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Article

New Explicit Oscillation Criteria for First-Order Differential Equations with Several Non-Monotone Delays

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Abstract: The oscillation of a first-order differential equation with several non-monotone delays is proposed. We extend the works of Kwong (1991) and Sficas and Stavroulakis (2003) for equations with several delays. Our results not only essentially improve but also generalize a large number of the existing ones. Using some numerical examples, we illustrate the applicability and effectiveness of our results over many known results in the literature.

Keywords: oscillation; differential equation; non-monotone delays

MSC: 34K11; 34K06

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1. Introduction

In this paper, we study the oscillation of the equation

$$x'(t) + \sum_{r=1}^{m} b_r(t)x(\tau_r(t)) = 0, \qquad t \geq t_0, \tag{1}$$

where $b_r, \tau_r \in C([t_0, \infty), [0, \infty))$ such that $\lim\limits_{t\to\infty} \tau_r(t) = \infty$, $r = 1, 2, \ldots, m$.

In the particular case $m = 1$, Equation (1) has the form

$$x'(t) + b(t)x(\tau(t)) = 0, \qquad t \geq t_0, \tag{2}$$

where $b, \tau \in C([t_0, \infty), [0, \infty))$, such that $\lim\limits_{t\to\infty} \tau(t) = \infty$.

By a solution of Equation (1), we mean a continuous function $x(t)$ on $[t_0 - \bar{t}, t_0]$, $\bar{t} = \inf\limits_{t \geq t_0}\{\tau_r(t), 1 \leq r \leq m\}$ that is continuously differentiable on $[t_0, \infty)$ and satisfies Equation (1) for all $t \geq t_0$. A solution $x(t)$ is called oscillatory if it has arbitrary large zeros in any interval $[t_1, \infty)$, $t_1 \geq t_0$; otherwise, it is called nonoscillatory. If Equation (1) has at least one eventually positive or eventually negative solution, it is called nonoscillatory; otherwise, it is called oscillatory.

It should be noted that the oscillatory behaviour for solutions of Equations (1) and (2) is totally different. In fact, all solutions of Equation (2) with τ and b as constants are oscillatory if and only if $b\tau > \frac{1}{e}$; see ([1] Thorem 2.2.3). However, the oscillation problem of Equation (1) in its simplest form (with constant delays and coefficients) is not complete. As a result, the oscillation theory is very interested in establishing necessary and/or sufficient oscillation conditions for Equation (1).

In the last few decades, the oscillation problem of functional differential equations has received much attention from mathematicians; see, for example, [1–37]. The reader is referred to [1,2,4–6,9,10,12,13,17,20,22,23,26] and [1,2,7,8,11,14,15,17–19,24,27,30–32,34–37] for the oscillation of Equations (1) and (2), respectively. The results on oscillation criteria of most of these works have iterative forms. Many sharp oscillation criteria for both Equations (1) and (2) with slowly varying coefficients have been established by [18–20,34]. Further new oscillation conditions for Equation (2) with a non-monotone delay have been obtained by [7,8]. These conditions are expressed in terms of the numbers $L^* = \limsup_{t\to\infty} \int_{h(t)}^{t} b(v)dv$ and $k^* = \liminf_{t\to\infty} \int_{h(t)}^{t} b(v)dv$, where $h(t)$ is a nondecreasing continuous function on $[t_1, \infty)$ for some $t_1 \geq t_0$ such that $\tau(t) \leq h(t)$ for all $t \geq t_1$; see, for a non-decreasing delay case, [15,24,30,32,35].

Several oscillation criteria for Equation (1) were established, but we will only highlight some of them. First, we need to define the following notation.

Assume that there exist nondecreasing continuous functions $\sigma_l(t)$ and $\sigma(t)$ on $[t_1, \infty)$ for some $t_1 \geq t_0$ such that $\tau_l(t) \leq \sigma_l(t) \leq \sigma(t) \leq t$, $l = 1, 2, \ldots, m$. Additionally, we define

$$\tau_{\max}(t) = \max_{1\leq l\leq m} \tau_l(t),$$

$$L(t) = \max_{1\leq l\leq m} L_l(t), \quad L_l(t) = \sup_{t_0\leq s\leq t} \tau_l(s), \quad l = 1, 2, \ldots, m, \tag{3}$$

$$\gamma = \liminf_{t\to\infty} \int_{\sigma(t)}^{t} \sum_{r=1}^{m} b_r(s)ds, \quad \gamma_l = \liminf_{t\to\infty} \int_{\sigma_l(t)}^{t} b_l(s)ds, \quad l = 1, 2, \ldots, m,$$

$$\eta = \liminf_{t\to\infty} \int_{\tau_{\max}(t)}^{t} \sum_{r=1}^{m} b_r(s)ds, \quad \eta_l = \liminf_{t\to\infty} \int_{\tau_l(t)}^{t} b_l(s)ds, \qquad l = 1, 2, \ldots, m,$$

$$D(\omega) = \begin{cases} 0, & \text{if} \quad \omega > 1/e, \\ \frac{1-\omega-\sqrt{1-2\omega-\omega^2}}{2}, & \text{if} \quad \omega \in \left[0, \frac{1}{\mathrm{e}}\right]. \end{cases} \tag{4}$$

Finally, let $\lambda(q)$ be the smaller real root of the transcendental equation $\lambda = \mathrm{e}^{\lambda q}$, $0 \leq q \leq \frac{1}{\mathrm{e}}$. Now, we mention some results from the literature that are related to our study.

Infante et al. [23] showed that if

$$\limsup_{t\to\infty} \prod_{i=1}^{m} \left[\prod_{i_1=1}^{m} \int_{\sigma_i(t)}^{t} b_{i_1}(s) \mathrm{e}^{\int_{\tau_{i_1}(s)}^{\sigma_{i_1}(t)} \sum_{l=1}^{m} b_l(s_1) \mathrm{e}^{\int_{\tau_l(s_1)}^{s_1} \sum_{r=1}^{m} b_r(s_2)ds_2} ds_1} ds \right]^{\frac{1}{m}} > \frac{1}{m^m}, \tag{5}$$

or

$$\limsup_{\epsilon\to 0^+} \left[\limsup_{t\to\infty} \prod_{i=1}^{m} \left[\prod_{i_1=1}^{m} \int_{\sigma_i(t)}^{t} b_{i_1}(s) \mathrm{e}^{\int_{\tau_{i_1}(s)}^{\sigma_{i_1}(t)} \sum_{l=1}^{m} (\lambda(\eta_l)-\epsilon) b_l(s_1)ds_1} ds \right]^{\frac{1}{m}} \right] > \frac{1}{m^m}, \tag{6}$$

then Equation (1) is oscillatory.

Koplatadze [26] established the condition

$$\limsup_{t\to\infty} \prod_{i=1}^{m} \left[\prod_{i_1=1}^{m} \int_{\sigma_i(t)}^{t} b_{i_1}(s) \mathrm{e}^{m \int_{\tau_{i_1}(s)}^{\sigma_{i_1}(t)} \left(\prod_{i_2=1}^{m} b_{i_2}(s_1) \right)^{\frac{1}{m}} \varpi_\ell(s_1)ds_1} ds \right]^{\frac{1}{m}} > \frac{1}{m^m} - \prod_{i=1}^{m} D(\gamma_i), \tag{7}$$

where $\varpi_1(t) = 0$ and $\varpi_\ell(t) = \mathrm{e}^{\sum_{l=1}^{m} \int_{\tau_l(t)}^{t} \left(\prod_{i=1}^{m} b_i(s) \right)^{\frac{1}{m}} \varpi_{\ell-1}(s)ds}$, $\ell = 2, 3, \ldots$.

Braverman et al. [10] introduced a recursive criterion, namely

$$\limsup_{t\to\infty} \int_{L(t)}^{t} \sum_{r=1}^{m} b_r(s) \varphi_l(L(t), \tau_r(s))ds > 1, \tag{8}$$

where

$$\varphi_1(t,s) = \exp\left\{\int_s^t \sum_{r=1}^m b_r(\zeta)d\zeta\right\},$$

$$\varphi_{l+1}(t,s) = \exp\left\{\int_s^t \sum_{r=1}^m b_r(\zeta)\varphi_l(\zeta,\tau_r(\zeta))d\zeta\right\}, \quad l \in \mathbb{N}. \tag{9}$$

Chatzarakis and Péics [12] obtained the sufficient condition

$$\limsup_{t\to\infty} \int_{L(t)}^t \sum_{r=1}^m b_r(s)\varphi_l(L(s),\tau_r(s))ds > \frac{1+\ln(\lambda(\eta))}{\lambda(\eta)} - D(\eta). \tag{10}$$

Attia et al. [5] introduced the condition

$$\limsup_{t\to\infty} \left(\prod_{i=1}^m \left(\prod_{i_1=1}^m \int_{\sigma_i(t)}^t Q_{i_1}(s)ds\right)^{\frac{1}{m}} + \frac{\prod_{i=1}^m D(\gamma_i)}{m^m} e^{\sum_{r=1}^m \int_{\sigma_r(t)}^t \sum_{r_1=1}^m b_{r_1}(s)ds}\right) > \frac{1}{m^m}, \tag{11}$$

where $0 < \gamma_l \leq \frac{1}{e}$, $l = 1,2,\ldots,m$, and

$$Q_{i_1}(s) = e^{\int_{\sigma_{i_1}(s)}^s \sum_{r=1}^m b_r(s_1)ds_1} \sum_{r_1=1}^m b_{r_1}(s) \int_{\tau_{r_1}(s)}^s b_{i_1}(s_1) e^{(\lambda(\gamma)-\epsilon)\int_{\tau_{i_1}(s_1)}^{\sigma_{i_1}(s)} \sum_{r_2=1}^m b_{r_2}(s_2)ds_2} ds_1,$$

for $\epsilon \in (0, \lambda(\gamma))$.

Bereketoglu et al. [9] defined the sequence $\{\varrho_\ell(t)\}_{\ell\geq 0}$ by

$$\begin{aligned}
\varrho_0(t) &= m\left(\prod_{i=1}^m b_i(t)\right)^{\frac{1}{m}} \\
\varrho_\ell(t) &= \sum_{r=1}^m b_r(t)\left[1 + m\left(\prod_{i=1}^m \int_{\sigma_r(t)}^t b_i(s) e^{\int_{\tau_i(s)}^t \varrho_{\ell-1}(s_1)ds_1} ds\right)^{\frac{1}{m}}\right], \quad \ell = 1,2,\ldots,
\end{aligned}$$

and obtained the condition

$$\limsup_{t\to\infty} \prod_{i=1}^m \left[\prod_{i_1=1}^m \int_{\sigma_i(t)}^t b_{i_1}(s) e^{\int_{\tau_{i_1}(s)}^{\sigma_{i_1}(t)} \varrho_\ell(s_1)ds_1} ds\right]^{\frac{1}{m}} > \frac{1}{m^m}\left(1 - \prod_{i=1}^m D(\gamma_i)\right), \tag{12}$$

where $\ell \in \mathbb{N}$.

Attia and El-Morshedy [6] improved (5) and (7) with $\ell = 3$ and proved that Equation (1) is oscillatory if

$$\limsup_{t\to\infty} \left(m\left(\prod_{i=1}^m D(\gamma_i)\right)^{1-\frac{1}{m}} \sum_{r=1}^m \bar{R}_r(t) + \sum_{r=2}^m m^r \left(\prod_{i=1}^m D(\gamma_i)\right)^{1-\frac{l}{m}} \prod_{i=1}^r \bar{R}_i(t)\right) > 1 - \prod_{i=1}^m D(\gamma_i), \tag{13}$$

where

$$\bar{R}_r(t) = \left(\prod_{i=1}^m \int_{\sigma_r(t)}^t b_i(u) e^{\int_{\tau_i(u)}^{\sigma_i(t)} \sum_{r_1=1}^m b_{r_1}(u_1) e^{(\lambda(\eta)-\epsilon)\int_{\tau_{r_1}(u_1)}^{u_1} \sum_{r_2=1}^m b_{r_2}(u_2)du_2} du_1} du\right)^{\frac{1}{m}} \tag{14}$$

and $r = 1,2,\ldots,m$, $\eta > 0$, $\epsilon \in (0, \lambda(\eta))$.

The purpose of this work is to improve and extend the method introduced by Kwong [29] for Equation (1) with non-monotone delays. Based on this, we obtain some new oscillation criteria that improve and generalize many existing ones reported in the literature. The significance of some of our results over the previous works is shown by using an illustrative example. In particular, it is shown that our results can examine the oscillation property, while many iterative oscillation criteria fail to do so for any number of iterations.

2. Main Results

In what follows, we will use the following notation:

$$M_r(t) = \max\{L_r(t), \ldots, L_m(t)\}, \quad r = 1, 2, \ldots, m, \quad t \geq t_0, \tag{15}$$

where $L_r(t)$ is defined by (3). It is clear that

$$M_r(t) \geq \tau_r(t), \tau_{r+1}(t), \ldots, \tau_m(t), \qquad r = 1, \ldots, m \tag{16}$$

and

$$M_i(t) \geq M_j(t), \qquad i \geq j,\ i, j = 1, \ldots, m.$$

Additionally, we define ρ_r and the sequence $\{\Omega_l(v,u)\}_{l=0}^{\infty}$, $v \geq u \geq t_0$ as follows:

$$\rho_r = \liminf_{t\to\infty} \int_{M_r(t)}^{t} \sum_{k=r}^{m} b_k(v)dv, \qquad \rho_r \leq \frac{1}{\mathrm{e}} \qquad r = 1, 2, \ldots, m \tag{17}$$

and

$$\Omega_l(v,u) = \mathrm{e}^{\int_u^v \sum_{i=1}^m b_i(\zeta)\Omega_{l-1}(\zeta,\tau_i(\zeta))d\zeta}, \quad l \in \mathbb{N},$$

with

$$\Omega_0(v,u) = \begin{cases} 1, & \rho_1 = 0, \\ \lambda(\rho_1) - \epsilon_1, & \rho_1 > 0, \qquad \epsilon_1 \in (0,\ \lambda(\rho_1)). \end{cases}$$

The proof of the following result follows from [37].

Lemma 1. *Let $x(t)$ be an eventually positive solution of Equation (1). Then,*

$$\liminf_{t\to\infty} \frac{x(t)}{x(M_r(t))} \geq D(\rho_r), \qquad r = 1, 2, \ldots, m.$$

Lemma 2. *Let $l \in \mathbb{N}$. Then,*

$$x(u) \geq x(v)\Omega_l(v,u), \qquad v \geq u,$$

where $x(t)$ is a positive solution of Equation (1).

Proof. Since $x(t)$ is a positive solution of Equation (1), then, $x(t)$ is eventually nonincreasing for all sufficiently large t. In view of (16), it follows from Equation (1) that

$$x'(t) + x(M_1(t)) \sum_{k=1}^{m} b_k(t) \leq 0, \qquad \text{for all sufficiently large } t.$$

Using ([17] Lemma 2.1.2) and the nonincreasing nature of $x(t)$, we obtain

$$\frac{x(M_1(t))}{x(t)} \geq \begin{cases} 1, & \rho_1 = 0, \\ \lambda(\rho_1) - \epsilon_1, & \rho_1 > 0, \end{cases}$$

where $\epsilon_1 > 0$ is sufficiently small.

Therefore,

$$\frac{x(M_1(t))}{x(t)} \geq \Omega_0(v,u) \qquad \text{for all sufficiently large } t, \text{ for } v \geq u \geq t_0. \tag{18}$$

Dividing Equation (1) by $x(t)$ and integrating from u to v, $v \geq u$, we have

$$\ln\left(\frac{x(u)}{x(v)}\right) = \int_u^v \sum_{k_1=1}^{m} b_{k_1}(v_1) \frac{x(\tau_{k_1}(v_1))}{x(v_1)} dv_1, \qquad v \geq u. \tag{19}$$

Consequently,

$$x(u) = x(v) \mathrm{e}^{\int_u^v \sum_{k_1=1}^{m} b_{k_1}(v_1) \frac{x(\tau_{k_1}(v_1))}{x(v_1)} dv_1}, \qquad v \geq u. \tag{20}$$

Then,

$$x(u) \geq x(v) \mathrm{e}^{\int_u^v \sum_{k_1=1}^{m} b_{k_1}(v_1) \frac{x(M_1(v_1))}{x(v_1)} dv_1}.$$

From this and (18), we obtain

$$x(u) \geq x(v) \mathrm{e}^{\int_u^v \sum_{k_1=1}^{m} b_{k_1}(v_1) \Omega_0(v_1, \tau_{k_1}(v_1)) dv_1} = x(v) \Omega_1(v, u).$$

Accordingly,

$$x(\tau_{k_1}(t)) \geq x(t) \Omega_1(t, \tau_{k_1}(t)), \quad k_1 = 1, 2, \ldots. \tag{21}$$

Substituting into (20), we have

$$x(u) \geq x(v) \mathrm{e}^{\int_u^v \sum_{k_1=1}^{m} b_{k_1}(v_1) \Omega_1(v_1, \tau_{k_1}(v_1)) dv_1} = x(v) \Omega_2(v, u), \qquad v \geq u.$$

Repeating this procedure l times, we derive

$$x(u) \geq x(v) \Omega_l(v, u) \qquad v \geq u.$$

The proof is complete. □

Lemma 3. *Assume that $B_r > 1$, $r \in \{1, 2, \ldots, m\}$ such that*

$$\liminf_{t \to \infty} \frac{x(M_r(t))}{x(t)} \geq B_r. \tag{22}$$

Then, for all sufficiently large t,

$$\int_{M_r(t)}^{t} \sum_{k=r}^{m} b_k(v) \mathrm{e}^{\int_{\tau_k(v)}^{M_r(v)} \sum_{k_1=1}^{m} b_{k_1}(v_1) \frac{x(\tau_{k_1}(v_1))}{x(v_1)} dv_1} dv \leq \frac{1 + \ln(B_r - \epsilon)}{B_r - \epsilon} - \frac{x(t)}{x(M_r(t))}, \tag{23}$$

where $\epsilon \in (0, B_r)$.

Proof. Clearly, $x(t)$ is eventually nonincreasing for all sufficiently large t. From (20), we have

$$x(u) = x(v) \mathrm{e}^{\int_u^v \sum_{k_1=1}^{m} b_{k_1}(v_1) \frac{x(\tau_{k_1}(v_1))}{x(v_1)} dv_1}, \qquad v \geq u. \tag{24}$$

By using (22), for sufficiently small ϵ, $0 < \epsilon < B_r$, we have

$$\frac{x(M_r(t))}{x(t)} > B_r - \epsilon > 1 \qquad \text{for all sufficiently large } t. \tag{25}$$

Then there exists $\bar{t} \in (M_r(t), t)$ such that

$$\frac{x(M_r(t))}{x(\bar{t})} = B_r - \epsilon.$$

Integrating Equation (1) from $\bar{t}$ to t, we obtain

$$x(t) - x(\bar{t}) + \int_{\bar{t}}^{t} \sum_{k=1}^{m} b_k(v) x(\tau_k(v)) dv = 0.$$

That is,

$$x(t) - x(\bar{t}) + \int_{\bar{t}}^{t} \sum_{k=1}^{r-1} b_k(v) x(\tau_k(v)) dv + \int_{\bar{t}}^{t} \sum_{k=r}^{m} b_k(v) x(\tau_k(v)) dv = 0. \tag{26}$$

In view of (16), it follows that

$$\tau_k(v) \le M_r(v), \qquad \bar{t} \le v \le t, \quad k = r, r+1, \ldots, m.$$

From this and (24), we obtain

$$x(\tau_k(v)) = x(M_r(v)) e^{\int_{\tau_k(v)}^{M_r(v)} \sum_{k_1=1}^{m} b_{k_1}(v_1) \frac{x(\tau_{k_1}(v_1))}{x(v_1)} dv_1}, \quad \bar{t} \le v \le t, \quad k = r, r+1, \ldots, m.$$

Substituting this into (26), we obtain

$$x(t) - x(\bar{t}) + \int_{\bar{t}}^{t} \sum_{k=r}^{m} x(M_r(v)) b_k(v) e^{\int_{\tau_k(v)}^{M_r(v)} \sum_{k_1=1}^{m} b_{k_1}(v_1) \frac{x(\tau_{k_1}(v_1))}{x(v_1)} dv_1} dv \le 0.$$

Using the nonincreasing nature of $x(t)$, we have

$$x(t) - x(\bar{t}) + x(M_r(t)) \int_{\bar{t}}^{t} \sum_{k=r}^{m} b_k(v) e^{\int_{\tau_r(v)}^{M_r(v)} \sum_{k_1=1}^{m} b_{k_1}(v_1) \frac{x(\tau_{k_1}(v_1))}{x(v_1)} dv_1} dv \le 0,$$

that is,

$$\begin{aligned} \int_{\bar{t}}^{t} \sum_{k=r}^{m} b_k(v) e^{\int_{\tau_r(v)}^{M_r(v)} \sum_{k_1=1}^{m} b_{k_1}(v_1) \frac{x(\tau_{k_1}(v_1))}{x(v_1)} dv_1} dv \le & \frac{x(\bar{t})}{x(M_r(t))} - \frac{x(t)}{x(M_r(t))} \\ = & \frac{1}{B_r - \epsilon} - \frac{x(t)}{x(M_r(t))}. \end{aligned} \tag{27}$$

By (19), we obtain

$$\ln\left(\frac{x(M_r(t))}{x(\bar{t})}\right) = \int_{M_r(t)}^{\bar{t}} \sum_{k=1}^{m} b_k(v) \frac{x(\tau_k(v))}{x(v)} dv \ge \int_{M_r(t)}^{\bar{t}} \sum_{k=r}^{m} b_k(v) \frac{x(M_r(v))}{x(v)} \frac{x(\tau_k(v))}{x(M_r(v))} dv.$$

From this, (24), and (25), we obtain

$$\int_{M_r(t)}^{\bar{t}} \sum_{k=r}^{m} b_k(v) e^{\int_{\tau_r(v)}^{M_r(v)} \sum_{k_1=1}^{m} b_{k_1}(v_1) \frac{x(\tau_{k_1}(v_1))}{x(v_1)} dv_1} dv \le \frac{\ln(B_r - \epsilon)}{B_r - \epsilon}.$$

Combining this with (27), we obtain

$$\int_{M_r(t)}^{t} \sum_{k=r}^{m} b_k(v) e^{\int_{\tau_r(v)}^{M_r(v)} \sum_{k_1=1}^{m} b_{k_1}(v_1) \frac{x(\tau_{k_1}(v_1))}{x(v_1)} dv_1} dv \le \frac{1 + \ln(B_r - \epsilon)}{B_r - \epsilon} - \frac{x(t)}{x(M_r(t))}.$$

The proof is complete. □

Remark 1. *It should be noted that when $\rho_r > 0$, the number B_r in the preceding lemma can be chosen as $\lambda(\rho_r)$ according to ([17] Lemma 2.1.2).*

Theorem 1. *Assume that* $r \in \{1, 2, \ldots, m\}$, $n_1 \in \mathbb{N}$ *and* $n_2 \in \mathbb{N}_0$. *If* $\rho_r > 0$ *and*

$$\limsup_{t\to\infty} \int_{M_r(t)}^{t} \sum_{k=r}^{m} b_k(v)\Omega_{n_1+1}(M_k(v), \tau_k(v))dv > \frac{1+\ln(\bar{B}_r^{n_2})}{\bar{B}_r^{n_2}} - D(\rho_r), \tag{28}$$

then every solution of Equation (1) is oscillatory, where

$$\bar{B}_r^{n_2} \le \begin{cases} \lambda(\rho_r), & n_2 = 0, \\ \liminf_{t\to\infty} \Omega_{n_2}(t, M_r(t)), & n_2 = 1, 2, \ldots . \end{cases}$$

Proof. If not, let $x(t)$ be a positive solution of Equation (1). From ([17] Lemma 2.1.2), we obtain

$$\liminf_{t\to\infty} \frac{x(M_r(t))}{x(t)} \ge \lambda(\rho_r) \ge \bar{B}_r^0.$$

This, together with Lemma 2, leads to

$$\liminf_{t\to\infty} \frac{x(M_r(t))}{x(t)} \ge \liminf_{t\to\infty} \Omega_{n_2}(t, M_r(t)) \ge \bar{B}_r^{n_2}$$

and

$$\frac{x(\tau_{k_1}(t))}{x(t)} \ge \Omega_{n_1}(t, \tau_{k_1}(t)) \qquad \text{for all sufficiently large } t, \qquad k_1 = 1, 2, \ldots, m. \tag{29}$$

Since $\rho_r > 0$, one can choose $B_r = \bar{B}_r^{n_2}$ in Lemma 3. Then, (23) implies that

$$\int_{M_r(t)}^{t} \sum_{k=r}^{m} b_k(v) e^{\int_{\tau_r(v)}^{M_r(v)} \sum_{k_1=1}^{m} b_{k_1}(v_1) \frac{x(\tau_{k_1}(v_1))}{x(v_1)} dv_1} dv \le \frac{1+\ln\left(\bar{B}_r^{n_2} - \epsilon\right)}{\bar{B}_r^{n_2} - \epsilon} - \frac{x(t)}{x(M_r(t))}. \tag{30}$$

By (29), we obtain

$$\int_{M_r(t)}^{t} \sum_{k=r}^{m} b_k(v) e^{\int_{\tau_r(v)}^{M_r(v)} \sum_{k_1=1}^{m} b_{k_1}(v_1)\Omega_{n_1}(v_1, \tau_{k_1}(v_1)) dv_1} dv \le \frac{1+\ln\left(\bar{B}_r^{n_2} - \epsilon\right)}{\bar{B}_r^{n_2} - \epsilon} - \frac{x(t)}{x(M_r(t))}.$$

In view of Lemma 1, we have

$$\limsup_{t\to\infty} \int_{M_r(t)}^{t} \sum_{k=r}^{m} b_k(v)\Omega_{n_1+1}(M_k(v), \tau_k(v))dv \le \frac{1+\ln\left(\bar{B}_r^{n_2} - \epsilon\right)}{\bar{B}_r^{n_2} - \epsilon} - D(\rho_r).$$

Letting $\epsilon \to 0$, we have a contradiction to (28). The proof is complete. □

Theorem 2. *Assume that* $r \in \{1, 2, \ldots, m\}$ *and* $n \in \mathbb{N}$. *If*

$$\limsup_{t\to\infty} \int_{M_r(t)}^{t} \sum_{k=r}^{m} b_k(v)\Omega_{n+1}(M_k(t), \tau_k(v))dv > 1 - D(\rho_r), \tag{31}$$

then every solution of Equation (1) is oscillatory.

Proof. Assume that $x(t)$ is a positive solution of Equation (1). Integrating Equation (1) from $M_r(t)$ to t, we have

$$x(t) - x(M_r(t)) + \int_{M_r(t)}^{t} \sum_{k=1}^{m} b_k(v) x(\tau_k(v))dv = 0. \tag{32}$$

Using (24) from the proof of Lemma 3, we obtain

$$x(\tau_k(v)) = x(M_k(t))e^{\int_{\tau_k(v)}^{M_k(t)} \sum_{k_1=1}^{m} b_{k_1}(v_1)\frac{x(\tau_{k_1}(v_1))}{x(v_1)}dv_1}, \qquad M_k(t) \le v \le t.$$

Substituting from this into (32), we obtain

$$x(t) - x(M_r(t)) + x(M_r(t)) \int_{M_r(t)}^{t} \sum_{k=1}^{m} b_k(v)e^{\int_{\tau_k(v)}^{M_k(t)} \sum_{k_1=1}^{m} b_{k_1}(v_1)\frac{x(\tau_{k_1}(v_1))}{x(v_1)}dv_1}dv = 0.$$

From this and Lemma 2, we obtain

$$\int_{M_r(t)}^{t} \sum_{k=1}^{m} b_k(v)e^{\int_{\tau_k(v)}^{M_k(t)} \sum_{k_1=1}^{m} b_{k_1}(v_1)\Omega_n\left(v_1,\tau_{k_1}(v_1)\right)dv_1}dv \le 1 - \frac{x(t)}{x(M_r(t))}.$$

This together with Lemma 1, implies that

$$\limsup_{t\to\infty} \int_{M_r(t)}^{t} \sum_{k=r}^{m} b_k(v)\Omega_{n+1}(M_k(t), \tau_k(v))dv \le 1 - D(\rho_r).$$

This contradiction completes the proof. □

Next, we introduce some corollaries for Equation (2). To this end, let $\bar{g}(t)$ and the sequence $\{\omega_l(v,u)\}_{l=0}^{\infty}$, $v \ge u \ge t_0$, be defined, respectively, by

$$\bar{g}(t) = \sup_{u\le t} \tau(u), \quad t \ge t_0$$

and

$$\begin{aligned} \omega_0(v,u) &= \begin{cases} 1, & \mu = 0, \\ \lambda(\mu) - \epsilon, & \mu > 0, \qquad \epsilon \in (0,\ \lambda(\mu)), \end{cases} \\ \omega_l(v,u) &= e^{\int_u^v b(\zeta)\omega_{l-1}(\zeta,\tau(\zeta))d\zeta}, \quad l \in \mathbb{N}, \end{aligned}$$

where

$$\mu = \liminf_{t\to\infty} \int_{\tau(t)}^{t} b(\zeta)d\zeta = \liminf_{t\to\infty} \int_{\bar{g}(t)}^{t} b(\zeta)d\zeta, \qquad \mu \le \frac{1}{e}.$$

According to Theorems 1 and 2, we have, respectively, the following corollaries:

Corollary 1. *Assume that* $n_1 \in \mathbb{N}$ *and* $n_2 \in \mathbb{N}_0$. *If* $\mu > 0$ *and*

$$\limsup_{t\to\infty} \int_{\bar{g}(t)}^{t} b(v)\omega_{n_1+1}(\bar{g}(v), \tau(v))dv > \frac{1 + \ln(\bar{\bar{B}}_{n_2})}{\bar{\bar{B}}_{n_2}} - D(\mu), \tag{33}$$

then every solution of Equation (2) is oscillatory, where

$$\bar{\bar{B}}_{n_2} \le \begin{cases} \lambda(\mu), & n_2 = 0, \\ \liminf_{t\to\infty} \omega_{n_2}(t, \bar{g}(t)), & n_2 = 1, 2, \dots . \end{cases}$$

Corollary 2. *Assume that* $n \in \mathbb{N}_0$. *If*

$$\limsup_{t\to\infty} \int_{\bar{g}(t)}^{t} b(v)\omega_{n+1}(\bar{g}(t), \tau(v))dv > 1 - D(\mu), \tag{34}$$

then every solution of Equation (2) is oscillatory.

Remark 2.

(1) *It should be noted that*

$$\Omega_1(v,u) = \exp\left\{ \int_u^v \sum_{i=1}^m b_i(\zeta)\Omega_0(\zeta,\tau_i(\zeta))d\zeta \right\} \geq \exp\left\{ \int_u^v \sum_{i=1}^m b_r(\zeta)d\zeta \right\} = \varphi_1(v,u).$$

Therefore, conditions (28) with $r = 1$ and $n_2 = 0$ and (31) with $r = 1$ improve (10) and (8), respectively.

(2) *Condition (33) with $n_2 = 0$ generalizes the condition*

$$\limsup_{t\to\infty} \int_{\tau(t)}^t b(v)dv > \frac{1+\ln(\lambda(\mu))}{\lambda(\mu)} - D(\mu),$$

due to Jaroš and Stavroulakis [24] when $\tau(t)$ is nondecreasing. Additionally, if there exits $n_2 \in \mathbb{N}_0$ such that $\bar{\bar{B}}_{n_2} \geq \lambda(\mu)$, then condition (33) improves the preceding condition.

3. Numerical Examples

The choice of $M_r(t)$ is necessary for the validity of conditions (28) and (31) when $r < m$. In fact, if $M_r(t)$ is replaced by $M_j(t)$, $r < m$, $r < j \leq m$, i.e., conditions (28) and (31) have, respectively, the form

$$\limsup_{t\to\infty} \int_{M_j(t)}^t \sum_{k=r}^m b_k(v)\Omega_{n_1+1}(M_k(v),\tau_k(v))dv > \frac{1+\ln(\bar{B}_r^{n_2})}{\bar{B}_r^{n_2}} - D(\rho_r)$$

and

$$\limsup_{t\to\infty} \int_{M_j(t)}^t \sum_{k=r}^m b_k(v)\Omega_{n_1+1}(M_k(t),\tau_k(v))dv > 1 - D(\rho_r), \tag{35}$$

then these conditions may not be sufficient for the oscillation. We show this fact in the following example:

Example 1. *Consider the differential equation*

$$x'(t) + \frac{1}{2\mathrm{e}}x(t-1) + \frac{1}{2\mathrm{e}^6}x(t-6) = 0.$$

This equation has the nonoscillatory solution $x(t) = \mathrm{e}^{-t}$. However, as we will show, condition (35) with $j = 2$ and $r = 1$ is satisfied. Let

$$b_1(t) = \frac{1}{2\mathrm{e}}, \quad \tau_1(t) = t-1, \quad b_2(t) = \frac{1}{2\mathrm{e}^6}, \quad \tau_2(t) = t-6.$$

Then,

$$M_1(t) = t-1, \qquad M_2(t) = t-6.$$

Clearly,

$$\begin{aligned} \limsup_{t\to\infty} \int_{M_2(t)}^t \sum_{k=1}^2 b_k(v)\Omega_{n_1+1}(M_k(v),\tau_k(v))dv &> \limsup_{t\to\infty} \int_{M_2(t)}^t \sum_{k=1}^2 b_k(v)dv \\ &= \frac{3}{\mathrm{e}} + \frac{3}{\mathrm{e}^6} > 1, \end{aligned}$$

and hence, condition (35) with $j = 2$ and $r = 1$ holds.

It is noticeable that the previous works give numerical examples to illustrate the effectiveness of their results over some special cases from earlier publications, especially the iterative conditions. In the following example, we prove the oscillation property, while all the previous iterative conditions fail to do so for any number of iterations.

Example 2. *Consider the differential equation*

$$x'(t) + b_1(t)x(\tau_1(t)) + b_2(t)x(\tau_2(t)) = 0, \quad t \geq 2, \tag{36}$$

where

$$b_1(t) = \begin{cases} 0 & \text{if } t \in [6i, 6i+2], \\ \frac{10}{9}(t-6i-2)\alpha & \text{if } t \in [6i+2, 6i+2.9], \\ \alpha & \text{if } t \in [6i+2.9, 6i+4], \\ \frac{-\alpha}{2}(t-6i-4)+\alpha & \text{if } t \in [6i+4, 6i+6], \end{cases} \quad i \in \mathbb{N}_0,$$

$b_2(t) = \beta$, $\tau_1(t) = t - \delta$ *where* $0 < \delta < 1$, *and*

$$\tau_2(t) = \begin{cases} t-1 & \text{if } t \in [3l, 3l+1], \\ -\frac{1}{2}t + \frac{9}{2}l + \frac{1}{2} & \text{if } t \in [3l+1, 3l+1.2], \\ \frac{7}{6}t - \frac{1}{2}l - \frac{3}{2} & \text{if } t \in [3l+1.2, 3l+3], \end{cases} \quad l \in \mathbb{N}_0.$$

It follows from (3) that $L_1(t) = \tau_1(t) = t - \delta$ *and*

$$L_2(t) = \begin{cases} t-1 & \text{if } t \in [3l, 3l+1], \\ 3l & \text{if } t \in [3l+1, 3l+\frac{9}{7}], \\ \frac{7}{6}t - \frac{1}{2}l - \frac{3}{2} & \text{if } t \in [3l+\frac{9}{7}, 3l+3], \end{cases} \quad l \in \mathbb{N}_0.$$

Please see Figure 1,

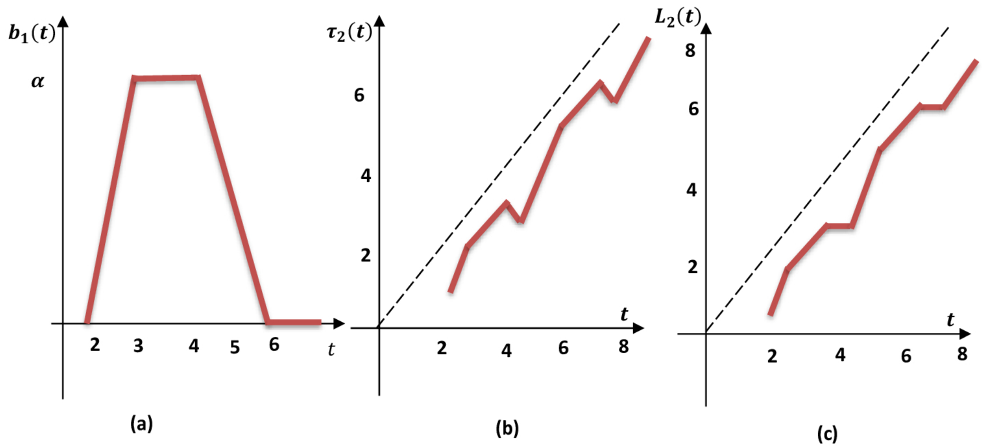

Figure 1. The graphs of the functions $b_1(t)$, $\tau_2(t)$, and $L_2(t)$ are shown in subfigures (**a**), (**b**), and (**c**), respectively.

Therefore,

$$M_1(t) = t - \delta \qquad \text{and} \qquad M_2(t) = L_2(t).$$

Clearly,

$$0 \leq b_1(t) \leq \alpha \qquad \text{and} \qquad t - 1.3 \leq \tau_2(t) \leq M_2(t) \leq t - 1.$$

Let

$$Z(t) = \int_{M_2(t)}^{t} b_2(v) e^{\int_{\tau_2(v)}^{M_2(v)} \sum_{k_1=1}^{2} b_{k_1}(v_1) dv_1} dv.$$

Therefore,

$$\begin{aligned} Z\left(3(2i+1)+\tfrac{9}{7}\right) &= \int_{M_2(3(2i+1)+\frac{9}{7})}^{3(2i+1)+\frac{9}{7}} b_2(v) e^{\int_{\tau_2(v)}^{M_2(v)} \sum_{k_1=1}^{2} b_{k_1}(v_1)dv_1} dv \\ &= \int_{3(2i+1)}^{3(2i+1)+1} \beta dv + \int_{3(2i+1)+1}^{3(2i+1)+1.2} \beta e^{\int_{-\frac{1}{2}v+\frac{9}{2}(2i+1)+\frac{1}{2}}^{3(2i+1)} (\alpha+\beta)dv_1} dv \\ &\quad + \int_{3(2i+1)+1.2}^{3(2i+1)+\frac{9}{7}} \beta e^{\int_{\frac{7}{6}v-\frac{1}{2}(2i+1)-\frac{3}{2}}^{3(2i+1)} (\alpha+\beta)dv_1} dv \\ &= \beta + \frac{20}{7} \frac{\beta\left(e^{\frac{1}{10}(\alpha+\beta)}-1\right)}{\alpha+\beta} > 0.6010602026 \end{aligned}$$

for $\beta = \frac{1}{e}$ *and* $\alpha = 13.9$. *In view of (17), we have*

$$\rho_2 = \liminf_{t\to\infty} \int_{M_2(t)}^{t} b_2(v)dv = \liminf_{t\to\infty} \int_{\tau_2(t)}^{t} b_2(v)dv = \lim_{l\to\infty} \int_{\tau_2(3l+1)}^{3l+1} \beta dv = \frac{1}{e},$$

and hence, $\lambda(\rho_2) = e$, *so one can choose* $\bar{B}_2^0 = \lambda(\rho_2) = e$. *Therefore,*

$$\frac{1+\ln(\bar{B}_2^0)}{\bar{B}_2^0} - D(\rho_2) < 0.59922.$$

Consequently,

$$\begin{aligned} \limsup_{t\to\infty} \int_{M_2(t)}^{t} b_2(v)\Omega_2(M_2(v),\tau_2(v))dv &\geq \limsup_{t\to\infty} Z(t) \\ &\geq \lim_{t\to\infty} Z\left(3(2i+1)+\tfrac{9}{7}\right) > 0.59993 \\ &> \frac{1+\ln(\lambda(\rho_2))}{\lambda(\rho_2)} - D(\rho_2). \end{aligned}$$

Then, according to Theorem 1 with $n_1 = 1$ *and* $n_2 = 0$, *Equation (36) is oscillatory for* $\beta = \frac{1}{e}$, $\alpha = 13.9$ *and for all* $0 < \delta < 1$. *However, all previous results cannot be applied to this equation, as we will show. It is clear that*

$$0 \le b_1(t) \le \alpha, \quad b_2(t) = \frac{1}{e}, \quad L_1(t) = \tau_1(t) = t - \delta$$

and

$$L_2(t) = M_2(t), \quad L(t) = L_1(t) \quad \text{and} \quad t - 1.3 \le \tau_2(t) \le M_2(t) \le t - 1,$$

where $L(t)$ *and* $L_i(t)$, $i = 1,2$ *are defined by (3). Next, we show that there exists a sequence of positive real numbers* $\{A_l\}_{l\geq 0}$ *such that* $A_0 = 1$ *and* $\varphi_l(t,\tau_i(t)) \le A_l$ *(that is defined by (1)) for some* $T > t_0$ *and all* $t \ge T$, $l = 1,2,\ldots$ *. Since*

$$\varphi_1(v,\tau_i(v)) = e^{\int_{\tau_i(v)}^{v} \sum_{k=1}^{2} b_k(u)du} \le e^{\int_{v-1.3}^{v}(\alpha+\beta)} = e^{1.3(\alpha+\beta)} = A_1 \qquad \text{for all } v.$$

$$\varphi_2(v,\tau_i(v)) = e^{\int_{\tau_i(v)}^{v} \sum_{k=1}^{2} b_k(u)\varphi_1(u,\tau_i(u))du} \le e^{\int_{v-1.3}^{v} \sum_{k=1}^{2} b_k(u)A_1 du} = e^{1.3(\alpha+\beta)A_1} = A_2 \ \text{for all } v.$$

Similarly, we obtain

$$\varphi_{l-1}(v,\tau_i(v)) \le e^{1.3(\alpha+\beta)A_{l-2}} = A_{l-1}, \qquad l = 2,3,\ldots \qquad \text{for all } v.$$

Clearly,

$$\begin{aligned}
\int_{L(t)}^{t}\sum_{i=1}^{2} b_i(u)\varphi_l(L(t),\tau_i(u))du &= \int_{\tau_1(t)}^{t}\sum_{i=1}^{2} b_i(u)e^{\int_{\tau_i(u)}^{L(t)}\sum_{k_1=1}^{2} b_{k_1}(u_1)\varphi_{l-1}(u_1,\tau_{k_1}(u_1))du_1}du \\
&\le \int_{t-\delta}^{t}\sum_{i=1}^{2} b_i(u)e^{\int_{\tau_2(u)}^{t}\sum_{k_1=1}^{2} b_{k_1}(u_1)\varphi_{l-1}(u_1,\tau_{k_1}(u_1))du_1}du \\
&\le \int_{t-\delta}^{t}\sum_{i=1}^{2} b_i(u)e^{\int_{t-2.3}^{t}\sum_{k_1=1}^{2} b_{k_1}(u_1)A_{l-1}du_1}du \\
&\le \delta(\alpha+\beta)e^{2.3(\alpha+\beta)A_{l-1}}
\end{aligned}$$

for some $T_1 > t_0$, *all* $t \ge T$ *and* $l \in \mathbb{N}$. *Then, for every* $l \in \mathbb{N}$, *one can choose* δ *sufficiently small such that* $\delta(\alpha+\beta)e^{2.3(\alpha+\beta)A_{l-1}} < 1$. *Consequently,*

$$\limsup_{t\to\infty}\int_{L(t)}^{t}\sum_{k=1}^{2} b_i(u)\varphi_l(L(t),\tau_i(u))du \le \delta(\alpha+\beta)e^{2.3(\alpha+\beta)A_{l-1}} < 1,$$

and hence (8) is not satisfied for every α,β *and* $l \in \mathbb{N}$.

Let

$$I(t) = \prod_{i=1}^{2}\left[\prod_{i_1=1}^{2}\int_{\sigma_i(t)}^{t} b_{i_1}(s)e^{\int_{\tau i_1(s)}^{\sigma_{i_1}(t)}\sum_{l=1}^{2}(\lambda(\eta_l)-\epsilon)b_l(s_1)ds_1}ds\right]^{\frac{1}{2}}.$$

In view of $\tau_1(t) \le \tau_2(t)$, $\sigma_l(t) \le t$ *and* $\lambda(\eta_l) \le e$, $l = 1,2$, *it follows that*

$$I(t) \le \prod_{i=1}^{2}\left[\prod_{i_1=1}^{2}\int_{\sigma_i(t)}^{t} b_{i_1}(s)e^{e(\alpha+\beta)(t-\tau_2(s))}ds\right]^{\frac{1}{2}}.$$

Using the fact that $\sigma_i(t) \ge t-1.3$, $i = 1,2$, *we obtain*

$$\begin{aligned}
I(t) &\le \left[e^{5.2\,e(\alpha+\beta)}\prod_{i_1=1}^{2}\int_{t-\delta}^{t} b_{i_1}(s)ds\right]^{\frac{1}{2}} \times \left[e^{5.2\,e(\alpha+\beta)}\prod_{i_1=1}^{2}\int_{t-1.3}^{t} b_{i_1}(s)ds\right]^{\frac{1}{2}} \\
&\le \left[\delta^2 e^{5.2\,e(\alpha+\beta)}(\max\{\alpha,\beta\})^2\right]^{\frac{1}{2}} \times \left[(1.3)^2 e^{5.2\,e(\alpha+\beta)}(\max\{\alpha,\beta\})^2\right]^{\frac{1}{2}} \\
&= 1.3\,\delta\, e^{5.2\,e(\alpha+\beta)}(\max\{\alpha,\beta\})^2.
\end{aligned}$$

Therefore, one can choose a δ *sufficiently small such that* $1.3\,\delta\, e^{5.2\,e(\alpha+\beta)}(\max\{\alpha,\beta\})^2 < \frac{1}{4}$, *so condition (6) cannot apply to Equation (36) for all* α *and* β.

Since

$$\begin{aligned}
\bar{R}_1(t) &= \left(\prod_{r=1}^{2}\int_{\sigma_1(t)}^{t} b_r(u)e^{\int_{\tau_r(u)}^{\sigma_r(t)}\sum_{l_1=1}^{2} b_{l_1}(u_1)e^{(\lambda(\eta)-\epsilon)\int_{\tau_{l_1}(u_1)}^{u_1}\sum_{l_2=1}^{2} b_{l_2}(u_2)du_2}du_1}du\right)^{\frac{1}{2}} \\
&\le \left(\prod_{r=1}^{2}\int_{t-\delta}^{t} b_r(u)e^{\int_{\tau_2(u)}^{t}\sum_{l_1=1}^{2} b_{l_1}(u_1)e^{e\int_{\tau_2(u_1)}^{u_1}(\alpha+\beta)du_2}du_1}du\right)^{\frac{1}{2}}.
\end{aligned}$$

then

$$\bar{R}_1(t) \le \delta\max\{\alpha,\beta\}e^{2.6\,(\alpha+\beta)e^{1.3e(\alpha+\beta)}}. \tag{37}$$

Additionally,

$$\bar{R}_2(t) = \left(\prod_{r=1}^{2} \int_{\sigma_2(t)}^{t} b_r(u) e^{\int_{\tau_r(u)}^{\sigma_r(t)} \sum_{l_1=1}^{2} b_{l_1}(u_1) e^{(\lambda(\gamma)-\epsilon) \int_{\tau_{l_1}(u_1)}^{u_1} \sum_{l_2=1}^{2} b_{l_2}(u_2) du_2} du_1} du \right)^{\frac{1}{2}} \tag{38}$$

$$\leq \left(\prod_{r=1}^{2} \int_{t-1.3}^{t} b_r(u) e^{\int_{\tau_2(u)}^{t} \sum_{l_1=1}^{2} b_{l_1}(u_1) e^{1.3\ \mathrm{e}(\alpha+\beta)} du_1} du \right)^{\frac{1}{2}} \tag{39}$$

$$\leq 1.3 \max\{\alpha, \beta\} e^{2.6(\alpha+\beta) e^{1.3\ \mathrm{e}(\alpha+\beta)}}. \tag{40}$$

In view of $\gamma_1 = 0$*, it follows that* $D(\gamma_1) = 0$*. From this, (37), and (38), we have*

$$2\left(\prod_{r=1}^{2} D(\gamma_r)\right)^{\frac{1}{2}} \sum_{l=1}^{2} \bar{R}_l(t) + 4\prod_{r=1}^{2} \bar{R}_r(t) = 4\prod_{r=1}^{2} \bar{R}_r(t) \leq 5.2\delta(\max\{\alpha, \beta\})^2 e^{5.2\ (\alpha+\beta) e^{1.3\mathrm{e}(\alpha+\beta)}}.$$

Hence, δ *can be chosen such that* $5.2\delta(\max\{\alpha, \beta\})^2 e^{5.2\ (\alpha+\beta) e^{1.3\mathrm{e}(\alpha+\beta)}} < 1$*, and so condition (13) is not satisfied for Equation (36) for all* α *and* β*. Similarly, we can show that all the mentioned iterative and non-iterative oscillation conditions cannot be applied to Equation (36) for all* α *and* β*.*

4. Conclusions

In this work, we obtained new sufficient oscillation criteria for Equation (1). These results extend and improve many known results in the literature. We showed that all solutions of Equation (36) are oscillatory, while all the previous iterative conditions cannot be applied to this equation for any number of iterations. Using the techniques given in this work, the oscillation property for difference equations with several non-monotone deviating arguments, as well as delay differential and difference equations with oscillating coefficients, can be studied.

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