, for $n \leq \alpha$ or $n \geq \beta$, then the long exact sequence shows that $\mathrm{Hom}_{\mathcal{D}(B)}(B, B[n]) \cong \mathrm{Hom}_{\mathcal{D}(A)}(P, Be[n])$ for $\alpha + 1 \leq n \leq \beta - 1$. Thus $\mathrm{Hom}_{\mathcal{D}(B)}(B, B[n]) \cong \mathrm{Hom}_{\mathcal{D}(A)}(P, Be[n])$ for all n. □

Remark 1. *If* $\alpha = -1$ *and we let* $j(M) = \min\{i \mid \mathrm{Ext}^i_{\mathcal{D}(B)}(M, B) \neq 0\}$ *for* $M \in \mathcal{D}(B)$, *then the condition (2) is equivalent to* $j(cone(\psi)) \geq \beta + 1$. *Thus, Corollary 3 is a dg version of ([8], Theorem 2.4).*

6. Conclusions

The Hopf dg Galois extension shows the relationship between dg algbras R and R^H, which relate to the equivalences of some derived categories. Since the Hopf dg Galois extension is compatible with the usual Hopf Galois extension, we can promote the propositions related to Hopf Galois extension, and relate these to derived categories in a similar way. For an H-comodule algebra and its subalgebras, there exists a kind of Hopf Galois extensions. These may be promoted to dg algebras and derived categories in some way.

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