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Duality Results for a Class of Constrained Robust Nonlinear Optimization Problems

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Abstract: In this paper, we establish various results of duality for a new class of constrained robust nonlinear optimization problems. For this new class of problems, involving functionals of (path-independent) curvilinear integral type and mixed constraints governed by partial derivatives of second order and uncertain data, we formulate and study Wolfe, Mond-Weir and mixed type robust dual optimization problems. In this regard, by considering the concept of *convex curvilinear integral vector functional*, determined by controlled second-order Lagrangians including uncertain data, and the notion of *robust weak efficient solution* associated with the considered problem, we create a new mathematical context to state and prove the duality theorems. Furthermore, an illustrative application is presented.

Keywords: multi-objective robust control problem; robust duality; uncertain data; robust feasible solution; robust weak efficient solution

MSC: 65K10; 26B25

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1. Introduction

Over time, from the desire to model several processes in science, nature or engineering, many researchers (for instance, the reader is directed to the works of Trélat and Zuazua [22], Mititelu and Treanţă [11], Treanţă [16], Jayswal and Preeti [3]) paid a particular attention in the study of certain ordinary differential equation, partial differential equation, partial differential inequation, or isoperimetric-type constrained optimization problems. As is well known, the (necessary and sufficient) optimality or efficiency conditions and the associated dual problems are essential in optimization theory. By using the duality theory, we can better understand the nature of the original (primal) problem from the perspective of a dual problem. In this regard, we make a dihonesty by mentioning only the notable works of Wolfe [27], Weir and Mond [26], Mishra et al. [10], Pham [12], Gao [2], Treanţă and Mititelu [15], Tung [23], Treanţă [17,20] and the references cited therein. To investigate some complex real-life phenomena or processes involving uncertain initial data, many researchers used several elements coming from interval analysis and robust control. In this respect, the reader can consult the following research papers of Jeyakumar et al. [4], Wei et al. [25], Liu and Yuan [8], Sun et al. [14], Du et al. [1], Treanţă [19], Lu et al. [9], Wang et al. [24]. For other different but connected ideas on this topic (robust control), the reader can consult Liu et al. [5–7]. Despite all the previous research works, our study has not been approached until now and we will present its totally novel elements in the following.

In this paper, motivated and inspired by the above mentioned papers, we introduce and study a new class of constrained robust nonlinear control problems, denoted by $(\mathcal{MRCP})$. For the new class of robust optimization problems involving curvilinear integral functionals (which are independent of the path), equality and inequality constraints including partial derivatives of second order and uncertain data, we formulate and investigate various robust dual optimization problems. To this aim, first we introduce the concept of convex curvilinear integral vector functional that is determined by controlled second-order Lagrangians with uncertain data. Then, by considering the notion of robust weak efficient solution associated with the problem $(\mathcal{MRCP})$, we formulate Wolfe, Mond-Weir and mixed type dual optimization results. Compared to other works published so far, the fundamental merits of this paper are the following: (i) by using closed controlled second-order Lagrange 1-forms with uncertain data, we introduce the notion of convexity for curvilinear integral-type vector functionals; (ii) construction of a mathematical setting determined by curvilinear integral-type vector functionals (containing partial derivatives of second order and uncertainty parameters) and infinite dimensional function spaces. These elements are completely new in the robust nonlinear optimization field. Furthermore, taking into account the physical importance (for instance, mechanical work) of the curvilinear integrals, the techniques developed in this paper can give rise to new ideas in many other research areas with applications in nature and engineering.

In the next section (see Section 2), we formulate the robust nonlinear optimization problem we intend to investigate, and some preliminary elements. Section 3 introduces Wolfe type robust dual optimization problem associated with the considered multi-objective robust nonliunear optimization problem $(\mathcal{MRCP})$. Robust weak, strong and strict converse duality results are provided here. Next, in Section 4, we formulate and study the Mond-Weir type robust dual optimization problem. Section 5 includes and characterizes the mixed type robust dual optimization problem. Furthermore, an illustrative real-life application is included here in order to validate the theoretical elements derived in the paper. The conclusions and a further research line of this paper are formulated in Section 6.

2. Problem Description

In this paper, we are considering the following notations and working hypotheses as in Treanţă and Das [21], and Treanţă [18,19]:

- consider $\mathbb{R}^p, \mathbb{R}^q, \mathbb{R}^r$ and $\mathbb{R}^n$ as Euclidean spaces, having the dimensions p, q, r and n, respectively;
- K is a compact set in $\mathbb{R}^p$, $t = (t^\alpha) \in K$, and $\Delta \subset K$ is a smooth curve that joins t_0 and t_1 in K;
- consider $\mathcal{S}$ is the space of state functions $s = (s^\tau) : K \to \mathbb{R}^q$, belonging to (almost averywhere) C^4-class, and the notations $s_\sigma := \dfrac{\partial s}{\partial t^\sigma}$, $s_{\alpha\beta} := \dfrac{\partial^2 s}{\partial t^\alpha \partial t^\beta}$;
- denote by $\mathcal{C}$ the space of all measurable control functions $\vartheta = (\vartheta^j) : K \to \mathbb{R}^r$;
- T denotes the transpose of a vector;
- consider the notations: $D_\sigma = \frac{\partial}{\partial t^\sigma}$, $D^2_{\alpha\beta} = \frac{\partial^2}{\partial t^\alpha \partial t^\beta}$;
- for two vectors $\rho, \varrho \in \mathbb{R}^n$, we use the following convention for inequalities and equalities:

(i) $\rho < \varrho \Leftrightarrow \rho_i < \varrho_i,\ \forall i = \overline{1, n}$,
(ii) $\rho = \varrho \Leftrightarrow \rho_i = \varrho_i,\ \forall i = \overline{1, n}$,
(iii) $\rho \leqq \varrho \Leftrightarrow \rho_i \leq \varrho_i,\ \forall i = \overline{1, n}$,
(iv) $\rho \leq \varrho \Leftrightarrow \rho_i \leq \varrho_i,\ \forall i = \overline{1, n}$ and $\rho_i < \varrho_i$ for some i.

The second-order PDE&PDI constrained *multi-objective robust control problem* (with data uncertainty in the objective and constraint functionals) we intend to investigate here is formulated as follows:

$$(\mathcal{MRCP}) \qquad \min_{(s(\cdot),\vartheta(\cdot))} \int_{\Delta} \phi_{\kappa}(t, s(t), s_{\sigma}(t), s_{\alpha\beta}(t), \vartheta(t), w) dt^{\kappa}$$

subject to

$$\varphi(t, s(t), s_{\sigma}(t), s_{\alpha\beta}(t), \vartheta(t), a) \leqq 0, \quad t \in K$$

$$\chi(t, s(t), s_{\sigma}(t), s_{\alpha\beta}(t), \vartheta(t), b) = 0, \quad t \in K$$

$$s(t_0) = s_0,\ s(t_1) = s_1,\ s_{\sigma}(t_0) = s_{\sigma 0},\ s_{\sigma}(t_1) = s_{\sigma 1},$$

where $\phi_{\kappa} = (\phi_{\kappa}^{1}, \ldots \phi_{\kappa}^{s}) = (\phi_{\kappa}^{\iota}) : J^2\Big(K, \mathbb{R}^q\Big) \times \mathcal{C} \times W_{\kappa}^{\iota} \to \mathbb{R}^s$, $\kappa = \overline{1,p}$, $\iota = \overline{1,s}$, $\varphi = (\varphi_1, \ldots, \varphi_m) = (\varphi_l) : J^2\Big(K, \mathbb{R}^q\Big) \times \mathcal{C} \times A_l \to \mathbb{R}^m$, $l = \overline{1,m}$, $\chi = (\chi_1, \ldots, \chi_n) = (\chi_{\zeta}) : J^2\Big(K, \mathbb{R}^q\Big) \times \mathcal{C} \times B_{\zeta} \to \mathbb{R}^n, \zeta = \overline{1,n}$, are functionals belonging to (almost averywhere) C^3-class, $w = (w_{\kappa}^{\iota}), a = (a_l)$ and $b = (b_{\zeta})$ represent the uncertainty parameters of the convex subsets $W = (W_{\kappa}^{\iota}) = W_{\kappa}^{1} \times \cdots \times W_{\kappa}^{s} \subset \mathbb{R}^s, A = (A_l) = A_1 \times \cdots \times A_m \subset \mathbb{R}^m$ and $B = (B_{\zeta}) = B_1 \times \cdots \times B_n \subset \mathbb{R}^n$, respectively, and $J^2\Big(K, \mathbb{R}^q\Big)$ is the jet bundle of second order for K and $\mathbb{R}^q$. Furthermore, assume that the previous multi-variate controlled Lagrangians of second order $\phi_{\kappa} = (\phi_{\kappa}^{\iota})$ provide closed controlled Lagrange 1-forms (with summation on the repeated indices)

$$\phi_{\kappa}^{\iota}(t, s(t), s_{\sigma}(t), s_{\alpha\beta}(t), \vartheta(t), w) dt^{\kappa}, \quad \iota = \overline{1,s},$$

which generates the following vector of controlled curvilinear integrals (which are independent of the path)

$$\left(\int_{\Delta} \phi_{\kappa}^{1}(t, s(t), s_{\sigma}(t), s_{\alpha\beta}(t), \vartheta(t), w_{\kappa}^{1}) dt^{\kappa}, \cdots, \int_{\Delta} \phi_{\kappa}^{s}(t, s(t), s_{\sigma}(t), s_{\alpha\beta}(t), \vartheta(t), w_{\kappa}^{s}) dt^{\kappa}\right).$$

The associated robust counterpart of the aforementioned multi-objective robust control problem $(\mathcal{MRCP})$ is defined as:

$$(\mathcal{RMRCP}) \qquad \min_{(s(\cdot),\vartheta(\cdot))} \int_{\Delta} \max_{w \in W} \phi_{\kappa}(t, s(t), s_{\sigma}(t), s_{\alpha\beta}(t), \vartheta(t), w) dt^{\kappa}$$

subject to

$$\varphi(t, s(t), s_{\sigma}(t), s_{\alpha\beta}(t), \vartheta(t), a) \leqq 0, \quad t \in K,\ a \in A$$

$$\chi(t, s(t), s_{\sigma}(t), s_{\alpha\beta}(t), \vartheta(t), b) = 0, \quad t \in K,\ b \in B$$

$$s(t_0) = s_0,\ s(t_1) = s_1,\ s_{\sigma}(t_0) = s_{\sigma 0},\ s_{\sigma}(t_1) = s_{\sigma 1}.$$

Next, we consider

$$X = \{(s, \vartheta) \in \mathcal{S} \times \mathcal{C} :\ \varphi(t, s(t), s_{\sigma}(t), s_{\alpha\beta}(t), \vartheta(t), a) \leqq 0,$$

$$\chi(t, s(t), s_{\sigma}(t), s_{\alpha\beta}(t), \vartheta(t), b) = 0,\ s(t_0) = s_0,\ s(t_1) = s_1,$$

$$s_{\sigma}(t_0) = s_{\sigma 0},\ s_{\sigma}(t_1) = s_{\sigma 1},\ t \in K, a \in A,\ b \in B\}$$

the feasible solution set in $(\mathcal{RMRCP})$, named the *robust feasible solution set* for the problem $(\mathcal{MRCP})$.

From now on, to simplify our presentation, we introduce some notations as follows: $\pi := (t, s(t), s_{\sigma}(t), s_{\alpha\beta}(t), \vartheta(t))$, $\eta := (t, y(t), y_{\sigma}(t), y_{\alpha\beta}(t), z(t))$.

In the following, we introduce the notion of an efficient solution for the considered class of constrained robust control problems.

Definition 1. *A robust feasible solution $(\bar{s}, \bar{\vartheta}) \in X$ is said to be a robust weak efficient solution to the multi-objective robust control problem $(\mathcal{MRCP})$ if there does not exist another point $(s, \vartheta) \in X$ such that*

$$\int_{\Delta} \max_{w \in W} \phi_{\kappa}(\pi, w) dt^{\kappa} < \int_{\Delta} \max_{w \in W} \phi_{\kappa}(\bar{\pi}, w) dt^{\kappa}.$$

To formulate the concept of *convexity* and the *robust necessary efficiency conditions* associated with the aforementioned multi-objective robust control problem, we will use the Saunders's multi-index notation (see Saunders [13], Treanţă [16]).

Definition 2. *A robust controlled vector functional of curvilinear integral type*

$$F(s, \vartheta, \bar{w}) = \int_{\Delta} \phi_{\kappa}(t, s(t), s_{\sigma}(t), s_{\alpha\beta}(t), \vartheta(t), \bar{w}) dt^{\kappa} = \int_{\Delta} \phi_{\kappa}(\pi, \bar{w}) dt^{\kappa}$$

is said to be convex (strict convex) at $(\bar{s}, \bar{\vartheta}) \in \mathcal{S} \times \mathcal{C}$ if the following inequality

$$F(s, \vartheta, \bar{w}) - F(\bar{s}, \bar{\vartheta}, \bar{w}) \geqq (>) \int_{\Delta} [s(t) - \bar{s}(t)] \frac{\partial \phi_{\kappa}}{\partial s}(\bar{\pi}, \bar{w}) dt^{\kappa} + \int_{\Delta} [s_{\sigma}(t) - \bar{s}_{\sigma}(t)] \frac{\partial \phi_{\kappa}}{\partial s_{\sigma}}(\bar{\pi}, \bar{w}) dt^{\kappa}$$

$$+ \frac{1}{n(\alpha, \beta)} \int_{\Delta} [s_{\alpha\beta}(t) - \bar{s}_{\alpha\beta}(t)] \frac{\partial \phi_{\kappa}}{\partial s_{\alpha\beta}}(\bar{\pi}, \bar{w}) dt^{\kappa} + \int_{\Delta} [\vartheta(t) - \bar{\vartheta}(t)] \frac{\partial \phi_{\kappa}}{\partial \vartheta}(\bar{\pi}, \bar{w}) dt^{\kappa}$$

holds for all $(s, \vartheta) \in \mathcal{S} \times \mathcal{C}$.

In accordance with Treanţă [16], we formulate the following theorem that provides the robust necessary efficiency conditions for the constrained multi-objective robust control problem $(\mathcal{MRCP})$.

Theorem 1. *Let $(\bar{s}, \bar{\vartheta}) \in X$ be a robust weak efficient solution to the problem $(\mathcal{MRCP})$. Further assume that $\max_{w \in W} \phi_{\kappa}(\pi, w) = \phi_{\kappa}(\pi, \bar{w})$. If the constraint conditions (for the existence of the multipliers) hold, then there exist the scalar vector $\bar{\mu} \in \mathbb{R}^s$, the piecewise smooth Lagrange multipliers $\bar{v} = (\bar{v}_l(t)) \in \mathbb{R}^m$, $\bar{\gamma} = (\bar{\gamma}_{\zeta}(t)) \in \mathbb{R}^n$, and the uncertain parameters $\bar{a} \in A$, $\bar{b} \in B$ such that $(\bar{s}, \bar{\vartheta})$ satisfies the following conditions:*

$$\bar{\mu}^T \frac{\partial \phi_{\kappa}}{\partial s}(\bar{\pi}, \bar{w}) + \bar{v}^T \varphi_s(\bar{\pi}, \bar{a}) + \bar{\gamma}^T \chi_s(\bar{\pi}, \bar{b}) - D_{\sigma}\left[\bar{\mu}^T \frac{\partial \phi_{\kappa}}{\partial s_{\sigma}}(\bar{\pi}, \bar{w}) + \bar{v}^T \varphi_{s_{\sigma}}(\bar{\pi}, \bar{a}) + \bar{\gamma}^T \chi_{s_{\sigma}}(\bar{\pi}, \bar{b})\right]$$

$$+ \frac{1}{n(\alpha, \beta)} D^2_{\alpha\beta}\left[\bar{\mu}^T \frac{\partial \phi_{\kappa}}{\partial s_{\alpha\beta}}(\bar{\pi}, \bar{w}) + \bar{v}^T \varphi_{s_{\alpha\beta}}(\bar{\pi}, \bar{a}) + \bar{\gamma}^T \chi_{s_{\alpha\beta}}(\bar{\pi}, \bar{b})\right] = 0, \quad \kappa = \overline{1, p}, \tag{1}$$

$$\bar{\mu}^T \frac{\partial \phi_{\kappa}}{\partial \vartheta}(\bar{\pi}, \bar{w}) + \bar{v}^T \varphi_{\vartheta}(\bar{\pi}, \bar{a}) + \bar{\gamma}^T \chi_{\vartheta}(\bar{\pi}, \bar{b}) = 0, \quad \kappa = \overline{1, p}, \tag{2}$$

$$\bar{v}^T \varphi(\bar{\pi}, \bar{a}) = 0, \ \bar{v} \geqq 0, \tag{3}$$

$$\bar{\mu} \geq 0, \tag{4}$$

hold for all $t \in K$, except at discontinuities.

Remark 1. *The conditions (1)–(4) are known as robust necessary efficiency conditions for the constrained multi-objective robust control problem $(\mathcal{MRCP})$.*

3. Robust Duality of Wolfe Type

In this section, in accordance with Wolfe [27], we formulate Wolfe type robust dual problem for the constrained multi-objective robust control problem, with data uncertainty in the objective and constraint functionals $(\mathcal{MRCP})$, as follows:

$$(\mathcal{W}-\mathcal{MRCP}) \quad \max_{(y(\cdot),z(\cdot))} \int_{\Delta} \Big\{ \phi_{\kappa}(\eta,w) + \nu^{T}\varphi(\eta,a)e + \gamma^{T}\chi(\eta,b)e \Big\} dt^{\kappa}$$

$$\text{subject to} \quad \mu^{T}\frac{\partial\phi_{\kappa}}{\partial s}(\eta,w) + \nu^{T}\varphi_{s}(\eta,a) + \gamma^{T}\chi_{s}(\eta,b) - D_{\sigma}\Big[\mu^{T}\frac{\partial\phi_{\kappa}}{\partial s_{\sigma}}(\eta,w) + \nu^{T}\varphi_{s_{\sigma}}(\eta,a) + \gamma^{T}\chi_{s_{\sigma}}(\eta,b)\Big] + \frac{1}{n(\alpha,\beta)}D^{2}_{\alpha\beta}\Big[\mu^{T}\frac{\partial\phi_{\kappa}}{\partial s_{\alpha\beta}}(\eta,w) + \nu^{T}\varphi_{s_{\alpha\beta}}(\eta,a) + \gamma^{T}\chi_{s_{\alpha\beta}}(\eta,b)\Big] = 0, \tag{5}$$

$$\mu^{T}\frac{\partial\phi_{\kappa}}{\partial\vartheta}(\eta,w) + \nu^{T}\varphi_{\vartheta}(\eta,a) + \gamma^{T}\chi_{\vartheta}(\eta,b) = 0, \quad \kappa = \overline{1,p}, \tag{6}$$

$$y(t_0) = s_0,\ y(t_1) = s_1,\ y_{\sigma}(t_0) = s_{\sigma 0},\ y_{\sigma}(t_1) = s_{\sigma 1}, \tag{7}$$

$$\mu > 0,\ e^{T}\mu = 1,\ e = (1,\ldots,1) \in \mathbb{R}^{s}. \tag{8}$$

The associated robust counterpart for the problem $(\mathcal{W}-\mathcal{MRCP})$ is given as:

$$(\mathcal{RW}-\mathcal{MRCP}) \quad \max_{(y(\cdot),z(\cdot),w,a,b)} \int_{\Delta} \Big\{ \phi_{\kappa}(\eta,w) + \nu^{T}\varphi(\eta,a)e + \gamma^{T}\chi(\eta,b)e \Big\} dt^{\kappa}$$

$$\text{subject to} \quad \mu^{T}\frac{\partial\phi_{\kappa}}{\partial s}(\eta,w) + \nu^{T}\varphi_{s}(\eta,a) + \gamma^{T}\chi_{s}(\eta,b) - D_{\sigma}\Big[\mu^{T}\frac{\partial\phi_{\kappa}}{\partial s_{\sigma}}(\eta,w) + \nu^{T}\varphi_{s_{\sigma}}(\eta,a) + \gamma^{T}\chi_{s_{\sigma}}(\eta,b)\Big] + \frac{1}{n(\alpha,\beta)}D^{2}_{\alpha\beta}\Big[\mu^{T}\frac{\partial\phi_{\kappa}}{\partial s_{\alpha\beta}}(\eta,w) + \nu^{T}\varphi_{s_{\alpha\beta}}(\eta,a) + \gamma^{T}\chi_{s_{\alpha\beta}}(\eta,b)\Big] = 0,$$

$$\mu^{T}\frac{\partial\phi_{\kappa}}{\partial\vartheta}(\eta,w) + \nu^{T}\varphi_{\vartheta}(\eta,a) + \gamma^{T}\chi_{\vartheta}(\eta,b) = 0, \quad \kappa = \overline{1,p},$$

$$y(t_0) = s_0,\ y(t_1) = s_1,\ y_{\sigma}(t_0) = s_{\sigma 0},\ y_{\sigma}(t_1) = s_{\sigma 1},$$

$$\mu > 0,\ \mu^{T}e = 1,\ e = (1,\ldots,1) \in \mathbb{R}^{s},$$

for all $w \in W,\ a \in A$ and $b \in B$.

Further, we denote by $X_w = \{(y,z;\mu,\nu,\gamma,w,a,b)$: satisfying conditions (5)–(8)$\}$ the set of all feasible solutions to $(\mathcal{RW}-\mathcal{MRCP})$ and we say that it is the *robust feasible solution set* to the problem $(\mathcal{W}-\mathcal{MRCP})$.

Definition 3. *A point $(\bar{y},\bar{z};\bar{\mu},\bar{\nu},\bar{\gamma},\bar{w},\bar{a},\bar{b}) \in X_w$ is said to be robust weak efficient solution to the Wolfe type robust dual problem $(\mathcal{W}-\mathcal{MRCP})$ if there does not exist another point $(y,z;\mu,\nu,\gamma,w,a,b) \in X_w$ such that*

$$\int_{\Delta} \Big\{ \phi_{\kappa}(\eta,\bar{w}) + \nu^{T}\varphi(\eta,\bar{a})e + \gamma^{T}\chi(\eta,\bar{b})e \Big\} dt^{\kappa} > \int_{\Delta} \Big\{ \phi_{\kappa}(\bar{\eta},\bar{w}) + \bar{\nu}^{T}\varphi(\bar{\eta},\bar{a})e + \bar{\gamma}^{T}\chi(\bar{\eta},\bar{b})e \Big\} dt^{\kappa}.$$

Next, we establish the weak duality result for $(\mathcal{MRCP})$ under some convexity assumptions. More precisely, we state that the value attained by the objective functional of the dual problem over its feasible set does not exceed the value attained by the objective functional of the primal problem.

Theorem 2. (*Robust Weak Duality*) *Let $(\bar{s},\bar{\vartheta})$ and $(\bar{y},\bar{z};\bar{\mu},\bar{\nu},\bar{\gamma},\bar{w},\bar{a},\bar{b})$ be robust feasible solutions of $(\mathcal{MRCP})$ and $(\mathcal{W}-\mathcal{MRCP})$, respectively. Assume that* $\max_{w\in W}\phi_{\kappa}(\bar{\pi},w) = \phi_{\kappa}(\bar{\pi},\bar{w})$, *and $\int_{\Delta}\bar{\mu}^{T}\phi_{\kappa}(\cdot,\bar{w})dt^{\kappa}$, $\int_{\Delta}\bar{\nu}^{T}\varphi(\cdot,\bar{a})dt^{\kappa}$ and $\int_{\Delta}\bar{\gamma}^{T}\chi(\cdot,\bar{b})dt^{\kappa}$ are convex at $(\bar{y},\bar{z})$. Then the following inequality cannot hold*

$$\int_\Delta \max_{w\in W}\phi_\kappa(\bar\pi,w)dt^\kappa < \int_\Delta \Big\{\phi_\kappa(\bar\eta,\bar w)+\bar v^T\varphi(\bar\eta,\bar a)e+\bar\gamma^T\chi(\bar\eta,\bar b)e\Big\}dt^\kappa.$$

Proof. Contrary to the result, we assume that

$$\int_\Delta \max_{w\in W}\phi_\kappa(\bar\pi,w)dt^\kappa < \int_\Delta \Big\{\phi_\kappa(\bar\eta,\bar w)+\bar v^T\varphi(\bar\eta,\bar a)e+\bar\gamma^T\chi(\bar\eta,\bar b)e\Big\}dt^\kappa.$$

Since $\max_{w\in W}\phi_\kappa(\bar\pi,w)=\phi_\kappa(\bar\pi,\bar w)$, we have

$$\int_\Delta \phi_\kappa(\bar\pi,\bar w)dt^\kappa < \int_\Delta \Big\{\phi_\kappa(\bar\eta,\bar w)+\bar v^T\varphi(\bar\eta,\bar a)e+\bar\gamma^T\chi(\bar\eta,\bar b)e\Big\}dt^\kappa.$$

The above inequality together with the robust feasibility of $(\bar s,\bar\vartheta)$ to the problem $(\mathcal{MRCP})$ implies

$$\int_\Delta \Big\{\phi_\kappa(\bar\pi,\bar w)+\bar v^T\varphi(\bar\pi,\bar a)e+\bar\gamma^T\chi(\bar\pi,\bar b)e\Big\}dt^\kappa$$
$$< \int_\Delta \Big\{\phi_\kappa(\bar\eta,\bar w)+\bar v^T\varphi(\bar\eta,\bar a)e+\bar\gamma^T\chi(\bar\eta,\bar b)e\Big\}dt^\kappa.$$

As $\bar\mu>0$ and $\bar\mu^Te=1$, therefore, the above inequality can be written as

$$\int_\Delta \Big\{\bar\mu^T\phi_\kappa(\bar\pi,\bar w)+\bar v^T\varphi(\bar\pi,\bar a)+\bar\gamma^T\chi(\bar\pi,\bar b)\Big\}dt^\kappa$$
$$< \int_\Delta \Big\{\bar\mu^T\phi_\kappa(\bar\eta,\bar w)+\bar v^T\varphi(\bar\eta,\bar a)+\bar\gamma^T\chi(\bar\eta,\bar b)\Big\}dt^\kappa. \tag{9}$$

Now, since $\int_\Delta \bar\mu^T\phi_\kappa(\cdot,\bar w)dt^\kappa$, $\int_\Delta \bar v^T\varphi(\cdot,\bar a)dt^\kappa$ and $\int_\Delta \bar\gamma^T\chi(\cdot,\bar b)dt^\kappa$ are convex at $(\bar y,\bar z)$, we have

$$\int_\Delta \Big\{\bar\mu^T\phi_\kappa(\bar\pi,\bar w)-\bar\mu^T\phi_\kappa(\bar\eta,\bar w)\Big\}dt^\kappa \geqq \int_\Delta (\bar s-\bar y)\bar\mu^T\frac{\partial\phi_\kappa}{\partial\bar s}(\bar\eta,\bar w)dt^\kappa$$
$$+\int_\Delta (\bar s_\sigma-\bar y_\sigma)\bar\mu^T\frac{\partial\phi_\kappa}{\partial\bar s_\sigma}(\bar\eta,\bar w)dt^\kappa+\frac{1}{n(\alpha,\beta)}\int_\Delta (\bar s_{\alpha\beta}-\bar y_{\alpha\beta})\bar\mu^T\frac{\partial\phi_\kappa}{\partial\bar s_{\alpha\beta}}(\bar\eta,\bar w)dt^\kappa$$
$$+\int_\Delta (\bar\vartheta-\bar z)\bar\mu^T\frac{\partial\phi_\kappa}{\partial\bar\vartheta}(\bar\eta,\bar w)dt^\kappa, \tag{10}$$

$$\int_\Delta \Big\{\bar v^T\varphi(\bar\pi,\bar a)-\bar v^T\varphi(\bar\eta,\bar a)\Big\}dt^\kappa \geqq \int_\Delta (\bar s-\bar y)\bar v^T\frac{\partial\varphi}{\partial\bar s}(\bar\eta,\bar a)dt^\kappa$$
$$+\int_\Delta (\bar s_\sigma-\bar y_\sigma)\bar v^T\frac{\partial\varphi}{\partial\bar s_\sigma}(\bar\eta,\bar a)dt^\kappa+\frac{1}{n(\alpha,\beta)}\int_\Delta (\bar s_{\alpha\beta}-\bar y_{\alpha\beta})\bar v^T\frac{\partial\varphi}{\partial\bar s_{\alpha\beta}}(\bar\eta,\bar a)dt^\kappa$$
$$+\int_\Delta (\bar\vartheta-\bar z)\bar v^T\frac{\partial\varphi}{\partial\bar\vartheta}(\bar\eta,\bar a)dt^\kappa, \tag{11}$$

and

$$\int_\Delta \Big\{\bar\gamma^T\chi(\bar\pi,\bar b)-\bar\gamma^T\chi(\bar\eta,\bar b)\Big\}dt^\kappa \geqq \int_\Delta (\bar s-\bar y)\bar\gamma^T\frac{\partial\chi}{\partial\bar s}(\bar\eta,\bar b)dt^\kappa$$
$$+\int_\Delta (\bar s_\sigma-\bar y_\sigma)\bar\gamma^T\frac{\partial\chi}{\partial\bar s_\sigma}(\bar\eta,\bar b)dt^\kappa+\frac{1}{n(\alpha,\beta)}\int_\Delta (\bar s_{\alpha\beta}-\bar y_{\alpha\beta})\bar\gamma^T\frac{\partial\chi}{\partial\bar s_{\alpha\beta}}(\bar\eta,\bar b)dt^\kappa$$
$$+\int_\Delta (\bar\vartheta-\bar z)\bar\gamma^T\frac{\partial\chi}{\partial\bar\vartheta}(\bar\eta,\bar b)dt^\kappa. \tag{12}$$

On adding the inequalities (10)–(12), and by considering the robust feasibility of the point $(\bar{y}, \bar{z}; \bar{\mu}, \bar{v}, \bar{\gamma}, \bar{w}, \bar{a}, \bar{b})$ to the problem $(\mathcal{W} - \mathcal{MRCP})$, we obtain

$$\int_{\Delta} \left\{ \bar{\mu}^T \phi_\kappa(\bar{\pi}, \bar{w}) + \bar{v}^T \varphi(\bar{\pi}, \bar{a}) + \bar{\gamma}^T \chi(\bar{\pi}, \bar{b}) \right\} dt^\kappa \geqq \int_{\Delta} \left\{ \bar{\mu}^T \phi_\kappa(\bar{\eta}, \bar{w}) + \bar{v}^T \varphi(\bar{\eta}, \bar{a}) + \bar{\gamma}^T \chi(\bar{\eta}, \bar{b}) \right\} dt^\kappa,$$

which contradicts the inequality (9). This completes the proof. □

Now, we formulate and prove the strong duality result which states that duality gap is zero.

Theorem 3. (*Robust Strong Duality*) *Let $(\bar{s}, \bar{\vartheta})$ be a robust weak efficient solution to the problem $(\mathcal{MRCP})$. Assume that* $\max_{w \in W} \phi_\kappa(\bar{\pi}, w) = \phi_\kappa(\bar{\pi}, \bar{w})$ *and the constraint conditions (for the existence of multiplier) hold for $(\mathcal{MRCP})$. Then, there exist the scalar vector $\bar{\mu} \in \mathbb{R}^s_+$, the piecewise smooth Lagrange multipliers $\bar{v} = (\bar{v}_l(t)) \in \mathbb{R}^m_+$ and $\bar{\gamma} = (\bar{\gamma}_\zeta(t)) \in \mathbb{R}^n$, and the uncertain parameters $\bar{a} \in A, \bar{b} \in B$ such that $(\bar{s}, \bar{\vartheta}; \bar{\mu}, \bar{v}, \bar{\gamma}, \bar{w}, \bar{a}, \bar{b})$ is a robust feasible solution to the problem $(\mathcal{W} - \mathcal{MRCP})$. Further, if the Robust Weak Duality* (*see Theorem 2*) *holds, then $(\bar{s}, \bar{\vartheta}; \bar{\mu}, \bar{v}, \bar{\gamma}, \bar{w}, \bar{a}, \bar{b})$ is a robust weak efficient solution to the problem $(\mathcal{W} - \mathcal{MRCP})$.*

Proof. Since $(\bar{s}, \bar{\vartheta})$ is a robust weak efficient solution to the problem $(\mathcal{MRCP})$, by Theorem 1, there exist the scalar vector $\bar{\mu} \in \mathbb{R}^s_+$, the piecewise smooth Lagrange multiplies $\bar{v} \in \mathbb{R}^m_+$, $\bar{\gamma} \in \mathbb{R}^n$, and the uncertain parameters $\bar{a} \in A, \bar{b} \in B$ such that the conditions (1)–(4) are satisfied at $(\bar{s}, \bar{\vartheta})$. This proves the robust feasibility of $(\bar{s}, \bar{\vartheta}; \bar{\mu}, \bar{v}, \bar{\gamma}, \bar{w}, \bar{a}, \bar{b})$ to the problem $(\mathcal{W} - \mathcal{MRCP})$ and the corresponding objective values are equal. If $(\bar{s}, \bar{\vartheta}; \bar{\mu}, \bar{v}, \bar{\gamma}, \bar{w}, \bar{a}, \bar{b})$ is not a robust weak efficient solution to the problem $(\mathcal{W} - \mathcal{MRCP})$, then there exists another point $(y, z; \bar{\mu}, \bar{v}, \bar{\gamma}, \bar{w}, \bar{a}, \bar{b})$ such that

$$\int_{\Delta} \left\{ \phi_\kappa(\bar{\pi}, \bar{w}) + \bar{v}^T \varphi(\bar{\pi}, \bar{a})e + \bar{\gamma}^T \chi(\bar{\pi}, \bar{b})e \right\} dt^\kappa < \int_{\Delta} \left\{ \phi_\kappa(\eta, \bar{w}) + \bar{v}^T \varphi(\eta, \bar{a})e + \bar{\gamma}^T \chi(\eta, \bar{b})e \right\} dt^\kappa.$$

From the condition (3), we get

$$\int_{\Delta} \phi_\kappa(\bar{\pi}, \bar{w}) dt^\kappa < \int_{\Delta} \left\{ \phi_\kappa(\eta, \bar{w}) + \bar{v}^T \varphi(\eta, \bar{a})e + \bar{\gamma}^T \chi(\eta, \bar{b})e \right\} dt^\kappa.$$

Since $\max_{w \in W} \phi_\kappa(\bar{\pi}, w) = \phi_\kappa(\bar{\pi}, \bar{w})$, we have

$$\int_{\Delta} \max_{w \in W} \phi_\kappa(\bar{\pi}, w) dt^\kappa < \int_{\Delta} \left\{ \phi_\kappa(\eta, \bar{w}) + \bar{v}^T \varphi(\eta, \bar{a})e + \bar{\gamma}^T \chi(\eta, \bar{b})e \right\} dt^\kappa,$$

which contradicts the Robust Weak Duality (see Theorem 2). In consequence, the point $(\bar{s}, \bar{\vartheta}; \bar{\mu}, \bar{v}, \bar{\gamma}, \bar{w}, \bar{a}, \bar{b})$ is a robust weak efficient solution to the problem $(\mathcal{W} - \mathcal{MRCP})$. □

Theorem 4. (*Robust Strict Converse Duality*) *Let $(\bar{y}, \bar{z}; \bar{\mu}, \bar{v}, \bar{\gamma}, \bar{w}, \bar{a}, \bar{b})$ be a robust feasible solution to the problem $(\mathcal{W} - \mathcal{MRCP})$. Assume that* $\max_{w \in W} \phi_\kappa(\pi, w) = \phi_\kappa(\pi, \bar{w})$, *and $\int_{\Delta} \bar{\mu}^T \phi_\kappa(\cdot, \bar{w}) dt^\kappa$, $\int_{\Delta} \bar{v}^T \varphi(\cdot, \bar{a}) dt^\kappa$ and $\int_{\Delta} \bar{\gamma}^T \chi(\cdot, \bar{b}) dt^\kappa$ are strict convex at $(\bar{y}, \bar{z})$. If $(\bar{s}, \bar{\vartheta}) \in X$ such that $\int_{\Delta} \phi_\kappa(\bar{\pi}, \bar{w}) dt^\kappa = \int_{\Delta} \phi_\kappa(\bar{\eta}, \bar{w}) dt^\kappa$, then $(\bar{s}, \bar{\vartheta})$ is a robust weak efficient solution to the problem $(\mathcal{MRCP})$.*

Proof. Since $(\bar{y}, \bar{z}; \bar{\mu}, \bar{v}, \bar{\gamma}, \bar{w}, \bar{a}, \bar{b})$ is a robust feasible solution to the problem $(\mathcal{W} - \mathcal{MRCP})$, on multiplying the inequality (5) and (6) by $(\hat{s} - \bar{y})$ and $(\hat{\vartheta} - \bar{z})$, respectively, and then integrate them, we get

$$
\begin{aligned}
&\int_\Delta (\hat{s}-\bar{y})\{\bar{\mu}^T\frac{\partial\phi_\kappa}{\partial\hat{s}}(\bar{\eta},\bar{w})+\bar{v}^T\varphi_{\hat{s}}(\bar{\eta},\bar{a})+\bar{\gamma}^T\chi_{\hat{s}}(\bar{\eta},\bar{b})\\
&-D_\sigma\Big[\bar{\mu}^T\frac{\partial\phi_\kappa}{\partial\hat{s}_\sigma}(\bar{\eta},\bar{w})+\bar{v}^T\varphi_{\hat{s}_\sigma}(\bar{\eta},\bar{a})+\bar{\gamma}^T\chi_{\hat{s}_\sigma}(\bar{\eta},\bar{b})\Big]\\
&+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}\Big[\bar{\mu}^T\frac{\partial\phi_\kappa}{\partial\hat{s}_{\alpha\beta}}(\bar{\eta},\bar{w})+\bar{v}^T\varphi_{\hat{s}_{\alpha\beta}}(\bar{\eta},\bar{a})+\bar{\gamma}^T\chi_{\hat{s}_{\alpha\beta}}(\bar{\eta},\bar{b})\Big]\}dt^\kappa\\
&+\int_\Delta(\hat{\vartheta}-\bar{z})\{\bar{\mu}^T\frac{\partial\phi_\kappa}{\partial\hat{\vartheta}}(\bar{\eta},\bar{w})+\bar{v}^T\varphi_{\hat{\vartheta}}(\bar{\eta},\bar{a})+\bar{\gamma}^T\chi_{\hat{\vartheta}}(\bar{\eta},\bar{b})\}dt^\kappa\\
&=\int_\Delta\Big[(\hat{s}-\bar{y})\{\bar{\mu}^T\frac{\partial\phi_\kappa}{\partial\hat{s}}(\bar{\eta},\bar{w})+\bar{v}^T\varphi_{\hat{s}}(\bar{\eta},\bar{a})+\bar{\gamma}^T\chi_{\hat{s}}(\bar{\eta},\bar{b})\}\\
&+(\hat{s}_\sigma-\bar{y}_\sigma)\{\bar{\mu}^T\frac{\partial\phi_\kappa}{\partial\hat{s}_\sigma}(\bar{\eta},\bar{w})+\bar{v}^T\varphi_{\hat{s}_\sigma}(\bar{\eta},\bar{a})+\bar{\gamma}^T\chi_{\hat{s}_\sigma}(\bar{\eta},\bar{b})\}\\
&+\frac{1}{n(\alpha,\beta)}(\hat{s}_{\alpha\beta}-\bar{y}_{\alpha\beta})\{\bar{\mu}^T\frac{\partial\phi_\kappa}{\partial\hat{s}_{\alpha\beta}}(\bar{\eta},\bar{w})+\bar{v}^T\varphi_{\hat{s}_{\alpha\beta}}(\bar{\eta},\bar{a})+\bar{\gamma}^T\chi_{\hat{s}_{\alpha\beta}}(\bar{\eta},\bar{b})\}\Big]dt^\kappa\\
&+\int_\Delta(\hat{\vartheta}-\bar{z})\{\bar{\mu}^T\frac{\partial\phi_\kappa}{\partial\hat{\vartheta}}(\bar{\eta},\bar{w})+\bar{v}^T\varphi_{\hat{\vartheta}}(\bar{\eta},\bar{a})+\bar{\gamma}^T\chi_{\hat{\vartheta}}(\bar{\eta},\bar{b})\}dt^\kappa=0,
\end{aligned}
\tag{13}
$$

where we used the formula of integration by parts, the divergence formula and the boundary conditions formulated in the considered problem.

Next, we proceed by contradiction and assume that $(\bar{s},\bar{\vartheta})$ is not a robust weak efficient solution to the problem $(\mathcal{MRCP})$. Therefore, there exists $(\hat{s},\hat{\vartheta})\in X$ such that

$$\int_\Delta \max_{w\in W}\phi_\kappa(\hat{\pi},w)dt^\kappa<\int_\Delta \max_{w\in W}\phi_\kappa(\bar{\pi},w)dt^\kappa.$$

Since $\max_{w\in W}\phi_\kappa(\pi,w)=\phi_\kappa(\pi,\bar{w})$, it follows

$$\int_\Delta\phi_\kappa(\hat{\pi},\bar{w})dt^\kappa<\int_\Delta\phi_\kappa(\bar{\pi},\bar{w})dt^\kappa.$$

By assumption, $\int_\Delta\phi_\kappa(\bar{\pi},\bar{w})dt^\kappa=\int_\Delta\phi_\kappa(\bar{\eta},\bar{w})dt^\kappa$. Therefore, the above inequality yields

$$\int_\Delta\phi_\kappa(\hat{\pi},\bar{w})dt^\kappa<\int_\Delta\phi_\kappa(\bar{\eta},\bar{w})dt^\kappa.$$

Since $\bar{\mu}\in\mathbb{R}^s_+$, we get

$$\int_\Delta\bar{\mu}^T\phi_\kappa(\hat{\pi},\bar{w})dt^\kappa<\int_\Delta\bar{\mu}^T\phi_\kappa(\bar{\eta},\bar{w})dt^\kappa. \tag{14}$$

On the other hand, from the assumption that $\int_\Delta\bar{\mu}^T\phi_\kappa(\cdot,\bar{w})dt^\kappa$ is strict convex at $(\bar{y},\bar{z})$, we have

$$
\begin{aligned}
&\int_\Delta\Big\{\bar{\mu}^T\phi_\kappa(\hat{\pi},\bar{w})-\bar{\mu}^T\phi_\kappa(\bar{\eta},\bar{w})\Big\}dt^\kappa>\int_\Delta(\hat{s}-\bar{y})\bar{\mu}^T\frac{\partial\phi_\kappa}{\partial\hat{s}}(\bar{\eta},\bar{w})dt^\kappa\\
&+\int_\Delta(\hat{s}_\sigma-\bar{y}_\sigma)\bar{\mu}^T\frac{\partial\phi_\kappa}{\partial\hat{s}_\sigma}(\bar{\eta},\bar{w})dt^\kappa+\frac{1}{n(\alpha,\beta)}\int_\Delta(\hat{s}_{\alpha\beta}-\bar{y}_{\alpha\beta})\bar{\mu}^T\frac{\partial\phi_\kappa}{\partial\hat{s}_{\alpha\beta}}(\bar{\eta},\bar{w})dt^\kappa\\
&+\int_\Delta(\hat{\vartheta}-\bar{z})\bar{\mu}^T\frac{\partial\phi_\kappa}{\partial\hat{\vartheta}}(\bar{\eta},\bar{w})dt^\kappa,
\end{aligned}
$$

which together with the inequality (14), gives

$$\int_\Delta(\hat{s}-\bar{y})\bar{\mu}^T\frac{\partial\phi_\kappa}{\partial\hat{s}}(\bar{\eta},\bar{w})dt^\kappa$$

$$+\int_{\Delta}(\hat{s}_{\sigma}-\bar{y}_{\sigma})\bar{\mu}^{T}\frac{\partial\phi_{\kappa}}{\partial\hat{s}_{\sigma}}(\bar{\eta},\bar{\omega})dt^{\kappa}+\frac{1}{n(\alpha,\beta)}\int_{\Delta}(\hat{s}_{\alpha\beta}-\bar{y}_{\alpha\beta})\bar{\mu}^{T}\frac{\partial\phi_{\kappa}}{\partial\hat{s}_{\alpha\beta}}(\bar{\eta},\bar{\omega})dt^{\kappa}$$

$$+\int_{\Delta}(\hat{\vartheta}-\bar{z})\bar{\mu}^{T}\frac{\partial\phi_{\kappa}}{\partial\hat{\vartheta}}(\bar{\eta},\bar{\omega})dt^{\kappa}<0. \tag{15}$$

Again, by assumption that $\int_{\Delta}\bar{v}^{T}\varphi(\cdot,\bar{a})dt^{\kappa}$ is strict convex at $(\bar{y},\bar{z})$, we get

$$\int_{\Delta}\Big\{\bar{v}^{T}\varphi(\hat{\pi},\bar{a})-\bar{v}^{T}\varphi(\bar{\eta},\bar{a})\Big\}dt^{\kappa}>\int_{\Delta}(\hat{s}-\bar{y})\bar{v}^{T}\varphi_{\hat{s}}(\bar{\eta},\bar{a})dt^{\kappa}$$

$$+\int_{\Delta}(\hat{s}_{\sigma}-\bar{y}_{\sigma})\bar{v}^{T}\varphi_{\hat{s}_{\sigma}}(\bar{\eta},\bar{a})dt^{\kappa}+\frac{1}{n(\alpha,\beta)}\int_{\Delta}(\hat{s}_{\alpha\beta}-\bar{y}_{\alpha\beta})\bar{v}^{T}\varphi_{\hat{s}_{\alpha\beta}}(\bar{\eta},\bar{a})dt^{\kappa}$$

$$+\int_{\Delta}(\hat{\vartheta}-\bar{z})\bar{v}^{T}\varphi_{\hat{\vartheta}}(\bar{\eta},\bar{a})dt^{\kappa}. \tag{16}$$

Since $(\hat{s},\hat{\vartheta})$ and $(\bar{s},\bar{\vartheta};\bar{\mu},\bar{v},\bar{\gamma},\bar{\omega},\bar{a},\bar{b})$ are the robust feasible solutions to the problem $(\mathcal{MRCP})$ and $(\mathcal{W}-\mathcal{MRCP})$, respectively, we obtain

$$\int_{\Delta}\bar{v}^{T}\varphi(\hat{\pi},\bar{a})dt^{\kappa}-\int_{\Delta}\bar{v}^{T}\varphi(\bar{\eta},\bar{a})dt^{\kappa}\leqq 0,$$

which, along with the inequality (16), involves

$$\int_{\Delta}(\hat{s}-\bar{y})\bar{v}^{T}\varphi_{\hat{s}}(\bar{\eta},\bar{a})dt^{\kappa}$$

$$+\int_{\Delta}(\hat{s}_{\sigma}-\bar{y}_{\sigma})\bar{v}^{T}\varphi_{\hat{s}_{\sigma}}(\bar{\eta},\bar{a})dt^{\kappa}+\frac{1}{n(\alpha,\beta)}\int_{\Delta}(\hat{s}_{\alpha\beta}-\bar{y}_{\alpha\beta})\bar{v}^{T}\varphi_{\hat{s}_{\alpha\beta}}(\bar{\eta},\bar{a})dt^{\kappa}$$

$$+\int_{\Delta}(\hat{\vartheta}-\bar{z})\bar{v}^{T}\varphi_{\hat{\vartheta}}(\bar{\eta},\bar{a})dt^{\kappa}<0. \tag{17}$$

Similarly, the functional $\int_{\Delta}\bar{\gamma}^{T}\chi(\cdot,\bar{b})dt^{\kappa}$ is also strict convex at $(\bar{y},\bar{z})$. The robust feasible solutions $(\hat{s},\hat{\vartheta})$ and $(\bar{s},\bar{\vartheta};\bar{\mu},\bar{v},\bar{\gamma},\bar{\omega},\bar{a},\bar{b})$ to the problem $(\mathcal{MRCP})$ and $(\mathcal{W}-\mathcal{MRCP})$, respectively, yields

$$\int_{\Delta}(\hat{s}-\bar{y})\bar{\gamma}^{T}\chi_{\hat{s}}(\bar{\eta},\bar{b})dt^{\kappa}$$

$$+\int_{\Delta}(\hat{s}_{\sigma}-\bar{y}_{\sigma})\bar{\gamma}^{T}\chi_{\hat{s}_{\sigma}}(\bar{\eta},\bar{b})dt^{\kappa}+\frac{1}{n(\alpha,\beta)}\int_{\Delta}(\hat{s}_{\alpha\beta}-\bar{y}_{\alpha\beta})\bar{\gamma}^{T}\chi_{\hat{s}_{\alpha\beta}}(\bar{\eta},\bar{b})dt^{\kappa}$$

$$+\int_{\Delta}(\hat{\vartheta}-\bar{z})\bar{\gamma}^{T}\chi_{\hat{\vartheta}}(\bar{\eta},\bar{b})dt^{\kappa}<0. \tag{18}$$

On adding the inequalities (15), (17) and (18), we obtain the following inequality

$$\int_{\Delta}\Big[(\hat{s}-\bar{y})\{\bar{\mu}^{T}\frac{\partial\phi_{\kappa}}{\partial\hat{s}}(\bar{\eta},\bar{\omega})+\bar{v}^{T}\varphi_{\hat{s}}(\bar{\eta},\bar{a})+\bar{\gamma}^{T}\chi_{\hat{s}}(\bar{\eta},\bar{b})\}$$

$$+(\hat{s}_{\sigma}-\bar{y}_{\sigma})\{\bar{\mu}^{T}\frac{\partial\phi_{\kappa}}{\partial\hat{s}_{\sigma}}(\bar{\eta},\bar{\omega})+\bar{v}^{T}\varphi_{\hat{s}_{\sigma}}(\bar{\eta},\bar{a})+\bar{\gamma}^{T}\chi_{\hat{s}_{\sigma}}(\bar{\eta},\bar{b})\}$$

$$+\frac{1}{n(\alpha,\beta)}(\hat{s}_{\alpha\beta}-\bar{y}_{\alpha\beta})\{\bar{\mu}^{T}\frac{\partial\phi_{\kappa}}{\partial\hat{s}_{\alpha\beta}}(\bar{\eta},\bar{\omega})+\bar{v}^{T}\varphi_{\hat{s}_{\alpha\beta}}(\bar{\eta},\bar{a})+\bar{\gamma}^{T}\chi_{\hat{s}_{\alpha\beta}}(\bar{\eta},\bar{b})\}\Big]dt^{\kappa}$$

$$+\int_{\Delta}(\hat{\vartheta}-\bar{z})\{\bar{\mu}^T\frac{\partial\phi_\kappa}{\partial\hat{\vartheta}}(\bar{\eta},\bar{w})+\bar{\nu}^T\varphi_{\hat{\vartheta}}(\bar{\eta},\bar{a})+\bar{\gamma}^T\chi_{\hat{\vartheta}}(\bar{\eta},\bar{b})\}dt^\kappa<0, \tag{19}$$

which contradicts the inequality (13). This completes the proof. □

4. Robust Duality of Mond-Weir Type

In this section, in accordance with Weir and Mond [26], we formulate the Mond-Weir type robust dual problem for the considered multi-objective nonlinear robust control problem $(\mathcal{MRCP})$, with data uncertainty in the objective and constraint functionals, as follows:

$$(\mathcal{MW}-\mathcal{MRCP})\quad \max_{(y(\cdot),z(\cdot))}\int_{\Delta}\phi_\kappa(\eta,w)dt^\kappa$$

$$\text{subject to}\quad \mu^T\frac{\partial\phi_\kappa}{\partial s}(\eta,w)+\nu^T\varphi_s(\eta,a)+\gamma^T\chi_s(\eta,b)$$
$$-D_\sigma\Big[\mu^T\frac{\partial\phi_\kappa}{\partial s_\sigma}(\eta,w)+\nu^T\varphi_{s_\sigma}(\eta,a)+\gamma^T\chi_{s_\sigma}(\eta,b)\Big]$$
$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}\Big[\mu^T\frac{\partial\phi_\kappa}{\partial s_{\alpha\beta}}(\eta,w)+\nu^T\varphi_{s_{\alpha\beta}}(\eta,a)+\gamma^T\chi_{s_{\alpha\beta}}(\eta,b)\Big]=0, \tag{20}$$
$$\mu^T\frac{\partial\phi_\kappa}{\partial\vartheta}(\eta,w)+\nu^T\varphi_\vartheta(\eta,a)+\gamma^T\chi_\vartheta(\eta,b)=0,\quad \kappa=\overline{1,p}, \tag{21}$$
$$\bar{\nu}^T\varphi(\eta,a)\geqq 0, \tag{22}$$
$$\chi(\eta,b)=0, \tag{23}$$
$$y(t_0)=s_0,\ y(t_1)=s_1,\ y_\sigma(t_0)=s_{\sigma 0},\ y_\sigma(t_1)=s_{\sigma 1}, \tag{24}$$
$$\mu\in\mathbb{R}^s_+,\ \mu^Te=1,\ \nu\in\mathbb{R}^m_+,\ \gamma\in\mathbb{R}^n. \tag{25}$$

The associated robust counterpart to the problem $(\mathcal{MW}-\mathcal{MRCP})$ is given as follows:

$$(\mathcal{RMW}-\mathcal{MRCP})\quad \max_{(y(\cdot),z(\cdot),w)}\int_{\Delta}\phi_\kappa(\eta,w)dt^\kappa$$

$$\text{subject to}\quad \mu^T\frac{\partial\phi_\kappa}{\partial s}(\eta,w)+\nu^T\varphi_s(\eta,a)+\gamma^T\chi_s(\eta,b)$$
$$-D_\sigma\Big[\mu^T\frac{\partial\phi_\kappa}{\partial s_\sigma}(\eta,w)+\nu^T\varphi_{s_\sigma}(\eta,a)+\gamma^T\chi_{s_\sigma}(\eta,b)\Big]$$
$$+\frac{1}{n(\alpha,\beta)}D^2_{\alpha\beta}\Big[\mu^T\frac{\partial\phi_\kappa}{\partial s_{\alpha\beta}}(\eta,w)+\nu^T\varphi_{s_{\alpha\beta}}(\eta,a)+\gamma^T\chi_{s_{\alpha\beta}}(\eta,b)\Big]=0,$$
$$\mu^T\frac{\partial\phi_\kappa}{\partial\vartheta}(\eta,w)+\nu^T\varphi_\vartheta(\eta,a)+\gamma^T\chi_\vartheta(\eta,b)=0,\quad \kappa=\overline{1,p},$$
$$\bar{\nu}^T\varphi(\eta,a)\geqq 0,$$
$$\chi(\eta,b)=0,$$
$$y(t_0)=s_0,\ y(t_1)=s_1,\ y_\sigma(t_0)=s_{\sigma 0},\ y_\sigma(t_1)=s_{\sigma 1},$$
$$\mu\in\mathbb{R}^s_+,\ \mu^Te=1,\ \nu\in\mathbb{R}^m_+,\ \gamma\in\mathbb{R}^n,$$

for all $w\in W,\ a\in A,\ b\in B$.

We denote by $X_{mw}=\{(y,z;\mu,\nu,\gamma,w,a,b):\ \text{satisfying conditions (20)–(25)}\}$ the set of all feasible solutions to $(\mathcal{RMW}-\mathcal{MRCP})$ and we say that it is the *robust feasible solution set* to the problem $(\mathcal{MW}-\mathcal{MRCP})$.

Now, under convexity hypotheses, we establish the robust weak and strong duality results for $(\mathcal{MRCP})$ and $(\mathcal{MW}-\mathcal{MRCP})$.

Theorem 5. (*Robust Weak Duality*) *Let* $(\bar{s},\bar{\vartheta})$ *and* $(\bar{y},\bar{z};\bar{\mu},\bar{\nu},\bar{\gamma},\bar{w},\bar{a},\bar{b})$ *be robust feasible solutions to the problem* $(\mathcal{MRCP})$ *and* $(\mathcal{MW}-\mathcal{MRCP})$*, respectively. Assume that* $\max_{w\in W}\phi_\kappa(\bar{\pi},w)=$

$\phi_\kappa(\bar{\pi},\bar{\varpi})$, and $\int_\Delta \bar{\mu}^T\phi_\kappa(\cdot,\bar{\varpi})dt^\kappa$, $\int_\Delta \bar{v}^T\varphi(\cdot,\bar{a})dt^\kappa$ and $\int_\Delta \bar{\gamma}^T\chi(\cdot,\bar{b})dt^\kappa$ are convex at $(\bar{y},\bar{z})$. Then the following inequality cannot hold

$$\int_\Delta \max_{w\in W}\phi_\kappa(\bar{\pi},w)dt^\kappa < \int_\Delta \phi_\kappa(\bar{\eta},\bar{\varpi})dt^\kappa.$$

Proof. Contrary to the result, we assume that

$$\int_\Delta \max_{w\in W}\phi_\kappa(\bar{\pi},w)dt^\kappa < \int_\Delta \phi_\kappa(\bar{\eta},\bar{\varpi})dt^\kappa.$$

Since $\max_{w\in W}\phi_\kappa(\bar{\pi},w) = \phi_\kappa(\bar{\pi},\bar{\varpi})$, we have

$$\int_\Delta \phi_\kappa(\bar{\pi},\bar{\varpi})dt^\kappa < \int_\Delta \phi_\kappa(\bar{\eta},\bar{\varpi})dt^\kappa. \tag{26}$$

By hypothesis, $\int_\Delta \bar{\mu}^T\phi_\kappa(\cdot,\bar{\varpi})dt^\kappa$, $\int_\Delta \bar{v}^T\varphi(\cdot,\bar{a})dt^\kappa$ and $\int_\Delta \bar{\gamma}^T\chi(\cdot,\bar{b})dt^\kappa$ are convex at $(\bar{y},\bar{z})$. Therefore, we have

$$\int_\Delta \Big\{\bar{\mu}^T\phi_\kappa(\bar{\pi},\bar{\varpi}) - \bar{\mu}^T\phi_\kappa(\bar{\eta},\bar{\varpi})\Big\}dt^\kappa \geqq \int_\Delta (\bar{s}-\bar{y})\bar{\mu}^T\frac{\partial\phi_\kappa}{\partial\bar{s}}(\bar{\eta},\bar{\varpi})dt^\kappa$$
$$+\int_\Delta (\bar{s}_\sigma-\bar{y}_\sigma)\bar{\mu}^T\frac{\partial\phi_\kappa}{\partial\bar{s}_\sigma}(\bar{\eta},\bar{\varpi})dt^\kappa + \frac{1}{n(\alpha,\beta)}\int_\Delta (\bar{s}_{\alpha\beta}-\bar{y}_{\alpha\beta})\bar{\mu}^T\frac{\partial\phi_\kappa}{\partial\bar{s}_{\alpha\beta}}(\bar{\eta},\bar{\varpi})dt^\kappa$$
$$+\int_\Delta (\bar{\vartheta}-\bar{z})\bar{\mu}^T\frac{\partial\phi_\kappa}{\partial\vartheta}(\bar{\eta},\bar{\varpi})dt^\kappa, \tag{27}$$

$$\int_\Delta \Big\{\bar{v}^T\varphi(\bar{\pi},\bar{a}) - \bar{v}^T\varphi(\bar{\eta},\bar{a})\Big\}dt^\kappa \geqq \int_\Delta (\bar{s}-\bar{y})\bar{v}^T\varphi_{\bar{s}}(\bar{\eta},\bar{a})dt^\kappa$$
$$+\int_\Delta (\bar{s}_\sigma-\bar{y}_\sigma)\bar{v}^T\varphi_{\bar{s}_\sigma}(\bar{\eta},\bar{a})dt^\kappa + \frac{1}{n(\alpha,\beta)}\int_\Delta (\bar{s}_{\alpha\beta}-\bar{y}_{\alpha\beta})\bar{v}^T\varphi_{\bar{s}_{\alpha\beta}}(\bar{\eta},\bar{a})dt^\kappa$$
$$+\int_\Delta (\bar{\vartheta}-\bar{z})\bar{v}^T\varphi_{\bar{\vartheta}}(\bar{\eta},\bar{a})dt^\kappa, \tag{28}$$

and

$$\int_\Delta \Big\{\bar{\gamma}^T\chi(\bar{\pi},\bar{b}) - \bar{\gamma}^T\chi(\bar{\eta},\bar{b})\Big\}dt^\kappa \geqq \int_\Delta (\bar{s}-\bar{y})\bar{\gamma}^T\chi_{\bar{s}}(\bar{\eta},\bar{b})dt^\kappa$$
$$+\int_\Delta (\bar{s}_\sigma-\bar{y}_\sigma)\bar{\gamma}^T\chi_{\bar{s}_\sigma}(\bar{\eta},\bar{b})dt^\kappa + \frac{1}{n(\alpha,\beta)}\int_\Delta (\bar{s}_{\alpha\beta}-\bar{y}_{\alpha\beta})\bar{\gamma}^T\chi_{\bar{s}_{\alpha\beta}}(\bar{\eta},\bar{b})dt^\kappa$$
$$+\int_\Delta (\bar{\vartheta}-\bar{z})\bar{\gamma}^T\chi_{\bar{\vartheta}}(\bar{\eta},\bar{b})dt^\kappa. \tag{29}$$

On adding the inequalities (27), (28) and (29), along with the robust feasibility of $(\bar{s},\bar{\vartheta})$ and $(\bar{y},\bar{z};\bar{\mu},\bar{v},\bar{\gamma},\bar{\varpi},\bar{a},\bar{b})$ to the problem $(\mathcal{MRCP})$ and (MW-MRCP), respectively, we have

$$\int_\Delta \phi_\kappa(\bar{\pi},\bar{\varpi})dt^\kappa \geqq \int_\Delta \phi_\kappa(\bar{\eta},\bar{\varpi})dt^\kappa,$$

which contradicts the inequality (26). This completes the proof. □

Theorem 6. (*Robust Strong Duality*) *Let $(\bar{s},\bar{\vartheta})$ be a robust weak efficient solution to the problem $(\mathcal{MRCP})$. Assume that* $\max_{w\in W}\phi_\kappa(\bar{\pi},w) = \phi_\kappa(\bar{\pi},\bar{\varpi})$ *and the constraint conditions (for the existence of multiplier) hold for $(\mathcal{MRCP})$. Then, there exist the scalar vector $\bar{\mu}\in\mathbb{R}^s_+$, the piecewise smooth Lagrange multipliers $\bar{v} = (\bar{v}_l(t)) \in \mathbb{R}^m_+$ and $\bar{\gamma} = (\bar{\gamma}_\zeta(t)) \in \mathbb{R}^n$, and the* *uncertain parameters* $\bar{a} \in A,\ \bar{b} \in B$ *such that* $(\bar{s}, \bar{\vartheta}; \bar{\mu}, \bar{\nu}, \bar{\gamma}, \bar{w}, \bar{a}, \bar{b})$ *is a robust feasible solution to the problem* (MW-MRCP). *Further, if the Robust Weak Duality* (*see Theorem* 5) *holds, then* $(\bar{s}, \bar{\vartheta}; \bar{\mu}, \bar{\nu}, \bar{\gamma}, \bar{w}, \bar{a}, \bar{b})$ *is a robust weak efficient solution to the problem* (MW-MRCP).

Proof. Since $(\bar{s}, \bar{\vartheta})$ is a robust weak efficient solution to the problem $(\mathcal{MRCP})$, by Theorem 1, there exist the scalar vector $\bar{\mu} \in \mathbb{R}^s_+$, the piecewise smooth Lagrange multipliers $\bar{\nu} \in \mathbb{R}^m_+,\ \bar{\gamma} \in \mathbb{R}^n$, and the uncertain $\bar{a} \in A,\ \bar{b} \in B$ such that the conditions (1)–(4) are satisfied at $(\bar{s}, \bar{\vartheta})$. This implies the robust feasibility of $(\bar{s}, \bar{\vartheta}; \bar{\mu}, \bar{\nu}, \bar{\gamma}, \bar{w}, \bar{a}, \bar{b})$ to the problem (MW-MRCP) and the corresponding objective values are equal. If $(\bar{s}, \bar{\vartheta}; \bar{\mu}, \bar{\nu}, \bar{\gamma}, \bar{w}, \bar{a}, \bar{b})$ is not a robust weak efficient solution to the problem (MW-MRCP), then there exists another point $(\bar{y}, \bar{z}; \bar{\mu}, \bar{\nu}, \bar{\gamma}, \bar{w}, \bar{a}, \bar{b})$ such that

$$\int_\Delta \max_{w \in W} \phi_\kappa(\bar{\pi}, w) dt^\kappa < \int_\Delta \phi_\kappa(\bar{\eta}, \bar{w}) dt^\kappa,$$

which contradicts the Robust Weak Duality (see Theorem 5). Hence, $(\bar{s}, \bar{\vartheta}; \bar{\mu}, \bar{\nu}, \bar{\gamma}, \bar{w}, \bar{a}, \bar{b})$ is a robust weak efficient solution to the problem (MW-MRCP). □

5. Robust Duality of Mixed Type

In this section, we formulate the mixed type robust dual problem for the multi-objective robust nonlinear control problem $(\mathcal{MRCP})$ as follows:

$$(\mathcal{M} - \mathcal{MRCP}) \quad \max_{(y(\cdot), z(\cdot))} \int_\Delta \Big\{ \phi_\kappa(\eta, w) + \nu^T \varphi(\eta, a) e + \gamma^T \chi(\eta, b) e \Big\} dt^\kappa$$

$$\begin{aligned}
\text{subject to} \quad & \mu^T \frac{\partial \phi_\kappa}{\partial s}(\eta, w) + \nu^T \varphi_s(\eta, a) + \gamma^T \chi_s(\eta, b) \\
& - D_\sigma \Big[\mu^T \frac{\partial \phi_\kappa}{\partial s_\sigma}(\eta, w) + \nu^T \varphi_{s_\sigma}(\eta, a) + \gamma^T \chi_{s_\sigma}(\eta, b) \Big] \\
& + \frac{1}{n(\alpha, \beta)} D^2_{\alpha\beta} \Big[\mu^T \frac{\partial \phi_\kappa}{\partial s_{\alpha\beta}}(\eta, w) + \nu^T \varphi_{s_{\alpha\beta}}(\eta, a) + \gamma^T \chi_{s_{\alpha\beta}}(\eta, b) \Big] = 0, && (30) \\
& \mu^T \frac{\partial \phi_\kappa}{\partial \vartheta}(\eta, w) + \nu^T \varphi_\vartheta(\eta, a) + \gamma^T \chi_\vartheta(\eta, b) = 0, \quad \kappa = \overline{1, p}, && (31) \\
& \bar{\nu}^T \varphi(\eta, a) \geq 0, && (32) \\
& \chi(\eta, b) = 0, && (33) \\
& y(t_0) = s_0,\ y(t_1) = s_1,\ y_\sigma(t_0) = s_{\sigma 0},\ y_\sigma(t_1) = s_{\sigma 1}, && (34) \\
& \mu \in \mathbb{R}^s_+,\ \mu^T e = 1,\ \nu \in \mathbb{R}^m_+,\ \gamma \in \mathbb{R}^n. && (35)
\end{aligned}$$

The associated robust counterpart to the problem $(\mathcal{M} - \mathcal{MRCP})$ is given as follows:

$$(\mathcal{RM} - \mathcal{MRCP}) \quad \max_{(y(\cdot), z(\cdot), w, a, b)} \int_\Delta \Big\{ \phi_\kappa(\eta, w) + \nu^T \varphi(\eta, a) e + \gamma^T \chi(\eta, b) e \Big\} dt^\kappa$$

$$\begin{aligned}
\text{subject to} \quad & \mu^T \frac{\partial \phi_\kappa}{\partial s}(\eta, w) + \nu^T \varphi_s(\eta, a) + \gamma^T \chi_s(\eta, b) \\
& - D_\sigma \Big[\mu^T \frac{\partial \phi_\kappa}{\partial s_\sigma}(\eta, w) + \nu^T \varphi_{s_\sigma}(\eta, a) + \gamma^T \chi_{s_\sigma}(\eta, b) \Big] \\
& + \frac{1}{n(\alpha, \beta)} D^2_{\alpha\beta} \Big[\mu^T \frac{\partial \phi_\kappa}{\partial s_{\alpha\beta}}(\eta, w) + \nu^T \varphi_{s_{\alpha\beta}}(\eta, a) + \gamma^T \chi_{s_{\alpha\beta}}(\eta, b) \Big] = 0, \\
& \mu^T \frac{\partial \phi_\kappa}{\partial \vartheta}(\eta, w) + \nu^T \varphi_\vartheta(\eta, a) + \gamma^T \chi_\vartheta(\eta, b) = 0, \quad \kappa = \overline{1, p}, \\
& \bar{\nu}^T \varphi(\eta, a) \geq 0, \\
& \chi(\eta, b) = 0, \\
& y(t_0) = s_0,\ y(t_1) = s_1,\ y_\sigma(t_0) = s_{\sigma 0},\ y_\sigma(t_1) = s_{\sigma 1}, \\
& \mu \in \mathbb{R}^s_+,\ \mu^T e = 1,\ \nu \in \mathbb{R}^m_+,\ \gamma \in \mathbb{R}^n,
\end{aligned}$$

for all $w \in W,\ a \in A,\ b \in B$.

We denote by $D_m = \{(y, z; \mu, \nu, \gamma, w, a, b) :$ satisfying conditions (30)–(35)$\}$ the set of all feasible solutions to $(\mathcal{RM} - \mathcal{MRCP})$ and we say that it is the *robust feasible solution set* to the problem $(\mathcal{M} - \mathcal{MRCP})$.

Theorem 7. (*Robust Weak Duality*) *Let* $(\bar{s}, \bar{\vartheta})$ *and* $(\bar{y}, \bar{z}; \bar{\mu}, \bar{\nu}, \bar{\gamma}, \bar{w}, \bar{a}, \bar{b})$ *be robust feasible solutions to the problem* $(\mathcal{MRCP})$ *and* $(\mathcal{M} - \mathcal{MRCP})$*, respectively. Furthermore, we ssume that* $\max_{w \in W} \phi_\kappa(\bar{\pi}, w) = \phi_\kappa(\bar{\pi}, \bar{w})$*, and* $\int_\Delta \bar{\mu}^T \phi_\kappa(\cdot, \bar{w}) dt^\kappa$*,* $\int_\Delta \bar{\nu}^T \varphi(\cdot, \bar{a}) dt^\kappa$ *and* $\int_\Delta \bar{\gamma}^T \chi(\cdot, \bar{b}) dt^\kappa$ *are convex at* $(\bar{y}, \bar{z})$*. Then the following inequality cannot be valid*

$$\int_\Delta \max_{w \in W} \phi_\kappa(\bar{\pi}, w) dt^\kappa < \int_\Delta \Big\{ \phi_\kappa(\bar{\eta}, \bar{w}) + \bar{\nu}^T \varphi(\bar{\eta}, \bar{a}) e + \bar{\gamma}^T \chi(\bar{\eta}, \bar{b}) e \Big\} dt^\kappa.$$

Proof. The proof follows in the same manner as in Theorem 2. Consequently, we omit it. □

Theorem 8. (*Robust Strong Duality*) *Let* $(\bar{s}, \bar{\vartheta})$ *be a robust weak efficient solution to the problem* $(\mathcal{MRCP})$*. Assume that* $\max_{w \in W} \phi_\kappa(\bar{\pi}, w) = \phi_\kappa(\bar{\pi}, \bar{w})$ *and the constraint conditions (for the existence of multiplier) hold for* $(\mathcal{MRCP})$*. Then, there exist the scalar vector* $\bar{\mu} \in \mathbb{R}^s_+$*, the piecewise smooth Lagrange multipliers* $\bar{\nu} = (\bar{\nu}_l(t)) \in \mathbb{R}^m_+$ *and* $\bar{\gamma} = (\bar{\gamma}_\zeta(t)) \in \mathbb{R}^n$*, and the uncertain parameters* $\bar{a} \in A,\ \bar{b} \in B$ *such that* $(\bar{s}, \bar{\vartheta}; \bar{\mu}, \bar{\nu}, \bar{\gamma}, \bar{w}, \bar{a}, \bar{b})$ *is a robust feasible solution to the problem* $(\mathcal{M} - \mathcal{MRCP})$*. Further, if the Robust Weak Duality* (*see Theorem 7*) *holds, then* $(\bar{s}, \bar{\vartheta}; \bar{\mu}, \bar{\nu}, \bar{\gamma}, \bar{w}, \bar{a}, \bar{b})$ *is a robust weak efficient solution to the problem* $(\mathcal{M} - \mathcal{MRCP})$*.*

Proof. The proof follows in the same manner as in Theorem 3. As consequence, we skip it. □

In the following, we present an illustrative application to validate, for example, Theorem 2. The next concrete problem can be solved exclusively by using the theoretical results derived in this paper.

Example 1. *Let us extremize the mechanical work provided by the variable forces*

$$\bar{V}_1\left(s^2 w_1^1 + \frac{5}{4}\vartheta,\ s^2 w_2^1 + \frac{5}{4}\vartheta\right), \quad \bar{V}_2\left(\frac{\vartheta^2}{w_1^2}, \frac{\vartheta^2}{w_2^2}\right),$$

with (uncertain parameters) $w_\kappa^1 \in [0,1],\ w_\kappa^2 \in \left[\frac{1}{2}, 1\right],\ \kappa = 1,2$*, to move its application point along the piecewise smooth curve* Δ*, that is included in* $K = \left[0, \frac{1}{4}\right]^2 = \left[0, \frac{1}{4}\right] \times \left[0, \frac{1}{4}\right]$ *and joins* $t_0 = (0,0)$ *and* $t_1 = \left(\frac{1}{4}, \frac{1}{4}\right)$*, so that*

$$\begin{aligned}
& as(s-3) \leq 0, \\
& \frac{\partial s}{\partial t^1} = b_1 - \vartheta, \\
& \frac{\partial s}{\partial t^2} = b_2 - \vartheta, \\
& s(0,0) = 0,\ s\left(\frac{1}{4}, \frac{1}{4}\right) = \frac{1}{2},
\end{aligned}$$

(with $t = (t^1, t^2) \in K = \left[0, \frac{1}{4}\right]^2$*, for all* $w = (w_1^1, w_2^1, w_1^2, w_2^2) \in W = W_1^1 \times W_2^1 \times W_1^2 \times W_2^2 = [0,1]^2 \times [\frac{1}{2}, 1]^2,\ a \in A = [0,1],\ b = (b_1, b_2) \in B = B_1 \times B_2 = [1,2]^2$*) is satisfied.*

In order to solve the above practical problem, let us consider $p = 2,\ q = r = 1,\ s = 2,\ W_1^1 = W_2^1 = [0,1],\ W_1^2 = W_2^2 = [\frac{1}{2},1],\ A = [0,1],\ B_1 = B_2 = [1,2]$ *and* $K = \left[0, \dfrac{1}{4}\right] \times \left[0, \dfrac{1}{4}\right]$ *is fixed by the diagonally opposite points* $t_0 = (t_0^1, t_0^2) = (0,0)$ *and* $t_1 = (t_1^1, t_1^2) = (\frac{1}{4}, \frac{1}{4})$ *in* $\mathbb{R}^2$. *Now, we formulate the following constrained multi-objective robust nonlinear control problem:*

$$
\begin{aligned}
\text{(P)} \quad & \min_{(s(\cdot),\vartheta(\cdot))} \left(\int_\Delta (s^2 w_1^1 + \frac{5}{4}\vartheta)\, dt^1 + (s^2 w_2^1 + \frac{5}{4}\vartheta)\, dt^2,\ \int_\Delta \frac{\vartheta^2}{w_1^2}\, dt^1 + \frac{\vartheta^2}{w_2^2}\, dt^2 \right) \\
\text{subject to} \quad & as(s-3) \leq 0, \\
& \frac{\partial s}{\partial t^1} = b_1 - \vartheta, \\
& \frac{\partial s}{\partial t^2} = b_2 - \vartheta, \\
& s(0,0) = 0,\ s\Big(\frac{1}{4}, \frac{1}{4}\Big) = \frac{1}{2},
\end{aligned}
$$

where $t = (t^1, t^2) \in K$.

Let $(\bar{s}, \bar{\vartheta}) = (t^1 + t^2,\ 1)$ *be a robust feasible solution to the problem* (P).

The robust counterpart of (P) *is defined as:*

$$
\begin{aligned}
\text{(RP)} \quad & \min_{(s(\cdot),\vartheta(\cdot))} \Big(\int_\Delta \max_{w_1^1 \in W_1^1} \Big(s^2 w_1^1 + \frac{5}{4}\vartheta\Big)\, dt^1 + \max_{w_2^1 \in W_2^1} \Big(s^2 w_2^1 + \frac{5}{4}\vartheta\Big)\, dt^2, \\
& \int_\Delta \max_{w_1^2 \in W_1^2} \frac{\vartheta^2}{w_1^2}\, dt^1 + \max_{w_2^2 \in W_2^2} \frac{\vartheta^2}{w_2^2}\, dt^2 \Big) \\
\text{subject to} \quad & as(s-3) \leq 0,\ \forall a \in A, \\
& \frac{\partial s}{\partial t^1} = b_1 - \vartheta,\ \forall b_1 \in B_1, \\
& \frac{\partial s}{\partial t^2} = b_2 - \vartheta,\ \forall b_2 \in B_2, \\
& s(0,0) = 0,\ s\Big(\frac{1}{4}, \frac{1}{4}\Big) = \frac{1}{2},
\end{aligned}
$$

where $t = (t^1, t^2) \in K$.

The Wolfe type robust dual problem associated with (P) *is defined as follows:*

$$
\begin{aligned}
\text{(WP)} \quad & \max_{(y(\cdot),z(\cdot))} \Big(\int_\Delta \Big\{ (y^2 w_1 + \frac{5}{4}z) + \nu a y(y-3) + \gamma_1(\frac{\partial y}{\partial t^1} - b_1 + z) \\
& \quad + \gamma_2(\frac{\partial y}{\partial t^2} - b_2 + z) \Big\}\, (dt^1 + dt^2),\ \int_\Delta \Big\{ \frac{z^2}{w_2} + \nu a y(y-3) \\
& \quad + \gamma_1(\frac{\partial y}{\partial t^1} - b_1 + z) + \gamma_2(\frac{\partial y}{\partial t^2} - b_2 + z) \Big\}\, (dt^1 + dt^2) \Big)
\end{aligned}
$$

$$
\text{subject to} \quad 2\mu_1 w_1 y + \nu a(2y-3) - \frac{\partial \gamma_1}{\partial t^1} - \frac{\partial \gamma_2}{\partial t^2} = 0, \tag{36}
$$

$$
\frac{5}{4}\mu_1 + \frac{2z}{w_2}\mu_2 + \gamma_1 + \gamma_2 = 0, \tag{37}
$$

$$
y(0,0) = 0,\ y\Big(\frac{1}{4}, \frac{1}{4}\Big) = \frac{1}{2}, \tag{38}
$$

$$
\mu^T > 0,\ e^T\mu = 1,\ e = (1,1) \in \mathbb{R}^2, \tag{39}
$$

where we denoted $w_1 := w_1^1 (= w_2^1)$ *and* $w_2 := w_1^2 (= w_2^2)$.

The robust counterpart to the problem (WP) *is given as:*

$$
\begin{aligned}
\text{(RWP)} \quad \max_{(y(\cdot),z(\cdot),w,a,b)} & \Big(\int_\Delta \Big\{ (y^2 w_1 + \frac{5}{4} z) + \nu a y(y-3) + \gamma_1 (\frac{\partial y}{\partial t^1} - b_1 + z) \\
& + \gamma_2 (\frac{\partial y}{\partial t^2} - b_2 + z) \Big\} \, (dt^1 + dt^2), \int_\Delta \Big\{ \frac{z^2}{w_2} + \nu a y(y-3) \\
& + \gamma_1 (\frac{\partial y}{\partial t^1} - b_1 + z) + \gamma_2 (\frac{\partial y}{\partial t^2} - b_2 + z) \Big\} \, (dt^1 + dt^2) \Big) \\
\text{subject to} \quad & 2\mu_1 w_1 y + \nu a (2y-3) - \frac{\partial \gamma_1}{\partial t^1} - \frac{\partial \gamma_2}{\partial t^2} = 0, \\
& \frac{5}{4}\mu_1 + \frac{2z}{w_2}\mu_2 + \gamma_1 + \gamma_2 = 0, \\
& y(0,0) = 0, \ y\Big(\frac{1}{4}, \frac{1}{4}\Big) = \frac{1}{2}, \\
& \mu^T > 0, \ e^T \mu = 1, \ e = (1,1) \in \mathbb{R}^2,
\end{aligned}
$$

for all $w = (w_1, w_2) \in W = W_1 \times W_2 = [0,1] \times [\frac{1}{2}, 1]$, $a \in A = [0,1]$, $b = (b_1, b_2) \in B = B_1 \times B_2 = [1,2]^2$.

We note that $D_w = \{(y, z, \mu, \nu, \gamma, w, a, b)$ *satisfying conditions* (36)–(39)$\}$ *is the robust feasible solution set to the* (WP). *Let us consider* $\bar{y} = t^1 + t^2, \bar{z} = -t^1 t^2 - \frac{5}{16}$, $\bar{\mu} = (\bar{\mu}_1, \bar{\mu}_2) = (\frac{1}{2}, \frac{1}{2})$, $\bar{\nu} = 0$, $\bar{\gamma} = (\bar{\gamma}_1, \bar{\gamma}_2) = (t^1 t^2, t^1 t^2)$, $\bar{w} = (\bar{w}_1, \bar{w}_2) = (1, \frac{1}{2})$, $\bar{a} = 1$, $\bar{b} = (\bar{b}_1, \bar{b}_2) = (2, 2)$. *Then* $(\bar{y}, \bar{z}, \bar{\mu}, \bar{\nu}, \bar{\gamma}, \bar{w}, \bar{a}, \bar{b})$ *is a robust feasible solution to* (WP). *Further, it can be easily verified that all the involved functionals are convex at* $(\bar{y}, \bar{z})$. *Furthermore, the following inequality*

$$
\begin{aligned}
& \int_\Delta \phi_\kappa(\bar{\pi}, \bar{w}) \, dt^\kappa - \int_\Delta \Big\{ \phi_\kappa(\bar{\eta}, \bar{w}) + \bar{\nu}^T \varphi(\bar{\eta}, \bar{a}) e + \bar{\gamma}^T \chi(\bar{\eta}, \bar{b}) e \Big\} \, dt^\kappa \\
& = \int_\Delta \Big((\bar{s}^2 \bar{w}_1 + \frac{5}{4}\bar{\vartheta}) \, (dt^1 + dt^2), \ \frac{\bar{\vartheta}^2}{\bar{w}_2} \, (dt^1 + dt^2) \Big) - \int_\Delta \Big(((\bar{y}^2 \bar{w}_1 + \frac{5}{4}\bar{z}) + \bar{\nu} \bar{a} \bar{y}(\bar{y} - 3) \\
& \quad + \bar{\gamma}_1 (\frac{\partial \bar{y}}{\partial t^1} - \bar{b}_1 + \bar{z}) + \bar{\gamma}_2 (\frac{\partial \bar{y}}{\partial t^2} - \bar{b}_2 + \bar{z})) \, (dt^1 + dt^2), \ (\frac{\bar{z}^2}{\bar{w}_2} + \bar{\nu} \bar{a} \bar{y}(\bar{y} - 3) \\
& \quad + \bar{\gamma}_1 (\frac{\partial \bar{y}}{\partial t^1} - \bar{b}_1 + \bar{z}) + \bar{\gamma}_2 (\frac{\partial \bar{y}}{\partial t^2} - \bar{b}_2 + \bar{z})) \, (dt^1 + dt^2) \Big) \\
& = \int_\Delta \Big((t^1 + t^2 + \frac{5}{4}) \, (dt^1 + dt^2), \ 2 \, (dt^1 + dt^2) \Big) \\
& - \int_\Delta \Big((t^1 + t^2 - \frac{5}{4} t^1 t^2 - \frac{25}{64} - 2(t^1)^2 (t^2)^2 - \frac{21}{8} t^1 t^2) \, (dt^1 + dt^2), \\
& (2(t^1)^2 (t^2)^2 + \frac{5}{4} t^1 t^2 + \frac{25}{128} - 2(t^1)^2 (t^2)^2 - \frac{21}{8} t^1 t^2) \, (dt^1 + dt^2) \Big) \\
& = \int_\Delta \Big((t^1 + t^2 + \frac{5}{4}) \, (dt^1 + dt^2), \ 2 \, (dt^1 + dt^2) \Big) \\
& - \int_\Delta \Big((t^1 + t^2 - \frac{31}{8} t^1 t^2 - \frac{25}{64} - 2(t^1)^2 (t^2)^2) \, (dt^1 + dt^2), \ (\frac{25}{128} - \frac{11}{8} t^1 t^2) \, (dt^1 + dt^2) \Big) \\
& = \Big(\frac{5}{8}, 1 \Big) - \Big(\frac{-1033}{11520}, \frac{1}{12} \Big) \\
& = \Big(\frac{8233}{11520}, \frac{11}{12} \Big) > 0,
\end{aligned}
$$

shows that the duality gap is positive. In consequence, Theorem 2 (Robust Weak Duality) is verified.

6. Conclusions

In the current study, we have established various duality results for the new class of constrained robust nonlinear optimization problems ($\mathcal{MRCP}$). More concretely, we have established and characterized Wolfe, Mond-Weir and mixed type robust dual optimization problems. In addition, an illustrative real-life application was included in the paper in order to validate the theoretical elements. On the other hand, as a possible research line that this study can open (among many other aspects), is the formulating of the derived results by considering the concept of *variational/functional derivative*.

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