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Article

Improvement of Furuta's Inequality with Applications to Numerical Radius

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Abstract: In diverse branches of mathematics, several inequalities have been studied and applied. In this article, we improve Furuta's inequality. Subsequently, we apply this improvement to obtain new radius inequalities that not been reported in the current literature. Numerical examples illustrate the main findings.

Keywords: Dragomir extension; inner product; isometry; Kato inequality; linear and bounded operator; normed space; power-mean inequality; Schwarz inequality

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1. Introduction and Background

Let $\mathscr{O}(\mathscr{H})$ denote a C^*-algebra of linear and bounded operators defined on a separable complex Hilbert space $\mathscr{H}$. Let I denote the identity operator in $\mathscr{O}(\mathscr{H})$. In this framework, the numerical radius of $M \in \mathscr{O}(\mathscr{H})$ is defined by

$$w(M) := \sup\{|\langle Ma, a\rangle| \colon a \in \mathscr{H}, \parallel a \parallel = 1\},$$

where $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ are the inner product and its associated norm, respectively. The numerical radius w is a norm, which is tantamount to the operator norm $\| \cdot \|$ on $\mathscr{O}(\mathscr{H})$. Indeed, for any $M \in \mathscr{O}(\mathscr{H})$, we have

$$\|M\| \geq w(M) \geq \frac{1}{2}\|M\|. \tag{1}$$

For more details, see page 9 in [1–3]. Recent results pertaining to the numerical radius can be found in [4–12].

As another essential notion, the spectrum of an operator M, denoted by $\mathrm{sp}(M)$, corresponds to the set of all $\lambda \in \mathbb{C}$ for which the operator $\lambda I - M$ does not have a bounded linear inverse. The spectral radius of an operator M is defined by

$$r(M) = \sup\{|\lambda| \colon \lambda \in \mathrm{sp}(M)\}. \tag{2}$$

If $M \in \mathscr{O}(\mathscr{H})$ is positive, then, according to [13], we have

$$\langle Ma, a\rangle^{\alpha} \leq \langle M^{\alpha}a, a\rangle, \quad a \in \mathscr{H}, \quad \alpha \geq 1. \tag{3}$$

Note that the inequality formulated in (3) is reversed if $\alpha \in [0,1]$.

Aujla and Silva [14] showed that if f is non-negative real convex, and M, N are positive operators, then

$$\left\| \frac{f(M)+f(N)}{2} \right\| \geq \left\| f\left(\frac{M+N}{2}\right) \right\|. \tag{4}$$

Kittaneh [15,16] showed some refinements of the inequalities stated in (1). In particular, it was proved that

$$w(J) \leq \frac{1}{2}\||J|+|J^*|\| \leq \frac{1}{2}\left(\|J\| + \|J^2\|^{1/2}\right) \tag{5}$$

and

$$\frac{1}{4}\|J^*J + JJ^*\| \leq w^2(J) \leq \frac{1}{2}\|J^*J + JJ^*\|, \tag{6}$$

for any $J \in \mathscr{O}(\mathscr{H})$, where J^* denotes the corresponding adjoint operator. Furthermore, Kittaneh et al. [17] established some inequalities that can be presented as

$$\omega^{\alpha}(J) \leq \frac{1}{2}\left\| |J|^{2\alpha s} + |J^*|^{2\alpha(1-s)} \right\| \tag{7}$$

and

$$\omega^{2\alpha}(J) \leq \left\| s|J|^{2\alpha} + (1-s)|J^*|^{2\alpha} \right\|, \tag{8}$$

where $J \in \mathscr{O}(\mathscr{H})$, $0 \leq s \leq 1$, and $\alpha \geq 1$.

For the product of two Hilbert space operators, $M, N \in \mathscr{O}(\mathscr{H})$, Dragomir [18] proved the following numerical radius:

$$\omega^{\alpha}(M^*N) \leq \frac{1}{2}\left\| |M|^{2\alpha} + |N|^{2\alpha} \right\|, \quad \alpha \geq 1. \tag{9}$$

Moreover, the well-known Schwarz inequality asserts that

$$|\langle Ja, b\rangle|^2 \leq \langle Jb, b\rangle\langle Ja, a\rangle,$$

where $J \in \mathscr{O}(\mathscr{H})$ is positive and $a, b \in \mathscr{H}$. In general, a numerical radius is not submultiplicative, that is, $w(MN) \leq w(M)w(N)$, for all operators M and N. Hence, it is helpful to ask: when does this inequality hold? Note that the numerical radius is submultiplicative if $MN = NM$ and M is a normal operator.

Reid [19] demonstrated the Schwarz inequality given by

$$|\langle MNa, a\rangle| \leq \|M\|\langle Na, a\rangle,$$

where $a \in \mathscr{H}$, for all positive operators $M, N \in \mathscr{O}(\mathscr{H})$, such that $MN = N^*M^*$.

Kato [20] proposed a mixed Schwarz inequality, which is expressed as

$$|\langle Ja, b\rangle|^2 \leq \langle |J|^{2s}a, a\rangle\langle |J^*|^{2(1-s)}b, b\rangle, \quad 0 \leq s \leq 1,$$

where $|J| = (J^*J)^{1/2}$ for $J \in \mathscr{O}(\mathscr{H})$ and a, b are vectors in $\mathscr{H}$.

Kittaneh [21] showed an extension of the inequalities given in (5) and (9), proving that

$$|\langle NJa, b\rangle| \leq r(N)\|f(|J|)a\|\|g(|J^*|)b\|, \tag{10}$$

for all vectors $a, b \in \mathscr{H}$ and $N, M \in \mathscr{O}(\mathscr{H})$, such that $|N|M = M^*|N|$, where f, g are nonnegative continuous functions, with $f(t) = t/g(t)$, for $t \geq 0$, and $r(N)$ being the spectral radius of an operator N, as defined in (2).

Furuta [22] asserted another extension of the inequalities formulated in (6), stating that

$$|\langle M|M|^{s+t-1}a, b\rangle|^2 \leq \langle |M|^{2s}a, a\rangle\langle |M|^{2t}b, b\rangle, \tag{11}$$

for any $a, b \in \mathscr{H}$ and $s, t \in [0, 1]$, with $s + t \geq 1$.

Dragomir [23] established that, if $M, N, J \in \mathscr{O}(\mathscr{H})$, such that M, N are positive, for which $\|Ma\| \geq \|Ja\|$ and $\|Nb\| \geq \|J^*b\|$, then

$$|\langle Ja, b\rangle| \leq \|M^t a\| \|N^{1-t}b\|,$$

for all $a, b \in \mathscr{H}$ and $t \in [0, 1]$. Moreover, in [23], a Furuta-type inequality is given by

$$|\langle VMNUa, b\rangle|^2 \leq \langle U^*|N|^2 Ua, a\rangle\langle V|M^*|^2 V^*b, b\rangle, M, N, U, V \in \mathscr{O}(\mathscr{H}), \quad a, b \in \mathscr{H}. \tag{12}$$

Using the formula expressed in (10), again in [23], Dragomir generalized the inequality formulated in (7) by proving that

$$\omega^s(VMNU) \leq \frac{1}{2}\|(U^*|N|^2U)^s + (V|M^*|^2V^*)^s\|, \tag{13}$$

for all $M, N, U, V \in \mathscr{O}(\mathscr{H})$ and $s \geq 1$. Recently, Kittaneh et al. [24] introduced an inequality presented as

$$\omega^2(M^*N) \leq \frac{1}{6}\left\||N|^4 + |M|^4\right\| + \frac{1}{3}\omega(M^*N)\left\||M|^2 + |N|^2\right\| \leq \frac{1}{2}\left\||M|^4 + |N|^4\right\|, \tag{14}$$

where $M, N \in \mathscr{O}(\mathscr{H})$. When $\alpha = 2$, the inequality established in (14) refines the expression stated in (9). Now, observe that, for $J \in \mathscr{O}(\mathscr{H})$, the inequality expressed as

$$|\langle Ja, a\rangle| \leq \sqrt{\langle |J|a, a\rangle\langle |J^*|a, a\rangle} \tag{15}$$

is a special case of Kato's inequality obtained in (9), when $a = b \in \mathscr{H}$. In the same work [24], Kittaneh et al. proved a refinement of the inequality shown in (13), and presented as

$$|\langle Ja, a\rangle|^2 \leq \frac{2}{3}|\langle Ja, a\rangle|\sqrt{\langle |J|a, a\rangle\langle |J^*|a, a\rangle} + \frac{1}{3}\langle |J|a, a\rangle\langle |J^*|a, a\rangle \leq \langle |J|a, a\rangle\langle |J^*|a, a\rangle. \tag{16}$$

Note that the inequality given in (16) yields a refinement of (6) established as

$$\omega^2(J) \leq \frac{1}{3}\||J| + |J^*|\|\omega(J) + \frac{1}{6}\left\||J|^2 + |J^*|^2\right\| \leq \frac{1}{2}\left\||J^*|^2 + |J|^2\right\|, \quad J \in \mathscr{O}(\mathscr{H}). \tag{17}$$

To the best of our knowledge, general refinements of the Dragomir extension of Furuta's inequality stated in (10) have not been proved. Therefore, the objective of the present study is to improve Furuta's inequality as defined in (10). Then, we obtain stronger refinements of the results presented in (11), (12) and (15). We apply our results to numerical radius inequalities, which are supported by two numerical examples.

The plan for the rest of this article is as follows. In Section 2, we provide a refinement of the Cauchy–Schwarz inequality. Section 3 introduces an application of our results to numerical radius inequalities. The article finishes with some conclusions regarding the present study in Section 4.

2. Refinement of the Cauchy–Schwarz Inequality

We start this section with the following lemma, which generalizes and refines Kato's inequality stated in (11).

Lemma 1. *Let $M, N, U, V \in \mathscr{O}(\mathscr{H})$, $\lambda \in [0,1]$, and $\alpha \geq 1$. Then, we have*

$$\begin{aligned} |\langle VMNUa, b\rangle|^{2\alpha} &\leq \lambda\Big\langle \Big(U^*|N|^2U\Big)^\alpha a, a\Big\rangle\Big\langle \Big(V|M^*|^2V^*\Big)^\alpha b, b\Big\rangle \\ &\quad + (1-\lambda)|\langle VMNUa, b\rangle|^\alpha \sqrt{\Big\langle \Big(U^*|N|^2U\Big)^\alpha a, a\Big\rangle\Big\langle \Big(V|M^*|^2V^*\Big)^\alpha b, b\Big\rangle} \\ &\leq \Big\langle \Big(U^*|N|^2U\Big)^\alpha a, a\Big\rangle\Big\langle \Big(V|M^*|^2V^*\Big)^\alpha b, b\Big\rangle, \end{aligned} \tag{18}$$

for all $a, b \in \mathscr{H}$.

Proof. Using (3) and (12), one can easily obtain that

$$\begin{aligned} &\lambda\Big\langle \Big(U^*|N|^2U\Big)^\alpha a, a\Big\rangle\Big\langle \Big(V|M^*|^2V^*\Big)^\alpha b, b\Big\rangle \\ &\quad + (1-\lambda)|\langle VMNUa, b\rangle|^\alpha \sqrt{\Big\langle \Big(U^*|N|^2U\Big)^\alpha a, a\Big\rangle\Big\langle \Big(V|M^*|^2V^*\Big)^\alpha b, b\Big\rangle} \\ &\geq \lambda\Big\langle U^*|N|^2Ua, a\Big\rangle^\alpha \Big\langle V|M^*|^2V^*b, b\Big\rangle^\alpha \\ &\quad + (1-\lambda)|\langle VMNUa, b\rangle|^\alpha \Big\langle U^*|N|^2Ua, a\Big\rangle^{\frac{\alpha}{2}} \Big\langle V|M^*|^2V^*b, b\Big\rangle^{\frac{\alpha}{2}} \\ &\geq \lambda|\langle VMNUa, b\rangle|^{2\alpha} + (1-\lambda)|\langle VMNUa, b\rangle|^\alpha|\langle VMNUa, b\rangle|^\alpha \quad \text{–by (12)–} \\ &= |\langle VMNUa, b\rangle|^{2\alpha}, \end{aligned} \tag{19}$$

for all $\lambda \in [0,1]$ and $\alpha \geq 1$. In addition, we have

$$\begin{aligned} &\lambda\Big\langle \Big(U^*|N|^2U\Big)^\alpha a, a\Big\rangle\Big\langle \Big(V|M^*|^2V^*\Big)^\alpha b, b\Big\rangle \\ &\quad + (1-\lambda)|\langle VMNUa, b\rangle|^\alpha \sqrt{\Big\langle \Big(U^*|N|^2U\Big)^\alpha a, a\Big\rangle\Big\langle \Big(V|M^*|^2V^*\Big)^\alpha b, b\Big\rangle} \\ \leq&\lambda\Big\langle \Big(U^*|N|^2U\Big)^\alpha a, a\Big\rangle\Big\langle \Big(V|M^*|^2V^*\Big)^\alpha b, b\Big\rangle + (1-\lambda) \quad \text{–by (13)–} \\ &\quad \times \sqrt{\Big\langle \Big(U^*|N|^2U\Big)^\alpha a, a\Big\rangle\Big\langle \Big(V|M^*|^2V^*\Big)^\alpha b, b\Big\rangle}\sqrt{\Big\langle \Big(U^*|N|^2U\Big)^\alpha a, a\Big\rangle\Big\langle \Big(V|M^*|^2V^*\Big)^\alpha b, b\Big\rangle} \\ =&\lambda\Big\langle \Big(U^*|N|^2U\Big)^\alpha a, a\Big\rangle\Big\langle \Big(V|M^*|^2V^*\Big)^\alpha b, b\Big\rangle \\ &\quad + (1-\lambda)\Big\langle \Big(U^*|N|^2U\Big)^\alpha a, a\Big\rangle\Big\langle \Big(V|M^*|^2V^*\Big)^\alpha b, b\Big\rangle \\ =&\Big\langle \Big(U^*|N|^2U\Big)^\alpha a, a\Big\rangle\Big\langle \Big(V|M^*|^2V^*\Big)^\alpha b, b\Big\rangle. \end{aligned} \tag{20}$$

Combining (19) and (20), we infer that

$$\begin{aligned} |\langle VMNUa, b\rangle|^{2\alpha} &\leq \lambda\Big\langle \Big(U^*|N|^2U\Big)^\alpha a, a\Big\rangle\Big\langle \Big(V|M^*|^2V^*\Big)^\alpha b, b\Big\rangle \\ &\quad + (1-\lambda)|\langle VMNUa, b\rangle|^\alpha \sqrt{\Big\langle \Big(U^*|N|^2U\Big)^\alpha a, a\Big\rangle\Big\langle \Big(V|M^*|^2V^*\Big)^\alpha b, b\Big\rangle} \\ &\leq \Big\langle \Big(U^*|N|^2U\Big)^\alpha a, a\Big\rangle\Big\langle \Big(V|M^*|^2V^*\Big)^\alpha b, b\Big\rangle, \end{aligned}$$

for any $\alpha \geq 1$, which proves the inequality stated in (18). □

Corollary 1. *Let $M \in \mathscr{O}(\mathscr{H})$, $s, t \geq 0$, with $s + t \geq 1$, and $\alpha \geq 1$. Then, we have*

$$\left|\left\langle M|M|^{s+t-1}a, b\right\rangle\right|^{2\alpha} \leq \lambda\left\langle |M|^{2\alpha s}a, a\right\rangle\left\langle |M^*|^{2\alpha t}b, b\right\rangle + (1-\lambda)\left|\left\langle M|M|^{s+t-1}a, b\right\rangle\right|^{\alpha}\sqrt{\left\langle |M|^{2\alpha s}a, a\right\rangle\left\langle |M^*|^{2\alpha t}b, b\right\rangle} \leq \left\langle |M|^{2\alpha s}a, a\right\rangle\left\langle |M^*|^{2\alpha t}b, b\right\rangle, \tag{21}$$

for all $a, b \in \mathscr{H}$.

Proof. Let $M, Z \in \mathscr{O}(\mathscr{H})$ such that $Z|M|$ is the polar decomposition of M, where Z is partial isometry. Setting $V = Z$, $N = I$, $U = |M|^s$, and replacing M by $|M|^t$ such that $s + t \geq 1$ in (18), we obtain $VMNU = Z|M|^t|M|^s = Z|M||M|^{s+t-1} = M|M|^{s+t-1}$, and also $U^*|N|^2U = |M|^{2s}$, $V|M^*|^2V^* = Z|M|^{2t}Z^* = |M^*|^{2t}$. This completes the proof. □

Remark 1. *As an example, in Corollary 1, assume $s, t \in [0, 1]$ with $s + t = 1$. Then, the expression given in (21) reduces to Lemma 5 in [2], which refines the celebrated mixed Schwarz inequality stated in (11).*

In the next corollary, we show a refinement of the Cauchy–Schwarz inequality for arbitrary operators.

Corollary 2. *Let $N, M \in \mathscr{O}(\mathscr{H})$, $\lambda \in [0, 1]$, and $\alpha \geq 1$. Then, we get*

$$|\langle Na, Mb\rangle|^{2\alpha} \leq \lambda\left\langle |N|^{2\alpha}a, a\right\rangle\left\langle |M|^{2\alpha}b, b\right\rangle + (1-\lambda)|\langle Na, Mb\rangle|^{\alpha}\sqrt{\left\langle |M|^{2\alpha}b, b\right\rangle\left\langle |N|^{2\alpha}a, a\right\rangle} \leq \left\langle |N|^{2\alpha}a, a\right\rangle\left\langle |M|^{2\alpha}b, b\right\rangle,$$

for all a, b in $\mathscr{H}$.

Proof. The result follows by setting $V = U = I$ and replacing M by M^* in (18). □

3. Applications to Numerical Radius Inequalities

In this section, we present some applications of Corollary 1 with inequalities involving the operator norm and numerical radius. We start this section with the following theorem.

Theorem 1. *Let $M, N, U, V \in \mathscr{O}(\mathscr{H})$. Then, we arrive at*

$$\omega^{2\alpha}(VMNU) \leq \frac{1}{2}\lambda\left\|\left(U^*|N|^2U\right)^{2\alpha} + \left(V|M^*|^2V^*\right)^{2\alpha}\right\| + \frac{1}{2}(1-\lambda)\omega^{\alpha}(VMNU)\left\|\left(U^*|N|^2U\right)^{\alpha} + \left(V|M^*|^2V^*\right)^{\alpha}\right\| \leq \frac{1}{2}\left\|\left(U^*|N|^2U\right)^{2\alpha} + \left(V|M^*|^2V^*\right)^{2\alpha}\right\|, \tag{22}$$

for all $\alpha \geq 1$ and $\lambda \in [0, 1]$.

Proof. First, note that the well-known power-mean inequality states that

$$(\alpha u^p + (1-\alpha)v^p)^{\frac{1}{p}} \geq \alpha u + (1-\alpha)v \geq u^{\alpha}v^{1-\alpha}, \tag{23}$$

where $u, v > 0$, $\alpha \in [0, 1]$, and $p \geq 1$ [25].

Now, let $a = b$ and set $\alpha = 1$ in (18). Then, by applying the inequality (23), we obtain

$$\begin{aligned}
&|\langle (VMNU)a,a\rangle|^2 \leq \lambda\Big\langle \Big(U^*|N|^2U\Big)a,a\Big\rangle\Big\langle \Big(V|M^*|^2V^*\Big)a,a\Big\rangle \\
&\quad + (1-\lambda)|\langle VMNUa,a\rangle|\sqrt{\Big\langle \Big(U^*|N|^2U\Big)a,a\Big\rangle\Big\langle \Big(V|M^*|^2V^*\Big)a,a\Big\rangle} \\
&\leq \Bigg(\lambda\Big\langle \Big(U^*|N|^2U\Big)a,a\Big\rangle^{\alpha}\Big\langle \Big(V|M^*|^2V^*\Big)a,a\Big\rangle^{\alpha} \\
&\quad + (1-\lambda)|\langle VMNUa,a\rangle|^{\alpha}\sqrt{\Big\langle \Big(U^*|N|^2U\Big)a,a\Big\rangle^{\alpha}\Big\langle \Big(V|M^*|^2V^*\Big)a,a\Big\rangle^{\alpha}}\Bigg)^{\frac{1}{\alpha}}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&|\langle (VMNU)a,a\rangle|^{2\alpha} \leq \lambda\Big\langle \Big(U^*|N|^2U\Big)a,a\Big\rangle^{\alpha}\Big\langle \Big(V|M^*|^2V^*\Big)a,a\Big\rangle^{\alpha} \\
&\quad + (1-\lambda)|\langle VMNUa,a\rangle|^{\alpha}\sqrt{\Big\langle \Big(U^*|N|^2U\Big)a,a\Big\rangle^{\alpha}\Big\langle \Big(V|M^*|^2V^*\Big)a,a\Big\rangle^{\alpha}} \\
&\leq \lambda\Big\langle \Big(U^*|N|^2U\Big)^{\alpha}a,a\Big\rangle\Big\langle \Big(V|M^*|^2V^*\Big)^{\alpha}a,a\Big\rangle \quad \text{–by (3)–} \\
&\quad + (1-\lambda)|\langle VMNUa,a\rangle|^{\alpha}\Big\langle \Big(U^*|N|^2U\Big)^{\alpha}a,a\Big\rangle^{\frac{1}{2}}\Big\langle \Big(V|M^*|^2V^*\Big)^{\alpha}a,a\Big\rangle^{\frac{1}{2}} \\
&\leq \frac{\lambda\Big(\Big\langle \Big(U^*|N|^2U\Big)^{\alpha}a,a\Big\rangle + \Big\langle \Big(V|M^*|^2V^*\Big)^{\alpha}a,a\Big\rangle\Big)^2}{4} \quad \text{–by (23)–} \\
&\quad + \frac{(1-\lambda)}{2}|\langle VMNUa,a\rangle|^{\alpha}\Big(\Big\langle \Big(U^*|N|^2U\Big)^{\alpha}a,a\Big\rangle + \Big\langle \Big(V|M^*|^2V^*\Big)^{\alpha}a,a\Big\rangle\Big) \\
&\leq \frac{\lambda\Big\langle \Big(U^*|N|^2U\Big)^{2\alpha}a,a\Big\rangle + \Big\langle \Big(V|M^*|^2V^*\Big)^{2\alpha}a,a\Big\rangle}{2} \quad \text{–by (23)–} \\
&\quad + \frac{(1-\lambda)}{2}|\langle VMNUa,b\rangle|^{\alpha}\Big\langle \Big[\Big(U^*|N|^2U\Big)^{\alpha} + \Big(V|M^*|^2V^*\Big)^{\alpha}\Big]a,a\Big\rangle \\
&\leq \frac{\lambda}{2}\Big\langle \Big[\Big(U^*|N|^2U\Big)^{2\alpha} + \Big(V|M^*|^2V^*\Big)^{2\alpha}\Big]a,a\Big\rangle \\
&\quad + \frac{(1-\lambda)}{2}|\langle VMNUa,a\rangle|^{\alpha}\Big\langle \Big[\Big(U^*|N|^2U\Big)^{\alpha} + \Big(V|M^*|^2V^*\Big)^{\alpha}\Big]a,a\Big\rangle.
\end{aligned}$$

Hence, by obtaining the supremum over all unit vectors $a \in \mathscr{H}$, we reach the first inequality stated in (22). Moreover, to prove the second inequality in (22), we employ (8) on the first inequality, obtaining

$$\begin{aligned}
\omega^{2\alpha}(VMNU) &\leq \frac{\lambda}{2}\Big\|\Big(U^*|N|^2U\Big)^{2\alpha} + \Big(V|M^*|^2V^*\Big)^{2\alpha}\Big\| \\
&\quad + \frac{(1-\lambda)}{2}\omega^{\alpha}(VMNU)\Big\|\Big(U^*|N|^2U\Big)^{\alpha} + \Big(V|M^*|^2V^*\Big)^{\alpha}\Big\| \\
&\leq \frac{\lambda}{2}\Big\|\Big(U^*|N|^2U\Big)^{2\alpha} + \Big(V|M^*|^2V^*\Big)^{2\alpha}\Big\| \\
&\quad + \frac{(1-\lambda)}{4}\Big\|\Big(U^*|N|^2U\Big)^{\alpha} + \Big(V|M^*|^2V^*\Big)^{\alpha}\Big\|^2 \\
&= \frac{\lambda}{2}\Big\|\Big(U^*|N|^2U\Big)^{2\alpha} + \Big(V|M^*|^2V^*\Big)^{2\alpha}\Big\| \\
&\quad + \frac{(1-\lambda)}{4}\left\|\left(\frac{2\Big(U^*|N|^2U\Big)^{\alpha} + 2\Big(V|M^*|^2V^*\Big)^{\alpha}}{2}\right)^2\right\|
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\lambda}{2}\left\| \left(U^*|N|^2U\right)^{2\alpha} + \left(V|M^*|^2V^*\right)^{2\alpha} \right\| \\
&\quad + \frac{(1-\lambda)}{4}\left\| \frac{\left(2\left(U^*|N|^2U\right)^{\alpha}\right)^2 + \left(2\left(V|M^*|^2V^*\right)^{\alpha}\right)^2}{2} \right\| \quad \text{–by (4)–} \\
&\le \frac{\lambda}{2}\left\| \left(U^*|N|^2U\right)^{2\alpha} + \left(V|M^*|^2V^*\right)^{2\alpha} \right\| \\
&\quad + \frac{(1-\lambda)}{2}\left\| \left(U^*|N|^2U\right)^{2\alpha} + \left(V|M^*|^2V^*\right)^{2\alpha} \right\| \\
&= \frac{1}{2}\left\| \left(U^*|N|^2U\right)^{2\alpha} + \left(V|M^*|^2V^*\right)^{2\alpha} \right\|,
\end{aligned}
$$

which proves the second inequality stated in (22). □

Corollary 3. *Let $M \in \mathscr{O}(\mathscr{H})$, and $s,t \ge 0$, with $s+t \ge 1$. Then, we have*

$$
\begin{aligned}
\omega^{2\alpha}\left(M|M|^{s+t-1}\right) &\le \frac{\lambda}{2}\left\| |M|^{4\alpha s} + |M^*|^{4\alpha t} \right\| \\
&\quad + \frac{(1-\lambda)}{2}\omega^{\alpha}\left(M|M|^{s+t-1}\right)\left\| |M|^{2\alpha s} + |M^*|^{2\alpha t} \right\| \\
&\le \frac{1}{2}\left\| |M|^{4\alpha s} + |M^*|^{4\alpha t} \right\|,
\end{aligned}
$$

for all $\alpha \ge 1$ and $\lambda \in [0,1]$.

Proof. These inequalities are proved by the formula given in (22) and the corresponding technique presented in Corollary 1. □

Remark 2. *Setting $\lambda = 0$ in Corollary 3, the first inequality can be restated in a new form given by*

$$
\omega^{\alpha}\left(M|M|^{s+t-1}\right) \le \frac{1}{2}\left\| |M|^{2\alpha s} + |M^*|^{2\alpha t} \right\|,
$$

for all $s,t \ge 0$, such that $s+t \ge 1$, and $\alpha \ge 1$.

Theorem 2. *Let $M,N,U,V \in \mathscr{O}(\mathscr{H})$, $\alpha \ge 1$, and $\lambda \in [0,1]$. Then, we attain*

$$
\begin{aligned}
\omega^{2\alpha}(VMNU) &\le \frac{\lambda}{4}\left\| \left(U^*|N|^2U\right)^{\alpha} + \left(V|M^*|^2V^*\right)^{\alpha} \right\|^2 \\
&\quad + \frac{(1-\lambda)}{2}\omega^{\alpha}(VMNU)\left\| \left(U^*|N|^2U\right)^{\alpha} + \left(V|M^*|^2V^*\right)^{\alpha} \right\| \\
&\le \frac{\lambda}{2}\left\| \left(U^*|N|^2U\right)^{2\alpha} + \left(V|M^*|^2V^*\right)^{2\alpha} \right\| \\
&\quad + \frac{(1-\lambda)}{2}\omega^{\alpha}(VMNU)\left\| \left(U^*|N|^2U\right)^{\alpha} + \left(V|M^*|^2V^*\right)^{\alpha} \right\| \\
&\le \frac{1}{2}\left\| \left(U^*|N|^2U\right)^{2\alpha} + \left(V|M^*|^2V^*\right)^{2\alpha} \right\|.
\end{aligned} \tag{24}
$$

Proof. For all $\lambda \in [0,1]$, we have

$$\begin{aligned}\omega^{2\alpha}(VMNU) &= \lambda\omega^{2\alpha}(VMNU) + (1-\lambda)\omega^{2\alpha}(VMNU) \\ &= \lambda\omega^{2\alpha}(VMNU) + (1-\lambda)\omega^{\alpha}(VMNU)\omega^{\alpha}(VMNU) \\ &\le \frac{\lambda}{4}\left\|\left(U^*|N|^2U\right)^{\alpha} + \left(V|M^*|^2V^*\right)^{\alpha}\right\|^2 \\ &\quad + \frac{(1-\lambda)}{2}\omega^{\alpha}(VMNU)\left\|\left(U^*|N|^2U\right)^{\alpha} + \left(V|M^*|^2V^*\right)^{\alpha}\right\|. \quad \text{–by (8)–}\end{aligned}$$

Furthermore, by using (4), we reach

$$\begin{aligned}\omega^{2\alpha}(VMNU) &\le \frac{\lambda}{4}\left\|\left(U^*|N|^2U\right)^{\alpha} + \left(V|M^*|^2V^*\right)^{\alpha}\right\|^2 \\ &\quad + \frac{(1-\lambda)}{2}\omega^{\alpha}(VMNU)\left\|\left(U^*|N|^2U\right)^{\alpha} + \left(V|M^*|^2V^*\right)^{\alpha}\right\| \\ &= \frac{\lambda}{4}\left\|\left(\frac{2\left(U^*|N|^2U\right)^{\alpha} + 2\left(V|M^*|^2V^*\right)^{\alpha}}{2}\right)^2\right\| \\ &\quad + \frac{(1-\lambda)}{2}\omega^{\alpha}(VMNU)\left\|\left(U^*|N|^2U\right)^{\alpha} + \left(V|M^*|^2V^*\right)^{\alpha}\right\| \\ &\le \frac{\lambda}{4}\left\|\frac{\left(2\left(U^*|N|^2U\right)^{\alpha}\right)^2 + \left(2\left(V|M^*|^2V^*\right)^{\alpha}\right)^2}{2}\right\| \\ &\quad + \frac{(1-\lambda)}{2}\omega^{\alpha}(VMNU)\left\|\left(U^*|N|^2U\right)^{\alpha} + \left(V|M^*|^2V^*\right)^{\alpha}\right\| \\ &= \frac{\lambda}{2}\left\|\left(U^*|N|^2U\right)^{2\alpha} + \left(V|M^*|^2V^*\right)^{2\alpha}\right\| \\ &\quad + \frac{(1-\lambda)}{2}\omega^{\alpha}(VMNU)\left\|\left(U^*|N|^2U\right)^{\alpha} + \left(V|M^*|^2V^*\right)^{\alpha}\right\|, \quad \text{–by (22)–}\end{aligned}$$

which completes the proof. □

Corollary 4. *Let $M, N, U, V \in \mathcal{O}(\mathscr{H})$. Then, we get*

$$\begin{aligned}\omega^{2\alpha}(VMNU) &\le \frac{1}{12}\left\|\left(U^*|N|^2U\right)^{\alpha} + \left(V|M^*|^2V^*\right)^{\alpha}\right\|^2 \\ &\quad + \frac{1}{3}\omega^{\alpha}(VMNU)\left\|\left(U^*|N|^2U\right)^{\alpha} + \left(V|M^*|^2V^*\right)^{\alpha}\right\| \\ &\le \frac{1}{6}\left\|\left(U^*|N|^2U\right)^{2\alpha} + \left(V|M^*|^2V^*\right)^{2\alpha}\right\| \\ &\quad + \frac{1}{3}\omega^{\alpha}(VMNU)\left\|\left(U^*|N|^2U\right)^{\alpha} + \left(V|M^*|^2V^*\right)^{\alpha}\right\| \\ &\le \frac{1}{2}\left\|\left(U^*|N|^2U\right)^{2\alpha} + \left(V|M^*|^2V^*\right)^{2\alpha}\right\|.\end{aligned}$$

Proof. The results follows by setting $\lambda = 1/3$ in (24). □

Remark 3. *The inequality given in Corollary 4 is a refinement of* (14) *when $V = U = I$, $\alpha = 1$, $\lambda = 1/3$, and M is replaced by M^*.*

The following example shows that the inequality given in Corollary 4 is a refinement of (14) and hence of (9).

Example 1. *Let us set $V = U = I$, $\alpha = 1$, and replace M by M^* in Corollary 4. In addition, consider*

$$M = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix}.$$

Then, it is clear that $\omega(M^*N) = 3$, $\||N|^2 + |M|^2\| = 15.3007$, and $\||N|^4 + |M|^4\| = 129.3063$. Therefore, we have

$$\begin{aligned}
\omega^2(M^*N) &= 9 \\
&\le \frac{1}{12}\left\||N|^2 + |M|^2\right\|^2 + \frac{1}{3}\omega(M^*N)\left\||N|^2 + |M|^2\right\| \\
&= 34.80998504 \\
&\le \frac{1}{6}\left\||N|^4 + |M|^4\right\| + \frac{1}{3}\omega(M^*N)\left\||N|^2 + |M|^2\right\| \\
&= 36.85175667 \\
&\le \frac{1}{2}\left\||N|^4 + |M|^4\right\| \\
&= 64.65317000,
\end{aligned}$$

which is equivalent to write $3 = \omega(M^*N) \le 5.899999 \lneqq 6.070564 \lneqq 8.040719$.

Corollary 5. *Let $M \in \mathscr{O}(\mathscr{H})$ and $s, t \ge 0$, with $s + t \ge 1$. Then, we attain*

$$\begin{aligned}
\omega^{2\alpha}\left(M|M|^{s+t-1}\right) &\le \frac{\lambda}{4}\left\||M|^{2\alpha s} + |M^*|^{2\alpha t}\right\|^2 + \frac{(1-\lambda)}{2}\omega^{\alpha}\left(M|M|^{s+t-1}\right)\left\||M|^{2\alpha s} + |M^*|^{2\alpha t}\right\| \\
&\le \frac{\lambda}{2}\left\||M|^{4\alpha s} + |M^*|^{4\alpha t}\right\| + \frac{(1-\lambda)}{2}\omega^{\alpha}\left(M|M|^{s+t-1}\right)\left\||M|^{2\alpha t} + |M^*|^{2\alpha s}\right\| \\
&\le \frac{1}{2}\left\||M|^{4\alpha s} + |M^*|^{4\alpha t}\right\|,
\end{aligned} \tag{25}$$

for all $\alpha \ge 1$ and $\lambda \in [0, 1]$.

Proof. By using (24) and the technique employed in Corollary 1, we complete the proof. □

Corollary 6. *Let $M \in \mathscr{O}(\mathscr{H})$ and $s, t \ge 0$, with $s + t \ge 1$. Then, we have*

$$\begin{aligned}
\omega^{2\alpha}\left(M|M|^{s+t-1}\right) &\le \frac{1}{12}\left\||M|^{2\alpha s} + |M^*|^{2\alpha t}\right\|^2 + \frac{1}{3}\omega^{\alpha}\left(M|M|^{s+t-1}\right)\left\||M|^{2\alpha s} + |M^*|^{2\alpha t}\right\| \\
&\le \frac{1}{6}\left\||M|^{4\alpha s} + |M^*|^{4\alpha t}\right\| + \frac{1}{3}\omega^{\alpha}\left(M|M|^{s+t-1}\right)\left\||M|^{2\alpha s} + |M^*|^{2\alpha t}\right\| \\
&\le \frac{1}{2}\left\||M|^{4\alpha s} + |M^*|^{4\alpha t}\right\|,
\end{aligned} \tag{26}$$

for all $\alpha \ge 1$.

Proof. The result follows by setting $\lambda = 1/3$ in (25). □

Remark 4. *As a special case, assume $s, t \in [0, 1]$, with $s + t = 1$ in Corollary 6. Then, the first inequality given in (26) refines the inequality stated in (17).*

The following example shows that the first inequality given in (26) refines the inequality stated in (17).

Example 2. *Let*

$$M = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}.$$

Then, it is clear that $\omega(M) = 1$. By employing the inequalities presented in Corollary 6, with $s = t = 1/2$ and $\alpha = 1$, we arrive at $\||M|^2 + |M^*|^2\| = 4$, $\||M| + |M^*|\| = 2$. Thus, we have

$$\begin{aligned}
\omega^2(M) &= 1 \\
&\leq \frac{1}{12}\||M| + |M^*|\|^2 + \frac{1}{3}\omega(M)\||M| + |M^*|\| \\
&= 1 \\
&\lneqq \frac{1}{6}\left\||M|^2 + |M^*|^2\right\| + \frac{1}{3}\omega(M)\||M| + |M^*|\| \\
&= 1.33 \\
&\leq \frac{1}{2}\left\||M|^2 + |M^*|^2\right\| \\
&= 2.
\end{aligned}$$

This is equivalent to write $\omega(M) = 1 \lneqq 1.1547 \lneqq 1.41421$. Note that the first inequality gives the exact value of $\omega(M)$.

4. Concluding Remarks

In this work, we have improved Furuta's inequality. From this improvement, we have been able to obtain new radius inequalities. We have used some known inequalities to prove our results. Two numerical examples have illustrated our findings. We believe that the new inequalities obtained in this article can serve as the basis for further applications [26].

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