mathematics

Article

Lifts of a Quarter-Symmetric Metric Connection from a Sasakian Manifold to Its Tangent Bundle

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Abstract: The objective of this paper is to explore the complete lifts of a quarter-symmetric metric connection from a Sasakian manifold to its tangent bundle. A relationship between the Riemannian connection and the quarter-symmetric metric connection from a Sasakian manifold to its tangent bundle was established. Some theorems on the curvature tensor and the projective curvature tensor of a Sasakian manifold with respect to the quarter-symmetric metric connection to its tangent bundle were proved. Finally, locally ϕ-symmetric Sasakian manifolds with respect to the quarter-symmetric metric connection to its tangent bundle were studied.

Keywords: complete lift; tangent bundle; quarter-symmetric metric connection; Sasakian manifold; curvature tensor; projective curvature tensor; locally ϕ-symmetric

MSC: 53C05; 53C25; 58A30

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Citation: Khan, M.N.I.; De, U.C.; Velimirović, L.S. Lifts of a Quarter-Symmetric Metric Connection from a Sasakian Manifold to Its Tangent Bundle. *Mathematics* **2023**, *11*, 53. https://doi.org/10.3390/math11010053

Academic Editor: Adara M. Blaga

Received: 29 November 2022
Revised: 16 December 2022
Accepted: 19 December 2022
Published: 23 December 2022

1. Introduction

The study of the tangent bundle is a powerful method in geometry that allows us to retrieve effective results while studying various connections and geometric structures, such as a quarter-symmetric metric connection, a semi-symmetric connection, an almost complex structure and a contact structure on the manifold M admitting lifts to its tangent bundle TM. Peyghan et al. [1] studied the members of a golden structure on TM with a Riemannian metric and established the integrability condition of such a structure on TM. The complete lifts of connections such as quarter-symmetric metric connection and quarter-symmetric non-metric connection from the manifold M to TM have been studied by Akpinar [2], Altunbas et al. ([3,4]), Kazan and Karadag [5], Khan [6]. For the recent studies on lifts of connections and geometric structures, we refer to ([7–11]) and many more.

The definition and discussion of a quarter-symmetric connection on a Riemannian manifold, on the other hand, were provided by Golab [12].

A linear connection $\tilde{\nabla}$ on a Riemannian manifold M (dim$=n$) with a Reimannian metric g is called a quarter-symmetric connection if its torsion tensor T of the connection $\tilde{\nabla}$

$$T(\zeta_1,\zeta_2) = \tilde{\nabla}_{\zeta_1}\zeta_2 - \tilde{\nabla}_{\zeta_2}\zeta_1 - [\zeta_1,\zeta_2] \tag{1}$$

satisfies

$$T(\zeta_1,\zeta_2) = \eta(\zeta_2)\phi\zeta_1 - \eta(\zeta_1)\phi\zeta_2, \tag{2}$$

where η is a 1-form and ϕ is a tensor field of type (1,1).

In addition, if $\tilde{\nabla}$ fulfills

$$(\tilde{\nabla}_{\zeta_1}g)(\zeta_2,\zeta_3) = 0, \tag{3}$$

$\forall \zeta_1, \zeta_2, \zeta_3 \in \Im_0^1(M)$, then $\widetilde{\nabla}$ is called a quarter-symmetric metric connection; otherwise, it is called a quarter-symmetric non-metric connection ([13–15]). The quarter-symmetric metric connections on different manifolds such as Riemannian, Hermitian, Kaehlerian, Kenmotsu and Sasakian manifolds have been studied by Mondol and De [16], Mishra and Pandey [17], Mukhopadhyay et al. [18], Bahadir [19], Sular et al. [20] and many more.

We established certain curvature properties on TM and explored the lifts of a quarter-symmetric metric connection from a Sasakian manifold to TM. The results of this paper are given as:

- We established a relationship between the Riemannian connection and the quarter-symmetric metric connection from a Sasakian manifold to TM.
- We derived the expression of the curvature tensor of a Sasakian manifold equipped with a quarter-symmetric metric connection to TM.
- We studied a ξ-projectively flat Sasakian manifold endowed with a quarter-symmetric metric connection to TM.
- We locally characterized a ϕ-symmetric Sasakian manifold admitting a quarter-symmetric metric connection to TM.

2. Preliminaries

Let us consider TM to be the tangent bundle of a manifold M. The set of all tensor fields of type (r,s) that are of contravariant degree r and covariant degree s in M and TM are denoted by $\Im_s^r(M)$ and $\Im_s^r(TM)$, respectively. Let the function, a 1-form, a vector field and a tensor field of type (1,1) be symbolized as f, η, ζ_1 and ϕ, respectively. The complete and vertical lifts of f, η, ζ_1, ϕ are symbolized as $f^C, \eta^C, \zeta_1^C, \phi^C$ and $f^V, \eta^V, \zeta_1^V, \phi^V$, respectively. The following operations on f, η, ζ_1 and ϕ are defined by [21,22]

$$(f\zeta_1)^V = f^V\zeta_1^V, (f\zeta_1)^C = f^C\zeta_1^V + f^V\zeta_1^C, \tag{4}$$

$$\zeta_1^V f^V = 0, \zeta_1^V f^C = \zeta_1^C f^V = (\zeta_1 f)^V, \zeta_1^C f^C = (\zeta_1 f)^C, \tag{5}$$

$$\eta^V(f^V) = 0, \eta^V(\zeta_1^C) = \eta^C(\zeta_1^V) = \eta(\zeta_1)^V, \eta^C(\zeta_1^C) = \eta(\zeta_1)^C, \tag{6}$$

$$\phi^V\zeta_1^C = (\phi\zeta_1)^V, \phi^C\zeta_1^C = (\phi\zeta_1)^C, \tag{7}$$

$$[\zeta_1,\zeta_2]^V = [\zeta_1^C,\zeta_2^V] = [\zeta_1^V,\zeta_2^C], [\zeta_1,\zeta_2]^C = [\zeta_1^C,\zeta_2^C]. \tag{8}$$

$$\nabla^C_{\zeta_1^C}\zeta_2^C = (\nabla_{\zeta_1}\zeta_2)^C, \qquad \nabla^C_{\zeta_1^C}\zeta_2^V = (\nabla_{\zeta_1}\zeta_2)^V, \tag{9}$$

where ∇ is the Levi–Civita connection.

Let M be a contact metric manifold of dimension n with a contact metric structure (ϕ, ξ, η, g) fulfilling the conditions [23]

$$\eta(\zeta_1) = g(\zeta_1,\xi), \ \phi^2 = -\zeta_1 + \eta(\zeta_1)\xi, \tag{10}$$

$$\phi\xi = 0, \quad \eta(\xi) = 1, \quad \eta.\phi = 0, \tag{11}$$

$$g(\phi\zeta_1,\phi\zeta_2) = g(\zeta_1,\zeta_2) - \eta(\zeta_1)\eta(\zeta_2), \tag{12}$$

where ϕ is a (1,1) tensor, ξ is a vector field, called the characteristic vector field, and η is a 1-form. If M satisfies

$$(\nabla_{\zeta_1}\phi)\zeta_2 = g(\zeta_1,\zeta_2)\xi - \eta(\zeta_2)\zeta_1, \tag{13}$$

then M is named a Sasakian manifold. In addition, the following properties hold on a Sasakian manifold M:

$$
\begin{aligned}
\nabla_{\zeta_1}\xi &= -\phi\zeta_1, & (14)\\
(\nabla_{\zeta_1}\eta)\zeta_2 &= g(\zeta_1,\phi\zeta_2), & (15)\\
R(\zeta_1,\zeta_2)\xi &= \eta(\zeta_2)\zeta_1-\eta(\zeta_1)\zeta_2, & (16)\\
R(\xi,\zeta_1)\zeta_2 &= (\nabla_{\zeta_1}\phi)\zeta_2), & (17)\\
S(\zeta_1,\xi) &= (n-1)\eta(\zeta_1), & (18)\\
S(\phi\zeta_1,\phi\zeta_2) &= S(\zeta_1,\zeta_2)-(n-1)\eta(\zeta_1)\eta(\zeta_2), & (19)
\end{aligned}
$$

where $\forall\zeta_1,\zeta_2\in\Im^1_0(M)$, R and S indicate the curvature tensor and the Ricci tensor, respectively.

3. Complete Lifts from a Sasakian Manifold to Its Tangent Bundle

Let us consider TM to be the tangent bundle of a Sasakian manifold M. Taking complete lifts on both sides of Equations (1), (2) and (10)–(32), we infer that

$$
\begin{aligned}
T^C(\zeta_1^C,\zeta_2^C) &= \tilde{\nabla}^C_{\zeta_1^C}\zeta_2^C-\tilde{\nabla}^C_{\zeta_2^C}\zeta_1^C-[\zeta_1^C,\zeta_2^C], & (20)\\
T^C(\zeta_1^C,\zeta_2^C) &= \pi^C(\zeta_2^C)(\phi\zeta_1)^V+\pi^V(\zeta_2^C)(\phi\zeta_1)^C\\
&- \pi^C(\zeta_1^C)(\phi\zeta_2)^V-\pi^V(\zeta_1^C)(\phi\zeta_2)^C, & (21)\\
\pi^C(\zeta_1^C) &= g^C(\xi^C,\zeta_1^C), & (22)\\
(d\eta(\zeta_1,\zeta_2)^C &= g^C((\phi\zeta_1)^C,\zeta_2^C),\quad \eta^C(\zeta_1^C)=g^C(\zeta_1^C,\xi^C)\\
(\phi^2)^C\zeta_1 &= -\zeta_1^C+\eta^C(\zeta_1^C)\xi^V+\eta^V(\zeta_1^C)\xi^C, & (23)\\
\phi^C\xi^V &= \phi^V\xi^C=\phi^V\xi^V=\phi^C\xi^C=0,\\
\eta^C\xi^C &= \eta^V\xi^V=0,\ \ \eta^C\xi^V=\eta^V\xi^C=1\\
\eta^V\circ\phi^C &= \eta^C\circ\phi^V=\eta^C\circ\phi^C=\eta^V\circ\phi^V=0, & (24)
\end{aligned}
$$

$$
\begin{aligned}
g((\phi\zeta_1)^C,(\phi\zeta_2)^C) &= g^C(\zeta_1^C,\zeta_2^C)-\eta^C(\zeta_1^C)\eta^V(\zeta_2^C)\\
&- \eta^V(\zeta_1^C)\eta^C(\zeta_2^C), & (25)\\
(\nabla^C_{\zeta_1^C}\phi^C)\zeta_2^C &= g^C(\zeta_1^C,\zeta_2^C)\xi^V+g^C(\zeta_1^V,\zeta_2^C)\xi^C\\
&- \eta^C(\zeta_2^C)\zeta_1^V-\eta^V(\zeta_2^C)\zeta_1^C, & (26)\\
\nabla^C_{\zeta_1^C}\xi^C &= -(\phi\zeta_1)^C, & (27)\\
(\nabla^C_{\zeta_1^C}\eta^C)\zeta_2^C &= g^C(\zeta_1^C,(\phi\zeta_2)^C), & (28)\\
R^C(\zeta_1^C,\zeta_2^C)\xi^C &= \eta^C(\zeta_2^C)\zeta_1^V+\eta^V(\zeta_2^C)\zeta_1^C-\eta^C(\zeta_1^C)\zeta_2^V\\
&- \eta^V(\zeta_1^C)\zeta_2^C, & (29)\\
R^C(\xi^C,\zeta_1^C)\zeta_2^C &= (\nabla^C_{\zeta_1^C}\phi^C)\zeta_2^C, & (30)\\
S^C(\zeta_1^C,\xi^C) &= (n-1)\eta^C(\zeta_1^C), & (31)\\
S^C((\phi\zeta_1)^C,(\phi\zeta_2)^C) &= S^C(\zeta_1^C,\zeta_2^C)-(n-1)\{\eta^C(\zeta_1^C)\eta^V(\zeta_2^C)\\
&+ \eta^V(\zeta_1^C)\eta^C(\zeta_2^C)\}, & (32)
\end{aligned}
$$

where $\forall\zeta_1^C,\zeta_2^C,\xi^C\in\Im^1_0(TM),\phi^C\in\Im^1_1(TM)$.

4. Relation between the Riemannian Connection and the Quarter-Symmetric Metric Connection from a Sasakian Manifold to Its Tangent Bundle

Assuming that M is an almost contact metric manifold, let $\widetilde{\nabla}$ be a linear connection and ∇ be a Riemannian connection. Then,

$$\widetilde{\nabla}_X Y=\nabla_X Y+U(\zeta_1,\zeta_2), \tag{33}$$

where $\forall\zeta_1,\zeta_2\in\Im^1_0(M),U\in\Im^1_2(M)$. Let $\widetilde{\nabla}$ be a quarter-symmetric metric connection in M. Then [12],

$$U(\zeta_1, \zeta_2) = \frac{1}{2}[T(\zeta_1, \zeta_2) + T'(\zeta_1, \zeta_2) + T'(\zeta_2, \zeta_1), \tag{34}$$

where T' is a (1,2) tensor; that is, $T' \in \Im^1_2(M)$ such that

$$g(T'(\zeta_1, \zeta_2), \zeta_3) = g(T'(\zeta_3, \zeta_1), \zeta_2). \tag{35}$$

Taking complete lifts on both sides of Equations (34)–(36), we infer that

$$\widetilde{\nabla}^C_{\zeta_1^C} \zeta_2^C = \nabla^C_{\zeta_1^C} \zeta_2^C + U^C(\zeta_1^C, \zeta_2^C), \tag{36}$$

$$U^C(\zeta_1^C, \zeta_2^C) = \frac{1}{2}[T^C(\zeta_1^C, \zeta_2^C) + T'^C(\zeta_1^C, \zeta_2^C) + T'^C(\zeta_2^C, \zeta_1^C)], \tag{37}$$

$$g^C(T'^C(\zeta_1^C, \zeta_2^C), \zeta_3^C) = g^C(T'^C(\zeta_3^C, \zeta_1^C), \zeta_2^C), \tag{38}$$

where U^C, ∇^C, T^C and T'^C are complete lifts of U, ∇, T and T', respectively.
From (21) and (38), we infer that

$$T'^C(\zeta_1^C, \zeta_2^C) = g^C((\phi\zeta_2)^C, \zeta_1^C)\xi^V + g^C((\phi\zeta_2)^V, \zeta_1^C)\xi^C - \eta^C(\zeta_1^C)(\phi\zeta_2)^V - \eta^V(\zeta_1^C)(\phi\zeta_2)^C. \tag{39}$$

Using (21) and (39) in (37), we provide

$$U^C(\zeta_1^C, \zeta_2^C) = -\eta^C(\zeta_1^C)(\phi\zeta_2)^V - \eta^V(\zeta_1^C)(\phi\zeta_2)^C.$$

Hence, a quarter-symmetric metric connection $\widetilde{\nabla}^C$ on a Sasakian manifold on TM is defined by

$$\widetilde{\nabla}^C_{\zeta_1^C} \zeta_2^C = \nabla^C_{\zeta_1^C} \zeta_2^C - \eta^C(\zeta_1^C)(\phi\zeta_2)^V - \eta^V(\zeta_1^C)(\phi\zeta_2)^C. \tag{40}$$

In contrast, we demonstrate that a linear connection $\widetilde{\nabla}$ on a Sasakian manifold defined by

$$\widetilde{\nabla}^C_{\zeta_1^C} \zeta_2^C = \nabla^C_{\zeta_1^C} \zeta_2^C - \eta^C(\zeta_1^C)(\phi\zeta_2)^V - \eta^V(\zeta_1^C)(\phi\zeta_2)^C. \tag{41}$$

denotes a quarter-symmetric metric connection on TM.
In view of (41), the torsion tensor of the connection $\widetilde{\nabla}^C$ on TM is defined by

$$T^C(\zeta_1^C, \zeta_2^C) = \eta^C(\zeta_2^C)(\phi\zeta_1)^V + \eta^V(\zeta_2^C)(\phi\zeta_1)^C - \eta^C(\zeta_1^C)(\phi\zeta_2)^V - \eta^V(\zeta_1^C)(\phi\zeta_2)^C. \tag{42}$$

The Equation (42) implies that $\widetilde{\nabla}^C$ is a quarter-symmetric connection on TM . Further, we infer that

$$(\widetilde{\nabla}^C_{\zeta_1^C} g^C)(\zeta_2^C, \zeta_3^C) = \zeta_1^C g^C(\zeta_2^V, \zeta_3^C) + \zeta_1^V g^C(\zeta_2^C, \zeta_3^C) - g^C(\widetilde{\nabla}^C_{\zeta_1^C} \zeta_2^C, \zeta_3^C). \tag{43}$$

In view of (42) and (43), $\widetilde{\nabla}^C$ is a quarter-symmetric metric connection on TM. The relationship between the Riemannian connection and the quarter-symmetric metric connection on a Sasakian manifold on TM is given by (41).

5. Expression of the Curvature Tensor of a Sasakian Manifold to Its Tangent Bundle

The two curvature tensors R and $\tilde{R}$ corresponding to the connections $\widetilde{\nabla}$ and ∇, respectively, are related by the formula [24]

$$\widetilde{R}(\zeta_1, \zeta_2)\zeta_3 = R(\zeta_1, \zeta_2)\zeta_3 - 2d\eta(\zeta_1, \zeta_2)\phi\zeta_3 + \eta(\zeta_1)g(\zeta_2, \zeta_3)\xi - \eta(\zeta_2)g(\zeta_1, \zeta_3)\xi + \{\eta(\zeta_2)\zeta_1 - \eta(\zeta_1)\zeta_2\}\eta(\zeta_3), \tag{44}$$

where $R(\zeta_1,\zeta_2)\zeta_3$ indicates the Riemannian curvature of M.

Taking complete lifts on both sides of (44), we infer that

$$\begin{aligned}\widetilde{R}^C(\zeta_1^C,\zeta_2^C)\zeta_3^C &= R^C(\zeta_1^C,\zeta_2^C)\zeta_3^C - 2d\eta^C(\zeta_1^C,\zeta_2^C)(\phi\zeta_3)^V \\ &- 2d\eta^V(\zeta_1^C,\zeta_2^C)(\phi\zeta_3)^C + \eta^C(\zeta_1^C)g^C(\zeta_2^C,\zeta_3^C)\xi^V \\ &+ \eta^C(\zeta_1^C)g^C(\zeta_2^V,\zeta_3^C)\xi^C + \eta^V(\zeta_1^C)g^C(\zeta_2^C,\zeta_3^C)\xi^C \\ &- \eta^C(\zeta_2^C)g^C(\zeta_1^C,\zeta_3^C)\xi^V - \eta^C(\zeta_2^C)g^C(\zeta_1^V,\zeta_3^C)\xi^C \\ &- \eta^V(\zeta_2^C)g^C(\zeta_1^C,\zeta_3^C)\xi^C + \eta^C(\zeta_2^C)\eta^C(\zeta_3^C)\zeta_1^V \\ &+ \eta^C(\zeta_2^C)\eta^V(\zeta_3^C)\zeta_1^C + \eta^V(\zeta_2^C)\eta^C(\zeta_3^C)\zeta_1^C \\ &- \{\eta^C(\zeta_1^C)\eta^C(\zeta_3^C)\zeta_2^V + \eta^C(\zeta_1^C)\eta^V(\zeta_3^C)\zeta_2^C \\ &+ \eta^V(\zeta_1^C)\eta^C(\zeta_3^C)\zeta_2^C\},\end{aligned} \tag{45}$$

where $R^C(\zeta_1^C,\zeta_2^C)\zeta_3^C$ is the complete lift of $R(\zeta_1,\zeta_2)\zeta_3$.

On contracting (45), we infer that

$$\begin{aligned}\widetilde{S}^C(\zeta_2^C,\zeta_3^C) &= S^C(\zeta_2^C,\zeta_3^C) - 2d\eta^C((\phi\zeta_3)^C,\zeta_2^C) + g^C(\zeta_2^C,\zeta_3^C) \\ &+ (n-2)\{\eta^C(\zeta_2^C)\eta^V(\zeta_3^C) + \eta^V(\zeta_2^C)\eta^C(\zeta_3^C)\},\end{aligned} \tag{46}$$

where $\widetilde{S}^C$ and S^C are the complete lifts of the Ricci tensors $\widetilde{S}$ and S of the connections $\widetilde{\nabla}$ and ∇, respectively. From (46), we infer that the Ricci tensor with regard to $\widetilde{\nabla}^C$ on a Sasakian manifold on TM is symmetric.

Again contracting (46), we infer that

$$\widetilde{r}^C = r^C + 2(n-1),$$

where $\widetilde{r}^C$ and r^C on TM are the complete lifts of the scalar curvatures $\widetilde{r}$ and r of the connections $\widetilde{\nabla}$ and ∇, respectively.

6. Expression of the Projective Curvature Tensor of a Sasakian Manifold to Its Tangent Bundle

The projective curvature tensor of a Sasakian manifold with regard to $\widetilde{\nabla}$ is given by [17]

$$\begin{aligned}\widetilde{P}(\zeta_1,\zeta_2)\zeta_3 &= \widetilde{R}(\zeta_1,\zeta_2)\zeta_3 + \frac{1}{n+1}[\widetilde{S}(\zeta_1,\zeta_2)\zeta_3 - \widetilde{S}(\zeta_2,\zeta_1)\zeta_3] \\ &+ \frac{1}{n^2-1}[\{n\widetilde{S}(\zeta_1,\zeta_3) + \widetilde{S}(\zeta_3,\zeta_1)\}\zeta_2 \\ &- \{n\widetilde{S}(\zeta_2,\zeta_3) + \widetilde{S}(\zeta_3,\zeta_2)\}\zeta_1].\end{aligned} \tag{47}$$

Due to the symmetric property of the Ricci tensor $\widetilde{S}$ of M with regard to $\widetilde{\nabla}$, the projective curvature tensor $\widetilde{P}$ becomes

$$\widetilde{P}(\zeta_1,\zeta_2)\zeta_3 = \widetilde{R}(\zeta_1,\zeta_2)\zeta_3 + \frac{1}{n+1}[\widetilde{S}(\zeta_1,\zeta_2)\zeta_3 - \widetilde{S}(\zeta_2,\zeta_1)\zeta_3]. \tag{48}$$

Taking complete lifts on both sides of (48), we acquire

$$\begin{aligned}\widetilde{P}^C(\zeta_1^C,\zeta_2^C)\zeta_3^C &= \widetilde{R}^C(\zeta_1^C,\zeta_2^C)\zeta_3^C \\ &+ \frac{1}{n+1}[\widetilde{S}^C(\zeta_1^C,\zeta_2^C)\zeta_3^V + \widetilde{S}^V(\zeta_1^C,\zeta_2^C)\zeta_3^C \\ &- \widetilde{S}^C(\zeta_2^C,\zeta_1^C)\zeta_3^V - \widetilde{S}^V(\zeta_2^C,\zeta_1^C)\zeta_3^C].\end{aligned} \tag{49}$$

Using (45) and (46), (49) reduces to

$$
\begin{aligned}
\widetilde{P}^C(\zeta_1^C,\zeta_2^C)\zeta_3^C &= P^C(\zeta_1^C,\zeta_2^C)\zeta_3^C - 2d\eta^C(\zeta_1^C,\zeta_2^C)(\phi\zeta_3)^V \\
&- 2d\eta^V(\zeta_1^C,\zeta_2^C)(\phi\zeta_3)^C + \eta^C(\zeta_1^C)g^C(\zeta_2^C,\zeta_3^C)\xi^V \\
&+ \eta^C(\zeta_1^C)g^C(\zeta_2^V,\zeta_3^C)\xi^C + \eta^V(\zeta_1^C)g^C(\zeta_2^C,\zeta_3^C)\xi^C \\
&- \eta^C(\zeta_2^C)g^C(\zeta_1^C,\zeta_3^C)\xi^V - \eta^C(\zeta_2^C)g^C(\zeta_1^V,\zeta_3^C)\xi^C \\
&- \eta^V(\zeta_2^C)g^C(\zeta_1^C,\zeta_3^C)\xi^C + \frac{2}{n-1}[d\eta^C((\phi\zeta_3)^C,\zeta_2^C)\zeta_1^V \\
&+ d\eta^V((\phi\zeta_3)^C,\zeta_2^C)\zeta_1^C] - d\eta^C((\phi\zeta_3)^C,\zeta_1^C)\zeta_2^V \\
&+ d\eta^V((\phi\zeta_3)^C,\zeta_1^C)\zeta_2^C + \frac{1}{n-1}\{\eta^C(\zeta_2^C)\eta^C(\zeta_3^C)\zeta_1^V \\
&+ \eta^C(\zeta_2^C)\eta^V(\zeta_3^C)\zeta_1^C + \eta^V(\zeta_2^C)\eta^C(\zeta_3^C)\zeta_1^C \\
&- \eta^C(\zeta_1^C)\eta^C(\zeta_3^C)\zeta_2^V - \eta^C(\zeta_1^C)\eta^V(\zeta_3^C)\zeta_2^C \\
&- \eta^V(\zeta_1^C)\eta^C(\zeta_3^C)\zeta_2^C - g^C(\zeta_2^C,\zeta_3^C)\zeta_1^V \\
&- g^C(\zeta_2^V,\zeta_3^C)\zeta_1^C + g^C(\zeta_1^C,\zeta_3^C)\zeta_2^V \\
&+ g^C(\zeta_1^V,\zeta_3^C)\zeta_2^C\},
\end{aligned} \tag{50}
$$

where P^C is the complete lift of the projective curvature tensor P defined by

$$P(\zeta_1,\zeta_2)\zeta_3 = R(\zeta_1,\zeta_2)\zeta_3 - \frac{1}{n-1}\{S(\zeta_2,\zeta_3)\zeta_1 - S(\zeta_1,\zeta_3)\zeta_2\}. \tag{51}$$

Mondol and De [16] defined that "A Sasakian manifold M is called ξ-projectively flat if the condition $P(\zeta_1,\zeta_2)\xi = 0$ holds on M".

According to the above definition, from (50), we acquire $\widetilde{P}(\zeta_1,\zeta_2)\xi = P(\zeta_1,\zeta_2)\xi$.

Hence, we conclude the following:

Theorem 1. *Let TM be the tangent bundle of a Sasakian manifold M with the Riemannian connection ∇. The Riemannian connection ∇^C on TM is ξ^C-projectively flat if and only if $\widetilde{\nabla}^C$ is so.*

Özgür [25] defined that "a Sasakian manifold fulfilling

$$\phi^2 P(\phi\zeta_1,\phi\zeta_2)\phi\zeta_3 = 0 \tag{52}$$

is called ϕ-projectively flat".

In the case of the quarter-symmetric metric connection $\widetilde{\nabla}$, we see that $\phi^2\widetilde{P}(\phi\zeta_1,\phi\zeta_2)\phi\zeta_3 = 0$ remain invariant if and only if

$$g(\widetilde{P}(\phi\zeta_1,\phi\zeta_2)\phi\zeta_3,\phi\zeta_4) = 0, \tag{53}$$

for $\zeta_1,\zeta_2,\zeta_3,\zeta_4 \in \Im_0^1(M)$.

In view of (49) and (53), ϕ-projectively flat means

$$
\begin{aligned}
& g^C(\widetilde{R}^C((\phi\zeta_1)^C,(\phi\zeta_2)^C)(\phi\zeta_3)^C,(\phi\zeta_4)^C) \\
= \; & \frac{1}{n-1}\{\widetilde{S}^C((\phi\zeta_2)^C,(\phi\zeta_3)^C)g^V((\phi\zeta_1)^C,(\phi\zeta_4)^C) \\
+ \; & \widetilde{S}^V((\phi\zeta_2)^C,(\phi\zeta_3)^C)g^C((\phi\zeta_1)^C,(\phi\zeta_4)^C) \\
- \; & -\widetilde{S}^C((\phi\zeta_1)^C,(\phi\zeta_3)^C)g^V((\phi\zeta_2)^C,(\phi\zeta_4)^C) \\
- \; & \widetilde{S}^V((\phi\zeta_1)^C,(\phi\zeta_3)^C)g^C((\phi\zeta_2)^C,(\phi\zeta_4)^C)\}.
\end{aligned} \tag{54}
$$

If $(e_1^C,e_2^C,......,e_{n-1}^C,\xi^C) \in TM$, then
$((\phi e_1)^C,(\phi e_2)^C,......,(\phi e_{n-1})^C,\xi^C) \in TM$.

Substituting $\zeta_1 = \zeta_4 = e_i$ into (54) and summing up with regard to $i = 1,2,......,n-1$, we acquire

$$
\begin{aligned}
& g^C(\widetilde{R}^C((\phi e_i)^C,(\phi\zeta_2)^C)(\phi\zeta_3)^C,(\phi e_i)^C) \\
= & \frac{1}{n-1}\{\widetilde{S}^C((\phi\zeta_2)^C,(\phi\zeta_3)^C)g^V((\phi e_i)^C,(\phi e_i)^C) \\
+ & \widetilde{S}^V((\phi\zeta_2)^C,(\phi\zeta_3)^C)g^C((\phi e_i)^C,(\phi e_i)^C) \\
- & \widetilde{S}^C((\phi e_i)^C,(\phi\zeta_3)^C)g^V((\phi\zeta_2)^C,(\phi e_i)^C) \\
- & \widetilde{S}^V((\phi e_i)^C,(\phi\zeta_3)^C)g^C((\phi\zeta_2)^C,(\phi e_i)^C)\}. \qquad (55)
\end{aligned}
$$

Using (23), (24), (28) and (46), the following equations are obtained:

$$
\begin{aligned}
& g^C(\widetilde{R}^C((\phi e_i)^C,(\phi\zeta_2)^C)(\phi\zeta_3)^C,(\phi e_i)^C) \qquad (56) \\
= & g^C(\widetilde{R}^C((\phi e_i)^C,(\phi\zeta_2)^C)(\phi\zeta_3)^C,(\phi e_i)^C) \\
- & 2g^C((\phi\zeta_2)^C,(\phi\zeta_3)^C) \\
= & S^C(\zeta_2^C,\zeta_3^C)-R^C(\xi^C,\zeta_2^C,\zeta_3^C,\xi^C) \\
- & (n-1)\{\eta^C(\zeta_2^C)\eta^V(\zeta_3^C)+\eta^V(\zeta_2^C)\eta^C(\zeta_3^C)\} \\
- & 2g^C((\phi\zeta_2)^C,(\phi\zeta_3)^C) \\
= & \widetilde{S}^C(\zeta_2^C,\zeta_3^C)-6g^C(\zeta_2^C,\zeta_3^C) \\
- & 2(n-4)\{\eta^C(\zeta_2^C)\eta^V(\zeta_3^C)+\eta^V(\zeta_2^C)\eta^C(\zeta_3^C)\}, \qquad (57)
\end{aligned}
$$

$$
\sum_{i=1}^{n-1} g^C((\phi e_i)^C,(\phi e_i)^C) = n-1, \qquad (58)
$$

$$
\sum_{i=1}^{n-1}(\widetilde{S}(\phi e_i,\phi\zeta_3)g(\phi\zeta_2,\phi e_i))^C = \widetilde{S}^C((\phi\zeta_2)^C,(\phi\zeta_3)^C). \qquad (59)
$$

In view of (56), (58) and (59), Equation (55) becomes

$$
\begin{aligned}
\widetilde{S}^C(\zeta_2^C,\zeta_3^C)-6g^C(\zeta_2^C,\zeta_3^C) & - 2(n-4)\{\eta^C(\zeta_2^C)\eta^V(\zeta_3^C)+\eta^V(\zeta_2^C)\eta^C(\zeta_3^C)\} \\
& = \frac{n-2}{n-1}\widetilde{S}^C((\phi\zeta_2)^C,(\phi\zeta_3)^C). \qquad (60)
\end{aligned}
$$

In view of (31) and (46), (60) becomes

$$
\widetilde{S}^C(\zeta_2^C,\zeta_3^C)=6g^C(\zeta_2^C,\zeta_3^C)-4(n-1)\{\eta^C(\zeta_2^C)\eta^V(\zeta_3^C)+\eta^V(\zeta_2^C)\eta^C(\zeta_3^C)\}. \qquad (61)
$$

Hence, we conclude the following:

Theorem 2. *Let TM be the tangent bundle of a Sasakian manifold M with regard to $\widetilde{\nabla}$. If a Sasakian manifold M on TM is ϕ^C-projectively flat with regard to $\widetilde{\nabla}^C$, then the manifold is an η^C-Einstein manifold with regard to $\widetilde{\nabla}^C$ on TM.*

7. Locally ϕ-Symmetric Sasakian Manifold with regard to the Quarter-Symmetric Metric Connection to its Tangent Bundle

Takahashi [26] defined that "A Sasakian manifold is said to be locally ϕ-symmetric if

$$
\phi^2(\nabla_{\zeta_4}R)(\zeta_1,\zeta_2)\zeta_3=0, \qquad (62)
$$

for all vector fields $\zeta_4,\zeta_1,\zeta_2,\zeta_3$ orthogonal to ξ, where ξ is the characteristic vector field of the Sasakian manifold M." Further, Mondal and De [16] defined locally ϕ-symmetric Sasakian manifold with regard to $\widetilde{\nabla}$ as

$$
\phi^2(\widetilde{\nabla}_{\zeta_4}\widetilde{R})(\zeta_1,\zeta_2)\zeta_3=0, \qquad (63)
$$

where $\zeta_4,\zeta_1,\zeta_2,\zeta_3$ are orthogonal to ξ. In view of (40), we infer that

$$((\widetilde{\nabla}_{\zeta_4}\widetilde{R})(\zeta_1,\zeta_2)\zeta_3)^C = ((\nabla_{\zeta_4}\widetilde{R})(\zeta_1,\zeta_2)\zeta_3)^C - \eta^C(\zeta_4^C)(\phi\widetilde{R}(\zeta_1,\zeta_2)\zeta_3)^V - \eta^V(\zeta_4^C)(\phi\widetilde{R}(\zeta_1,\zeta_2)\zeta_3)^C. \tag{64}$$

Now, differentiating (44) with regard to ζ_4, we infer that

$$\begin{aligned}((\widetilde{\nabla}_{\zeta_4}\widetilde{R})(\zeta_1,\zeta_2)\zeta_3)^C &= ((\nabla_{\zeta_4}\widetilde{R})(\zeta_1,\zeta_2)\zeta_3)^C - 2d\eta^C(\zeta_1^C,\zeta_2^C)((\nabla_{\zeta_4}\phi)\zeta_3)^V \\
&- 2d\eta^V(\zeta_1^C,\zeta_2^C)((\nabla_{\zeta_4}\phi)\zeta_3)^C + (\nabla_{\zeta_4}\eta)^C(\zeta_1^C)g^C(\zeta_2^C,\zeta_3^C)\xi^V \\
&+ (\nabla_{\zeta_4}\eta)^C(\zeta_1^C)g^C(\zeta_2^V,\zeta_3^C)\xi^C + (\nabla_{\zeta_4}\eta)^V(\zeta_1^C)g^C(\zeta_2^C,\zeta_3^C)\xi^C \\
&- (\nabla_{\zeta_4}\eta)^C(\zeta_2^C)g^C(\zeta_1^C,\zeta_3^C)\xi^V - (\nabla_{\zeta_4}\eta)^C(\zeta_2^C)g^C(\zeta_1^V,\zeta_3^C)\xi^C \\
&- (\nabla_{\zeta_4}\eta)^V(\zeta_2^C)g^C(\zeta_1^C,\zeta_3^C)\xi^C - \eta^C(\zeta_2^C)g^C(\zeta_1^C,\zeta_3^C)(\nabla_{\zeta_4}\xi)^V \\
&- \eta^C(\zeta_2^C)g^C(\zeta_1^V,\zeta_3^C)(\nabla_{\zeta_4}\xi)^C - \eta^V(\zeta_2^C)g^C(\zeta_1^C,\zeta_3^C)(\nabla_{\zeta_4}\xi)^C \\
&- \eta^C(\zeta_1^C)g^C(\zeta_2^C,\zeta_3^C)(\nabla_{\zeta_4}\xi)^V - \eta^C(\zeta_1^C)g^C(\zeta_2^V,\zeta_3^C)(\nabla_{\zeta_4}\xi)^C \\
&- \eta^V(\zeta_1^C)g^C(\zeta_2^C,\zeta_3^C)(\nabla_{\zeta_4}\xi)^C + (\nabla_{\zeta_4}\eta)^C(\zeta_2^C)\eta^C(\zeta_3^C)\zeta_1^V \\
&+ (\nabla_{\zeta_4}\eta)^C(\zeta_2^C)\eta^V(\zeta_3^C)\zeta_1^V + (\nabla_{\zeta_4}\eta)^V(\zeta_2^C)\eta^C(\zeta_3^C)\zeta_1^C \\
&+ (\nabla_{\zeta_4}\eta)^C(\zeta_3^C)\eta^C(\zeta_2^C)\zeta_1^V + (\nabla_{\zeta_4}\eta)^C(\zeta_3^C)\eta^V(\zeta_2^C)\zeta_1^V \\
&+ (\nabla_{\zeta_4}\eta)^V(\zeta_3^C)\eta^C(\zeta_2^C)\zeta_1^C - (\nabla_{\zeta_4}\eta)^C(\zeta_1^C)\eta^C(\zeta_3^C)\zeta_2^V \\
&- (\nabla_{\zeta_4}\eta)^C(\zeta_1^C)\eta^V(\zeta_3^C)\zeta_2^V - (\nabla_{\zeta_4}\eta)^V(\zeta_1^C)\eta^C(\zeta_3^C)\zeta_2^C \\
&- (\nabla_{\zeta_4}\eta)^C(\zeta_3^C)\eta^C(\zeta_1^C)\zeta_2^V + (\nabla_{\zeta_4}\eta)^C(\zeta_3^C)\eta^V(\zeta_1^C)\zeta_2^V \\
&+ (\nabla_{\zeta_4}\eta)^V(\zeta_3^C)\eta^C(\zeta_1^C)\zeta_2^C.\end{aligned} \tag{65}$$

Using (25), (26) and (27), we infer that

$$\begin{aligned}((\widetilde{\nabla}^C_{\zeta_4^C}\widetilde{R})(\zeta_1,\zeta_2)\zeta_3)^C &= ((\nabla_{\zeta_4}\widetilde{R})(\zeta_1,\zeta_2)\zeta_3)^C - 2d\eta^C(\zeta_1^C,\zeta_2^C)g^C(\zeta_3^C,\zeta_4^C)\xi^V \\
&- 2d\eta^C(\zeta_1^C,\zeta_2^C)g^C(\zeta_3^V,\zeta_4^C)\xi^C \\
&- 2d\eta^C(\zeta_1^C,\zeta_2^C)g^C(\zeta_3^C,\zeta_4^C)\xi^C \\
&+ 4d\eta^C(\zeta_1^C,\zeta_2^C)g^C((\phi\zeta_1)^C,\zeta_2^C)\eta^C(\zeta_3^C)\zeta_4^V \\
&+ 4d\eta^C(\zeta_1^C,\zeta_2^C)g^C((\phi\zeta_1)^C,\zeta_2^C)\eta^V(\zeta_3^C)\zeta_4^C \\
&+ 4d\eta^C(\zeta_1^C,\zeta_2^C)g^C((\phi\zeta_1)^V,\zeta_2^C)\eta^C(\zeta_3^C)\zeta_4^C \\
&+ 4d\eta^V(\zeta_1^C,\zeta_2^C)g^C((\phi\zeta_1)^C,\zeta_2^C)\eta^C(\zeta_3^C)\zeta_4^C \\
&+ g^C((\phi\zeta_4)^C,\zeta_2^C)g^C(\zeta_1^C,\zeta_3^C)\xi^V\end{aligned}$$

$$
\begin{aligned}
&+ \quad g^C((\phi\zeta_4)^C,\zeta_2^C)g^C(\zeta_1^V,\zeta_3^C)\xi^C \\
&+ \quad g^C((\phi\zeta_4)^V,\zeta_2^C)g^C(\zeta_1^C,\zeta_3^C)\xi^C \\
&- \quad g^C((\phi\zeta_4)^C,\zeta_1^C)g^C(\zeta_2^C,\zeta_3^C)\xi^V \\
&- \quad g^C((\phi\zeta_4)^C,\zeta_1^C)g^C(\zeta_2^V,\zeta_3^C)\xi^C \\
&- \quad g^C((\phi\zeta_4)^V,\zeta_1^C)g^C(\zeta_2^C,\zeta_3^C)\xi^C \\
&+ \quad \eta^C(\zeta_2^C)g^C(\zeta_1^C,\zeta_3^C)(\phi\zeta_4)^V \\
&+ \quad \eta^C(\zeta_2^C)g^C(\zeta_1^V,\zeta_3^C)(\phi\zeta_4)^C+\eta^V(\zeta_2^C)g^C(\zeta_1^C,\zeta_3^C)(\phi\zeta_4)^C \\
&- \quad \eta^C(\zeta_1^C)g^C(\zeta_2^C,\zeta_3^C)(\phi\zeta_4)^V-\eta^C(\zeta_1^C)g^C(\zeta_2^V,\zeta_3^C)(\phi\zeta_4)^C \\
&- \quad \eta^V(\zeta_1^C)g^C(\zeta_2^C,\zeta_3^C)(\phi\zeta_4)^C-g^C((\phi\zeta_4)^C,\zeta_2^C)\eta^C(\zeta_3^C)\zeta_1^V \\
&- \quad g^C((\phi\zeta_4)^C,\zeta_2^C)\eta^V(\zeta_3^C)\zeta_1^C-g^C((\phi\zeta_4)^V,\zeta_2^C)\eta^C(\zeta_3^C)\zeta_1^C \\
&- \quad g^C((\phi\zeta_4)^C,\zeta_3^C)\eta^C(\zeta_2^C)\zeta_1^V-g^C((\phi\zeta_4)^C,\zeta_3^C)\eta^V(\zeta_2^C)\zeta_1^C \\
&- \quad g^C((\phi\zeta_4)^V,\zeta_3^C)\eta^C(\zeta_1^C)\zeta_1^C \\
&+ \quad g^C((\phi\zeta_4)^C,\zeta_1^C)\eta^C(\zeta_3^C)\zeta_2^V+g^C((\phi\zeta_4)^C,\zeta_1^C)\eta^V(\zeta_3^C)\zeta_2^C \\
&+ \quad g^C((\phi\zeta_4)^V,\zeta_1^C)\eta^C(\zeta_3^C)\zeta_2^C \\
&+ \quad g^C((\phi\zeta_4)^C,\zeta_3^C)\eta^C(\zeta_1^C)\zeta_2^V+g^C((\phi\zeta_4)^C,\zeta_3^C)\eta^V(\zeta_1^C)\zeta_2^C \\
&+ \quad g^C((\phi\zeta_4)^V,\zeta_3^C)\eta^C(\zeta_1^C)\zeta_2^C.
\end{aligned} \tag{66}
$$

Using (66) and (24) in (64), we infer that

$$
\begin{aligned}
(\phi^2(\widetilde{\nabla}_{\zeta_4}\widetilde{R})(\zeta_1,\zeta_2)\zeta_3)^C \quad = \quad & (\phi^2(\nabla_W R)(\zeta_1,\zeta_2)\zeta_3)^C-2d\eta^C(\zeta_1^C,\zeta_2^C)\eta^C(\zeta_3^C)\zeta_4^V \\
- \quad & 2d\eta^C(\zeta_1^C,\zeta_2^C)\eta^V(\zeta_3^C)\zeta_4^{-}2d\eta^V(\zeta_1^C,\zeta_2^C)\eta^C(\zeta_3^C)\zeta_4^C \\
+ \quad & 2d\eta^C(\zeta_1^C,\zeta_2^C)\eta^C(\zeta_3^C)\eta^C(\zeta_4^C)\xi^V \\
+ \quad & 2d\eta^C(\zeta_1^C,\zeta_2^C)\eta^C(\zeta_3^C)\eta^V(\zeta_4^C)\xi^C \\
+ \quad & 2d\eta^C(\zeta_1^C,\zeta_2^C)\eta^V(\zeta_3^C)\eta^C(\zeta_4^C)\xi^C \\
+ \quad & 2d\eta^V(\zeta_1^C,\zeta_2^C)\eta^C(\zeta_3^C)\eta^C(\zeta_4^C)\xi^C \\
- \quad & \eta^C(\zeta_2^C)g^C(\zeta_1^C,\zeta_3^C)(\phi\zeta_4)^V \\
- \quad & \eta^C(\zeta_2^C)g^C(\zeta_1^V,\zeta_3^C)(\phi\zeta_4)^C-\eta^V(\zeta_2^C)g^C(\zeta_1^C,\zeta_3^C)(\phi\zeta_4)^C \\
+ \quad & \eta^C(\zeta_1^C)g^C(\zeta_2^C,\zeta_3^C)(\phi\zeta_4)^V+\eta^C(\zeta_1^C)g^C(\zeta_2^V,\zeta_3^C)(\phi\zeta_4)^C \\
+ \quad & \eta^V(\zeta_1^C)g^C(\zeta_2^C,\zeta_3^C)(\phi\zeta_4)^C+g^C((\phi\zeta_4)^C,\zeta_2^C)\eta^C(\zeta_3^C)\zeta_1^V \\
+ \quad & g^C((\phi\zeta_4)^C,\zeta_2^C)\eta^V(\zeta_3^C)\zeta_1^C+g^C((\phi\zeta_4)^V,\zeta_2^C)\eta^C(\zeta_3^C)\zeta_1^C \\
- \quad & g^C((\phi\zeta_4)^C,\zeta_2^C)\eta^C(\zeta_3^C)\eta^C(\zeta_1^C)\xi^V \\
- \quad & g^C((\phi\zeta_4)^C,\zeta_2^C)\eta^C(\zeta_3^C)\eta^V(\zeta_1^C)\xi^C \\
- \quad & g^C((\phi\zeta_4)^C,\zeta_2^C)\eta^V(\zeta_3^C)\eta^C(\zeta_1^C)\xi^C \\
- \quad & g^C((\phi\zeta_4)^V,\zeta_2^C)\eta^C(\zeta_3^C)\eta^C(\zeta_1^C)\xi^C \\
+ \quad & g^C((\phi\zeta_4)^C,\zeta_3^C)\eta^C(\zeta_2^C)\zeta_1^V+g^C((\phi\zeta_4)^C,\zeta_3^C)\eta^V(\zeta_2^C)\zeta_1^C \\
+ \quad & g^C((\phi\zeta_4)^V,\zeta_3^C)\eta^C(\zeta_2^C)\zeta_1^C-g^C((\phi\zeta_4)^C,\zeta_1^C)\eta^C(\zeta_3^C)\zeta_2^V \\
- \quad & g^C((\phi\zeta_4)^C,\zeta_1^C)\eta^V(\zeta_3^C)\zeta_2^C-g^C((\phi\zeta_4)^V,\zeta_1^C)\eta^C(\zeta_3^C)\zeta_2^C \\
+ \quad & g^C((\phi\zeta_4)^C,\zeta_1^C)\eta^C(\zeta_3^C)\eta^C(\zeta_2^C)\xi^V \\
+ \quad & g^C((\phi\zeta_4)^C,\zeta_1^C)\eta^C(\zeta_3^C)\eta^V(\zeta_2^C)\xi^C \\
+ \quad & g^C((\phi\zeta_4)^C,\zeta_1^C)\eta^V(\zeta_3^C)\eta^C(\zeta_2^C)\xi^C \\
+ \quad & g^C((\phi\zeta_4)^V,\zeta_1^C)\eta^C(\zeta_3^C)\eta^C(\zeta_2^C)\xi^C \\
- \quad & g^C((\phi\zeta_4)^C,\zeta_3^C)\eta^C(\zeta_1^C)\zeta_2^V-g^C((\phi\zeta_4)^C,\zeta_3^C)\eta^V(\zeta_1^C)\zeta_2^C \\
- \quad & g^C((\phi\zeta_4)^V,\zeta_3^C)\eta^C(\zeta_1^C)\zeta_2^C-\eta^C(\zeta_4^C)(\phi^2(\phi\widetilde{R})(\zeta_1,\zeta_2)\zeta_3)^V \\
- \quad & \eta^V(\zeta_4^C)(\phi^2(\phi\widetilde{R})(\zeta_1,\zeta_2)\zeta_3)^C.
\end{aligned} \tag{67}
$$

If we take $\zeta_4, \zeta_1, \zeta_2, \zeta_3$ orthogonal to ξ, (67) reduces to

$$(\phi^2(\widetilde{\nabla}_{\zeta_4}\widetilde{R})(\zeta_1,\zeta_2)\zeta_3)^C = (\phi^2(\nabla_W R)(\zeta_1,\zeta_2)\zeta_3)^C.$$

Hence, the following theorem can be stated as:

Theorem 3. *Let TM be the tangent bundle of a Sasakian manifold M. Then, ∇^C is locally ϕ^C-symmetric on TM if and only if $\widetilde{\nabla}^C$ on TM is so.*

8. Example

Let us consider a three-dimensional differentiable manifold $M = \{(u,v,w) : u,v,w \in \Re^3, z \neq 0\}$, where $\Re$ is a set of real numbers and TM its tangent bundle. Let e_1, e_2, e_3 be linearly independent vector fields on M given by

$$e_1 = -u\frac{\partial}{\partial u}, e_2 = u\frac{\partial}{\partial v}, e_3 = u\frac{\partial}{\partial w}.$$

Let g be the Riemannian metric and η be a 1-form on M given by

$$g(e_1,e_2) = g(e_1,e_3) = g(e_2,e_3) = 0, g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1$$

and

$$\eta(\zeta_3) = g(\zeta_3, e_1), \ \ \zeta_3 \in \Im^1_0(M).$$

Let ϕ be the (1,1) tensor field defined by $\phi e_1 = 0, \phi e_2 = e_2, \phi e_3 = e_3$.. Using the linearity of ϕ and g, we acquire $\eta(e_1) = 1$, $\phi^2\zeta_3 = -\zeta_3 + \eta(\zeta_3)e_1$ and $g(\phi\zeta_1, \phi\zeta_2) = g(\zeta_1,\zeta_2) - \eta(\zeta_1)\eta(\zeta_2)$.

Thus, for $e_1 = \xi$, the (ϕ,ξ,η,g) is a contact metric structure on M and M is called a contact metric manifold. In addition, M satisfies

$$(\nabla_{\zeta_1}\phi)\zeta_2 = g(\zeta_1,\zeta_2)e_1 - \eta(\zeta_2)\zeta_1.$$

Hence, for $e_1 = \xi$, M is a Sasakian manifold.

Let e_1^C, e_2^C, e_3^C and e_1^V, e_2^V, e_3^V be the complete and vertical lifts on TM of e_1, e_2, e_3 on M. Let g^C be the complete lift of a Riemannian metric g on TM such that

$$g^C(\zeta_1^V, e_1^C) = (g^C(\zeta_1,e_1))^V = (\eta(\zeta_1))^V, \tag{68}$$

$$g^C(\zeta_1^C, e_1^C) = (g^C(\zeta_1,e_1))^C = (\eta(\zeta_1))^C, \tag{69}$$

$$g^C(e_1^C, e_1^C) = 1, \ g^V(\zeta_1^V, e_1^C) = 0, \ g^V(e_1^V, e_1^V) = 0$$

and so on. Let ϕ^C and ϕ^V be the complete and vertical lifts of the (1,1) tensor field ϕ defined by

$$\phi^V(e_1^V) = \phi^C(e_1^C) = 0,$$
$$\phi^V(e_2^V) = e_2^V, \ \phi^C(e_2^C) = e_2^C,$$
$$\phi^V(e_3^V) = e_3^V, \ \phi^C(e_3^C) = e_3^C.$$

By using the linearity of ϕ and g, we infer that

$$(\phi^2\zeta_1)^C = -\zeta_1^C + \eta^V(\zeta_1)e_1^C + \eta^C(\zeta_1)e_1^V, \tag{70}$$

$$\begin{aligned} g^C((\phi e_1)^C, (\phi e_2)^C) &= g^C(e_1^C, e_2^C) - (\eta(e_1))^C(\eta(e_2))^V \\ &- (\eta(e_1))^V(\eta(e_2))^C. \end{aligned}$$

Thus, for $e_1 = \xi$ in (68)–(70), the structure $(\phi^C, \xi^C, \eta^C, g^C)$ is a contact metric structure on TM and satisfies the relation

$$\begin{aligned}(\nabla^C_{e_1^C}\phi^C)e_2^C &= g^C(e_1^C, e_2^C)\xi^V + g^C(e_1^V, e_2^C)\xi^C \\ &- \eta^C(e_2^C)e_1^V - e^V(e_2^C)e_1^C,\end{aligned}$$

Then, $(\phi^C, \xi^C, \eta^C, g^C, TM)$ is a Sasakian manifold.

Author Contributions: Conceptualization, M.N.I.K., U.C.D.; funding acquisition, M.N.I.K.; investigation, M.N.I.K., L.S.V.; methodology, U.C.D., L.S.V.; project administration, U.C.D.; writing—original draft, M.N.I.K., U.C.D., L.S.V.; writing—review editing, M.N.I.K., U.C.D., L.S.V. All authors have read and agreed to the published version of the manuscript.

Funding: The researchers would like to thank the Deanship of Scientific Research, Qassim University for funding the publication of this project.

Data Availability Statement: Not Applicable.

Acknowledgments: The researchers would like to thank the Deanship of Scientific Research, Qassim University for funding the publication of this project. The authors would like to thank the reviewers and the editor for reviewing the paper carefully and for their valuable comments to improve the overall quality of the paper.

Conflicts of Interest: The authors declare no conflict of interest in this paper.

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