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Geometry of Tangent Poisson–Lie Groups

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Abstract: Let G be a Poisson–Lie group equipped with a left invariant contravariant pseudo-Riemannian metric. There are many ways to lift the Poisson structure on G to the tangent bundle TG of G. In this paper, we induce a left invariant contravariant pseudo-Riemannian metric on the tangent bundle TG, and we express in different cases the contravariant Levi-Civita connection and curvature of TG in terms of the contravariant Levi-Civita connection and the curvature of G. We prove that the space of differential forms $\Omega^*(G)$ on G is a differential graded Poisson algebra if, and only if, $\Omega^*(TG)$ is a differential graded Poisson algebra. Moreover, we show that G is a pseudo-Riemannian Poisson–Lie group if, and only if, the Sanchez de Alvarez tangent Poisson–Lie group TG is also a pseudo-Riemannian Poisson–Lie group. Finally, some examples of pseudo-Riemannian tangent Poisson–Lie groups are given.

Keywords: Poisson geometry; Riemannian geometry; Lie group; Lie algebra

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1. Introduction

The Riemannian geometry of tangent bundles and cotangent bundles of smooth manifolds is an important area in physics, classical mechanics and geometrical optics. If M is the configuration space of a mechanical system, then each point of the cotangent bundle T^*M of M determines a state of the system and T^*M is called the phase space [1]. Moreover, Poisson manifolds play a fundamental role in Hamiltonian dynamics, where they serve as a phase space. For this reason, there is some interest on how structures and, more generally, properties of M carry down to T^*M. Furthermore, if M is equipped with a pseudo-Riemannian metric compatible with the Poisson structure on M [2,3], it would be interesting to see if the compatibility remains fulfilled on the tangent bundle TM. First, recall that the notion of compatibility between a Poisson structure Π_M and a contravariant pseudo-Riemannian metric $\langle , \rangle^*_M$ on a smooth manifold M was first introduced by M.Boucetta in [2]. A triplet $(M, \Pi_M, \langle , \rangle^*_M)$ is compatible in the sense of M.Boucetta [2,4] and is a so-called pseudo-Riemannian Poisson manifold if, for any $\alpha, \beta, \gamma \in \Omega^1(M)$:

$$\mathcal{D}^M\Pi_M(\alpha,\beta,\gamma) = \Pi^\sharp_M(\alpha)\Pi_M(\beta,\gamma) - \Pi_M(\mathcal{D}^M_\alpha\beta,\gamma) - \Pi_M(\beta,\mathcal{D}^M_\alpha\gamma) = 0,$$

where $\mathcal{D}^M$ is the contravariant Levi-Civita connection associated with the couple $(\Pi_M, \langle , \rangle^*_M)$.

In [3,5], Hawkins showed that, if a deformation of the graded algebra $\Omega^*(M)$ of differential forms on a pseudo-Riemannian manifold $(M, \langle , \rangle_M)$ comes from a spectral triple describing the pseudo-Riemannian structure, then the Poisson tensor Π_M on M (which characterizes the deformation) and the pseudo-Riemannian metric $\langle , \rangle_M$ satisfy the following compatibility conditions:

(H_1) The metric contravariant connection $\mathcal{D}^M$ associated with $(\Pi_M, \langle , \rangle_M)$ is flat.

(H_2) The metacurvature $\mathcal{M}^M$ of $\mathcal{D}^M$ is zero, i.e., the connection $\mathcal{D}^M$ is metaflat.

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The metric contravariant connection $\mathcal{D}^M$ naturally associated with $(\Pi_M, \langle,\rangle_M)$ is exactly the Levi-Civita contravariant connection.

A triplet $(M, \Pi_M, \langle,\rangle_M)$ satisfying conditions H_1 and H_2 is said to be compatible in the sense of Hawkins. A deformation of the differential graded algebra of differential forms $\Omega^*(M)$ defines a generalized Poisson bracket on this space. Moreover, a generalized Poisson bracket making $\Omega^*(M)$ a differential graded Poisson algebra exists if, and only if, $(M, \Pi_M, \langle,\rangle_M)$ is compatible in the sense of Hawkins [3].

An important class of Poisson manifolds equipped with pseudo-Riemannian metrics is the family of Poisson–Lie groups equipped with left invariant pseudo-Riemannian metrics.

The notion of the Poisson–Lie group was first introduced by Drinfel'd [6,7] and Semenov–Tian–Shansky [8]. Semenov, Kosmann–Schwarzbach and Magri [9] used Poisson–Lie groups to understand the Hamiltonian structure of the group of dressing transformations of certain integrable systems. These Poisson–Lie groups play the role of symmetry groups.

In [10], M.Boumaiza and N.Zaalani showed that if (G, Π_G) is a Poisson–Lie group, then the tangent bundle (TG, Π_{TG}) of G, with its tangent Poisson structure Π_{TG} defined in the sense of Sanchez de Alvarez [11], is a Poisson–Lie group. This Poisson–Lie group (TG, Π_{TG}) is called a Sanchez de Alvarez tangent Poisson–Lie group of G [12].

The second author and N. Zaalani [12] have studied the compatibility between the Sanchez de Alvarez Poisson structure and the natural left invariant Riemannian metric. The non-compatibility between the Sanchez de Alvarez Poisson structure and the natural Riemannian metric (except in the trivial case $\Pi_G = 0$) on TG leads us to define another metric on the tangent Lie group TG which is compatible with the Sanchez de Alvarez Poisson structure.

In this paper, we equip G with a Poisson structure and a pseudo-Riemannian metric. Then, we lift these structures on the tangent bundle TG of G, and we study the Riemannian geometry of G and its relations with the geometry of TG.

This paper is organized as follows: In Section 2, we recall basic definitions and facts about contravariant connections, curvatures, metacurvatures, generalized Poisson brackets and pseudo-Riemannian Poisson–Lie groups. In Section 3, we induce a left invariant contravariant pseudo-Riemannian metric $\langle,\rangle^*_{TG}$ on the tangent Poisson–Lie group (TG, Π_{TG}) and we express in different cases the Levi-Civita connection and curvature of $(TG, \Pi_{TG}, \langle,\rangle^*_{TG})$ in terms of the Levi-Civita connection and curvature of $(G, \Pi_G, \langle,\rangle^*_G)$. In the case where the tangent bundle TG is equipped with the Sanchez de Alvarez Poisson structure, we show that the space of differential forms $\Omega^*(TG)$ on TG is a differential graded Poisson algebra if, and only if, $\Omega^*(G)$ is a differential graded Poisson algebra. In Section 4, we show that $(G, \Pi_G, \langle,\rangle^*_G)$ is a pseudo-Riemannian Poisson–Lie group if, and only if, the Sanchez de Alvarez tangent Poisson–Lie group $(TG, \Pi_{TG}, \langle,\rangle^*_{TG})$ is also a pseudo-Riemannian Poisson–Lie group. In Section 5, we give some examples of pseudo-Riemannian tangent Poisson–Lie groups .

2. Preliminaries

2.1. Contravariant Connections and Curvatures

Contravariant connections on Poisson manifolds were defined by Vaisman [13] and studied in detail by Fernandes [14]. This notion appears extensively in the context of noncommutative deformations [3,5].

Let (M, Π_M) be a Poisson manifold. We associate the Poisson tensor Π_M with the anchor map $\Pi_M^\sharp : T^*M \to TM$ defined by $\beta(\Pi_M^\sharp(\alpha)) = \Pi_M(\alpha, \beta)$ and the Koszul bracket $[,]_M$ on the space of differential 1-forms $\Omega^1(M)$ given by:

$$[\alpha, \beta]_M \quad = \quad \mathcal{L}_{\Pi_M^\sharp(\alpha)}\beta - \mathcal{L}_{\Pi_M^\sharp(\beta)}\alpha - d(\Pi_M(\alpha, \beta)).$$

A contravariant connection on M, with respect to Π_M, is an $\mathbb{R}$-bilinear map

$$\mathcal{D}^M : \Omega^1(M) \times \Omega^1(M) \to \Omega^1(M), \quad (\alpha, \beta) \mapsto \mathcal{D}^M_\alpha \beta,$$

such that for all $f \in \mathcal{C}^\infty(M)$,

$$\mathcal{D}^M_{f\alpha}\beta = f\mathcal{D}^M_\alpha \beta \text{ and } \mathcal{D}^M_\alpha(f\beta) = f\mathcal{D}^M_\alpha \beta + \Pi^\sharp_M(\alpha)(f)\beta.$$

The torsion $\mathcal{T}^M$ and the curvature R^M of a contravariant connection $\mathcal{D}^M$ are formally identical to the usual ones:

$$\mathcal{T}^M(\alpha, \beta) = \mathcal{D}^M_\alpha \beta - \mathcal{D}^M_\beta \alpha - [\alpha, \beta]_M$$

$$R^M(\alpha, \beta)\gamma = \mathcal{D}^M_\alpha \mathcal{D}^M_\beta \gamma - \mathcal{D}^M_\beta \mathcal{D}^M_\alpha \gamma - \mathcal{D}^M_{[\alpha,\beta]_M}\gamma. \quad (1)$$

These are (2,1) and (3,1)-type tensor fields, respectively. When $\mathcal{T}^M \equiv 0$ (resp., $R^M \equiv 0$), $\mathcal{D}^M$ is called torsion-free (resp., flat).

Let (M, Π_M) be a Poisson manifold. Let $\langle , \rangle_M$ be a covariant Riemannian metric on M and $\langle , \rangle^*_M$ the contravariant Riemannian metric associated with $\langle , \rangle_M$. The metric contravariant connection associated with $(\Pi_M, \langle , \rangle^*_M)$ is the unique contravariant connection $\mathcal{D}^M$ such that $\mathcal{D}^M$ is torsion-free and the metric $\langle , \rangle^*_M$ is parallel with respect to $\mathcal{D}^M$, i.e.,

$$\Pi^\sharp_M(\alpha)\langle \beta, \gamma\rangle^*_M = \langle \mathcal{D}^M_\alpha \beta, \gamma\rangle^*_M + \langle \beta, \mathcal{D}^M_\alpha \gamma\rangle^*_M. \quad (2)$$

The connection $\mathcal{D}^M$ is called the Levi-Civita contravariant connection associated with $(\Pi_M, \langle , \rangle^*_M)$ and can be defined by the Koszul formula:

$$\begin{aligned} 2\langle \mathcal{D}^M_\alpha \beta, \gamma\rangle^*_M &= \Pi^\sharp_M(\alpha)\langle \beta, \gamma\rangle^*_M + \Pi^\sharp_M(\beta)\langle \alpha, \gamma\rangle^*_M - \Pi^\sharp_M(\gamma)\langle \alpha, \beta\rangle^*_M \\ &+ \langle [\alpha, \beta]_M, \gamma\rangle^*_M + \langle [\gamma, \alpha]_M, \beta\rangle^*_M + \langle [\gamma, \beta]_M, \alpha\rangle^*_M. \end{aligned} \quad (3)$$

We say that $\mathcal{D}^M$ is locally symmetric if $\mathcal{D}^M R^M = 0$, i.e., if for any $\alpha, \beta, \gamma, \delta \in \Omega^1(M)$, we have:

$$\begin{aligned} (\mathcal{D}^M_\alpha R^M)(\beta, \gamma)\delta &:= \mathcal{D}^M_\alpha(R^M(\beta, \gamma)\delta) - R^M(\mathcal{D}^M_\alpha \beta, \gamma)\delta - R^M(\beta, \gamma)\mathcal{D}^M_\alpha \delta \\ &- R^M(\beta, \mathcal{D}^M_\alpha \gamma)\delta = 0. \end{aligned} \quad (4)$$

2.2. Generalized Poisson Bracket on the Space of Differential forms $\Omega^*(M)$

Let (M, Π_M) be a Poisson manifold and $\mathcal{D}^M$ a torsion-free and flat connection with respect to Π_M. In [3], E.Hawkins showed that such a connection defines an $\mathbb{R}$-bilinear bracket on the space of differential forms $\Omega^*(M)$, also denoted by $\{ , \}_M$, such that :

1. The bracket $\{ , \}_M$ is antisymmetric, i.e.,

$$\{\sigma, v\}_M = -(-1)^{\deg(\sigma)\deg(v)}\{v, \sigma\}_M;$$

2. $\{ , \}_M$ satisfies the product rule, i.e.,

$$\{\sigma, v \wedge \nu\}_M = \{\sigma, v\}_M \wedge \nu + (-1)^{\deg(\sigma)\deg(v)} v \wedge \{\sigma, \nu\}_M;$$

3. The exterior differential d is a derivation with respect to $\{ , \}_M$, i.e.,

$$d\{\sigma, v\}_M = \{d\sigma, v\}_M + (-1)^{\deg(\sigma)}\{\sigma, dv\}_M;$$

4. For any $f_1, f_2 \in \mathcal{C}^\infty(M)$ and for any $\sigma \in \Omega^*(M)$, the bracket $\{f_1, f_2\}_M$ coincides with the initial Poisson bracket on M and

$$\{f_1, \sigma\}_M = \mathcal{D}^M_{df}\sigma.$$

This bracket is given for any $\alpha, \beta \in \Omega^1(M)$ by [15]:

$$\{\alpha, \beta\}_M = -\mathcal{D}^M_\alpha d\beta - \mathcal{D}^M_\beta d\alpha + d\mathcal{D}^M_\beta \alpha + [\alpha, d\beta]_M, \tag{5}$$

where $[,]_M$ is the generalized Koszul bracket on $\Omega^*(M)$ satisfying the Leibnuz identity, i.e.,

$$[\sigma, v \wedge \nu]_M = [\sigma, v]_M \wedge \nu + (-1)^{(\deg(\sigma)-1)\deg(v)} v \wedge [\sigma, \nu]_M. \tag{6}$$

Note that the generalized Koszul bracket for the differential forms is analogous to the Schouten–Nijenhuis bracket for the multivector fields (for more details, see [16] page 44).

We call this bracket $\{,\}_M$ a generalized pre-Poisson bracket associated with the contravariant connection $\mathcal{D}^M$. E.Hawkins showed that there exists a (2,3) tensor $\mathcal{M}^M$ that is symmetrical in the contravariant indices and antisymmetrical in the covariant indices such that the generalized pre-Poisson bracket satisfies the graded Jacobi identity

$$\{\sigma, \{v, \nu\}_M\}_M - \{\{\sigma, v\}_M, \nu\}_M - (-1)^{\deg(\sigma)\deg(v)}\{v, \{\sigma, \nu\}_M\}_M = 0,$$

if, and only if, $\mathcal{M}^M$ is identically zero.

$\mathcal{M}^M$ is called metacurvature of $\mathcal{D}^M$ and is given by

$$\mathcal{M}^M(df_1, \alpha, \beta) = \{f_1, \{\alpha, \beta\}_M\}_M - \{\{f_1, \alpha\}_M, \beta\}_M - \{\{f_1, \beta\}_M, \alpha\}_M. \tag{7}$$

If $\mathcal{M}^M$ vanishes identically, the contravariant connection $\mathcal{D}^M$ is called metaflat and the bracket $\{,\}_M$ is called the generalized Poisson bracket associated with $\mathcal{D}^M$, making $\Omega^*(M)$ a differential graded Poisson algebra (for more details, see [3]).

2.3. Pseudo-Riemannian Poisson–Lie Group

An important class of Poisson manifolds is the family of Poisson–Lie groups. A Lie group G is called a Poisson–Lie group if it is also a Poisson manifold such that the product

$$m : G \times G \to G : (g, h) \mapsto gh$$

is a Poisson map, where $G \times G$ is equipped with the product Poisson structure.

Let G be a Poisson Lie group with Lie algebra $(\mathfrak{g}, [,]_{\mathfrak{g}})$ and Π_G the Poisson tensor on G. Pulling Π_G back to the identity element e of G by the left translations, we obtain a map $\Pi^l_G : G \to \mathfrak{g} \wedge \mathfrak{g}$, defined by $\Pi^l_G(g) = (L_{g^{-1}})_*\Pi_G(g)$, where $(L_g)_*$ denotes the tangent map of the left translation L_g of G by g. The intrinsic derivative

$$\xi := d_e\Pi^l_G : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$$

of Π^l_G at e is a 1-cocycle relative to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g}$. The dual map of ξ is a Lie bracket $[,]_{\mathfrak{g}^*} : \mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}^*$ on $\mathfrak{g}^*$. It is well-known that $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra.

Let (G, Π_G) be a Poisson–Lie group with Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$. Let $\langle,\rangle_{\mathfrak{g}^*}$ be a bilinear, symmetric and non-degenerate form on $\mathfrak{g}^*$ and let $\langle,\rangle^*_G$ be the contravariant pseudo-Riemannian given by $\langle,\rangle^*_G = (L_g)_*\langle,\rangle_{\mathfrak{g}^*}$. We say that $(G, \Pi_G, \langle,\rangle^*_G)$ is a pseudo-Riemannian Poisson–Lie group if, and only if, the Poisson tensor Π_G and the metric $\langle,\rangle^*_G$ are compatible in the sense given by M.Boucetta in [4,17], as follows:

$$[Ad^*_g(A_\alpha\gamma + ad^*_{\Pi^l_G(g)(\alpha)}\gamma), Ad^*_g(\beta)]_{\mathfrak{g}^*} + [Ad^*_g(\alpha), Ad^*_g(A_\beta\gamma + ad^*_{\Pi^l_G(g)(\beta)}\gamma)]_{\mathfrak{g}^*} = 0, \tag{8}$$

for any $g \in G$ and for any $\alpha, \beta, \gamma \in \mathfrak{g}^*$, where A is the infinitesimal Levi-Civita connection associated with $([,]_{\mathfrak{g}^*}, \langle,\rangle_{\mathfrak{g}^*})$.

Note that the infinitesimal Levi-Civita connection A is the restriction of the Levi-Civita contravariant connection $\mathcal{D}^G$ to $\mathfrak{g}^* \times \mathfrak{g}^*$ and is given for any $\alpha, \beta, \gamma \in \mathfrak{g}^*$, by:

$$2\langle A_\alpha\beta, \gamma\rangle_{\mathfrak{g}^*} = \langle[\alpha, \beta]_{\mathfrak{g}^*}, \gamma\rangle_{\mathfrak{g}^*} + \langle[\gamma, \alpha]_{\mathfrak{g}^*}, \beta\rangle_{\mathfrak{g}^*} + \langle[\gamma, \beta]_{\mathfrak{g}^*}, \alpha\rangle_{\mathfrak{g}^*}. \tag{9}$$

In [4], M.Boucetta showed that if $(G, \Pi_G, \langle , \rangle_G^*)$ is a pseudo-Riemannian Poisson–Lie group, then its dual Lie algebra $(\mathfrak{g}^*, [,]_{\mathfrak{g}^*}, \langle , \rangle_{\mathfrak{g}^*})$ equipped with the form $\langle , \rangle_{\mathfrak{g}^*}$ is a pseudo-Riemannian Lie algebra, i.e, for any $\alpha, \beta, \gamma \in \mathfrak{g}^*$, we have

$$[A_\alpha \beta, \gamma]_{\mathfrak{g}^*} + [\alpha, A_\gamma \beta]_{\mathfrak{g}^*} = 0. \tag{10}$$

3. Pseudo-Riemannian Geometry of Tangent Poisson–Lie Group

Let G be a n-dimensional Lie group with multiplication $m : G \times G \to G : (g, h) \mapsto gh$ and with Lie algebra $(\mathfrak{g}, [,]_{\mathfrak{g}})$. We denote by $L_g : G \to G : h \mapsto gh$, the left translation and $R_g : G \to G : h \mapsto hg$, the right translation of G by g.

The tangent map Tm of m,

$$Tm : TG \times TG \mapsto TG : (X_g, Y_h) \mapsto T_h L_g Y_h + T_g R_h X_g, \tag{11}$$

defines a Lie group structure on TG with identity element $(e, 0)$ and with Lie algebra the semi-direct product of Lie algebra $\mathfrak{g} \rtimes \mathfrak{g}$, with bracket [10,18]:

$$[(X, Y), (X', Y')]_{\mathfrak{g} \rtimes \mathfrak{g}} = ([X, X']_{\mathfrak{g}}, [X, Y']_{\mathfrak{g}} + [Y, X']_{\mathfrak{g}}), \tag{12}$$

where $(X, X'), (Y, Y') \in \mathfrak{g} \rtimes \mathfrak{g}$.

Let (G, Π_G) be a Poisson–Lie group with Lie bilagebra $(\mathfrak{g}, \mathfrak{g}^*)$ and let TG be the tangent bundle of G. According to M.Boumaiza and N.Zaalani [10], the tangent bundle TG of G with the multiplication (11) and with its tangent Poisson structure Π_{TG}, defined in the sense of Sanchez de Alvarez [11], is a Poisson–Lie group with Lie bialgebra $(\mathfrak{g} \rtimes \mathfrak{g}, \mathfrak{g}^* \ltimes \mathfrak{g}^*)$, where $\mathfrak{g}^* \ltimes \mathfrak{g}^*$ is the semi-direct product Lie algebra with bracket:

$$[(\alpha, \beta), (\alpha', \beta')]_{\mathfrak{g}^* \ltimes \mathfrak{g}^*} = ([\alpha, \beta']_{\mathfrak{g}^*} + [\beta, \alpha']_{\mathfrak{g}^*}, [\beta, \beta']_{\mathfrak{g}^*}), \tag{13}$$

where $(\alpha, \beta),\ (\alpha', \beta') \in \mathfrak{g}^* \times \mathfrak{g}^*$.

On the other hand, if (G, Π_G) is a Poisson–Lie group, there exists a linear Poisson structure $\Pi_{\mathfrak{g}}$ on $\mathfrak{g}$, whose value at $X \in \mathfrak{g}$ is given by $\Pi_{\mathfrak{g}}(X) = d_e \Pi_G(X)$. The linear Poisson structure $\Pi_{\mathfrak{g}}$ on $\mathfrak{g} = T_e G$ makes $(\mathfrak{g}, \Pi_{\mathfrak{g}})$ an abelian Poisson–Lie group with Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ such that the Lie bracket of $\mathfrak{g}$ is zero and the Lie bracket of $\mathfrak{g}^*$ is $[,]_{\mathfrak{g}^*}$.

If we identify the tangent bundle $TG \equiv G \times \mathfrak{g}$ with the direct product Poisson–Lie group of (G, Π_G) and $(\mathfrak{g}, \Pi_{\mathfrak{g}})$; then, $(TG \equiv G \times \mathfrak{g}, \Pi_{G \times \mathfrak{g}})$ is a Poisson–Lie group, with Lie-bialgebra $(\mathfrak{g} \times \mathfrak{g}, \mathfrak{g}^* \times \mathfrak{g}^*)$, where $\mathfrak{g} \times \mathfrak{g}$ is the direct product Lie algebra with bracket:

$$[(X, Y), (X', Y')]_{\mathfrak{g} \times \mathfrak{g}} = ([X, X']_{\mathfrak{g}}, 0), \quad (X, Y),\ (X', Y') \in \mathfrak{g} \times \mathfrak{g}, \tag{14}$$

and $\mathfrak{g}^* \times \mathfrak{g}^*$ is the direct product Lie algebra with bracket:

$$[(\alpha, \beta), (\alpha', \beta')]_{\mathfrak{g}^* \times \mathfrak{g}^*} = ([\alpha, \alpha']_{\mathfrak{g}^*}, [\beta, \beta']_{\mathfrak{g}^*}), \ \ (\alpha, \beta),\ (\alpha', \beta') \in \mathfrak{g}^* \times \mathfrak{g}^*. \tag{15}$$

Now, we equip G with a left invariant pseudo-Riemannian metric, and we lift this metric to the tangent bundle TG.

Let $\pi : TG \to G : (g, X) \mapsto g$, be the natural projection. The differential mapping $d\pi(e, 0)$ at the point $(e, 0)$ is given by:

$$d\pi(e, 0) : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : (X, Y) \mapsto X,$$

and the vertical subspace $\mathcal{V}_{(e,0)}$ of $\mathfrak{g} \times \mathfrak{g}$ is given by $\mathcal{V}_{(e,0)} = \ker(d\pi(e, 0)) = \{0\} \times \mathfrak{g}$.

It has been shown that the complete and vertical lifts of any left invariant vector fields of G are left invariant fields on the tangent Lie group TG (see proposition 1.3 page 183 of [19] or theorems 1.2.2 and 1.2.3 of [20]). In fact, if $(X_1, \dots, X_n)$ is a basis for the Lie algebra $\mathfrak{g}$ of G, then $\{X_1^v = (0, X_1), \dots, X_n^v = (0, X_n), X_1^c = (X_1, 0), \dots, X_n^c = (X_n, 0)\}$ is a basis for the Lie algebra $\mathfrak{g} \times \mathfrak{g}$ of TG, where $X_1^v = (0, X_1)$ (resp., $X_1^c = (X_1, 0)$) is the vertical lift (resp., the complete lift) of the vector field X_1 on G to TG.

Let $\langle , \rangle_G$ be a left invariant pseudo-Riemannian metric on G. Then, we define a left invariant pseudo-Riemannian metric $\langle , \rangle_{TG}$ on TG as follows :

$$\begin{array}{rcl} \langle (0,Y),(0,Y') \rangle_{TG}(e,0) & = & 0, \\ \langle (X,0),(0,Y') \rangle_{TG}(e,0) & = & \langle X,Y' \rangle_G(e) \\ \langle (X,0),(X',0) \rangle_{TG}(e,0) & = & \langle X,X' \rangle_G(e), \end{array} \tag{16}$$

where $(X,Y),(X',Y') \in \mathfrak{g} \times \mathfrak{g}$.

The left invariant contravariant pseudo-Riemannian metric $\langle , \rangle_{TG}^*$ on TG associated with $\langle , \rangle_{TG}$ is given for any $(\alpha,\beta),(\alpha',\beta') \in \mathfrak{g}^* \times \mathfrak{g}^*$ by:

$$\begin{array}{rcl} \langle (\alpha,0),(\alpha',0) \rangle_{TG}^*(e,0) & = & 0, \\ \langle (\alpha,0),(0,\beta') \rangle_{TG}^*(e,0) & = & \langle \alpha,\beta' \rangle_G^*(e) \\ \langle (0,\beta),(0,\beta') \rangle_{TG}^*(e,0) & = & \langle \beta,\beta' \rangle^*(e), \end{array} \tag{17}$$

where $\alpha^v = (\alpha, 0)$ (resp., $\alpha^c = (0, \alpha)$) is the vertical lift (resp., the complete lift) of the 1-form α on G to TG. (for more details on lift tensor fields, see [19]).

3.1. Pseudo-Riemannian Geometry of Product Poisson Structure on TG

In this subsection, we consider the left invariant contravariant pseudo-Riemannian metric $\langle , \rangle_{TG}^*$ defined as above on the tangent bundle $(TG, \Pi_{G\times\mathfrak{g}})$ equipped with the product Poisson structure. Then, we study the geometry of the triplet $(TG, \Pi_{G\times\mathfrak{g}}, \langle , \rangle_{TG}^*)$ and its relations with the geometry of $(G, \Pi_G, \langle , \rangle_G^*)$.

First of all, we note that if we denote by $\mathcal{D}^g$ the Levi-Civita connection associated with $(\Pi_{\mathfrak{g}}, \langle , \rangle_G(e))$ and by $R^{\mathfrak{g}}$ the curvature of $\mathcal{D}^{\mathfrak{g}}$, then the restriction of $\mathcal{D}^G$ to $\mathfrak{g}^* \times \mathfrak{g}^*$ coincides with $\mathcal{D}^{\mathfrak{g}}$ and the restriction of the curvature R^G of $\mathcal{D}^G$ to $\mathfrak{g}^*$ coincides with $R^{\mathfrak{g}}$, i.e.,

$$\mathcal{D}_\alpha^G \beta = \mathcal{D}_\alpha^{\mathfrak{g}} \beta, \quad R^G(\alpha,\beta)\gamma = R^{\mathfrak{g}}(\alpha,\beta)\gamma,$$

for any $\alpha, \beta, \gamma \in \mathfrak{g}^*$.

Proposition 1. *Let $\mathcal{D}^G$ and $\mathcal{D}^{G\times\mathfrak{g}}$ be the Levi-Civita contravariant connections associated with $(\Pi_G, \langle , \rangle_G^*)$ and $(\Pi_{G\times\mathfrak{g}}, \langle , \rangle_{TG}^*)$, respectively. Then, for any $(\alpha,\beta),(\alpha',\beta'),(\alpha'',\beta'') \in \mathfrak{g}^* \times \mathfrak{g}^*$, we have:*

1. $\langle \mathcal{D}_{(\alpha,0)}^{G\times\mathfrak{g}}(\alpha',0),(\alpha'',0) \rangle_{TG}^* = 0,$
2. $\langle \mathcal{D}_{(\alpha,0)}^{G\times\mathfrak{g}}(\alpha',0),(0,\beta'') \rangle_{TG}^* = \frac{1}{2}\langle ([\alpha,\alpha']_{\mathfrak{g}^*},0),(0,\beta'') \rangle_{TG}^*,$
3. $\langle \mathcal{D}_{(\alpha,0)}^{G\times\mathfrak{g}}(0,\beta'),(\alpha'',0) \rangle_{TG}^* = -\frac{1}{2}\langle (0,\mathrm{ad}_\alpha^t \beta'),(\alpha'',0) \rangle_{TG}^*,$
4. $\langle \mathcal{D}_{(\alpha,0)}^{G\times\mathfrak{g}}(0,\beta'),(0,\beta'') \rangle_{TG}^* = -\frac{1}{2}\langle (\mathrm{ad}_{\beta'}^t \alpha,0),(0,\beta'') \rangle_{TG}^*,$
5. $\langle \mathcal{D}_{(0,\beta)}^{G\times\mathfrak{g}}(\alpha',0),(\alpha'',0) \rangle_{TG}^* = -\frac{1}{2}\langle (0,\mathrm{ad}_{\alpha'}^t \beta),(\alpha'',0) \rangle_{TG}^*,$
6. $\langle \mathcal{D}_{(0,\beta)}^{G\times\mathfrak{g}}(\alpha',0),(0,\beta'') \rangle_{TG}^* = -\frac{1}{2}\langle (\mathrm{ad}_\beta^t \alpha',0),(0,\beta'') \rangle_{TG}^*,$
7. $\langle \mathcal{D}_{(0,\beta)}^{G\times\mathfrak{g}}(0,\beta'),(\alpha'',0) \rangle_{TG}^* = \frac{1}{2}\langle (0,[\beta,\beta']_{\mathfrak{g}^*}),(\alpha'',0) \rangle_{TG}^*,$
8. $\langle \mathcal{D}_{(0,\beta)}^{G\times\mathfrak{g}}(0,\beta'),(0,\beta'') \rangle_{TG}^* = \langle (0,\mathcal{D}_\beta^G \beta'),(0,\beta'') \rangle_{TG}^*,$

where ad_α^t denotes the transpose of ad_α with respect to $\langle , \rangle_G^*$.

Proof. According to Equations (9), (15) and (17), for example for (5) we obtain:

$$\begin{aligned}
2\langle \mathcal{D}^{TG}_{(0,\beta)}(\alpha',0),(\alpha'',0)\rangle^*_{TG} &= \langle [(0,\beta),(\alpha',0)]_{\mathfrak{g}^*\times\mathfrak{g}^*},(\alpha'',0)\rangle^*_{TG} \\
&+ \langle [(\alpha'',0),(0,\beta)]_{\mathfrak{g}^*\times\mathfrak{g}^*},(\alpha',0)\rangle^*_{TG} \\
&+ \langle [(\alpha'',0),(\alpha',0)]_{\mathfrak{g}^*\times\mathfrak{g}^*},(0,\beta)\rangle^*_{TG} \\
&= \langle [\alpha'',\alpha']_{\mathfrak{g}^*},(\beta,0)\rangle^*_{G} \\
&= -\langle \alpha'',\mathrm{ad}^t_{\alpha'}\beta\rangle^*_G \\
&= -\langle (0,\mathrm{ad}^t_{\alpha'}\beta),(\alpha'',0)\rangle^*_{TG}.
\end{aligned}$$

□

Lemma 1. *For any $(\alpha,\beta),(\alpha',\beta')\in\mathfrak{g}^*\times\mathfrak{g}^*$, we have :*

1. $\mathcal{D}^{G\times\mathfrak{g}}_{(\alpha,0)}(\alpha',0)=\frac{1}{2}([\alpha,\alpha']_{\mathfrak{g}^*},0)$,
2. $\mathcal{D}^{G\times\mathfrak{g}}_{(\alpha,0)}(0,\beta')=\frac{1}{2}(\mathrm{ad}^t_\alpha\beta'-\mathrm{ad}^t_{\beta'}\alpha,-\mathrm{ad}^t_\alpha\beta')$,
3. $\mathcal{D}^{G\times\mathfrak{g}}_{(0,\beta)}(\alpha',0)=\frac{1}{2}(\mathrm{ad}^t_{\alpha'}\beta-\mathrm{ad}^t_{\beta}\alpha',-\mathrm{ad}^t_{\alpha'}\beta)$,
4. $\mathcal{D}^{G\times\mathfrak{g}}_{(0,\beta)}(0,\beta')=\frac{1}{2}(\mathcal{D}^G_\beta\beta'+\mathcal{D}^G_{\beta'}\beta,[\beta,\beta']_{\mathfrak{g}^*})$.

Proof. Using the previous proposition we obtain:

(1)

$$\begin{aligned}
\langle \mathcal{D}^{G\times\mathfrak{g}}_{(\alpha,0)}(\alpha',0),(\alpha'',\beta'')\rangle^*_{TG} &= \langle \mathcal{D}^{G\times\mathfrak{g}}_{(\alpha,0)}(\alpha',0),(\alpha'',0)\rangle^*_{TG}+\langle \mathcal{D}^{G\times\mathfrak{g}}_{(\alpha,0)}(\alpha',0),(0,\beta'')\rangle^*_{TG} \\
&= \tfrac{1}{2}\langle([\alpha,\alpha']_{\mathfrak{g}^*},0),(0,\beta'')\rangle^*_{TG} \\
&= \langle \tfrac{1}{2}([\alpha,\alpha']_{\mathfrak{g}^*},0),(\alpha'',\beta'')\rangle^*_{TG},
\end{aligned}$$

then, $\mathcal{D}^{G\times\mathfrak{g}}_{(\alpha,0)}(\alpha',0)=\frac{1}{2}([\alpha,\alpha']_{\mathfrak{g}^*},0)$.

(2)

$$\begin{aligned}
\langle \mathcal{D}^{G\times\mathfrak{g}}_{(\alpha,0)}(0,\beta'),(\alpha'',\beta'')\rangle^*_{TG} &= \langle \mathcal{D}^{G\times\mathfrak{g}}_{(\alpha,0)}(0,\beta'),(\alpha'',0)\rangle^*_{TG}+\langle \mathcal{D}^{G\times\mathfrak{g}}_{(\alpha,0)}(0,\beta'),(0,\beta'')\rangle^*_{TG} \\
&= -\tfrac{1}{2}\langle(0,\mathrm{ad}^t_\alpha\beta'),(\alpha'',0)\rangle^*_{TG}+-\tfrac{1}{2}\langle(\mathrm{ad}^t_{\beta'}\alpha,0),(0,\beta'')\rangle^*_{TG} \\
&= \langle \tfrac{1}{2}(\mathrm{ad}^t_\alpha\beta'-\mathrm{ad}^t_{\beta'}\alpha,-\mathrm{ad}^t_\alpha\beta'),(\alpha'',\beta'')\rangle^*_{TG},
\end{aligned}$$

then, $\mathcal{D}^{G\times\mathfrak{g}}_{(\alpha,0)}(0,\beta')=\frac{1}{2}(\mathrm{ad}^t_\alpha\beta'-\mathrm{ad}^t_{\beta'}\alpha,-\mathrm{ad}^t_\alpha\beta')$.
In the same way, we can obtain (3) and (4). □

Theorem 1. *Let R^G and $R^{G\times\mathfrak{g}}$ be the curvatures of $\mathcal{D}^G$ and $\mathcal{D}^{G\times\mathfrak{g}}$ respectively. Then for any $(\alpha,\beta),(\alpha',\beta'),(\alpha'',\beta'')\in\mathfrak{g}^*\times\mathfrak{g}^*$, we have:*

1.

$$\begin{aligned}
R^{G\times\mathfrak{g}}((\alpha,0),(\alpha',0))(\alpha'',0) &= \tfrac{1}{4}\Big([\alpha,[\alpha',\alpha'']_{\mathfrak{g}^*}]_{\mathfrak{g}^*}+[\alpha',[\alpha'',\alpha]_{\mathfrak{g}^*}]_{\mathfrak{g}^*} \\
&+ 2[\alpha'',[\alpha,\alpha']_{\mathfrak{g}^*}]_{\mathfrak{g}^*},0\Big).
\end{aligned}$$

2.

$$\begin{aligned}
R^{G\times\mathfrak{g}}((\alpha,0),(\alpha',0))(0,\beta'') = \frac{1}{4}\Big(&[\alpha,\mathrm{ad}^t_{\alpha'}\beta''-\mathrm{ad}^t_{\beta''}\alpha']_{\mathfrak{g}^*}-[\alpha',\mathrm{ad}^t_\alpha\beta''-\mathrm{ad}^t_{\beta''}\alpha]_{\mathfrak{g}^*} \\
&- \mathrm{ad}^t_\alpha\mathrm{ad}^t_{\alpha'}\beta''+\mathrm{ad}^t_{\mathrm{ad}^t_{\alpha'}\beta''}\alpha+\mathrm{ad}^t_{\alpha'}\mathrm{ad}^t_\alpha\beta''-\mathrm{ad}^t_{\mathrm{ad}^t_\alpha\beta''}\alpha' \\
&- 2\mathrm{ad}^t_{[\alpha,\alpha']_{\mathfrak{g}^*}}\beta''+2\mathrm{ad}^t_{\beta''}[\alpha,\alpha']_{\mathfrak{g}^*},\mathrm{ad}^t_\alpha\mathrm{ad}^t_{\alpha'}\beta''-\mathrm{ad}^t_{\alpha'}\mathrm{ad}^t_\alpha\beta'' \\
&+ 2\mathrm{ad}^t_{[\alpha,\alpha']_{\mathfrak{g}^*}}\beta''\Big).
\end{aligned}$$

3.

$$R^{G\times\mathfrak{g}}((0,\beta),(\alpha',0))(\alpha'',0) = \frac{1}{4}\Big(\mathrm{ad}^t_{[\alpha',\alpha'']_{\mathfrak{g}^*}}\beta - \mathrm{ad}^t_\beta[\alpha',\alpha'']_{\mathfrak{g}^*} - [\alpha',\mathrm{ad}^t_{\alpha''}\beta - \mathrm{ad}^t_\beta\alpha'']_{\mathfrak{g}^*} + \mathrm{ad}^t_{\alpha'}\mathrm{ad}^t_{\alpha''}\beta - \mathrm{ad}^t_{\mathrm{ad}^t_{\alpha''}\beta}\alpha', -\mathrm{ad}^t_{[\alpha',\alpha'']_{\mathfrak{g}^*}}\beta - \mathrm{ad}^t_{\alpha'}\mathrm{ad}^t_{\alpha''}\beta\Big).$$

4.

$$R^{G\times\mathfrak{g}}((0,\beta),(\alpha',0))(0,\beta'') = \frac{1}{4}\Big(\mathrm{ad}^t_{\mathrm{ad}^t_{\alpha'}\beta''-\mathrm{ad}^t_{\beta''}\alpha'}\beta - \mathrm{ad}^t_\beta\mathrm{ad}^t_{\alpha'}\beta'' + \mathrm{ad}^t_\beta\mathrm{ad}^t_{\beta''}\alpha' - \mathcal{D}^G_\beta\mathrm{ad}^t_{\alpha'}\beta'' - \mathcal{D}^G_{\mathrm{ad}^t_{\alpha'}\beta''}\beta - [\alpha',\mathcal{D}^G_\beta\beta'' + \mathcal{D}^G_{\beta''}\beta]_{\mathfrak{g}^*} - \mathrm{ad}^t_{\alpha'}[\beta,\beta'']_{\mathfrak{g}^*} + \mathrm{ad}^t_{[\beta,\beta'']_{\mathfrak{g}^*}}\alpha', -\mathrm{ad}^t_{\mathrm{ad}^t_{\alpha'}\beta'-\mathrm{ad}^t_{\beta''}\alpha'}\beta - [\beta,\mathrm{ad}^t_{\alpha'}\beta'']_{\mathfrak{g}^*} - \mathrm{ad}^t_{\alpha'}[\beta,\beta']_{\mathfrak{g}^*}\Big).$$

5.

$$R^{G\times\mathfrak{g}}((0,\beta),(0,\beta'))(\alpha'',0) = \frac{1}{4}\Big(\mathrm{ad}^t_{\mathrm{ad}^t_{\alpha''}\beta'-\mathrm{ad}^t_{\beta'}\alpha''}\beta - \mathrm{ad}^t_\beta\mathrm{ad}^t_{\alpha''}\beta' + \mathrm{ad}^t_\beta\mathrm{ad}^t_{\beta'}\alpha'' - \mathcal{D}^G_\beta\mathrm{ad}^t_{\alpha''}\beta' - \mathcal{D}^G_{\mathrm{ad}^t_{\alpha''}\beta}\beta' - \mathrm{ad}^t_{\mathrm{ad}^t_{\alpha''}\beta-\mathrm{ad}^t_\beta\alpha''}\beta' + \mathrm{ad}^t_{\beta'}\mathrm{ad}^t_{\alpha''}\beta - \mathrm{ad}^t_{\beta'}\mathrm{ad}^t_\beta\alpha'' + \mathcal{D}^G_{\beta'}\mathrm{ad}^t_{\alpha''}\beta + \mathcal{D}^G_{\mathrm{ad}^t_{\alpha''}\beta}\beta' - 2\mathrm{ad}^t_{\alpha''}[\beta,\beta']_{\mathfrak{g}^*} + 2\mathrm{ad}^t_{[\beta,\beta']_{\mathfrak{g}^*}}\alpha'', -\mathrm{ad}^t_{\mathrm{ad}^t_{\alpha''}\beta'-\mathrm{ad}^t_{\beta'}\alpha''}\beta - [\beta,\mathrm{ad}^t_{\alpha''}\beta']_{\mathfrak{g}^*} + \mathrm{ad}^t_{\mathrm{ad}^t_{\alpha''}\beta-\mathrm{ad}^t_\beta\alpha''}\beta' + [\beta',\mathrm{ad}^t_{\alpha''}\beta]_{\mathfrak{g}^*} - 2\mathrm{ad}^t_{\alpha''}[\beta,\beta']_{\mathfrak{g}^*}\Big).$$

6.

$$R^{G\times\mathfrak{g}}((0,\beta),(0,\beta'))(0,\beta'') = \frac{1}{4}\Big(R^G(\beta,\beta')\beta'' - \mathcal{D}^G_\beta\mathcal{D}^G_{\beta''}\beta' + \mathcal{D}^G_\beta\mathcal{D}^G_{\beta''}\beta' - \mathcal{D}^G_{[\beta,\beta']_{\mathfrak{g}^*}}\beta'' + \mathrm{ad}^t_{(\mathcal{D}^G_{\beta'}\beta''+\mathcal{D}^G_{\beta''}\beta')}\beta - \mathrm{ad}^t_\beta(\mathcal{D}^G_{\beta'}\beta'' + \mathcal{D}^G_{\beta''}\beta') - \mathrm{ad}^t_{(\mathcal{D}^G_\beta\beta''+\mathcal{D}^G_{\beta''}\beta)}\beta' + \mathrm{ad}^t_{\beta'}(\mathcal{D}^G_\beta\beta'' + \mathcal{D}^G_{\beta''}\beta) - \mathcal{D}^G_{[\beta,\beta'']_{\mathfrak{g}^*}}\beta' + 2\mathcal{D}^G_{\beta''}[\beta,\beta']_{\mathfrak{g}^*}, -\mathrm{ad}^t_{(\mathcal{D}^G_{\beta'}\beta''+\mathcal{D}^G_{\beta''}\beta')}\beta + \mathrm{ad}^t_{(\mathcal{D}^G_\beta\beta''+\mathcal{D}^G_{\beta''}\beta)}\beta' + [\beta'',[\beta,\beta']_{\mathfrak{g}^*}]_{\mathfrak{g}^*}\Big).$$

Proof. Using the Equation (1) and the Lemma 1, for example, for (3), we find:

$$\begin{aligned}
R^{G\times\mathfrak{g}}((0,\beta),(\alpha',0))(\alpha'',0) &= \mathcal{D}^{G\times\mathfrak{g}}_{(0,\beta)}\mathcal{D}^{G\times\mathfrak{g}}_{(\alpha',0)}(\alpha'',0) - \mathcal{D}^{G\times\mathfrak{g}}_{(\alpha',0)}\mathcal{D}^{G\times\mathfrak{g}}_{(0,\beta)}(\alpha'',0)\\
&- \mathcal{D}^{G\times\mathfrak{g}}_{[(0,\beta),(\alpha',0)]_{\mathfrak{g}^*\times\mathfrak{g}^*}}(\alpha'',0)\\
&= \mathcal{D}^{G\times\mathfrak{g}}_{(0,\beta)}\tfrac{1}{2}([\alpha',\alpha'']_{\mathfrak{g}^*},0) - \mathcal{D}^{G\times\mathfrak{g}}_{(\alpha',0)}\tfrac{1}{2}(\mathrm{ad}^t_{\alpha''}\beta - \mathrm{ad}^t_{\beta}\alpha'', -\mathrm{ad}^t_{\alpha''}\beta)\\
&= \tfrac{1}{4}\Big(\mathrm{ad}_{[\alpha',\alpha'']_{\mathfrak{g}^*}}\beta - \mathrm{ad}^t_{\beta}[\alpha',\alpha'']_{\mathfrak{g}^*}, -\mathrm{ad}^t_{[\alpha',\alpha'']_{\mathfrak{g}^*}}\beta\Big)\\
&- \tfrac{1}{4}\Big([\alpha', \mathrm{ad}^t_{\alpha''}\beta - \mathrm{ad}^t_{\beta}\alpha'']_{\mathfrak{g}^*}, 0\Big)\\
&+ \tfrac{1}{4}\Big(\mathrm{ad}^t_{\alpha'}\mathrm{ad}^t_{\alpha''}\beta - \mathrm{ad}^t_{\mathrm{ad}^t_{\alpha''}\beta}\alpha', -\mathrm{ad}^t_{\alpha'}\mathrm{ad}^t_{\alpha''}\beta\Big)\\
&= \tfrac{1}{4}\Big(\mathrm{ad}^t_{[\alpha',\alpha'']_{\mathfrak{g}^*}}\beta - \mathrm{ad}^t_{\beta}[\alpha',\alpha'']_{\mathfrak{g}^*} - [\alpha', \mathrm{ad}^t_{\alpha''}\beta - \mathrm{ad}^t_{\beta}\alpha'']_{\mathfrak{g}^*}\\
&+ \mathrm{ad}^t_{\alpha'}\mathrm{ad}^t_{\alpha''}\beta - \mathrm{ad}^t_{\mathrm{ad}^t_{\alpha''}\beta}\alpha', -\mathrm{ad}^t_{[\alpha',\alpha'']_{\mathfrak{g}^*}}\beta - \mathrm{ad}^t_{\alpha'}\mathrm{ad}^t_{\alpha''}\beta\Big).
\end{aligned}$$

□

If $\langle,\rangle^*_G$ is a bi-invariant pseudo-Riemannian metric on a Poisson–Lie group (G,Π_G), then as a consequence of Formula (9), we have $\mathcal{D}^G_\alpha\beta = \frac{1}{2}[\alpha,\beta]_{\mathfrak{g}^*}$ and $\mathrm{ad}^t_\alpha = -\mathrm{ad}_\alpha$, for any $\alpha,\beta\in\mathfrak{g}^*$. (For more details in the covariant case, see [21]).

Corollary 1. *If we let $\langle,\rangle^*_G$ be a bi-invariant contravariant pseudo-Riemannian metric on a Poisson–Lie group (G,Π_G), then for any $(\alpha,\beta),(\alpha',\beta')\in\mathfrak{g}^*\times\mathfrak{g}^*$, we have :*

1. $\mathcal{D}^{G\times\mathfrak{g}}_{(\alpha,0)}(\alpha',0) = (\mathcal{D}^G_\alpha\alpha',0)$;
2. $\mathcal{D}^{G\times\mathfrak{g}}_{(\alpha,0)}(0,\beta') = (2\mathcal{D}^G_{\beta'}\alpha, \mathcal{D}^G_\alpha\beta')$;
3. $\mathcal{D}^{G\times\mathfrak{g}}_{(0,\beta)}(\alpha',0) = (2\mathcal{D}^G_\beta\alpha', \mathcal{D}^G_{\alpha'}\beta)$;
4. $\mathcal{D}^{G\times\mathfrak{g}}_{(0,\beta)}(0,\beta') = (0,\mathcal{D}^G_\beta\beta')$.

Proof. Since $\langle,\rangle^*_G$ is bi-invariant, then using Lemma 1, for example, for (2), we find:

$$\begin{aligned}
\mathcal{D}^{G\times\mathfrak{g}}_{(\alpha,0)}(0,\beta') &= \tfrac{1}{2}(\mathrm{ad}^t_\alpha\beta' - \mathrm{ad}^t_{\beta'}\alpha, -\mathrm{ad}^t_\alpha\beta')\\
&= \tfrac{1}{2}(-\mathrm{ad}_\alpha\beta' + \mathrm{ad}_{\beta'}\alpha, \mathrm{ad}_\alpha\beta')\\
&= ([\beta',\alpha]_{\mathfrak{g}^*}, \tfrac{1}{2}[\alpha,\beta']_{\mathfrak{g}^*})\\
&= (2\mathcal{D}^G_{\beta'}\alpha, \mathcal{D}^G_\alpha\beta').
\end{aligned}$$

□

Corollary 2. *If we let $\langle,\rangle^*_G$ be a bi-invariant contravariant pseudo-Riemannian metric on a Poisson–Lie group (G,Π_G)., then for any $(\alpha,\beta),(\alpha',\beta'),(\alpha'',\beta'')\in\mathfrak{g}^*\times\mathfrak{g}^*$, we have:*

1. $R^{G\times\mathfrak{g}}((\alpha,0),(\alpha',0))(\alpha'',0) = (R^G(\alpha,\alpha')\alpha'',0)$;
2. $$\begin{aligned}R^{G\times\mathfrak{g}}((\alpha,0),(\alpha',0))(0,\beta'') &= \Big(2(R^G(\alpha,\beta'')\alpha' + R^G(\beta'',\alpha')\alpha) - \mathcal{D}^G_{\mathcal{D}^G_{\beta''}\alpha}\alpha'\\ &+ \mathcal{D}^G_{\mathcal{D}^G_{\beta''}\alpha'}\alpha, R^G(\alpha,\alpha')\beta''\Big)\end{aligned};$$
3. $$\begin{aligned}R^{G\times\mathfrak{g}}((0,\beta),(\alpha',0))(\alpha'',0) &= \Big(2(\mathcal{D}^G_\beta\mathcal{D}^G_{\alpha'}\alpha'' - \mathcal{D}^G_{\alpha'}\mathcal{D}^G_\beta\beta'' - \mathcal{D}^G_{\mathcal{D}^G_{\alpha''}\beta}\alpha'), \mathcal{D}^G_{\mathcal{D}^G_{\alpha'}\alpha''}\beta\\ &- \mathcal{D}^G_{\alpha'}\mathcal{D}^G_{\alpha''}\beta\Big)\end{aligned};$$

4. $R^{G\times\mathfrak{g}}((0,\beta),(\alpha',0))(0,\beta'') = \left(4\mathcal{D}^G_\beta\mathcal{D}^G_{\beta''}\alpha' - 2\mathcal{D}^G_{\mathcal{D}^G_\beta\beta''}\alpha', 2\mathcal{D}^G_{\mathcal{D}^G_{\beta''}\alpha'}\beta - \mathcal{D}^G_\beta\mathcal{D}^G_{\alpha'}\beta''\right);$

5.

$$\begin{array}{rcl} R^{G\times\mathfrak{g}}((0,\beta),(0,\beta'))(\alpha'',0) &=& \Big(4R^G(\beta,\beta')\alpha'' + 2\mathcal{D}^G_{[\beta,\beta']_{\mathfrak{g}^*}}\alpha'', \mathcal{D}^G_\beta\mathcal{D}^G_{\alpha''}\beta' \\ &-& \mathcal{D}^G_{\beta'}\mathcal{D}^G_{\alpha''}\beta + 2\mathcal{D}^G_{\mathcal{D}^G_{\beta'}\alpha''}\beta - 2\mathcal{D}^G_{\mathcal{D}^G_\beta\alpha''}\beta' - \mathcal{D}^G_{\alpha''}[\beta,\beta']_{\mathfrak{g}^*}\Big) \end{array};$$

6. $R^{G\times\mathfrak{g}}((0,\beta),(0,\beta'))(0,\beta'') = \left(0, R^G(\beta,\beta')\beta''\right).$

According to the Theorem 1 if the connection D^G is flat, then the connection $D^{G\times\mathfrak{g}}$ is not necessarily flat. So, in this case, we cannot study the generalized Poisson bracket on the space of differential forms $\Omega^*(TG)$. For this reason, we focus on the Sanchez de Alvarez Poisson structure on the tangent bundle TG in the following subsection.

3.2. Pseudo-Riemannian Geometry of Sanchez de Alvarez Tangent Poisson–Lie Group

In this subsection, we consider the left invariant contravariant pseudo-Riemannian metric $\langle,\rangle^*_{TG}$ on the Sanchez de Alvarez Poisson–Lie group (TG,Π_{TG}), and we study the geometry of the triplet $(TG,\Pi_{TG},\langle,\rangle^*_{TG})$ and its relations with the geometry of $(G,\Pi_G,\langle,\rangle^*_G)$.

Proposition 2. *Let $(G,\Pi_G,\langle,\rangle^*_G)$ be a Poisson–Lie group equipped with the left invariant contravariant pseudo-Riemannian metric $\langle,\rangle^*_G$ and $(TG,\Pi_{TG},\langle,\rangle^*_{TG})$ the Sanchez de Alvarez tangent Poisson–Lie group of G equipped with the left invariant pseudo-Riemannian metric $\langle,\rangle^*_{TG}$ associated with $\langle,\rangle^*_G$. Let $\mathcal{D}^{TG}$ and $\mathcal{D}^G$ be the Levi-Civita contravariant connections associated with $(\Pi_{TG},\langle,\rangle^*_{TG})$ and $(\Pi_G,\langle,\rangle^*_G)$, respectively. Then, for any $(\alpha,\beta),(\alpha',\beta')\in\mathfrak{g}^*\times\mathfrak{g}^*$, we have:*

$$\mathcal{D}^{TG}_{(\alpha,\beta)}(\alpha',\beta') = \left(\mathcal{D}^G_\alpha\beta' + \mathcal{D}^G_\beta\alpha', \mathcal{D}^G_\beta\beta'\right).$$

Proof. According to Equations (9), (13) and (17), we obtain:

$$\begin{array}{rcl} 2\langle\mathcal{D}^{TG}_{(\alpha,\beta)}(\alpha',\beta'),(\alpha'',\beta'')\rangle^*_{TG} &=& \langle[(\alpha,\beta),(\alpha',\beta')]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*},(\alpha'',\beta'')\rangle^*_{TG} \\ &+& \langle[(\alpha'',\beta''),(\alpha,\beta)]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*},(\alpha',\beta')\rangle^*_{TG} \\ &+& \langle[(\alpha'',\beta''),(\alpha',\beta')]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*},(\alpha,\beta)\rangle^*_{TG} \\ &=& \langle([\alpha,\beta']_{\mathfrak{g}^*}+[\beta,\alpha']_{\mathfrak{g}^*},[\beta,\beta']_{\mathfrak{g}^*}),(\alpha'',\beta'')\rangle^*_{TG} \\ &+& \langle([\alpha'',\beta]_{\mathfrak{g}^*}+[\beta'',\alpha]_{\mathfrak{g}^*},[\beta'',\beta]_{\mathfrak{g}^*}),(\alpha',\beta')\rangle^*_{TG} \\ &+& \langle[\alpha'',\beta']_{\mathfrak{g}^*}+[\beta'',\alpha']_{\mathfrak{g}^*},[\beta'',\beta']_{\mathfrak{g}^*}),(\alpha,\beta)\rangle^*_{TG} \\ &=& 2\langle(\mathcal{D}^G_\alpha\beta'+\mathcal{D}^G_\beta\alpha',\mathcal{D}^G_\beta\beta'),(\alpha'',\beta'')\rangle^*_{TG}. \end{array}$$

□

Lemma 2. *Let R^{TG} and R^G be the curvatures of $\mathcal{D}^{TG}$ and $\mathcal{D}^G$, respectively. Then, for any $(\alpha,\beta),(\alpha',\beta'),(\alpha'',\beta'')\in\mathfrak{g}^*\times\mathfrak{g}^*$, we have :*

$$R^{TG}((\alpha,\beta),(\alpha',\beta'))(\alpha'',\beta'') = \left(R^G(\alpha,\beta')\beta'' + R^G(\beta,\alpha')\beta'' + R^G(\beta,\beta')\alpha'', R^G(\beta,\beta')\beta''\right).$$

Proof. Using the definition of the curvature tensor (1) and Proposition 2, we obtain:

$$\begin{aligned} R^{TG}((\alpha,\beta),(\alpha',\beta'))(\alpha'',\beta'') &= \mathcal{D}^{TG}_{(\alpha,\beta)}\mathcal{D}^{TG}_{(\alpha',\beta')}(\alpha'',\beta'') - \mathcal{D}^{TG}_{(\alpha',\beta')}\mathcal{D}^{TG}_{(\alpha,\beta)}(\alpha'',\beta'') \\ &- \mathcal{D}^{TG}_{[(\alpha,\beta),(\alpha',\beta')]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*}}(\alpha'',\beta'') \\ &= \mathcal{D}^{TG}_{(\alpha,\beta)}(\mathcal{D}^G_{\alpha'}\beta'' + \mathcal{D}^G_{\beta'}\alpha'', \mathcal{D}^G_{\beta'}\beta'') \\ &- \mathcal{D}^{TG}_{(\alpha',\beta')}(\mathcal{D}^G_{\alpha}\beta'' + \mathcal{D}^G_{\beta}\alpha'', \mathcal{D}^G_{\beta}\beta'') \\ &- \mathcal{D}^{TG}_{([\alpha,\beta']_{\mathfrak{g}^*}+[\beta,\alpha']_{\mathfrak{g}^*},[\beta,\beta']_{\mathfrak{g}^*})}(\alpha'',\beta'') \\ &= \Big(\mathcal{D}^G_{\alpha}\mathcal{D}^G_{\beta'}\beta'' + \mathcal{D}^G_{\beta}\mathcal{D}^G_{\alpha'}\beta'' + \mathcal{D}^G_{\beta}\mathcal{D}^G_{\beta'}\alpha'', \mathcal{D}^G_{\beta}\mathcal{D}^G_{\beta'}\beta''\Big) \\ &- \Big(\mathcal{D}^G_{\alpha'}\mathcal{D}^G_{\beta}\beta'' + \mathcal{D}^G_{\beta'}\mathcal{D}^G_{\alpha}\beta'' + \mathcal{D}^G_{\beta'}\mathcal{D}^G_{\beta}\alpha'', \mathcal{D}^G_{\beta'}\mathcal{D}^G_{\beta}\beta''\Big) \\ &- \Big(\mathcal{D}^G_{[\alpha,\beta']_{\mathfrak{g}^*}}\beta'' + \mathcal{D}^G_{[\beta,\alpha']_{\mathfrak{g}^*}}\beta'' + \mathcal{D}^G_{[\beta,\beta']_{\mathfrak{g}^*}}\alpha'', \mathcal{D}^G_{[\beta,\beta']_{\mathfrak{g}^*}}\beta''\Big) \\ &= \Big(R^G(\alpha,\beta')\beta'' + R^G(\beta,\alpha')\beta'' + R^G(\beta,\beta')\alpha'', R^G(\beta,\beta')\beta''\Big) \end{aligned}$$

□

Proposition 3. *The Levi-Civita contravariant connection $\mathcal{D}^G$ is locally symmetric if and only if the connection $\mathcal{D}^{TG}$ is locally symmetric.*

Proof. For any $(\alpha,\beta),(\alpha',\beta'),(\alpha'',\beta''),(\alpha''',\beta''')\in\mathfrak{g}^*\times\mathfrak{g}^*$ we obtain $H = (\mathcal{D}^{TG}_{(\alpha,\beta)}R^{TG})((\alpha',\beta'),(\alpha'',\beta''))(\alpha''',\beta''')$. According to Equation (4), Proposition 2 and Lemma 2, we obtain:

$$\begin{aligned} H &= \mathcal{D}^{TG}_{(\alpha,\beta)}(R^{TG}((\alpha',\beta'),(\alpha'',\beta''))(\alpha''',\beta''') - R^{TG}(\mathcal{D}^{TG}_{(\alpha,\beta)}(\alpha',\beta'),(\alpha'',\beta''))(\alpha''',\beta''') \\ &- R^{TG}((\alpha',\beta'),(\alpha'',\beta''))\mathcal{D}^{TG}_{(\alpha,\beta)}(\alpha''',\beta''') - R^{TG}((\alpha',\beta'),\mathcal{D}^{TG}_{(\alpha,\beta)}(\alpha'',\beta''))(\alpha''',\beta''') \\ &= \mathcal{D}^{TG}_{(\alpha,\beta)}\Big((R^G(\alpha',\beta'')\beta''' + R^G(\beta',\alpha'')\beta''' + R^G(\beta',\beta'')\alpha''', R^G(\beta',\beta'')\beta'''\Big) \\ &- R^{TG}((\mathcal{D}^G_{\alpha}\beta' + \mathcal{D}^G_{\beta}\alpha', \mathcal{D}^G_{\beta}\beta'),(\alpha'',\beta''))(\alpha''',\beta''') \\ &- R^{TG}((\alpha',\beta'),(\alpha'',\beta''))(\mathcal{D}^G_{\alpha}\beta''' + \mathcal{D}^G_{\beta}\alpha''', \mathcal{D}^G_{\beta}\beta''') \\ &- R^{TG}((\alpha',\beta'),(\mathcal{D}^G_{\alpha}\beta'' + \mathcal{D}^G_{\beta}\alpha'', \mathcal{D}^G_{\beta}\beta''))(\alpha''',\beta''') \end{aligned}$$

By developing again with Proposition 2 and Lemma 2, we obtain :

$$\begin{aligned} (\mathcal{D}^{TG}_{(\alpha,\beta)}R^{TG})((\alpha',\beta'),(\alpha'',\beta''))(\alpha''',\beta''') &= \Big((\mathcal{D}^G_{\alpha}R^G)(\beta',\beta'')\beta''' + (\mathcal{D}^G_{\beta}R^G)(\alpha',\beta'')\beta''' \\ &+ (\mathcal{D}^G_{\beta}R^G)(\beta',\alpha'')\beta''' \\ &+ (\mathcal{D}^G_{\beta}R^G)(\beta',\beta'')\alpha''', (\mathcal{D}^G_{\beta}R^G)(\beta',\beta'')\beta'''\Big) \end{aligned}$$

If $\mathcal{D}^G R^G = 0$, then $\mathcal{D}^{TG}R^{TG} = (0,0)$. Conversely, if $\mathcal{D}^{TG}R^{TG} = (0,0)$, then for any $\beta,\beta',\beta'',\beta'''\in\mathfrak{g}^*$, we have

$$\mathcal{D}^G_{\beta}R^G(\beta',\beta'')\beta''' = 0.$$

Hence, $\mathcal{D}^G$ is locally symmetric. □

Lemma 3. *Let $[,]_G$ and $[,]_{TG}$ be the generalized Koszul brackets on $\Omega^*(G)$ and $\Omega^*(TG)$, respectively. Then, for any $(\alpha,\beta),(\alpha',\beta')\in\mathfrak{g}^*\times\mathfrak{g}^*$, we have:*

$$\begin{aligned} [(\alpha,\beta), d(\alpha',\beta')]_{TG} &= ([\alpha,d\beta']_G + [\beta,d\alpha']_G, [\beta,d\beta']_G) \\ &= ([\alpha,d\beta']_G + [\beta,d\alpha']_G)^v + ([\beta,d\beta']_G)^c, \end{aligned}$$

where $([\alpha,d\beta']_G)^v$ (resp., $([\beta,d\beta']_G)^c$) is the vertical lift (resp., the complete lift) of the 2-form $[\alpha,d\beta']_G$ (resp., $[\beta,d\beta']_G$) on G to TG.

Proof. Let (x_i) be local coordinates of G in a neighborhood of e and (x_i, y_i) be the correspondent local coordinates of TG, in a neighborhood of $(e, 0)$. Let $\alpha = \sum_i \alpha_i dx_i$ and $\beta^{'} = \sum_i \beta_i^{'} dx_i$ be elements of $\mathfrak{g}^*$. We write $(\alpha, 0) = \sum_i \alpha_i dx_i$ and $(0, \beta^{'}) = \sum_i \beta_i^{'} dy_i$. Then, using Equations (6) and (13), for example, for $[(\alpha, 0), d(0, \beta^{'})]_{TG}$, we have:

$$\begin{array}{rcl}
[(\alpha,0), d(0,\beta^{'})]_{TG} & = & \sum_i [(\alpha,0), d(0,\beta_i^{'} dx_i)]_{TG} = \sum_i [(\alpha,0),(0,d\beta_i^{'} \wedge dx_i)]_{TG} \\
& = & \sum_i \Big([(\alpha,0),(0,d\beta_i^{'}) \wedge (dx_i,0) + (d\beta_i^{'},0) \wedge (0,dx_i)]_{TG} \Big) \\
& = & \sum_i \Big([(\alpha,0),(0,d\beta_i^{'})]_{TG} \wedge (dx_i,0) + (0,d\beta_i^{'}) \wedge [(\alpha,0),(dx_i,0)]_{TG} \\
& + & [(\alpha,0),(d\beta_i^{'},0)]_{TG} \wedge (0,dx_i) + (d\beta_i^{'},0) \wedge [(\alpha,0),(0,dx_i)]_{TG} \Big) \\
& = & \sum_i \Big(([\alpha, d\beta_i^{'}]_G, 0) \wedge (dx_i,0) + (d\beta_i^{'},0) \wedge ([\alpha, dx_i]_G, 0) \Big) \\
& = & \sum_i \Big(([\alpha, d\beta_i^{'}]_G \wedge dx_i, 0) + (d\beta_i^{'} \wedge [\alpha, dx_i]_G, 0) \Big) \\
& = & \sum_i ([\alpha, d\beta_i^{'} \wedge dx_i]_G, 0) \\
& = & ([\alpha, d\beta^{'}]_G, 0) \\
& = & ([\alpha, d\beta^{'}]_G)^v.
\end{array}$$

Considering all the possible cases

$$([(0,\beta), d(0,\beta^{'})]_{TG}, [(0,\beta), d(\alpha^{'},0)]_{TG}, [(\alpha,0), d(0,\beta^{'})]_{TG}, [(\alpha,0), d(\alpha^{'},0)]_{TG}),$$

we obtain the following lemma. □

Proposition 4. *Let $\{,\}_{TG}$ and $\{,\}_G$ be the Hawkins generalized pre-Poisson brackets of the Levi-Civita contravariant connections $\mathcal{D}^{TG}$ and $\mathcal{D}^G$, respectively. Then, for any $(\alpha, \beta), (\alpha^{'}, \beta^{'}) \in \mathfrak{g}^* \times \mathfrak{g}^*$, we have :*

$$\begin{array}{rcl}
\{(\alpha,\beta),(\alpha^{'},\beta^{'})\}_{TG} & = & (\{\alpha,\beta^{'}\}_G + \{\beta,\alpha^{'}\}_G, \{\beta,\beta^{'}\}_G) \\
& = & (\{\alpha,\beta^{'}\}_G + \{\beta,\alpha^{'}\}_G)^v + (\{\beta,\beta^{'}\}_G)^c.
\end{array}$$

Proof. Note that the Levi-Civita contravariant connections $\mathcal{D}^G$ and $\mathcal{D}^{TG}$ naturally extend to $\Omega^2(G)$ and $\Omega^2(TG)$, respectively. Using Equation (5), Proposition 2 and Lemma 3, we obtain

$$\begin{array}{rcl}
\{(\alpha,\beta),(\alpha^{'},\beta^{'})\}_{TG} & = & -\mathcal{D}^{TG}_{(\alpha,\beta)} d(\alpha^{'},\beta^{'}) - \mathcal{D}^{TG}_{(\alpha^{'},\beta^{'})} d(\alpha,\beta) + d\mathcal{D}^{TG}_{(\alpha^{'},\beta^{'})}(\alpha,\beta) \\
& + & [(\alpha,\beta), d(\alpha^{'},\beta^{'})]_{TG} \\
& = & -(\mathcal{D}^G_\alpha d\beta^{'} + \mathcal{D}^G_\beta d\alpha^{'}, \mathcal{D}^G_\beta d\beta^{'}) - (\mathcal{D}^G_{\alpha^{'}} d\beta + \mathcal{D}^G_{\beta^{'}} d\alpha, \mathcal{D}^G_{\beta^{'}} d\beta) \\
& + & (d\mathcal{D}^G_{\alpha^{'}}\beta + d\mathcal{D}^G_{\beta^{'}}\alpha, d\mathcal{D}^G_{\beta^{'}}\beta) + ([\alpha, d\beta^{'}]_G + [\beta, d\alpha^{'}]_G, [\beta, d\beta^{'}]_G) \\
& = & \Big(-\mathcal{D}^G_\alpha d\beta^{'} - \mathcal{D}^G_\beta d\alpha^{'} - \mathcal{D}^G_{\alpha^{'}} d\beta - \mathcal{D}^G_{\beta^{'}} d\alpha + d\mathcal{D}^G_{\alpha^{'}}\beta + d\mathcal{D}^G_{\beta^{'}}\alpha \\
& + & [\alpha, d\beta^{'}]_G + [\beta, d\alpha^{'}]_G, -\mathcal{D}^G_\beta d\beta^{'} - \mathcal{D}^G_{\beta^{'}} d\beta + d\mathcal{D}^G_{\beta^{'}}\beta + [\beta, d\beta^{'}]_G \Big) \\
& = & \Big(\{\alpha,\beta^{'}\}_G + \{\beta,\alpha^{'}\}_G, \{\beta,\beta^{'}\}_G \Big) \\
& = & (\{\alpha,\beta^{'}\}_G + \{\beta,\alpha^{'}\}_G)^v + (\{\beta,\beta^{'}\}_G)^c.
\end{array}$$

□

Lemma 4. *Let $\mathcal{M}^G$ and $\mathcal{M}^{TG}$ be the metacurvatures of the Levi-Civita contravariant connections $\mathcal{D}^G$ and $\mathcal{D}^{TG}$, respectively. Then, for any $(\alpha,\beta), (\alpha^{'},\beta^{'}), (\alpha^{''},\beta^{''}) \in \mathfrak{g}^* \times \mathfrak{g}^*$, we have:*

1. $\mathcal{M}^{TG}((\alpha,0),(\alpha^{'},0),(\alpha^{''},0)) = 0$;

2. $\mathcal{M}^{TG}((\alpha,0),(\alpha',0),(0,\beta''))=0$;
3. $\mathcal{M}^{TG}((\alpha,0),(0,\beta'),(0,\beta''))=(\mathcal{M}^{G}(\alpha,\beta',\beta''),0)$;
4. $\mathcal{M}^{TG}((0,\beta),(0,\beta'),(0,\beta''))=(0,\mathcal{M}^{G}(\beta,\beta',\beta''))$;
5. $\mathcal{M}^{TG}((0,\beta),(0,\beta'),(\alpha'',0))=(\mathcal{M}^{G}(\beta,\beta',\alpha''),0)$;
6. $\mathcal{M}^{TG}((0,\beta),(\alpha',0),(\alpha'',0))=0$.

Proof. Let (x_i) be local coordinates of G in a neighborhood of e and let (x_i,y_i) be the correspondent local coordinates of TG in a neighborhood of $(e,0)$. Let $\alpha=\sum_i \alpha_i dx_i$ and $\beta=\sum_i \beta_i dx_i$ be elements of $\mathfrak{g}^*$. We write $(\alpha,0)=\sum_i \alpha_i dx_i$ and $(0,\beta)=\sum_i \beta_i dy_i$. Using Equation (7) and Propositions 2 and 4, then—for example, for 3)—we obtain:

$$\begin{aligned}
\mathcal{M}^{TG}((\alpha,0),(0,\beta'),(0,\beta'')) &= \sum_i \alpha_i\Big(\{x_i\circ\pi,\{(0,\beta'),(0,\beta'')\}_{TG}\}_{TG} \\
&- \{\{x_i\circ\pi,(0,\beta')\}_{TG},(0,\beta'')\}_{TG} \\
&- \{\{x_i\circ\pi,(0,\beta'')\}_{TG},(0,\beta')\}_{TG}\Big) \\
&= \sum_i \alpha_i\Big(\mathcal{D}^{TG}_{(dx_i,0)}(0,\{\beta',\beta''\}_G)-\{\mathcal{D}^{TG}_{(dx_i,0)}(0,\beta'),(0,\beta'')\}_{TG} \\
&- \{\mathcal{D}^{TG}_{(dx_i,0)}(0,\beta''),(0,\beta')\}_{TG}\Big) \\
&= \sum_i \alpha_i\Big((\mathcal{D}^{G}_{dx_i}\{\beta',\beta''\}_G,0)-(\{\mathcal{D}^{G}_{dx_i}\beta',\beta''\}_G,0) \\
&- (\{\mathcal{D}^{G}_{dx_i}\beta'',\beta'\}_G,0)\Big) \\
&= \sum_i \alpha_i\Big((\{x_i,\{\beta',\beta''\}_G\}_G,0)-(\{\{x_i,\beta'\}_G,\beta''\}_G,0) \\
&- (\{\{x_i,\beta''\}_G,\beta'\}_G,0)\Big) \\
&= (\mathcal{M}^{G}(\alpha,\beta',\beta''),0).
\end{aligned}$$

□

Theorem 2. *Let $(G,\Pi_G,\langle,\rangle^*_G)$ be a Poisson–Lie group equipped with the left invariant contravariant pseudo-Riemannian metric $\langle,\rangle^*_G$ and $(TG,\Pi_{TG},\langle,\rangle^*_{TG})$ the Sanchez de Alvarez tangent Poisson–Lie group of G equipped with the left invariant pseudo-Riemannian metric $\langle,\rangle^*_{TG}$ associated with $\langle,\rangle^*_G$. Then, the space of the differential form $\Omega^*(G)$ is a differential graded Poisson algebra if, and only if, $\Omega^*(TG)$ is a differential graded Poisson algebra.*

Proof. According to Lemma 2, if $R^G=0$, then $R^{TG}=(0,0)$. We now assume that $R^{TG}=(0,0)$; then, for any $\beta,\beta',\beta''\in\mathfrak{g}^*$, we have

$$R^G(\beta,\beta')\beta''=0.$$

Then, $\mathcal{D}^G$ is flat if, and only if, $\mathcal{D}^{TG}$ is flat.

Moreover, According to Lemma 4, for any $(\alpha,\beta),(\alpha',\beta'),(\alpha'',\beta'')\in\mathfrak{g}^*\times\mathfrak{g}^*$, we obtain:

$$\begin{aligned}
\mathcal{M}^{TG}((\alpha,\beta),(\alpha',\beta'),(\alpha'',\beta'')) &= \Big(\mathcal{M}^G(\alpha,\beta',\beta'')+\mathcal{M}^G(\beta,\alpha',\beta'') \\
&+ \mathcal{M}^G(\beta,\beta',\alpha''),\mathcal{M}^G(\beta,\beta',\beta'')\Big).
\end{aligned}$$

So, if $\mathcal{M}^G=0$, then $\mathcal{M}^{TG}=(0,0)$. We now assume that $\mathcal{M}^{TG}=(0,0)$; then, for any $\beta,\beta',\beta''\in\mathfrak{g}^*$, we have

$$\mathcal{M}^G(\beta,\beta')\beta''=0.$$

Then, $\mathcal{D}^G$ is metaflat if, and only if, $\mathcal{D}^{TG}$ is metaflat.

Hence, we deduce that the connection $\mathcal{D}^G$ defines a generalized Poisson bracket $\{,\}_G$ on $\Omega^*(G)$ if, and only if, the connection $\mathcal{D}^{TG}$ defines a generalized Poisson bracket $\{,\}_{TG}$ on $\Omega^*(TG)$. □

4. Pseudo-Riemannian Sanchez de Alvarez Tangent Poisson–Lie Group

The second author and N.Zaalani [12] showed that the Sanchez de Alvarez tangent Poisson–Lie group (TG, Π_{TG}) equipped with the natural left invariant Riemannian metric is a Riemannian Poisson–Lie group if, and only if, (G, Π_G) is a trivial Poisson–Lie group. In this section, we study the compatibility in the sense of M.Boucetta between the Sanchez de Alvarez Poisson–Lie structure Π_{TG} and the pseudo-Riemannian metric $\langle,\rangle^*_{TG}$ given in (17).

Let $\langle,\rangle_{\mathfrak{g}^*}$ be a bilinear, symmetric and non-degenerate form on $\mathfrak{g}^*$. We define a bilinear, symmetric and non-degenerate form $\langle,\rangle_{\mathfrak{g}^*\times\mathfrak{g}^*}$ on $\mathfrak{g}^* \times \mathfrak{g}^*$, which is analogous to (17), as follows:

$$\begin{aligned}
\langle(\alpha,0),(\alpha',0)\rangle_{\mathfrak{g}^*\times\mathfrak{g}^*} &= 0,\\
\langle(\alpha,0),(0,\beta')\rangle_{\mathfrak{g}^*\times\mathfrak{g}^*} &= \langle\alpha,\beta'\rangle_{\mathfrak{g}^*}\\
\langle(0,\beta),(0,\beta')\rangle_{\mathfrak{g}^*\times\mathfrak{g}^*} &= \langle\beta,\beta'\rangle_{\mathfrak{g}^*}.
\end{aligned}$$

where $(\alpha,\beta),(\alpha',\beta') \in \mathfrak{g}^* \times \mathfrak{g}^*$.

Let $\langle,\rangle^*_G$ be the left invariant contravariant pseudo-Riemannian metric associated with $\langle,\rangle_{\mathfrak{g}^*}$ and let $\langle,\rangle^*_{TG}$ be the metric associated with $\langle,\rangle_{\mathfrak{g}^*\times\mathfrak{g}^*}$.

Remark 1. *If $(G, \Pi_G, \langle,\rangle^*_G)$ is a pseudo-Riemannian Poisson Lie group, then its dual Lie algebra $(\mathfrak{g}^*, [,]_{\mathfrak{g}^*}, \langle,\rangle_{\mathfrak{g}^*})$ equipped with the form $\langle,\rangle_{\mathfrak{g}^*}$ is a pseudo-Riemannian Lie algebra and the abelian Poisson–Lie group $(\mathfrak{g}, \Pi_{\mathfrak{g}}, \langle,\rangle_{\mathfrak{g}})$ equipped with the form $\langle,\rangle_{\mathfrak{g}}$ associated with $\langle,\rangle_{\mathfrak{g}^*}$ is a pseudo-Riemannian Poisson–Lie group [4].*

Theorem 3. *Let $(G, \Pi_G, \langle,\rangle^*_G)$ be a Poisson–Lie group equipped with the left invariant contravariant pseudo-Riemannian metric $\langle,\rangle^*_G$ and let $(TG, \Pi_{TG}, \langle,\rangle^*_{TG})$ be the Sanchez de Alvarez tangent Poisson–Lie group of G equipped with the left invariant pseudo-Riemannian metric $\langle,\rangle^*_{TG}$. Then, $(G, \Pi_G, \langle,\rangle^*_G)$ is a pseudo-Riemannian Poisson–Lie group if, and only if, $(TG, \Pi_{TG}, \langle,\rangle^*_{TG})$ is a pseudo-Riemannian Poisson–Lie group.*

Proof. Note that the linear transformation $Ad^*_g : \mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie algebra automorphism [22].

The infinitesimal Levi-Civita connection B associated with $([,]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*}, \langle,\rangle_{\mathfrak{g}^*\times\mathfrak{g}^*})$ is given for any $(\alpha,\alpha'),(\gamma,\gamma') \in \mathfrak{g}^* \times \mathfrak{g}^*$ by:

$$B_{(\alpha,\alpha')}(\gamma,\gamma') = (A_\alpha\gamma' + A_{\alpha'}\gamma, A_{\alpha'}\gamma'),$$

where A is the infinitesimal Levi-Civita connection associated with $([,]_{\mathfrak{g}^*}, \langle,\rangle_{\mathfrak{g}^*})$, respectively.

For any $(X,Y) \in \mathfrak{g} \rtimes \mathfrak{g}$ and $(\gamma,\gamma') \in \mathfrak{g}^* \ltimes \mathfrak{g}^*$,

$$ad^*_{(X,Y)}(\gamma,\gamma') = (ad^*_X\gamma + ad^*_Y\gamma', ad^*_X\gamma').$$

Let (x_i) be local coordinates of G in a neighborhood of e and let (x_i, y_i) be the correspondent local coordinates of TG. The Poisson tensors of G and TG are expressed by [10]:

$$\Pi_G = \sum_{i,j} \Pi^{ij}_G \frac{\partial}{\partial_{x_i}} \wedge \frac{\partial}{\partial_{x_j}},$$

and

$$\Pi_{TG} = \sum_{i,j,k} \Pi^{ij}_G \frac{\partial}{\partial_{x_i}} \wedge \frac{\partial}{\partial_{y_j}} + y_k \frac{\partial \Pi^{ij}_G}{\partial_{x_k}} \frac{\partial}{\partial_{y_i}} \wedge \frac{\partial}{\partial_{y_j}}, \tag{18}$$

respectively. Then, for any $(g,X)\in TG$ and for any $(\alpha,\alpha^{'})\in\mathfrak{g}^*\times\mathfrak{g}^*$, we have

$$\Pi^l_{TG}(g,X)(\alpha,\alpha^{'})=(\Pi^l_G(g)(\alpha^{'}),\Pi^l_G(g)(\alpha)+\Pi_{\mathfrak{g}}(X)(\alpha^{'})),$$

where $\Pi_{\mathfrak{g}}$ is the linear Poisson structure on $\mathfrak{g}$ associated with Π_G.

Then, for any $(\alpha,\alpha^{'}),(\beta,\beta^{'}),(\gamma,\gamma^{'})\in\mathfrak{g}^*\times\mathfrak{g}^*$, we obtain:

$$\begin{aligned}
& [B_{(\alpha,\alpha^{'})}(\gamma,\gamma^{'})+ad^*_{\Pi^l_{TG}(g,X)(\alpha,\alpha^{'})}(\gamma,\gamma^{'}),(\beta,\beta^{'})]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*}\\
+\;& [(\alpha,\alpha^{'}),B_{(\beta,\beta^{'})}(\gamma,\gamma^{'})+ad^*_{\Pi^l_{TG}(g,X)(\beta,\beta^{'})}(\gamma,\gamma^{'})]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*}\\
=\;& [(A_\alpha\gamma^{'}+A_{\alpha^{'}}\gamma,A_{\alpha^{'}}\gamma^{'})+ad^*_{(\Pi^l_G(g)(\alpha^{'}),\Pi^l_G(g)(\alpha)+\Pi_{\mathfrak{g}}(X)(\alpha^{'}))}(\gamma,\gamma^{'}),(\beta,\beta^{'})]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*}\\
+\;& [(\alpha,\alpha^{'}),(A_\beta\gamma^{'}+A_{\beta^{'}}\gamma,A_{\beta^{'}}\gamma^{'})+ad^*_{(\Pi^l_G(g)(\beta^{'}),\Pi^l_G(g)(\beta)+\Pi_{\mathfrak{g}}(X)(\beta^{'}))}(\gamma,\gamma^{'})]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*}\\
=\;& [(A_\alpha\gamma^{'}+A_{\alpha^{'}}\gamma+ad^*_{\Pi^l_G(g)(\alpha^{'})}\gamma+ad^*_{\Pi^l_G(g)(\alpha)+\Pi_{\mathfrak{g}}(X)(\alpha^{'})}\gamma^{'},A_{\alpha^{'}}\gamma^{'}\\
+\;& ad^*_{\Pi^l_G(g)(\alpha^{'})}\gamma^{'}),(\beta,\beta^{'})]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*}\\
+\;& [(\alpha,\alpha^{'}),(A_\beta\gamma^{'}+A_{\beta^{'}}\gamma+ad^*_{\Pi^l_G(g)(\beta^{'})}\gamma+ad^*_{\Pi^l_G(g)(\beta)+\Pi_{\mathfrak{g}}(X)(\beta^{'})}\gamma^{'},A_{\beta^{'}}\gamma^{'}\\
+\;& ad^*_{\Pi^l_G(g)(\beta^{'})}\gamma^{'})]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*}\\
=\;& \Big([A_\alpha\gamma^{'}+A_{\alpha^{'}}\gamma+ad^*_{\Pi^l_G(g)(\alpha^{'})}\gamma+ad^*_{\Pi^l_G(g)(\alpha)}\gamma^{'}+ad^*_{\Pi_{\mathfrak{g}}(X)(\alpha^{'})}\gamma^{'},\beta^{'}]_{\mathfrak{g}^*}\\
+\;& [A_{\alpha^{'}}\gamma^{'}+ad^*_{\Pi^l_G(g)(\alpha^{'})}\gamma^{'},\beta]_{\mathfrak{g}^*},[A_{\alpha^{'}}\gamma^{'}+ad^*_{\Pi^l_G(g)(\alpha^{'})}\gamma^{'},\beta]_{\mathfrak{g}^*}\Big)\\
+\;& \Big([\alpha,A_{\beta^{'}}\gamma^{'}+ad^*_{\Pi^l_G(g)(\beta^{'})}\gamma^{'}]_{\mathfrak{g}^*}+[\alpha,A_\beta\gamma^{'}+A_{\beta^{'}}\gamma+ad^*_{\Pi^l_G(g)(\beta^{'})}\gamma+ad^*_{\Pi^l_G(g)(\beta)}\gamma^{'}\\
+\;& ad^*_{\Pi_{\mathfrak{g}}(X)(\beta^{'})}\gamma^{'}]_{\mathfrak{g}^*},[\alpha^{'},A_{\beta^{'}}\gamma^{'}+ad^*_{\Pi^l_G(g)(\beta^{'})}\gamma^{'}\Big)\\
=\;& \Big([A_\alpha\gamma^{'}+ad^*_{\Pi^l_G(g)(\alpha)}\gamma^{'},\beta^{'}]_{\mathfrak{g}^*}+[\alpha,A_{\beta^{'}}\gamma^{'}+ad^*_{\Pi^l_G(g)(\beta^{'})}\gamma^{'}]_{\mathfrak{g}^*}+[A_{\alpha^{'}}\gamma+ad^*_{\Pi^l_G(g)(\alpha^{'})}\gamma,\beta^{'}]_{\mathfrak{g}^*}\\
+\;& [\alpha^{'},A_{\beta^{'}}\gamma+ad^*_{\Pi^l_G(g)(\beta^{'})}\gamma]_{\mathfrak{g}^*}+[A_{\alpha^{'}}\gamma^{'}+ad^*_{\Pi^l_G(g)(\alpha^{'})}\gamma^{'},\beta]_{\mathfrak{g}^*}+[\alpha^{'},A_\beta\gamma^{'}+ad^*_{\Pi^l_G(g)(\beta)}\gamma^{'}]_{\mathfrak{g}^*}\\
+\;& [ad^*_{\Pi_{\mathfrak{g}}(X)(\alpha^{'})}\gamma^{'},\beta^{'}]_{\mathfrak{g}^*}+[\alpha^{'},ad^*_{\Pi_{\mathfrak{g}}(X)(\beta^{'})}\gamma^{'}]_{\mathfrak{g}^*},[A_{\alpha^{'}}\gamma^{'}+ad^*_{\Pi^l_G(g)(\alpha^{'})}\gamma^{'},\beta^{'}]_{\mathfrak{g}^*}\\
+\;& [\alpha^{'},A_{\beta^{'}}\gamma^{'}+ad^*_{\Pi^l_G(g)(\beta^{'})}\gamma^{'}]_{\mathfrak{g}^*}\Big).
\end{aligned}$$

Then, using Remark 1, if $(G,\Pi_G,\langle,\rangle^*_G)$ is a pseudo-Riemannian Poisson–Lie group, then $(TG,\Pi_{TG},\langle,\rangle^*_{TG})$ is a pseudo-Riemannian Poisson–Lie group. Conversely, if $(TG,\Pi_{TG},\langle,\rangle^*_{TG})$ is a pseudo-Riemannian Poisson–Lie group, then for any $x\in G$ and for any $\alpha^{'},\beta^{'},\gamma^{'}\in\mathfrak{g}^*$, we have

$$[A_{\alpha^{'}}\gamma^{'}+ad^*_{\Pi^l_G(g)(\alpha^{'})}\gamma^{'},\beta^{'}]_{\mathfrak{g}^*}+[\alpha^{'},A_{\beta^{'}}\gamma^{'}+ad^*_{\Pi^l_G(g)(\beta^{'})}\gamma^{'}]_{\mathfrak{g}^*}=0.$$

Therefore, $(G,\Pi_G,\langle,\rangle^*_G)$ is a pseudo-Riemannian Poisson–Lie group. □

Corollary 3. *The semi-direct product Lie algebra $(\mathfrak{g}^*\ltimes\mathfrak{g}^*,[,]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*},\langle,\rangle_{\mathfrak{g}^*\times\mathfrak{g}^*})$ equipped with the form $\langle,\rangle_{\mathfrak{g}^*\times\mathfrak{g}^*}$ is a pseudo-Riemannian Lie algebra if, and only if, $(\mathfrak{g}^*,[,]_{\mathfrak{g}^*},\langle,\rangle_{\mathfrak{g}^*})$ is a pseudo-Riemannian Lie algebra.*

Proof. According to Equation (10), for any $(\alpha,\beta),(\alpha^{'},\beta^{'}),(\alpha^{''},\beta^{''})\in\mathfrak{g}^*\ltimes\mathfrak{g}^*$, we obtain :

$$\begin{aligned}
& [B_{(\alpha,\beta)}(\alpha^{'},\beta^{'}),(\alpha^{''},\beta^{''})]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*}+[(\alpha,\beta),B_{(\alpha^{''},\beta^{''})}(\alpha^{'},\beta^{'})]_{\mathfrak{g}^*\ltimes\mathfrak{g}^*}\\
=\;& \Big([A_\alpha\beta^{'},\beta^{''}]_{\mathfrak{g}^*}+[\alpha,A_{\beta^{''}}\beta^{'}]_{\mathfrak{g}^*}+[A_\beta\alpha^{'},\beta^{''}]_{\mathfrak{g}^*}+[\beta,A_{\beta^{''}}\alpha^{'}]_{\mathfrak{g}^*}+[A_\beta\beta^{'},\alpha^{''}]_{\mathfrak{g}^*}\\
& +\;[\beta,A_{\alpha^{''}}\beta^{'}]_{\mathfrak{g}},[A_\beta\beta^{'},\beta^{''}]_{\mathfrak{g}^*}+[\beta,A_{\beta^{''}}\beta^{'}]_{\mathfrak{g}^*}\Big),
\end{aligned}$$

Then we obtain the corollary. □

5. Examples

Let (x_i) be local coordinates of G in a neighborhood of e and let (x_i, y_i) be the correspondent local coordinates of TG. The pseudo-Riemannian metrics on G and TG are expressed by:

$$\langle , \rangle_G = \sum_{i,j} g^{ij} dx_i \otimes dx_j$$

and

$$\langle , \rangle_{TG} = \sum_{i,j,k} y_k \frac{\partial g^{ij}}{\partial_{x^k}} dx_i \otimes dx_j + g^{ij} dx_i \otimes dy_j + g^{ij} dy_i \otimes dx_j, \tag{19}$$

respectively.

1. Let (e_1, e_2, e_3) be an orthonormal basis of $\mathbb{R}^3$. The Lie algebra $\mathbb{R}^3$ with the bracket

$$[e_1, e_2]_{\mathbb{R}^3} = \lambda e_3, \quad [e_1, e_3]_{\mathbb{R}^3} = -\lambda e_2, \quad [e_2, e_3]_{\mathbb{R}^3} = 0, \quad \lambda < 0,$$

is a Riemannian Lie algebra [4]. The infinitesimal situation can be integrated, and we obtain that the triplet $(\mathbb{R}^3, \Pi_{\mathbb{R}^3}, \langle , \rangle_{\mathbb{R}^3})$ is a Riemannian Poisson Lie group, where $\mathbb{R}^3$ is equipped with its abelian Lie group structure, $\langle , \rangle_{\mathbb{R}^3}$ its canonical Euclidian metric and

$$\Pi_{\mathbb{R}^3} = \lambda \frac{\partial}{\partial_x} \wedge \left(z \frac{\partial}{\partial_y} - y \frac{\partial}{\partial_z}\right), \quad \lambda < 0.$$

Using Equations (18) and (19), the six-dimensional Sanchez de Alvarez tangent Poisson–Lie group $(T\mathbb{R}^3 \equiv \mathbb{R}^6, \Pi_{\mathbb{R}^6}, \langle , \rangle_{TG})$, where $\mathbb{R}^6$ is equipped with its abelian Lie group structure with coordinate (x, y, z, u, v, w),

$$\Pi_{\mathbb{R}^6} = \lambda \frac{\partial}{\partial_x} \wedge \left(z \frac{\partial}{\partial_v} - y \frac{\partial}{\partial_w}\right) + \lambda \frac{\partial}{\partial_u} \wedge \left(w \frac{\partial}{\partial_v} - v \frac{\partial}{\partial_w}\right)$$

and

$$\langle , \rangle_{\mathbb{R}^6} = dxdu + dydv + dzdw + dudx + dvdy + dwdz,$$

is a pseudo-Riemannian Poisson–Lie group.

2. The Poisson–Lie group $(\mathbb{R}^4, \Pi_{\mathbb{R}^4}, \langle , \rangle_{\mathbb{R}^4})$, where

$$\Pi_{\mathbb{R}^4} = \frac{\partial}{\partial_x} \wedge \left(z \frac{\partial}{\partial_t} - t \frac{\partial}{\partial_z}\right), \quad \langle , \rangle_{\mathbb{R}^4} = dx^2 + dy^2 + dz^2 + dt^2,$$

is compatible in the sense of Hawkins and is also a Riemannian Poisson–Lie group [22]. Then, the eight-dimensional tangent Poisson–Lie group $(T\mathbb{R}^4 = \mathbb{R}^8, \Pi_{\mathbb{R}^8}, \langle , \rangle_{\mathbb{R}^8})$, with coordinates (x, y, z, t, u, v, w, s),

$$\Pi_{\mathbb{R}^8} = \frac{\partial}{\partial_x} \wedge \left(z \frac{\partial}{\partial_s} - t \frac{\partial}{\partial_w}\right) + \frac{\partial}{\partial_u} \wedge \left(w \frac{\partial}{\partial_s} - s \frac{\partial}{\partial_w}\right)$$

and

$$\langle , \rangle_{\mathbb{R}^8} = dxdu + dydv + dzdw + dtds + dudx + dvdy + dwdz + dsdt$$

is also compatible in the sense of Hawkins and a pseudo-Riemannian Poisson–Lie group.

3. By [22], the four-dimensional torus $(\mathbb{T}^4 = \mathbb{R}^4/\mathbb{Z}^4, \Pi_{\mathbb{T}^4}, \langle , \rangle_G)$, is a Riemannian Poisson–Lie group (resp., compatible in the sense of Hawkins), where

$$\mathbb{T}^4 = \{(e^{ix}, e^{iy}, e^{iz}, e^{it}) \ / \ x, y, z, t \in [0, 2\pi[\ \},$$

$$\Pi_{\mathbb{T}^4} = \frac{\partial}{\partial_x} \wedge \left(z \frac{\partial}{\partial_t} - t \frac{\partial}{\partial_z}\right) \text{ and } g = dx^2 + dy^2 + dz^2 + dt^2.$$

Then, the eight-dimensional tangent Poisson–Lie group $(T\mathbb{T}^4, \Pi_{T\mathbb{T}^4}, \langle , \rangle_{TG})$, with coordinates (x, y, z, t, u, v, w, s),

$$\Pi_{T\mathbb{T}^4} = \frac{\partial}{\partial_x} \wedge (z\frac{\partial}{\partial_s} - t\frac{\partial}{\partial_w}) + \frac{\partial}{\partial_u} \wedge (w\frac{\partial}{\partial_s} - s\frac{\partial}{\partial_w})$$

and

$$\langle , \rangle_{T\mathbb{T}^4} = dxdu + dydv + dzdw + dtds + dudx + dvdy + dwdz + dsdt$$

is also a pseudo-Riemannian Poisson–Lie group (resp., compatible in the sense of Hawkins).

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