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Review

A Survey on the Study of Generalized Schrödinger Operators along Curves

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Abstract: In this survey, we review the historical development for the Carleson problem about the a.e. pointwise convergence in five aspects: the a.e. convergence for generalized Schrödinger operators along vertical lines; a.e. convergence for Schrödinger operators along arbitrary single curves; a.e. convergence for Schrödinger operators along a family of restricted curves; upper bounds of p for the L^p-Schrödinger maximal estimates; and a.e. convergence rate for generalized Schrödinger operators along curves. Finally, we list some open problems which need to be addressed.

Keywords: Schrödinger operator; pointwise convergence; maximal estimate; convergence rate; tangential curves

MSC: 42B20; 42B25; 35S10

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1. Overview of History

The Shrödinger equation is one of the pillars of non-relativistic quantum mechanics, which models the evolution of the quantum state of a quantum system. It was initially proposed by Schrödinger [1] in 1926. On the one hand, a lot of mathematicians are attracted to the theoretical research on it and the related nonlinear variants; see the comprehensive monograph in [2,3]. On the other hand, there are many traditional computational methods for solving the linear and nonlinear Shrödinger equations, including the finite difference methods, finite element methods, split-step methods, and pseudo-spectral methods, etc., for which one can see a more detailed description in [4–6].

In this survey, we focus on investigating the pointwise convergence of the solution of the free Schrödinger equation, which describes the continuity of the solutions for the free Schrödinger equations to the initial data. This convergence problem was first raised by Carleson [7] in 1980 and has been highly considered by many experts in the field of harmonic analysis and partial differential equations, such as Bourgain, Tao, Guth, Sjölin, Vega, etc. With the help of the tools produced in the development of the well-known restriction estimates in the harmonic analysis, recently such a convergence problem can be completely solved except the endpoints. So, now we are more concerned about several variants of such a pointwise convergence problem because they also play an important role in the study of the pointwise convergence for the solutions of some important equations, such as the Schrödinger equation for the quantum harmonic oscillator, see [8]. In fact, there are still many open problems to be solved. This survey will sort out some existing results and methods for the convergence of generalized Schrödinger operators along curves and list some open problems in this field.

Let $P(\xi)$ be a continuous real-valued function defined on $\mathbb{R}^n$. The solution of generalized Schrödinger equation with initial datum

$$\begin{cases} \partial_t u(x,t) - iP(D)u(x,t) = 0 & x \in \mathbb{R}^n, t \in \mathbb{R}^+, \\ u(x,0) = f(x) \end{cases} \tag{1}$$

can be formally written as

$$e^{itP(D)}f(x) := \int_{\mathbb{R}^n} e^{ix\cdot\xi + itP(\xi)} \hat{f}(\xi) d\xi, \tag{2}$$

where $\hat{f}(\xi)$ denotes the Fourier transform of f.

It is well known that the related pointwise convergence problem is to determine the optimal exponent s for which the following statements (A)–(C) are true whenever $f \in H^s(\mathbb{R}^n)$:

(**A**) The a.e. convergence for generalized Schrödinger operators along vertical lines, i.e.,

$$\lim_{t\to 0^+} e^{itP(D)}f(x) = f(x), \quad a.e.\ x \in \mathbb{R}^n. \tag{3}$$

(**B**) The a.e. convergence for Schrödinger operators along arbitrary single curves instead of the above vertical lines, i.e.,

$$\lim_{t\to 0^+} e^{itP(D)}f(\gamma(x,t)) = f(x), \quad a.e.\ x \in \mathbb{R}^n, \tag{4}$$

where $\gamma : \mathbb{R}^n \times [-1,1] \to \mathbb{R}^n$, with $\gamma(x,0) = x$.

(**C**) The a.e. convergence for Schrödinger operator along a family of restricted curves in $\mathbb{R}^n \times \mathbb{R}$: precisely, suppose that Θ is a given compact set in $\mathbb{R}^n$, and γ is a map from $\mathbb{R}^n \times [0,1] \times \Theta$ to $\mathbb{R}^n$, we consider the relationship between the fractal dimension of Θ and the optimal exponent s for which

$$\lim_{\substack{(y,t)\to(x,0) \\ y\in\Gamma_{x,t}}} e^{itP(D)}f(y) = f(x) \quad a.e.\ x \in \mathbb{R}^n, \tag{5}$$

where $\Gamma_{x,t} = \{\gamma(x,t,\theta) : \theta \in \Theta\}$.

According to the celebrated Stein's maximal principle, all of the above pointwise convergence problems can be deduced to establish the corresponding L^p-maximal estimates. Hence,

(**D**) It is interesting to look for the optimal p and s so that the L^p-Schrödinger maximal estimate holds.

The problems on the a.e. convergence rate of some important operators (such as Fourier multipliers, certain integral means, and summability means Fourier integrals) were investigated in a lot of works [9–14], etc. Therefore,

(**E**) It is also attractive to consider the relationship between the smoothness of the functions f and a.e. convergence rate for generalized Schrödinger operators along curves in $\mathbb{R}^n \times \mathbb{R}$.

Concretely, assume that $|P(\xi)| \le C|\xi|^m$ (ξ is large enough) and the curve $\gamma(x,t)$ satisfies bi-Lipschitz in x and Hölder condition of order α in t. If the corresponding L^p-maximal estimates for generalized Schrödinger operators along curves hold when $s > s_0$, one can look for an optimal convergence rate $I(\alpha,\delta,m)$ for all $f \in H^{s+\delta}(\mathbb{R}^n)$, $0 \le \delta < m$, such that

$$e^{itP(D)}(f)(\gamma(x,t)) - f(x) = o(t^{I(\alpha,\delta,m)}), \quad a.e.\ \ x \in B(x_0,r) \quad \text{as} \quad t \to 0^+. \tag{6}$$

It is worthy to study all of the above problems in depth. Next, we will overview their history separately and list some of the open problems related to this subject later.

1.1. Case (A): a.e. Convergence along Vertical Lines

This case will refer to a very long history; here, we only list the latest developments as far as we know. A more complete description can be found in [13]. We clarify the existing results according to $P(\xi)$ so that the readers may notice the influence of the phase function on the convergence results of the generalized Schrödinger operators $e^{itP(D)}f$.

(1) Elliptic case: $P(\xi) = |\xi|^2$. It was Carleson [7] who first proved the convergence for $s \geq 1/4$ when $n = 1$. Later on, Dahlberg–Kenig [15] constructed a counterexample which shows that Carleson's result is indeed sharp. For $n \geq 2$, Bourgain [16] gave a counterexample showing that it is false if $s < \frac{n}{2(n+1)}$. An alternative counterexample was proposed by Lucà–Rogers in [17]. Recently, using decoupling and polynomial decomposition, Du–Guth–Li [18] and Du–Zhang [19] obtained the sharp pointwise convergence up to the endpoint for $n = 2$ and $n \geq 3$, respectively. Therefore, in this case, the pointwise convergence problem raised by Carleson was completely solved.

(2) Non-elliptic case: $P(\xi) = \xi_1^2 - \xi_2^2 \pm \xi_3^2 \pm \cdots \pm \xi_n^2$. When $n = 2$, it corresponds to the pointwise convergence of the solution to the non-elliptic Schrödinger equation, i.e., $i\partial_t u + (\partial_x^2 - \partial_y^2)u = 0$. Rogers–Vargas–Vega [20] proved that the pointwise convergence holds if and only if $s \geq 1/2$ when $f \in H^s(\mathbb{R}^2)$. Obviously, in the non-elliptic case, the convergence results are worse than that in the elliptic case. When $n \geq 3$, similar results were also established except the endpoint, see [20].

(3) Fractional case:

(I) $P(\xi) = |\xi|^\alpha$. When $\alpha > 1$, Sjölin [21] showed that the convergence of $e^{it\triangle^{\frac{\alpha}{2}}}f$ for $s \geq 1/4$ if $n = 1$, and Cho–Ko [22,23] obtained an almost everywhere convergence for $s > \frac{n}{2(n+1)}, n \geq 2$. When $0 < \alpha < 1$, Walther [24] proved that the corresponding pointwise convergence holds for $s > \alpha/4$ when $n = 1$, and $s > \alpha/4$ in a higher dimension assuming that the initial value f is radial. Zhang [25] demonstrated the convergence for $f \in H^s(\mathbb{R}^n)$ when $s > n\alpha/4$.

(II) $P(\xi) = \sum_{k=1}^n \pm|\xi_k|^\alpha$ and $\alpha \neq 2$. When $\alpha > 1$, the pointwise convergence of $e^{it\sum_{k=1}^n \pm D_k^\alpha}$ was proved in [20] if $f \in H^s(\mathbb{R}^n)$, $s > 1/2$, and $n \geq 2$. The convergence property fails for $s < 1/4$ by the result of Sjölin in [26]. This left the convergence problem open in the range $1/4 \leq s \leq 1/2$ for $n \geq 2$. Recently, An–Chu–Pierce [27] developed a flexible new method to approach such problems and proved that if the Schrödinger maximal operator with the phase function $P(\xi) = \sum_{k=1}^n |\xi_k|^\alpha (\alpha \geq 3, \alpha \in \mathbb{Z})$ is bounded from $H^s(\mathbb{R}^n)$ to $L^1(B(0,1))$, then $s \geq \frac{1}{4} + \frac{n-1}{4((\alpha-1)n+1)}$, which is the first result that exceeds a long-standing barrier at $1/4$. When $0 < \alpha < 1$, Zhang [25] proved that convergence for $f \in H^s(\mathbb{R}^n)$ holds if $s > n\alpha/4$.

(4) More general case:

(I) A class of very important operators with a physical background, such as Boussinesq operators $P(\xi) = |\xi|\sqrt{1+|\xi|^2}$ and Beam operators $P(\xi) = \sqrt{1+|\xi|^4}$. We observed that from the phase function point of view, this class of operators can be seen as a perturbation of elliptic Schrödinger operators from case(A)-(1). In [28], Li–Li obtained the sharp convergence for Boussinesq operators when $n = 1$. However, when $n \geq 2$, limited to decoupling techniques, we could not expect to deal with Boussinesq operators or Beam operators by the method that obtains the maximal estimates for elliptic Schrödinger operators. Li–Wang [29] established the transference principle, i.e., if the absolute value of the difference in phase functions $P(\xi)$ and $Q(\xi)$ is a bound constant when $|\xi|$ is large enough, the L^p-maximal estimate for one of the corresponding Schrödinger operators will imply the other. Therefore, they employed the maximal estimates for phase function $P(\xi)$ from the results of case(A)-(1) and obtained the convergence results for Boussinesq operators and Beam operators, which are also sharp when $n = 2$ by the counterexample in [29].

(II) When $P(\xi) = \xi_1\xi_2 + \xi_1^m$, $m \in \mathbb{N}^+$, the corresponding equations are higher-order dispersive equations. When $P(\xi) = \xi_1\xi_2 + |\xi_1|^m$, $m \in \mathbb{R}^+$ with $1 < m < 2$, the corresponding equations are non-elliptic Schrödinger equations with fractional-order perturbations. With the help of Theorem 4.1 in [30], Li–Wang [29] proved that the corresponding maximal estimates hold for $s > 1/2$ and also gave a counterexample to show that it is sharp for the case $1 < m < 2$.

1.2. Case (B): a.e. Convergence for Schrödinger Operator along Arbitrary Single Curves

Now, we will consider two cases: smooth curves and tangential curves. While for smooth curves the considered problem can often be reduced to Case (A), for tangential curves the situation is much more complicated, especially in multidimensional spaces.

(1) Smooth curves: In the study of the pointwise convergence problem of the Schrödinger equation for the harmonic oscillator, Lee–Rogers [8] turned to prove (4) for $P(\xi) = |\xi|^2$ and any $\gamma \in C^1(\mathbb{R}^d \times [-1,1] \to \mathbb{R}^d)$ (such as $\gamma(x,t) = x - (t^\kappa, 0, \cdots, 0)$ with $\kappa \geq 1$). Furthermore, they showed that this pointwise convergence is essentially equivalent to the vertical results in Case(A)-(1).

(2) Tangential curves: The curves $(\gamma(x,t), t)$ are called tangential curves because as $t \to 0$, $(\gamma(x,t), t)$ approaches $(x, 0)$ tangentially to the hyperplane $\{(y,t) \in \mathbb{R}^n \times \mathbb{R} : t = 0\}$.

Let $P(\xi)$ satisfy $|D_\xi^\beta P(\xi)| \lesssim |\xi|^{m-|\beta|}, |\nabla P(\xi)| \sim |\xi|^{m-1}$ for $|\xi| \gg 1$, and $m \geq 2$. Assume that $\gamma(x,t)$ is bi-Lipschitz in x and Hölder with order $\frac{1}{m-1}$ in t. By [31] [Proposition 4.3], the pointwise convergence (4) follows from the corresponding maximal estimates along the vertical line (x,t). Obviously, some convergence results along tangential curves for generalized Schrödinger operators can be obtained when $m > 2$.

For the elliptical case $m = 2$ and $n = 1$, Cho–Lee–Vargas [31] proved the pointwise convergence along the curve $(\gamma(x,t), t)$ holds for $s > \max\{1/2 - \alpha, 1/4\}$ if the function γ satisfies Hölder condition of order α, $0 < \alpha \leq 1$ in t and bi-Lipschitz in x. Ding–Niu [32] used the linearization method and improved to $s \geq 1/4$, if $1/2 \leq \alpha \leq 1$.

For the fractional case $P(\xi) = |\xi|^m$, $m > 1$, Cho–Schiraki [33] proved that the pointwise convergence along tangential curves holds for $s > \max\{1/4, (1 - m\alpha)/2\}$. Meanwhile, they also estimated the capacitary dimension of the divergence set. Later, Yuan–Zhao [34] extended the result for $m > 1$ to the case $0 < m < 1$, which is sharp up to the endpoint.

Comparing with the case in $\mathbb{R} \times \mathbb{R}$, much less is known about the convergence problem for Schrödinger operators along tangential curves in a higher-dimensional case ($\mathbb{R}^n \times \mathbb{R}, n \geq 2$). Let $\Gamma_\alpha := \{\gamma : [0,1] \to \mathbb{R}^2 : \text{for each } t, t' \in [0,1], |\gamma(t) - \gamma(t')| \leq C_\alpha |t - t'|^\alpha\}, C_\alpha \geq 1$. For example, $\gamma(t) = t^\alpha \mu, \alpha \in [1/2, 1)$ and μ is a bounded vector in $\mathbb{R}^2$. For $s > 3/8$, Li–Wang [13] employed the broad–narrow argument and polynomial partitioning, then obtained the pointwise convergence of the Schrödinger operators along the curves $(x + \gamma(t), t) \in \mathbb{R}^2 \times \mathbb{R}$ with $\gamma \in \Gamma_\alpha$ for some $\alpha \in [1/2, 1)$.

Unlike the one-dimensional case, the TT^*-method adopted in Cho–Lee–Vargas [31] fails in a higher-dimensional case. One might expect to solve the convergence problem along tangential curves using the argument in [18,19]. However, there are several technical challenges to overcome.

First of all, the maximal estimates for Schrödinger operators along curves are no longer translation invariant in the t-direction, which makes it difficult to apply the induction-on-scale method in the physical space. In order to solve this problem, Li–Wang [13] used induction on both the physical radius R and curves γ, where $R \gg 1$.

What is more, after parabolic rescaling, it can be found that the support for the Fourier transform of $e^{it\Delta} f(x + R\gamma(\frac{t}{R^2}))$ is not clear, then many nice properties do not work well any more, which plays a fundamental role in the study of Case(A)-(1) when $n = 2$. In order to overcome such difficulties, Li–Wang [13] gave a substitution for the local constant property. They also constructed the counterexample to show that this convergence result may not be sharp. However, one still does not know whether the decoupling method can improve the exponent $s > 3/8$ to $s > 1/3$ or not.

1.3. Case (C): a.e. Convergence for Schrödinger Operator along a Family of Restricted Curves

Similar with Section 1.2, we will consider a family of restricted smooth curves and tangential curves, respectively. This is attributed to the fact that the convergence results along a family of restricted curves depend heavily on the results along a single curve.

(1) A family of smooth curves and $P(\xi) = |\xi|^2$.

(I) $(x,t) \in \mathbb{R} \times \mathbb{R}$: If $\Gamma_{x,t} = \{x + t\theta : \theta \in \Theta\}$, where $t \in [-1,1]$, Θ is a given compact set in $\mathbb{R}$. In [31], Cho–Lee–Vargas proved that the corresponding non-tangential convergence result holds for $s > \frac{\beta(\Theta)+1}{4}$; here, $\beta(\Theta)$ denotes the upper Minkowski dimension of Θ. Recently, Shiraki [35] generalized this result to the fractional Schrödinger equation with $P(\xi) = |\xi|^m$, $m > 1$.

(II) $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$: If the function $\gamma(x,t,\theta)$ satisfies bi-Lipschitz in x, and Lipschitz in t and θ, Li–Wang–Yan [36] employed the sharp L^p-maximal estimates from [18] and proved that (5) holds whenever $f \in H^s(\mathbb{R}^2)$ for each $s > \frac{\beta(\Theta)+1}{3}$. Furthermore, they showed that this convergence result is sharp up to the endpoint.

(III) $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, $n \geq 3$: Combining the maximal estimate from [19], Li–Wang–Yan [36] showed that if $s > \frac{\beta(\Theta)}{2} + \frac{n}{2(n+1)}$, then (5) holds whenever $f \in H^s(\mathbb{R}^n)$. However, this result is not sharp because the upper bound of p in L^p-maximal estimates in [19] is still open. Hence, in [36], they combined with the counterexample given by Sjölin–Sjögren [37] and obtained an upper bound for $p \leq \frac{2(n+1)}{n}$. If one can establish the L^p-maximal estimate for $p \leq \frac{2(n+1)}{n}$, then the convergence (5) holds whenever $f \in H^s(\mathbb{R}^n)$ for each $s > \frac{\beta(\Theta)}{p} + \frac{n}{2(n+1)}$.

(2) A family of tangential curves.

$(x,t) \in \mathbb{R} \times \mathbb{R}$: If the function $\gamma(x,t,\theta)$ satisfies bi-Lipschitz in x, and Lipschitz in θ, but Hölder with order α in t $(0 < \alpha < 1)$, Li–Wang [13] obtained the convergence results along such a family of tangential curves, which is sharp when $\beta(\Theta) = 0$ (see [31]) and $\beta(\Theta) = 1$ (see [37]). The necessity for the case $0 < \beta(\Theta) < 1$ is still open. Moreover, Fan–Li–Wang [38] extended these convergence results to the fractional case, $P(\xi) = |\xi|^m$, $m \neq 1$.

In the one-dimensional case, Cho–Lee–Vargas [31] adopted the TT^* method and time localizing lemma to prove the non-tangential convergence result for the Schrödinger operators stated in Case(C)-(1)-(I). Noting that the time localizing lemma is invalid for the fractional Schrödinger operator with $P(\xi) = |\xi|^m$ as $m \to 1$, Shiraki [35] improved the method in [31] so that the time localizing lemma is no longer necessary, then generalized the result in [31] to the fractional Schrödinger operator.

In a higher-dimensional case, the TT^*-method can no longer be applied to obtain the corresponding results described in Case(C)-(1)-(II) and Case(C)-(1)-(III). Using the time localizing lemma and Fourier expansion, Li–Wang–Yan [36] discovered the relationship between the maximal estimates for Schrödinger operators along a family of smooth curves and the maximal estimates for Schrödinger operators along a single vertical line. Then, the convergence results in Case(C)-(1)-(II) and Case(C)-(1)-(III) follow from the L^p-maximal estimates for the Schrödinger operators by [18,19].

1.4. Case (D): The Upper Bounds of p for L^p-Schrödinger Maximal Estimate

Next, we will observe two cases: free Schrödinger operators and Schrödinger operators along curves. For the L^p-maximal estimates of free Schrödinger operators, the upper bound of p depends on the spatial dimension. However, for the L^p-maximal estimates of Schrödinger operators along curves, the smoothness of the (tangential) curves will also affect the upper bound of p.

(1) $\gamma(x,t) = x$: When the spatial dimension $n = 1$ and $P(\xi) = |\xi|^m$ $(m > 1)$, Sjölin [39] studied the upper bound for L^p Schrödinger maximal estimates. When the spatial dimension $n = 2$ and $p(\xi) = |\xi|^2$, Du–Guth–Li [18] proved the sharp L^p-estimates for all $p \leq 3$ and $s > 1/3$. When the spatial dimension $n \geq 3$, Du–Zhang [19] proved the sharp L^2-estimate with $s > n/2(n+1)$, but the sharp L^p-estimate of the Schrödinger maximal operator is still unknown for $p > 2$. The partial results on this problem were obtained by using polynomial partitioning and refined Strichartz estimates in [40–42]. In [36], Li–Wang–

Yan showed that if there exists $p \geq 2$ such that for any $s > \frac{n}{2(n+1)}$, L^p-maximal estimates for Schrödinger operators hold whenever $f \in H^s(\mathbb{R}^n)$, then $p \leq \frac{2(n+1)}{n}$.

(2) General curve $\gamma(x,t)$ and $(x,t) \in \mathbb{R} \times \mathbb{R}$: Li–Wang [13] considered the sharp upper bounds for L^p-Schrödinger maximal estimates when $\gamma(x,t)$ satisfies bi-Lipschitz in x and Hölder with order α in t. Recently, Fan–Li–Wang [38] extended this result to fractional Schrödinger maximal estimates along tangential curves.

In addition to its own interests, the optimal upper bound for p in the L^p-Schrödinger maximal estimate is closely related to the non-tangential convergence results as described in case(C). Therefore, new developments in this area have been expected.

1.5. Case (E): a.e. Convergence Rate for Generalized Schrödinger Operators along Tangential Curves

In [29], Li–Wang proved that if L^p-maximal estimates hold for celebrated Schrödinger operators for each $f \in H^s(\mathbb{R}^n)$ $(s > s_0)$, the corresponding convergence rate is $I(\alpha, \delta, m) = \alpha\delta/m$. Recently, Li–Wang [13] improved it to sharp results: if $1/m \leq \alpha < 1$, $(\delta, I) \in \{(x,y) : x \geq 0, y \geq 0, y \leq x/m, y < \alpha\}$; if $0 < \alpha < 1/m$, $(\delta, I) \in \{(x,y) : x \geq 0, y \geq 0, y \leq \alpha x, y < \alpha\}$.

In [29], Li–Wang improved the previous convergence rate result established by Cao–Fan–Wang [9]. Compared with the method in [9], Li–Wang [29] adopted a more effective time–frequency decomposition. Moreover, some discussion of necessity can also be found in [29].

2. Open Problems

Based on the results on this topic mentioned above, we list the following open problems which are well worth considering.

(1) There is no positive or negative convergence results for Case(A)-(1) when $s = \frac{n}{2(n+1)}$.

(2) The optimal exponent s of the pointwise convergence is still open for fractional Schrödinger operator Case(A)-(3)-(II).

(3) The convergence problem is still open for Schrödinger operators (or fractional Schrödinger operators $P(\xi) = |\xi|^m$, $0 < m < 1$ and $m = 2$) along tangential curves in the higher-dimensional case $\mathbb{R}^n \times \mathbb{R}, n \geq 2$. Specially, for $\mathbb{R}^2 \times \mathbb{R}$, there are still two open problems that need to be solved:

(I) Is it possible to improve the regularity index $s > 3/8$ to $1/3$?

(II) For the tangential curve $\gamma(x,t)$ which satisfies bi-Lipschitz in x and Hölder with order α in t, $\alpha \in (0, 1/2)$, the corresponding convergence problem is still completely open.

(4) How to estimate the capacitary dimension of the divergence set for Schrödinger operators (or fractional Schrödinger operators $P(\xi) = |\xi|^m, 0 < m < 1$ and $m > 1$) along tangential curves in the higher-dimensional case $\mathbb{R}^n \times \mathbb{R}, n \geq 2$ is still an open question.

(5) In Case(C), the necessity for all the convergence results when $0 < \beta(\Theta) < n$ is still open.

(6) In Case(C)-(2), the a.e. convergence problem remains open for Schrödinger operators (or fractional Schrödinger operators $P(\xi) = |\xi|^m$, $0 < m < 1$, and $m > 1$) along a family of restricted tangential curves in $\mathbb{R}^n \times \mathbb{R}, n \geq 2$.

(7) In Case(D), for $n \geq 3$, there is no sharp L^p-estimates for Schrödinger maximal operators for $2 < p \leq \frac{2(n+1)}{n}$.

(8) In Case(D), for $n \geq 2$, it is still unknown for the upper bound of p for L^p-Schrödinger (or fractional Schrödinger) maximal estimates along tangential curves.

(9) In Case(E), it is still unknown about the a.e. convergence rate for fractional Schrödinger operators $(P(\xi) = |\xi|^m, 0 < m < 1)$ along curves.

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