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Article

Studying the Harmonic Functions Associated with Quantum Calculus

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Abstract: Using the derivative operators' $\mathfrak{q}$-analogs values, a wide variety of holomorphic function subclasses, $\mathfrak{q}$-starlike, and $\mathfrak{q}$-convex functions have been researched and examined. With the aid of fundamental ideas from the theory of $\mathfrak{q}$-calculus operators, we describe new $\mathfrak{q}$-operators of harmonic function $\mathrm{H}^{\gamma}_{\varrho,\chi;\mathfrak{q}}\mathfrak{F}(\varpi)$ in this work. We also define a new harmonic function subclass related to the Janowski and $\mathfrak{q}$-analog of Le Roy-type functions Mittag–Leffler functions. Several important properties are assigned to the new class, including necessary and sufficient conditions, the covering Theorem, extreme points, distortion bounds, convolution, and convex combinations. Furthermore, we emphasize several established remarks for confirming our primary findings presented in this study, as well as some applications of this study in the form of specific outcomes and corollaries

Keywords: convolutions; $\mathfrak{q}$-Mittag–Leffler; Le Roy-type Mittag–Leffler; harmonic function; holomorphic functions; univalent function; $\mathfrak{q}$-calculus

MSC: 30C45; 30C50

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1. Introduction

Quantum calculus, also known as ($\mathfrak{q}$-calculus), is a branch of mathematics that extends classical calculus to the quantum realm. It is an effective tool for studying the behavior of quantum systems and has numerous applications in mathematics, engineering, physics, and finance. Quantum calculus has recently grown in importance as a tool for understanding harmonic functions, which are essential to the study of mathematics, physics, and engineering. The study of harmonic functions using $\mathfrak{q}$-calculus offers fresh and creative approaches to comprehending and simulating physical and engineering systems. It was demonstrated that the study of fractional harmonic functions, which are a generalization of classical harmonic functions, makes excellent use of $\mathfrak{q}$-calculus.

To determine coefficient estimates and inclusion relations, harmonic classes of holomorphic functions have recently been created and investigated. Yousef et al. [1] defined a new subclass of univalent functions and acquire a few geometrical properties using a generalized linear operator. Using a certain convolution $\mathfrak{q}$-operator, Srivastava et al. [2] introduced two new families of harmonic meromorphically functions and conducted investigations into the inclusion features.

Srivastava et al. [3] developed and investigated a new class of harmonic functions involving Janowski functions using a $\mathfrak{q}$-derivative operator. Khan et al. (2007) [4], on the other hand, employed a new class of harmonic functions involving a symmetric Sălăgean $\mathfrak{q}$-derivative operator. Studies on a novel class of harmonic functions related to starlike harmonic functions are carried out using the concepts of subordination and Ruscheweyh derivatives, see [5–9].

In 1990, Ismail et al. [10] established a class of complex-valued functions that are holomorphic on the open unit disk $\beth = \{\varpi : \varpi \in \mathbb{C}, |\varpi| < 1\}$, therefore incorporating $\mathfrak{q}$-calculus into the theory of holomorphic univalent functions with the normalizations $\mathfrak{F}(0) = \mathfrak{F}'(0) - 1 = 0$, and $|\mathfrak{F}(\varpi) \geq |\mathfrak{F}(\mathfrak{q}\varpi)|$ on $\beth$ for every $\mathfrak{q}$ $(0 < \mathfrak{q} < 1)$. Several authors employed the theory of holomorphic univalent functions and $\mathfrak{q}$-calculus as a result of the influence of these writers; see, for instance [11–16].

The complex-valued function $\mathfrak{F} = u + iv$ is said to be harmonic in $\beth$ if both v and u are real-valued harmonic functions in $\beth$. Furthermore, the complex-valued harmonic function $\mathfrak{F} = iv + u$ can be expressed as $\mathfrak{F} = \overline{\Im} + \hbar$, where $\Im$ and $\hbar$ are holomorphic in $\beth$. In particular, $\hbar$ is known as the holomorphic part, and $\Im$ is known as the co-holomorphic part of $\mathfrak{F}$.

Before using $\mathfrak{q}$-calculus and harmonic univalent functions, it is necessary to understand the notation and terminology for harmonic univalent functions.

A function $\mathfrak{F}(\varpi) = \hbar(\varpi) + \overline{\Im(\varpi)}$ where $\hbar$ and $\Im$, respectively, are the holomorphic and co-holomorphic parts, is locally univalent and sense preserving in $\beth$ if and only if $|\hbar'(\varpi)| > |\Im'(\varpi)|)$ in $\beth$.

The continuous complex-valued functions $\mathfrak{F}(\varpi) = \hbar(\varpi) + \overline{\Im(\varpi)}$ defined in $\beth$ with the following form

$$\overline{\Im(\varpi)} = \sum_{\Bbbk=1}^{\infty} \overline{b_{\Bbbk}\varpi^{\Bbbk}}, \qquad \hbar(\varpi) = \varpi + \sum_{\Bbbk=2}^{\infty} a_{\Bbbk}\varpi^{\Bbbk}, \;\; |b_1| < 1. \tag{1}$$

Let $\mathcal{H}$ and its subclass $\mathcal{U}_{\mathcal{H}}$ are defined as (see, for details [17–21]),

$$\mathcal{H} = \Big\{\mathfrak{F} : \beth \to \mathbb{C} \,|\, \mathfrak{F} \text{ of the form (1) is a sense preserving and locally univalent function in } \beth \Big\},$$
$$\mathcal{U}_{\mathcal{H}} = \Big\{\mathfrak{F} \in \mathcal{H} \,|\, \mathfrak{F} \text{ is univalent function in the open unit disk } \beth\Big\}.$$

If $\Im(\varpi) = 0$ in $\beth$, then class $\mathcal{U}_{\mathcal{H}}$ is reduced to the class $\mathcal{S}$ of normalised holomorphic functions which are univalent in $\beth$, (for more details, see [22]).

Ahuja et al. [23] presented the $\mathfrak{q}$-harmonic class of functions in $\beth$ denoted by $\mathcal{H}_{\mathfrak{q}}$.

Definition 1 ([23])**.** *A harmonic function $\mathfrak{F} = \hbar + \overline{\Im}$ as in (1) is said to be $\mathfrak{q}$-harmonic sense preserving and locally univalent in $\beth$, denoted by $\mathcal{H}_{\mathfrak{q}}$, iff the second dilatation $\xi(\varpi;\mathfrak{q})$ satisfies the condition*

$$|\xi(\varpi;\mathfrak{q})| = \left|\partial_{\mathfrak{q}}\Im(\varpi)(\partial_{\mathfrak{q}}\hbar(\varpi))^{-1}\right| < 1, \qquad (q \in (0,1),\, \varpi \in \beth). \tag{2}$$

Note that when $\mathfrak{q} \to 1^-$, then $\mathcal{H}_{\mathfrak{q}}$ reduces to the well known family $\mathcal{H}$.

Furthermore, The class $\overline{\mathcal{H}_{\mathfrak{q}}}$ consisting of functions $\mathfrak{F} = \overline{\Im} + \hbar$, where

$$\Im(\varpi) = -\sum_{\Bbbk=1}^{\infty} |b_{\Bbbk}|\varpi^{\Bbbk}, \quad \text{and} \quad \hbar(\varpi) = \varpi - \sum_{\Bbbk=2}^{\infty} |a_{\Bbbk}|\varpi^{\Bbbk}, \;(\varpi \in \beth). \tag{3}$$

2. Preliminaries and Definitions

In this section, we provide some fundamental definitions and properties of $\mathfrak{q}$ -calculus that are applied throughout this investigation. These are based on the assumption that $\mathfrak{q} \in (0,1)$.

Definition 2 ([24])**.** *Let $0 < \mathfrak{q} < 1$. Then the $[\Bbbk]_{\mathfrak{q}}$ denotes the basic (or $\mathfrak{q}$-) number, defined by*

$$[\Bbbk]_{\mathfrak{q}} = \begin{cases} \frac{1-\mathfrak{q}^{\Bbbk}}{1-\mathfrak{q}} & , \;\Bbbk \in \mathbb{C}\backslash\{0\} \\ 0 & , \;\Bbbk = 0 \\ 1 + \mathfrak{q} + \cdots + \mathfrak{q}^{n-1} = \sum_{i=0}^{n-1} \mathfrak{q}^i & , \;\Bbbk = n \in \mathbb{N}. \end{cases}$$

It is obvious from Definition 2 that $\lim_{\mathfrak{q}\to 1^-}[n]_{\mathfrak{q}} = \lim_{\mathfrak{q}\to 1^-}\frac{1-\mathfrak{q}^n}{1-\mathfrak{q}} = n$.

Definition 3 ([24])**.** *The q-derivative (or q-difference operator) of a function f is defined by*

$$\partial_{\mathfrak{q}} f(z) = \begin{cases} \frac{f(z)-f(\mathfrak{q}z)}{z-\mathfrak{q}z} & z \in \mathbb{C}\backslash\{0\} \\ f'(0) & z = 0. \end{cases}$$

We note that $\lim_{\mathfrak{q}\to 1^-}\partial_{\mathfrak{q}} f(z) = f'(z)$, *if f is differentiable at $z \in \mathbb{C}$.*

For a parameter $\chi, \varrho \in \mathbb{C}$ with $\Re e\{\chi\} > 0$ and $\Re e\{\varrho\} > 0$. Then, the generalized Mittag–Leffler-type function, introduced by Wiman [25] by

$$\mathcal{M}_{\varrho,\chi}(\varpi) = \sum_{\Bbbk=0}^{\infty} \frac{\varpi^{\Bbbk}}{\Gamma(\varrho\Bbbk+\chi)}, \qquad (\varpi \in \mathbb{C}). \tag{4}$$

Recently, Schneider [26] and independently Garra and Polito [27], introduced Le Roy-type Mittag–Leffler function, defined as

$$\mathcal{F}^{\gamma}_{\varrho,\chi}(\varpi) = \sum_{\Bbbk=0}^{\infty} \frac{\varpi^{\Bbbk}}{[\Gamma(\varrho\Bbbk+\chi)]^{\gamma}}, \qquad (\varrho,\chi,\gamma > 0, \varpi \in \mathbb{C}). \tag{5}$$

For $\Re e\{\chi\} > 0, \Re e\{\varrho\} > 0$, In 2014, Sharma and Jain [28] defined the $\mathfrak{q}$-Mittag–Leffler-type function, by

$$\mathcal{M}^{\gamma}_{\varrho,\chi}(z;\mathfrak{q}) = \sum_{\Bbbk=0}^{\infty} \frac{\varpi^{\Bbbk}}{\Gamma_{\mathfrak{q}}(\varrho\Bbbk+\chi)} \qquad (\varrho,\chi,\gamma \in \mathbb{C}), \tag{6}$$

where $|\mathfrak{q}| < 1$ and $\Gamma_{\mathfrak{q}}$ is the $\mathfrak{q}$-gamma function can also be defined by

$$\Gamma_{\mathfrak{q}}(1+\varpi) = (1-\mathfrak{q}^{\varpi})(1-\mathfrak{q})^{-1}\Gamma_{\mathfrak{q}}(\varpi), \qquad (\mathfrak{q} \in (0,1),\ \varpi \in \mathbb{C}).$$

Motivated by Gerhold [29] and Garra and Polito [27], we define the $\mathfrak{q}$-analog of Le Roy-type Mittag–Leffler function, by

$$\mathcal{M}^{\gamma}_{\varrho,\chi}(\varpi;\mathfrak{q}) = \sum_{\Bbbk=0}^{\infty} \frac{\varpi^{\Bbbk}}{[\Gamma_{\mathfrak{q}}(\varrho\Bbbk+\chi)]^{\gamma}}, \qquad (z \in \beth), \tag{7}$$

where $\Re e(\varrho) > 0, \chi \in \mathbb{C} \setminus \{0,-1,-2,\cdots\}$.

The normalization of the $\mathfrak{q}$-analog of Le Roy-type Mittag–Leffler function $\mathfrak{M}^{\gamma}_{\varrho,\chi}(z;\mathfrak{q})$ can be defined by

$$\mathfrak{M}^{\gamma}_{\varrho,\chi}(\varpi;\mathfrak{q}) = \varpi(\Gamma_{\mathfrak{q}}(\chi))^{\gamma}\mathcal{M}^{\gamma}_{\varrho,\chi}(\varpi;\mathfrak{q}) = \varpi + \sum_{\Bbbk=1}^{\infty}\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma}\varpi^{\Bbbk}, \tag{8}$$

where $\Re e(\varrho) > 0, \chi \in \mathbb{C} \setminus \{0,-1,-2,\cdots\}$.

Next, we define the following new $\mathfrak{q}$-operators of $\mathrm{H}^{\gamma}_{\varrho,\chi;\mathfrak{q}}\mathfrak{F}(\varpi)$ of harmonic function $\mathfrak{F} = \overline{\Im} + \hbar$ by

$$\mathrm{H}^{\gamma}_{\varrho,\chi;\mathfrak{q}}\mathfrak{F}(\varpi) = \mathrm{H}^{\gamma}_{\varrho,\chi;\mathfrak{q}}\hbar(\varpi) + \overline{\mathrm{H}^{\gamma}_{\varrho,\chi;\mathfrak{q}}\Im(\varpi)} \qquad (z \in \beth),$$

where

$$\mathrm{H}^{\gamma}_{\varrho,\chi;\mathfrak{q}}\hbar(\varpi) = \mathfrak{M}^{\gamma}_{\varrho,\chi}(\varpi;\mathfrak{q}) \circledast \hbar(\varpi) = \varpi + \sum_{\Bbbk=2}^{\infty}\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma} a_{\Bbbk}\varpi^{\Bbbk},$$

$$\mathrm{H}^{\gamma}_{\varrho,\chi;\mathfrak{q}}\Im(\varpi) = \mathfrak{M}^{\gamma}_{\varrho,\chi}(\varpi;\mathfrak{q}) \circledast \Im(\varpi) = \sum_{\Bbbk=1}^{\infty}\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma} b_{\Bbbk}\varpi^{\Bbbk}.$$

Below is an illustration that shows how we introduce the class of harmonic univalent functions by using the operator $\mathrm{H}^{\gamma}_{\varrho,\chi;\mathrm{q}}\mathfrak{F}(\varpi)$.

Definition 4. *For $\aleph \in [0,1)$, a function f as is in (1) is said to be in the class $\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_{\mathrm{q}}}(\varrho,\chi,\gamma,\aleph)$ of harmonic convex function of order $\aleph$ in $\beth$, if it satisfies the condition*

$$\Re e\left\{\frac{\varpi\partial_{\mathrm{q}}\left(\varpi\partial_{\mathrm{q}}\mathrm{H}^{\gamma}_{\varrho,\chi;\mathrm{q}}\hbar(\varpi)\right)+\overline{\varpi\partial_{\mathrm{q}}\left(\varpi\partial_{\mathrm{q}}\mathrm{H}^{\gamma}_{\varrho,\chi;\mathrm{q}}\Im(\varpi)\right)}}{\varpi\partial_{\mathrm{q}}(\mathrm{H}^{\gamma}_{\varrho,\chi;\mathrm{q}}\hbar(\varpi))-\overline{\varpi\partial_{\mathrm{q}}(\mathrm{H}^{\gamma}_{\varrho,\chi;\mathrm{q}}\Im(\varpi))}}\right\}\geq\aleph,\qquad(\varpi\in\beth).\tag{9}$$

Here, the class $\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_{\mathrm{q}}}(\varrho,\chi,\gamma,\aleph)}$ is defined by $\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_{\mathrm{q}}}(\varrho,\chi,\gamma,\aleph)}=\overline{\mathcal{H}_{\mathrm{q}}}\cap\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_{\mathrm{q}}}(\varrho,\chi,\gamma,\aleph)$.

Remark 1. *When $\gamma=0$, then $\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_{\mathrm{q}}}(\varrho,\chi,\gamma,\aleph)$ reduces to $\mathcal{H}^{\mathcal{C}}_{\mathrm{q}}(\aleph)$, known as q-harmonic convex functions of order $\aleph$ introduced by Ahuja et al. [23]. Furthermore, when $\gamma=0$ and $\mathrm{q}\to 1^{-}$, then $\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_{\mathrm{q}}}(\varrho,\chi,\gamma,\aleph)$ reduces to $\mathcal{U}^{\mathcal{C}}_{\mathcal{H}}(\aleph)$, known as harmonic convex functions of order $\aleph$ (see [30–32]). Furthermore, when $\gamma=0$, $\mathrm{q}\to 1$ and $\Im(\varpi)=0$, then $\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_{\mathrm{q}}}(\varrho,\chi,\gamma,\aleph)$ reduces to the traditional $\mathcal{S}^{\mathcal{C}}(\aleph)$, known as convex functions of order $\aleph$, studied in 1936 [33]. Moreover, when $\gamma=0$ and $\Im(\varpi)=0$, then $\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_{\mathrm{q}}}(\varrho,\chi,\gamma,\aleph)$ is reduced to $\mathcal{S}^{\mathcal{C}}_{\mathrm{q}}(\aleph)$ [34].*

3. Sufficient Coefficient Condition

In this section, we start with a sufficient coefficient condition for functions $\mathfrak{F}$ in the class $\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_{\mathrm{q}}}(\varrho,\chi,\gamma,\aleph)$.

Theorem 1. *Let $0\leq\aleph<1$ and $\mathrm{q}\in(0,1)$. If $\mathfrak{F}$ as is in (1) and satisfies the condition*

$$\sum_{\Bbbk=2}^{\infty}[\Bbbk]_{\mathrm{q}}\Big(([\Bbbk]_{\mathrm{q}}-\aleph)|a_{\Bbbk}|+([\Bbbk]_{\mathrm{q}}+\aleph|b_{\Bbbk}|\Big)\left(\frac{\Gamma_{\mathrm{q}}(\chi)}{\Gamma_{\mathrm{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma}\leq 2(1-\aleph),\tag{10}$$

where $a_1=1$. Then, $\mathfrak{F}$ is harmonic sense preserving and univalent in the open unit disk $\beth$ and so, $\mathfrak{F}\in\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_{\mathrm{q}}}(\varrho,\chi,\gamma,\aleph)$.

Proof. Let $\mathfrak{F}$ be as is in (1) and satisfies the condition (10), then $\mathfrak{F}$ is sense preserving in $\beth$ if it satisfies $|\partial_{\mathrm{q}}\hbar(\varpi)|>|\partial_{\mathrm{q}}\Im(\varpi)|$. Since

$$\begin{aligned}|\partial_{\mathrm{q}}\hbar(\varpi)|&=\left|1+\sum_{\Bbbk=2}^{\infty}[\Bbbk]_{\mathrm{q}}a_{\Bbbk}\varpi^{\Bbbk-1}\right|\geq 1-\sum_{\Bbbk=2}^{\infty}[\Bbbk]_{\mathrm{q}}|a_{\Bbbk}|r^{\Bbbk-1}\\&\geq 1-\sum_{\Bbbk=2}^{\infty}\left(\frac{[\Bbbk]_{\mathrm{q}}-\aleph}{1-\aleph}\right)[\Bbbk]_{\mathrm{q}}\left(\frac{\Gamma_{\mathrm{q}}(\chi)}{\Gamma_{\mathrm{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma}|a_{\Bbbk}|\\&\geq\sum_{\Bbbk=2}^{\infty}[\Bbbk]_{\mathrm{q}}\left(\frac{[\Bbbk]_{\mathrm{q}}+\aleph}{1-\aleph}\right)\left(\frac{\Gamma_{\mathrm{q}}(\chi)}{\Gamma_{\mathrm{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma}|b_{\Bbbk}|\\&\geq\sum_{\Bbbk=2}^{\infty}[\Bbbk]_{\mathrm{q}}|b_{\Bbbk}|\geq\sum_{\Bbbk=2}^{\infty}[\Bbbk]_{\mathrm{q}}|b_{\Bbbk}|r^{\Bbbk-1}>|\partial_{\mathrm{q}}\Im(\varpi)|,\end{aligned}$$

it follows that $\mathfrak{F}\in\mathcal{H}_{\mathrm{q}}$, by Definition 3.

To show $\mathfrak{F}$ is univalent in $\beth$, for $0 < |\varpi_1| \leq |\varpi_2| < 1$, we observe that

$$\begin{aligned}
\left|\frac{\mathfrak{F}(\varpi_1)-\mathfrak{F}(\varpi_2)}{\hbar(\varpi_1)-\hbar(\varpi_2)}\right| &\geq 1-\left|\frac{\Im(\varpi_1)-\Im(\varpi_2)}{\hbar(\varpi_1)-\hbar(\varpi_2)}\right| \\
&= 1-\left|\frac{\sum_{\Bbbk=1}^{\infty} b_{\Bbbk}\left(\varpi_1^{\Bbbk}-\varpi_2^{\Bbbk}\right)}{\varpi_1-\varpi_2+\sum_{\Bbbk=2}^{\infty} b_{\Bbbk}\left(\varpi_1^{\Bbbk}-\varpi_2^{\Bbbk}\right)}\right| \\
&> 1-\frac{\sum_{\Bbbk=1}^{\infty}[\Bbbk]_{\mathfrak{q}}|b_{\Bbbk}|}{1-\sum_{\Bbbk=2}^{\infty}[\Bbbk]_{\mathfrak{q}}|b_{\Bbbk}|} \\
&\geq 1-\frac{\sum_{\Bbbk=1}^{\infty}[\Bbbk]_{\mathfrak{q}}\left(\frac{[\Bbbk]_{\mathfrak{q}}+\aleph}{1-\aleph}\right)\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma}|b_{\Bbbk}|}{1-\sum_{\Bbbk=2}^{\infty}[\Bbbk]_{\mathfrak{q}}\left(\frac{[\Bbbk]_{\mathfrak{q}}-\aleph}{1-\aleph}\right)\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma}|a_{\Bbbk}|} \\
&\geq 0,
\end{aligned}$$

which proves the univalence.

Now, we show that $\mathfrak{F} \in \mathcal{U}_{\mathcal{H}_{\mathfrak{q}}}^{\mathcal{C}}(\varrho,\chi,\gamma,\aleph)$. Then, from Definition 4, we can write (9)

$$\frac{\varpi\partial_{\mathfrak{q}}\left(\varpi\partial_{\mathfrak{q}}\mathrm{H}_{\varrho,\chi;\mathfrak{q}}^{\gamma}\hbar(\varpi)\right)+\overline{\varpi\partial_{\mathfrak{q}}\left(\varpi\partial_{\mathfrak{q}}\mathrm{H}_{\varrho,\chi;\mathfrak{q}}^{\gamma}\Im(\varpi)\right)}}{\varpi\partial_{\mathfrak{q}}(\mathrm{H}_{\varrho,\chi;\mathfrak{q}}^{\gamma}\hbar(\varpi))-\overline{\varpi\partial_{\mathfrak{q}}(\mathrm{H}_{\varrho,\chi;\mathfrak{q}}^{\gamma}\Im(\varpi))}}=\frac{\eth(\varpi)}{\daleth(\varpi)},$$

where

$$\begin{aligned}
\eth(\varpi) &= \varpi\partial_{\mathfrak{q}}\left(\varpi\partial_{\mathfrak{q}}\mathrm{H}_{\varrho,\chi;\mathfrak{q}}^{\gamma}\hbar(\varpi)\right)+\overline{\varpi\partial_{\mathfrak{q}}\left(\varpi\partial_{\mathfrak{q}}\mathrm{H}_{\varrho,\chi;\mathfrak{q}}^{\gamma}\Im(\varpi)\right)} \\
&= \varpi+\sum_{\Bbbk=2}^{\infty}[\Bbbk]_{\mathfrak{q}}^{2}\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma}a_{\Bbbk}\varpi^{\Bbbk}+\sum_{\Bbbk=1}^{\infty}[\Bbbk]_{\mathfrak{q}}^{2}\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma}\overline{b_{\Bbbk}\varpi^{\Bbbk}}
\end{aligned}$$

and

$$\begin{aligned}
\daleth(\varpi) &= \varpi\partial_{\mathfrak{q}}(\mathrm{H}_{\varrho,\chi;\mathfrak{q}}^{\gamma}\hbar(\varpi))-\overline{\varpi\partial_{\mathfrak{q}}(\mathrm{H}_{\varrho,\chi;\mathfrak{q}}^{\gamma}\Im(\varpi))} \\
&= \varpi+\sum_{\Bbbk=2}^{\infty}[\Bbbk]_{\mathfrak{q}}\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma}a_{\Bbbk}\varpi^{\Bbbk}-\sum_{\Bbbk=1}^{\infty}[\Bbbk]_{\mathfrak{q}}\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma}\overline{b_{\Bbbk}\varpi^{\Bbbk}}.
\end{aligned}$$

It is suffices to demonstrate that

$$\left|\eth(\varpi)+(1-\aleph)\daleth(\varpi)\right|-\left|\eth(\varpi)-(1+\aleph)\daleth(\varpi)\right|\geq 0, \tag{11}$$

using the fact that $\Re e\{\omega\} \geq \aleph$ is true if and only if $|\omega-1-\aleph| \leq |\omega+1-\aleph|$.

Replacing for $\eth(\varpi)$ and $\daleth(\varpi)$ in (11), we obtain

$$
\begin{aligned}
&\left|\eth(\varpi)+(1-\aleph)\daleth(\varpi)\right|-\left|\eth(\varpi)-(1+\aleph)\daleth(\varpi)\right| \\
&=\left|(2-\aleph)\varpi+\sum_{\Bbbk=2}^{\infty}[\Bbbk]_{\mathfrak{q}}([\Bbbk]_{\mathfrak{q}}-\aleph+1)\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma}\right. \\
&\left.+\sum_{\Bbbk=1}^{\infty}[\Bbbk]_{\mathfrak{q}}([\Bbbk]_{\mathfrak{q}}-\aleph+1)\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma}a_{\Bbbk}\varpi^{\Bbbk}\right| \\
&-\left|-\aleph\varpi+\sum_{\Bbbk=2}^{\infty}[\Bbbk]_{\mathfrak{q}}([\Bbbk]_{\mathfrak{q}}-\aleph-1)\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma}\right. \\
&\left.+\sum_{\Bbbk=1}^{\infty}[\Bbbk]_{\mathfrak{q}}([\Bbbk]_{\mathfrak{q}}-\aleph-1)\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma}\overline{b_{\Bbbk}\varpi^{\Bbbk}}\right| \\
&\geq 2(1-\aleph)|\varpi|\left\{1-\sum_{\Bbbk=2}^{\infty}[\Bbbk]_{\mathfrak{q}}\left(\frac{[\Bbbk]_{\mathfrak{q}}-\aleph}{1-\aleph}\right)\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma}|a_{\Bbbk}||\varpi|^{\Bbbk-1}\right. \\
&\left.-\sum_{\Bbbk=2}^{\infty}[\Bbbk]_{\mathfrak{q}}\left(\frac{[\Bbbk]_{\mathfrak{q}}+\aleph}{1-\aleph}\right)\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma}|b_{\Bbbk}||\varpi|^{\Bbbk-1}\right\} \\
&=2(1-\aleph)|\varpi|\left\{1-\sum_{\Bbbk=2}^{\infty}[\Bbbk]_{\mathfrak{q}}\left[\left(\frac{[\Bbbk]_{\mathfrak{q}}-\aleph}{1-\aleph}\right)|a_{\Bbbk}|+\left(\frac{[\Bbbk]_{\mathfrak{q}}+\aleph}{1-\aleph}\right)|b_{\Bbbk}|\right]\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma}\right\}.
\end{aligned}
$$

Using the enquiringly (10), we see that the last expression is non-negative. This implies that $\mathfrak{F}\in\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_{\mathfrak{q}}}(\varrho,\chi,\gamma,\aleph)$ is obtained. □

Remark 2. *When $\gamma=1$, Theorem 1 reduces to the corresponding convolution condition obtained in [23]. Furthermore, for $\mathfrak{q}\to 1^{-}$, $\gamma=1$ and $\aleph=0$, Theorem 1 reduces to the matching convolution condition found in [35].*

Theorem 2. *Let $\mathfrak{F}\in\overline{\mathcal{H}_{\mathfrak{q}}}$ as is in (3). Then, $\mathfrak{F}\in\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_{\mathfrak{q}}}(\varrho,\chi,\gamma,\aleph)}$ if and only if*

$$
\sum_{\Bbbk=2}^{\infty}[\Bbbk]_{\mathfrak{q}}\Big(([\Bbbk]_{\mathfrak{q}}-\aleph)|a_{\Bbbk}|+([\Bbbk]_{\mathfrak{q}}+\aleph|b_{\Bbbk}|\Big)\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma}\leq 2(1-\aleph), \tag{12}
$$

where $a_1=1$ and $0\leq\aleph<1$.

Proof. Since $\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_{\mathfrak{q}}}(\varrho,\chi,\gamma,\aleph)}\subset\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_{\mathfrak{q}}}(\varrho,\chi,\gamma,\aleph)$, this Theorem 2 merely requires that we prove the "only if" clause. Due to this, for functions $\mathfrak{F}=\overline{\Im}+\hbar$ of the form (3), the condition (9) is equivalent to

$$
\Re e\left\{\frac{(1-\aleph)\varpi-\sum\limits_{\Bbbk=2}^{\infty}[\Bbbk]_{\mathfrak{q}}\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma}([\Bbbk]_{\mathfrak{q}}-\aleph)|a_{\Bbbk}|\varpi^{\Bbbk}-\sum\limits_{\Bbbk=1}^{\infty}[\Bbbk]_{\mathfrak{q}}\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma}([\Bbbk]_{\mathfrak{q}}+\aleph)|b_{\Bbbk}|\overline{\varpi}^{\Bbbk}}{\varpi-\sum\limits_{\Bbbk=2}^{\infty}[\Bbbk]_{\mathfrak{q}}\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma}|a_{\Bbbk}|\varpi^{\Bbbk}+\sum\limits_{\Bbbk=1}^{\infty}[\Bbbk]_{\mathfrak{q}}\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma}|b_{\Bbbk}|\overline{\varpi}^{\Bbbk}}\right\}\geq 0. \tag{13}
$$

For all values of $\varpi\in\beth$, $|\varpi|=\jmath<1$, the aforementioned necessary condition (13) has to be true. By choosing the values of ϖ on the positive real axis where $0\leq|\varpi|=\jmath<1$, we must have

$$
\frac{(1-\aleph)-\sum\limits_{\Bbbk=2}^{\infty}[\Bbbk]_{\mathfrak{q}}\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma}([\Bbbk]_{\mathfrak{q}}-\aleph)|a_{\Bbbk}|\jmath^{\Bbbk-1}-\sum\limits_{n=1}^{\infty}[\Bbbk]_{\mathfrak{q}}\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma}([\Bbbk]_{\mathfrak{q}}+\aleph)|b_{\Bbbk}|\jmath^{\Bbbk-1}}{1-\sum\limits_{\Bbbk=2}^{\infty}[\Bbbk]_{\mathfrak{q}}\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma}|a_{\Bbbk}|\jmath^{\Bbbk-1}+\sum\limits_{\Bbbk=1}^{\infty}[\Bbbk]_{\mathfrak{q}}\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\varrho(\Bbbk-1)+\chi)}\right)^{\gamma}|b_{\Bbbk}|\jmath^{\Bbbk-1}}\geq 0. \tag{14}
$$

The numerator in (14) is negative for $\jmath$ sufficiently near to 1 if condition (12) does not hold. Hence, there is a point $\varpi_0 = \jmath_0$ in $(0,1)$ for which the quotient in (14) is negative. As this conflicts with the necessary predicate for $\mathfrak{F} \in \overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_q}(\varrho,\chi,\gamma,\aleph)}$, the proof is complete. □

Next, we determine the extreme points of closed convex hulls of $\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_q}(\varrho,\chi,\gamma,\aleph)}$, denoted by $clco\,\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_q}(\varrho,\chi,\gamma,\aleph)}$.

4. Extreme Points

In this section, we determine extreme points for the class $clco\,\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_q}(\varrho,\chi,\gamma,\aleph)}$.

Theorem 3. *Let $\mathfrak{F} \in \overline{\mathcal{H}_q}$ as is in (3). Then, $\mathfrak{F} \in clco\,\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_q}(\varrho,\chi,\gamma,\aleph)}$ if and only if $\mathfrak{F}(\varpi) = \sum\limits_{\Bbbk=1}^{\infty}(\psi_\Bbbk \hbar_\Bbbk(\varpi) + \phi_\Bbbk \Im_\Bbbk(\varpi))$, where*

$$\hbar_1(\varpi) = \varpi, \qquad \hbar_\Bbbk(\varpi) = \varpi - \frac{1-\aleph}{([\Bbbk]_q - \aleph)[\Bbbk]_q}\left(\frac{\Gamma_q(\chi + \varrho(\Bbbk-1))}{\Gamma_q(\chi)}\right)^{\gamma}\varpi^{\Bbbk}, \qquad (\Bbbk \geq 2)$$

$$\Im_\Bbbk(\varpi) = \varpi - \frac{1-\aleph}{([\Bbbk]_q + \aleph)[\Bbbk]_q}\left(\frac{\Gamma_q(\chi + \varrho(\Bbbk-1))}{\Gamma_q(\chi)}\right)^{\gamma}\overline{\varpi}^{\Bbbk}, \qquad (\Bbbk \geq 1)$$

and $\sum\limits_{\Bbbk=1}^{\infty}(\psi_\Bbbk + \phi_\Bbbk) = 1$ where $\psi_\Bbbk \geq 0$ and $\phi_\Bbbk \geq 0$. The extreme points of $\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_q}(\varrho,\chi,\gamma,\aleph)}$ are specifically $\hbar_\Bbbk$ and $\Im_\Bbbk$.

Proof. For $\mathfrak{F}$ of the form $\mathfrak{F}(\varpi) = \sum\limits_{n=1}^{\infty}(\psi_\Bbbk \hbar_\Bbbk(\varpi) + \phi_\Bbbk \Im_\Bbbk(\varpi))$ where $\sum\limits_{\Bbbk=1}^{\infty}(\psi_\Bbbk + \phi_\Bbbk) = 1$, we have

$$\begin{aligned}\mathfrak{F}(\varpi) = \varpi &- \sum_{\Bbbk=2}^{\infty}\frac{1-\aleph}{([\Bbbk]_q - \aleph)[\Bbbk]_q}\left(\frac{\Gamma_q(\chi + \varrho(\Bbbk-1))}{\Gamma_q(\chi)}\right)^{\gamma}\psi_\Bbbk \varpi^{\Bbbk} \\ &- \sum_{\Bbbk=1}^{\infty}\frac{1-\aleph}{([\Bbbk]_q + \aleph)[\Bbbk]_q}\left(\frac{\Gamma_q(\chi + \varrho(\Bbbk-1))}{\Gamma_q(\chi)}\right)^{\gamma}\phi_\Bbbk \overline{z}^{\Bbbk}.\end{aligned}$$

Then, $\mathfrak{F} \in clco\,\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_q}(\varrho,\chi,\gamma,\aleph)}$ because

$$\begin{aligned}&\sum_{\Bbbk=2}^{\infty}\frac{([\Bbbk]_q - \aleph)[\Bbbk]_q}{1-\aleph}\left(\frac{\Gamma_q(\chi)}{\Gamma_q((\Bbbk-1)\varrho + \chi)}\right)^{\gamma}\left(\frac{1-\aleph}{[\Bbbk]_q([\Bbbk]_q - \aleph)}\left(\frac{\Gamma_q(\chi + \varrho(\Bbbk-1))}{\Gamma_q(\chi)}\right)^{\gamma}\psi_\Bbbk\right) \\ &\quad + \sum_{\Bbbk=1}^{\infty}\frac{([\Bbbk]_q - \aleph)[\Bbbk]_q}{1-\aleph}\left(\frac{\Gamma_q(\chi)}{\Gamma_q((\Bbbk-1)\varrho + \chi)}\right)^{\gamma}\left(\frac{1-\aleph}{[\Bbbk]_q([\Bbbk]_q + \aleph)}\left(\frac{\Gamma_q(\chi + \varrho(\Bbbk-1))}{\Gamma_q(\chi)}\right)^{\gamma}\phi_\Bbbk\right) \\ &= \sum_{\Bbbk=2}^{\infty}\psi_\Bbbk + \sum_{\Bbbk=1}^{\infty}\phi_\Bbbk = 1 - \psi_1 \leq 1.\end{aligned}$$

Conversely, suppose $\mathfrak{F} \in clco\,\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_q}(\varrho,\chi,\gamma,\aleph)}$. In view of (12), we have

$$|a_\Bbbk| \leq \frac{1-\aleph}{[\Bbbk]_q([\Bbbk]_q - \aleph)}\left(\frac{\Gamma_q(\chi + \varrho(n-1))}{\Gamma_q(\chi)}\right)^{\gamma} \text{ and } |b_\Bbbk| \leq \frac{1-\aleph}{[\Bbbk]_q([\Bbbk]_q + \aleph)}\left(\frac{\Gamma_q((\Bbbk-1)\varrho + \chi)}{\Gamma_q(\chi)}\right)^{\gamma}.$$

Set

$$\psi_\Bbbk = \frac{([\Bbbk]_q - \aleph)[\Bbbk]_q}{1-\aleph}\left(\frac{\Gamma_q(\chi)}{\Gamma_q(\chi + \varrho(\Bbbk-1))}\right)^{\gamma}|a_\Bbbk|, \qquad (\Bbbk \geq 2)$$

and

$$\phi_\Bbbk = \frac{([\Bbbk]_q + \aleph)[\Bbbk]_q}{1-\aleph}\left(\frac{\Gamma_q(\chi)}{\Gamma_q(\chi + \varrho(\Bbbk-1))}\right)^{\gamma}|b_\Bbbk|, \qquad (\Bbbk \geq 1).$$

By Theorem 2, $\sum_{\Bbbk=2}^{\infty}\psi_{\Bbbk}+\sum_{\Bbbk=1}^{\infty}\phi_{\Bbbk}\le 1$. Therefore we define $\psi_1=1-\sum_{\Bbbk=2}^{\infty}\psi_{\Bbbk}-\sum_{\Bbbk=1}^{\infty}\phi_{\Bbbk}\ge 0$. Consequently, we obtain $\mathfrak{F}(\varpi)=\sum_{\Bbbk=1}^{\infty}(\psi_{\Bbbk}h_{\Bbbk}(\varpi)+\phi_{\Bbbk}g_{\Bbbk}(\varpi))$ as required. □

5. Convolution and Convex Combinations

The Hadamard products (or convolution) of two functions $\mathfrak{F}(\varpi)=\varpi+\sum_{\Bbbk=2}^{\infty}a_{\Bbbk}\varpi^{\Bbbk}+\sum_{\Bbbk=1}^{\infty}\overline{b_{\Bbbk}\varpi^{\Bbbk}}$ and $F(\varpi)=\varpi+\sum_{\Bbbk=2}^{\infty}c_{\Bbbk}\varpi^{\Bbbk}+\sum_{\Bbbk=1}^{\infty}\overline{d_{\Bbbk}\varpi^{\Bbbk}}$ is defined by

$$\mathfrak{F}(\varpi)\circledast F(\varpi)=(\mathfrak{F}\circledast F)(\varpi)=\varpi+\sum_{\Bbbk=2}^{\infty}a_{\Bbbk}c_{\Bbbk}\varpi^{\Bbbk}+\sum_{\Bbbk=1}^{\infty}\overline{b_{\Bbbk}d_{\Bbbk}\varpi^{\Bbbk}},\qquad (z\in\daleth).$$

Next, we show that the class $\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_q}(\varrho,\chi,\gamma,\aleph)}$ is closed under convolution.

Theorem 4. *For $0\le\aleph_2<\aleph_1$, suppose $\mathfrak{F}\in\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_q}(\varrho,\chi,\gamma,\aleph_1)}$ and $F\in\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_q}(\varrho,\chi,\gamma,\aleph_2)}$. Then, $\mathfrak{F}\circledast F\in\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_q}(\varrho,\chi,\gamma,\aleph_1)}\subset\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_q}(\varrho,\chi,\gamma,\aleph_2)}$.*

Proof. Let $\mathfrak{F}(\varpi)=\varpi-\sum_{\Bbbk=2}^{\infty}|a_{\Bbbk}|\varpi^{\Bbbk}-\sum_{\Bbbk=1}^{\infty}|b_{\Bbbk}|\overline{\varpi}^{\Bbbk}$ and $F(\varpi)=\varpi-\sum_{\Bbbk=2}^{\infty}|c_{\Bbbk}|\varpi^{\Bbbk}+\sum_{\Bbbk=1}^{\infty}|d_{\Bbbk}|\overline{\varpi}^{\Bbbk}$, then

$$\mathfrak{F}(\varpi)\circledast F(\varpi)=\varpi+\sum_{\Bbbk=2}^{\infty}|a_{\Bbbk}||c_{\Bbbk}|\varpi^{\Bbbk}+\sum_{\Bbbk=1}^{\infty}|b_{\Bbbk}||d_{\Bbbk}|\overline{\varpi}^{\Bbbk}.$$

Since $F\in\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_q}(\varrho,\chi,\gamma,\aleph_2)}$, it follows from Theorem 2 that $|c_{\Bbbk}|\le 1$ and $|d_{\Bbbk}|\le 1$. Therefore, we have

$$\sum_{\Bbbk=2}^{\infty}\frac{[\Bbbk]_q([\Bbbk]_q-\aleph)}{1-\aleph}\left(\frac{\Gamma_q(\chi)}{\Gamma_q(\chi+\varrho(\Bbbk-1))}\right)^{\gamma}|a_{\Bbbk}||c_{\Bbbk}|+\sum_{\Bbbk=1}^{\infty}\frac{[\Bbbk]_q([\Bbbk]_q-\aleph)}{1-\aleph}\left(\frac{\Gamma_q(\chi)}{\Gamma_q(\chi+\varrho(\Bbbk-1))}\right)^{\gamma}|b_{\Bbbk}||d_{\Bbbk}|$$
$$\le\sum_{\Bbbk=2}^{\infty}\frac{[\Bbbk]_q([\Bbbk]_q-\aleph)}{1-\aleph}\left(\frac{\Gamma_q(\chi)}{\Gamma_q(\chi+\varrho(\Bbbk-1))}\right)^{\gamma}|a_{\Bbbk}|+\sum_{\Bbbk=1}^{\infty}\frac{[\Bbbk]_q([\Bbbk]_q-\aleph)}{1-\aleph}\left(\frac{\Gamma_q(\chi)}{\Gamma_q(\chi+\varrho(\Bbbk-1))}\right)^{\gamma}|b_{\Bbbk}|\le 2.$$

In view of Theorem 2, it follows that $\mathfrak{F}\circledast F\in\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_q}(\varrho,\chi,\gamma,\aleph_1)}\subset\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_q}(\varrho,\chi,\gamma,\aleph_2)}$. □

We now show that $\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_q}(\varrho,\chi,\gamma,\aleph)}$ is closed under a convex combination of its members.

Theorem 5. *The class $\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_q}(\varrho,\chi,\gamma,\aleph)}$ is closed under convex combination.*

Proof. Let $\mathfrak{F}_{\ell}\in\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_q}(\varrho,\chi,\gamma,\aleph)}$ for $\ell=1,2,3,\cdots$. where $\mathfrak{F}_{\ell}$ is given by

$$\mathfrak{F}_{\ell}(\varpi)=\varpi-\sum_{\Bbbk=2}^{\infty}|a_{\Bbbk_{\ell}}|\varpi^{\Bbbk}-\sum_{\Bbbk=1}^{\infty}|b_{\Bbbk_{\ell}}|\overline{\varpi}^{\Bbbk}.$$

Then, by Theorem 2, we have

$$\sum_{\Bbbk=2}^{\infty}\frac{[\Bbbk]_q([\Bbbk]_q-\aleph)}{1-\aleph}\left(\frac{\Gamma_q(\chi)}{\Gamma_q(\chi+\varrho(\Bbbk-1))}\right)^{\gamma}|a_{\Bbbk_{\ell}}|+\sum_{\Bbbk=2}^{\infty}\frac{[\Bbbk]_q([\Bbbk]_q-\aleph)}{1-\aleph}\left(\frac{\Gamma_q(\chi)}{\Gamma_q(\chi+\varrho(\Bbbk-1))}\right)^{\gamma}|b_{\Bbbk_{\ell}}|\le 2. \tag{15}$$

For $\sum_{\ell=1}^{\infty}\tau_{\ell}=1$ and $0\le\tau_{\ell}\le 1$, the convex combination of $\mathfrak{F}_{\ell}$ may be written as

$$\sum_{\ell=1}^{\infty}\tau_{\ell}\mathfrak{F}_{\ell}(\varpi)=\varpi-\sum_{\Bbbk=2}^{\infty}\sum_{\ell=1}^{\infty}\tau_{\ell}|a_{\Bbbk_{\ell}}|\varpi^{\Bbbk}-\sum_{\Bbbk=1}^{\infty}\sum_{\ell=1}^{\infty}\tau_{\ell}|b_{\Bbbk_{\ell}}|\overline{\varpi}^{\Bbbk}.$$

Using (15), we have

$$\begin{aligned}&\sum_{\Bbbk=1}^{\infty}[\Bbbk]_{\mathfrak{q}}\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\chi+\varrho(\Bbbk-1))}\right)^{\gamma}\left[\frac{([\Bbbk]_{\mathfrak{q}}-\aleph)}{1-\aleph}|a_{\Bbbk_\ell}|+\frac{([\Bbbk]_{\mathfrak{q}}-\aleph)}{1-\aleph}\sum_{\ell=1}^{\infty}\tau_\ell|b_{\Bbbk_\ell}|\right]\\&=\sum_{\ell=1}^{\infty}\tau_\ell\left(\sum_{\Bbbk=1}^{\infty}[\Bbbk]_{\mathfrak{q}}\left(\frac{\Gamma_{\mathfrak{q}}(\chi)}{\Gamma_{\mathfrak{q}}(\chi+\varrho(\Bbbk-1))}\right)^{\gamma}\left[\frac{([\Bbbk]_{\mathfrak{q}}-\aleph)}{1-\aleph}|a_{\Bbbk_\ell}|+\frac{([\Bbbk]_{\mathfrak{q}}-\aleph)}{1-\aleph}|b_{\Bbbk_\ell}|\right]\right)\\&\le 2\sum_{\ell=1}^{\infty}\tau_\ell=2.\end{aligned}$$

Furthermore, by Theorem 2, we have $\sum\limits_{\ell=1}^{\infty}\tau_\ell\mathfrak{F}_\ell\in\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_{\mathfrak{q}}}(\varrho,\chi,\gamma,\aleph)}$. □

6. Distortion Bounds and Covering Theorem

The following Theorem contains distortion bounds for functions $\mathfrak{F}$ in the class $\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_{\mathfrak{q}}}(\varrho,\chi,\gamma,\aleph)}$.

Theorem 6. *If* $\mathfrak{F}\in\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_{\mathfrak{q}}}(\varrho,\chi,\gamma,\aleph)}$ *then for* $|\varpi|=\jmath<1$, *we have*

$$\mathfrak{F}(\varpi)\ge(1-|b_1|)\jmath-\frac{(\Gamma_{\mathfrak{q}}(\varrho+\chi))^{\gamma}}{[2]_{\mathfrak{q}}(\Gamma_{\mathfrak{q}}(\chi))^{\gamma}}\left(\frac{1-\aleph}{[2]_{\mathfrak{q}}-\aleph}-\frac{1+\aleph}{[2]_{\mathfrak{q}}-\aleph}|b_1|\right)\jmath^2,\tag{16}$$

and

$$\mathfrak{F}(\varpi)\le(1+|b_1|)\jmath+\frac{(\Gamma_{\mathfrak{q}}(\varrho+\chi))^{\gamma}}{[2]_{\mathfrak{q}}(\Gamma_{\mathfrak{q}}(\chi))^{\gamma}}\left(\frac{1-\aleph}{[2]_{\mathfrak{q}}-\aleph}-\frac{1+\aleph}{[2]_{\mathfrak{q}}-\aleph}|b_1|\right)\jmath^2.\tag{17}$$

Proof. Let $\mathfrak{F}\in\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_{\mathfrak{q}}}(\varrho,\chi,\gamma,\aleph)}$. When we use $\mathfrak{F}$ absolute value, we obtain

$$\begin{aligned}\mathfrak{F}(\varpi)&\ge(1-|b_1|)\jmath-\sum_{\Bbbk=2}^{\infty}\Big(|a_\Bbbk|-|b_\Bbbk|\Big)\jmath^{\Bbbk}\\&\ge(1-|b_1|)\jmath-\sum_{\Bbbk=2}^{\infty}\Big(|a_\Bbbk|-|b_\Bbbk|\Big)\jmath^{2}\\&\ge(1-|b_1|)\jmath-\frac{\jmath^2(1-\aleph)(\Gamma_{\mathfrak{q}}(\varrho+\chi))^{\gamma}}{[2]_{\mathfrak{q}}([2]_{\mathfrak{q}}-\aleph)(\Gamma_{\mathfrak{q}}(\chi))^{\gamma}}\sum_{\Bbbk=2}^{\infty}\frac{[2]_{\mathfrak{q}}(\Gamma_{\mathfrak{q}}(\chi))^{\gamma}}{(1-\aleph)(\Gamma_{\mathfrak{q}}(\varrho+\chi))^{\gamma}}\times\\&\qquad\Big(([2]_{\mathfrak{q}}-\aleph)|a_\Bbbk|-([2]_{\mathfrak{q}}+\aleph)|b_\Bbbk|\Big)\\&\ge(1-|b_1|)\jmath-\frac{\jmath^2(1-\aleph)(\Gamma_{\mathfrak{q}}(\varrho+\chi))^{\gamma}}{[2]_{\mathfrak{q}}([2]_{\mathfrak{q}}-\aleph)(\Gamma_{\mathfrak{q}}(\chi))^{\gamma}}\sum_{\Bbbk=2}^{\infty}\frac{[\Bbbk]_{\mathfrak{q}}(\Gamma_{\mathfrak{q}}(\chi))^{\gamma}}{(1-\aleph)(\Gamma_{\mathfrak{q}}(\chi+\varrho(\Bbbk-1)))^{\gamma}}\times\\&\qquad\Big(([\Bbbk]_{\mathfrak{q}}-\aleph)|a_\Bbbk|-([\Bbbk]_{\mathfrak{q}}+\aleph)|b_\Bbbk|\Big)\\&\ge(1-|b_1|)\jmath-\frac{(1-\aleph)(\Gamma_{\mathfrak{q}}(\varrho+\chi))^{\gamma}}{[2]_{\mathfrak{q}}([2]_{\mathfrak{q}}-\aleph)(\Gamma_{\mathfrak{q}}(\chi))^{\gamma}}\left(1-\frac{1+\aleph}{1-\aleph}|b_1|\right)\jmath^2.\end{aligned}$$

This proves (16). The proof of (17) is omitted because it is comparable to that of (16). □

The inequality (16) leads to the covering conclusion shown below.

Corollary 1. *If* $\mathfrak{F}\in\overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_{\mathfrak{q}}}(\varrho,\chi,\gamma,\aleph)}$ *then*

$$\left\{\omega:|\omega|<\frac{[2]_{\mathfrak{q}}([2]_{\mathfrak{q}}-\aleph)\frac{(\Gamma_{\mathfrak{q}}(\chi))^{\gamma}}{(\Gamma_{\mathfrak{q}}(\varrho+\chi))^{\gamma}}-1+\aleph+\left(1+\aleph-[2]_{\mathfrak{q}}([2]_{\mathfrak{q}}-\aleph)\frac{(\Gamma_{\mathfrak{q}}(\chi))^{\gamma}}{(\Gamma_{\mathfrak{q}}(\varrho+\chi))^{\gamma}}\right)|b_1|}{[2]_{\mathfrak{q}}([2]_{\mathfrak{q}}-\aleph)\frac{(\Gamma_{\mathfrak{q}}(\chi))^{\gamma}}{(\Gamma_{\mathfrak{q}}(\varrho+\chi))^{\gamma}}}\right\}\subseteq f(\sqsupset).$$

Remark 3. *The result for $\mathfrak{q}$-harmonic convex functions of order $\aleph$ derived in [23] is the result for $\gamma = 0$, as shown by the covering Theorem 6 in Corollary 1. Moreover, the covering Theorem 6 in Corollary 1 offers the usual conclusion for harmonic convex functions found in [36] with the conditions $\gamma = 0$ and $\mathfrak{q} \to 1$. In addition, for $\gamma = 0$, $\mathfrak{q} \to 1^-$, $\aleph = b_1 = 0$, Corollary 1 yields the following result given in [36].*

Remark 4. *If $\mathfrak{F} \in \overline{\mathcal{U}^{\mathcal{C}}_{\mathcal{H}_\mathfrak{q}}(1,1,0,0)}$ then $\{\omega : |\omega| < \frac{3}{4}\} \subseteq \mathfrak{F}(\beth)$.*

7. Conclusions

Numerous academics have recently applied $\mathfrak{q}$-calculus to the study of geometric functions, creating new subclasses of $\mathfrak{q}$- starlike, $\mathfrak{q}$-convex, and harmonic functions. Using the concept of Le Roy-type Mittag–Leffler functions, a novel class of harmonic functions was established in this study. We established some novel results for this newly defined class, such as extreme points, convolution and convex combinations, distortion limitations, and the covering theorem, as well as necessary and sufficient conditions. Future studies on harmonic functions and symmetric $\mathfrak{q}$-calculus operators will be inspired by the findings of this study.

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