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Left (Right) Regular Elements of Some Transformation Semigroups

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Abstract: For a nonempty set X, let $T(X)$ be the total transformation semigroup on X. In this paper, we consider the subsemigroups of $T(X)$ which are defined by $T(X,Y) = \{\alpha \in T(X) : X\alpha \subseteq Y\}$ and $S(X,Y) = \{\alpha \in T(X) : Y\alpha \subseteq Y\}$ where Y is a non-empty subset of X. We characterize the left regular and right regular elements of both $T(X,Y)$ and $S(X,Y)$. Moreover, necessary and sufficient conditions for $T(X,Y)$ and $S(X,Y)$ to be left regular and right regular are given. These results are then applied to determine the numbers of left and right regular elements in $T(X,Y)$ for a finite set X.

Keywords: regular elements; magnifying elements; transformation semigroups

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1. Introduction and Preliminaries

Let S be a semigroup. An element x of S is called left regular if $x = yx^2$ for some $y \in S$. A right regular element is defined dually. Denote by $LReg(S)$ and $RReg(S)$ the sets of all left regular elements and right regular elements of S, respectively. Left and right regular elements are important in semigroup theory because they play a key role in the study of regular semigroups, which are semigroups in which every element is both left and right regular. Regular semigroups have many interesting properties and are used in various areas of mathematics, including algebra, topology and theoretical computer science. An element x of S is called left (right) magnifying if there is a proper subset M of S satisfying $xM = S$ ($Mx = S$). In 1963, Ljapin [1] studied the notion of right and left magnifying elements of a semigroup. Several years later, Migliorini introduced the concepts of the minimal subset related to a magnifying element of S in [2,3]. Gutan [4] researched semigroups with strong and nonstrong magnifying elements in 1996. Later, he proved that every semigroup with magnifying elements is factorizable in [5].

Let $T(X)$ be the total transformation semigroup on a nonempty set X. It is well known that $T(X)$ is a regular semigroup. Moreover, every semigroup is isomorphic to a subsemigroup of some total transformation semigroups. The most basic mathematical structures are transformation semigroups. In 1952, Malcev [6] characterized ideals of $T(X)$. Later, Miller and Doss [7] studied its group $\mathcal{H}$-classes and its Green's relations. The generalization of these studies is the focus of this paper.

In 1975, Symons [8] considered a subsemigroup of $T(X)$ defined by

$$T(X,Y) = \{\alpha \in T(X) : X\alpha \subseteq Y\}$$

where Y is a nonempty subset of X. He determined all the automorphisms of $T(X,Y)$. In 2005, Nenthein et al. [9] described regular elements in $T(X,Y)$ and counted the number of all regular elements of $T(X,Y)$ when X is a finite set. They described such a number in terms of $|X|$, $|Y|$ and their related Stirling numbers. A few years later, Sanwong and Sommanee [10] studied regularity and Green's relations for the semigroup $T(X,Y)$. They determined when $T(X,Y)$ becomes a regular semigroup. Moreover, they gave a class of maximal inverse subsemigroups of $T(X,Y)$ in 2008. After that, they proved that the set $F(X,Y) = \{\alpha \in T(X,Y) : X\alpha = Y\alpha\}$ is the largest regular subsemigroup of $T(X,Y)$ and determined its Green's relations. In [11], Sanwong described Green's relations and found all maximal regular subsemigroups of $F(X,Y)$. In 2009, maximal and minimal congruences on $T(X,Y)$ were considered. Sanwong et al. [12] found that $T(X,Y)$ has only one maximal congruence if X is a finite set. They generalized [13] Theorem 3.4 for Y being infinite. Furthermore, characterizations of all minimal congruences on $T(X,Y)$ were given. In the same year, Sun [14] proved that while the semigroup $T(X,Y)$ is not left abundant, it is right abundant. Later in 2016, Lei Sun and Junling Sun [15] investigated the natural partial order on $T(X,Y)$. Moreover, they determined the maximal elements and the minimal elements of $T(X,Y)$.

Consider the semigroup

$$S(X,Y) = \{\alpha \in T(X) : Y\alpha \subseteq Y\}.$$

of transformations that leave Y invariant. In 1966, Magill [16] constructed and discussed the semigroup $S(X,Y)$. In fact, if $Y = X$, then $S(X,Y) = T(X)$. Later in [8], automorphism groups of a semigroup $S(X,Y)$ were given by Symons. In 2005, Nenthein et al. [9] characterized regularity for $S(X,Y)$. In addition, they found the number of regular elements in $S(X,Y)$ for a finite set X. Honyam and Sanwong [17] studied its ideals, group $\mathcal{H}$-classes and Green's relations on $S(X,Y)$. Furthermore, they described when $S(X,Y)$ is isomorphic to $T(A)$ for some set A. A few years later, the left, right regular and intra-regular elements of a semigroup $S(X,Y)$ were discussed by Choomanee, Honyam and Sanwong [18]. Moreover, when X is finite, they calculated the number of left regular elements in $S(X,Y)$. In [19], natural partial orders on the semigroup $S(X,Y)$ were considered by Sun and Wang. Moreover, they investigated left and right compatible elements with respect to this partial oder. Finally, they described the abundance of $S(X,Y)$. In [20], all elements in the semigroup $S(X,Y)$ that are left compatible with the natural partial order were studied. Left and right magnifying elements of $S(X,Y)$ were given by Chiram and Baupradist in [21]. In a recent study, Punkumkerd and Honyam [22] provided a characterization of left and right magnifying elements on the semigroup $\overline{PT}(X,Y)$. $\overline{PT}(X,Y)$ denotes the set of all partial transformations α from a subset of X to X and $(dom\alpha \cap Y)\alpha \subseteq Y$, where $dom\alpha$ is the domain of α. Their results have shown to be more general than the previous findings from [21].

In Section 2, we consider left and right magnifying elements of $T(X,Y)$ and $S(X,Y)$. We prove that each left magnifying element in $T(X,Y)$ is not regular. Furthermore, we show that every left and right magnifying element in $S(X,Y)$ is regular. In Sections 3 and 4, we focus on left and right regularity on $T(X,Y)$ and $S(X,Y)$. We show that every left magnifying element in $T(X,Y)$ is a right regular element. Every right magnifying element is a left regular element. As [10] determined when $T(X,Y)$ becomes a regular semigroup, we also characterize whenever $T(X,Y)$ and $S(X,Y)$ is a left (right) regular semigroup.

Note that throughout this paper, we will write mappings from the right, $x\alpha$ rather than $\alpha(x)$ and compose that the left to the right, $x(\alpha\beta) = (x\alpha)\beta$ rather than $(\alpha\beta)(x) = \alpha(\beta(x))$ where $\alpha, \beta \in T(X)$ and $x \in X$. For each $\alpha \in T(X)$, we denote the set $\{z\alpha^{-1} : z \in X\alpha\}$ by $\pi(\alpha)$ and $\pi_Y(\alpha)$ is the set $\{P \in \pi(\alpha) : P \cap Y \neq \emptyset\}$ for a subset $Y \subseteq X$. Then, it is obvious that $\pi(\alpha)$ is a partition of X.

2. Magnifying Elements

In this section, we focus on characterizations of left magnifying elements and right magnifying elements in $T(X,Y)$ and $S(X,Y)$. The relationships between magnifying elements and regular elements are given.

Theorem 1 ([23])**.** *Let $\alpha \in T(X,Y)$. Then, α is right magnifying if and only if α is surjective but not injective and is such that $y\alpha^{-1} \cap Y \neq \emptyset$ for all $y \in Y$ and $|y\alpha^{-1} \cap Y| > 1$ for some $y \in Y$.*

Theorem 2 ([23])**.** *A mapping $\alpha \in T(X)$ is left magnifying if and only if α is injective but not surjective.*

Theorem 3 ([23])**.** *Let $\alpha \in T(X,Y)$. If $|Y| = |X|$ and $X \neq Y$, then α is left magnifying if and only if α is injective.*

Remark 1. *For each $\alpha \in T(X,Y)$, we note from Theorem 1 that α is right magnifying if and only if $Y\alpha = Y$ and $\alpha|_Y$ is not injective.*

Theorem 4. *Let $\alpha \in T(X,Y)$ and $X \neq Y$. Then, α is left magnifying in a semigroup $T(X,Y)$ if and only if α is injective.*

Proof. Assume that α is left magnifying. Suppose that M is a proper subset of $T(X,Y)$ satisfying $\alpha M = T(X,Y)$. Let $a, b \in X$ be such that $a\alpha = b\alpha$. If $|Y| = 1$, then $T(X,Y)$ contains exactly one element. Thus, $T(X,Y)$ has no proper subset M such that $\alpha M = T(X,Y)$. This is a contradiction. Therefore, $|Y| > 1$. Let $y \in Y \setminus \{a\alpha\}$. Define $\gamma : X \to X$ by

$$x\gamma = \begin{cases} a\alpha & \text{if } x = a, \\ y & \text{otherwise.} \end{cases}$$

It is verifiable that $\gamma \in T(X,Y)$. From $\alpha M = T(X,Y)$, there exists $\beta \in M$ such that $\alpha\beta = \gamma$. Suppose that $a \neq b$. Then, $a\alpha = a\gamma = a\alpha\beta = b\alpha\beta = b\gamma = y$, which is a contradiction. Hence, $a = b$ and so α is injective.

Suppose that α is injective. Then, we choose $y \in Y$ and let M be the set $\{\gamma \in T(X,Y) : (X \setminus Y)\gamma = \{y\}\}$. From $X \neq Y$, we have $M \neq \emptyset$. It follows from our assumption that every $x \in X\alpha$, there is a unique $x' \in X$ satisfying $x'\alpha = x$. Let $\beta \in T(X,Y)$. We define $\gamma : X \to X$ by

$$x\gamma = \begin{cases} x'\beta & \text{if } x \in X\alpha, \\ y & \text{otherwise.} \end{cases}$$

It is verifiable that $\gamma \in M$. Now, let $x \in X$. Thus, $x\alpha\gamma = (x\alpha)'\beta$. From α being injective and $(x\alpha)'\alpha = x\alpha$, we obtain $(x\alpha)' = x$. Therefore, $x\alpha\gamma = (x\alpha)'\beta = x\beta$. Hence, $\beta = \alpha\gamma$ and so $\alpha M = T(X,Y)$. This implies that α is left magnifying. □

The set of natural numbers is represented by the letter $\mathbb{N}$. Additionally, we denote the set of even natural numbers and the set of all odd natural numbers greater than 3 by $2\mathbb{N}$ and $2\mathbb{N} + 1$, respectively.

Example 1. *Let $X = \mathbb{N}$ and $Y = 2\mathbb{N}$. Define $\alpha : X \to Y$ by $x\alpha = 2x$ for all $x \in X$. Then, $X\alpha = Y$. Clearly, α is injective. From Theorem 4, we obtain that α is left magnifying. We will show that α is not a regular element. Suppose that α is a regular element in $T(X,Y)$. Thus, there exists $\beta \in T(X,Y)$ such that $\alpha\beta\alpha = \alpha$. Consider $3\alpha\beta\alpha = 3\alpha = 6$. Thus, $3\alpha\beta \in 6\alpha^{-1} = \{3\}$. Hence, $3\alpha\beta = 3$, which is a contradiction. So α is not regular element.*

From the above example, we will verify that in $T(X,Y)$, each left magnifying element is not a regular element.

Theorem 5. *If $X \neq Y$, then every left magnifying element of $T(X,Y)$ is not regular.*

Proof. Assume that $X \neq Y$. Let α be a left magnifying element. From Theorem 4, we obtain that α is injective. Suppose that α is a regular element in $T(X,Y)$. Then, there exists

$\beta \in T(X,Y)$ such that $\alpha = \alpha\beta\alpha$. Since $X \neq Y$, we choose $x \in X \setminus Y$. Therefore, $x\alpha\beta\alpha = x\alpha$ and then $x\alpha\beta \in (x\alpha)\alpha^{-1}$. Since α is injective, we have $x\alpha\beta = x \notin Y$. This is a contradiction with $X\beta \subseteq Y$. So α is not regular. □

Theorem 6. *Every right magnifying element of a semigroup $T(X,Y)$ is regular.*

Proof. Let α be a right magnifying element of $T(X,Y)$. By Remark 1, we obtain that $Y\alpha = Y$. It follows that $X\alpha \subseteq Y = Y\alpha$. From $Y\alpha \subseteq X\alpha$, we have $X\alpha = Y\alpha$. This means that $\alpha \in F(X,Y)$ and so α is a regular element of $T(X,Y)$. □

The following example shows that there exists an element in some $T(X,Y)$ which is regular but it is not right magnifying.

Example 2. *Let $X = \mathbb{N}$ and $Y = 2\mathbb{N}$. Define $\alpha : X \to Y$ by*

$$x\alpha = \begin{cases} x+2 & \text{if } x \in 2\mathbb{N}, \\ x+3 & \text{otherwise.} \end{cases}$$

Note that $\alpha \in T(X,Y)$. It is easy to see that $Y\alpha = 2\mathbb{N} \setminus \{2\} \neq Y$ and $X\alpha = 2\mathbb{N} \setminus \{2\} = Y\alpha$. Thus, α is regular but it is not right magnifying.

In the rest of this section, we consider magnifying elements in $S(X,Y)$.

Lemma 1 ([21])**.** *Let $\alpha \in S(X,Y)$. Then, α is a right magnifying element if and only if α is surjective but not injective such that $y\alpha^{-1} \cap Y \neq \emptyset$ for all $y \in Y$.*

Lemma 2 ([21])**.** *Let $\alpha \in S(X,Y)$. Then, α is a left magnifying element if and only if α is injective but not surjective such that $y\alpha^{-1} \subseteq Y$ for all $y \in Y \cap X\alpha$.*

Lemma 3 ([9])**.** *Let $\alpha \in S(X,Y)$. Then, α is a regular element if and only if $Y\alpha = X\alpha \cap Y$.*

Theorem 7. *Every left magnifying element of a semigroup $S(X,Y)$ is regular.*

Proof. Suppose that α is left magnifying. We will show that $X\alpha \cap Y = Y\alpha$. Clearly, $Y\alpha \subseteq X\alpha \cap Y$. Let $y \in X\alpha \cap Y$. Then, there exists $y' \in X$ such that $y = y'\alpha$. Thus, $y' \in y\alpha^{-1} \subseteq Y$ by Lemma 2. This implies that $X\alpha \cap Y = Y\alpha$. From Lemma 3, we obtain that α is regular. □

Theorem 8. *Every right magnifying element of a semigroup $S(X,Y)$ is regular.*

Proof. Suppose that α is right magnifying. We will show that $X\alpha \cap Y = Y\alpha$. Clearly, $Y\alpha \subseteq X\alpha \cap Y$. Let $y \in X\alpha \cap Y$. By Lemma 1, we have $y\alpha^{-1} \cap Y \neq \emptyset$. Thus, there exists $y' \in y\alpha^{-1} \cap Y$. Hence, $y = y'\alpha \in Y\alpha$. Therefore, $X\alpha \cap Y = Y\alpha$. From Lemma 3, we obtain α is regular. □

Example 3. *Let α be defined in Example 2. It is clear that $\alpha \in S(X,Y)$ and α is neither injective nor surjective. Since $X\alpha = Y\alpha$, this means that $X\alpha \cap Y = Y\alpha$. Hence, α is regular, while it is neither a left nor right magnifying element in $S(X,Y)$.*

3. Left Regular and Right Regular Elements in $T(X,Y)$

Now, we start with the characterizations of left regular and right regular elements in $T(X,Y)$. Moreover, we determine whenever $T(X,Y)$ becomes a left regular semigroup and a right regular semigroup, respectively.

Theorem 9. *Let $\alpha \in T(X,Y)$. Then, the following statements are equivalent.*

(1) α is left regular.

(2) $Y\alpha^2 = X\alpha$.
(3) *for any* $P \in \pi(\alpha)$, $Y\alpha \cap P \neq \emptyset$.

Proof. (1) $\Rightarrow$ (2). Suppose that α is left regular. Thus, there exists $\beta \in T(X, Y)$ satisfying $\alpha = \beta\alpha^2$. This implies that $Y\alpha^2 \subseteq X\alpha = X\beta\alpha^2 \subseteq Y\alpha^2$. Thus, $Y\alpha^2 = X\alpha$.

(2) $\Rightarrow$ (3). Suppose that $Y\alpha^2 = X\alpha$ and let $P \in \pi(\alpha)$. Thus, there is $y \in X\alpha$ satisfying $y\alpha^{-1} = P$. By assumption, we have $y \in Y\alpha^2$. Thus, there exists $z \in Y$ such that $y = z\alpha^2$; that is, $z\alpha \in y\alpha^{-1} = P$. It follows that $z\alpha \in Y\alpha \cap P$. Therefore, $Y\alpha \cap P \neq \emptyset$.

(3) $\Rightarrow$ (1). Assume that (3) holds. For $x \in X$, there is a unique $P_x \in \pi(\alpha)$ satisfying $x \in P_x$. From assumption, we have $Y\alpha \cap P_x \neq \emptyset$. So there exists $x' \in Y$ such that $x'\alpha \in P_x = (x\alpha)\alpha^{-1}$. Define $\beta : X \to X$ by $x\beta = x'$ for all $x \in X$. It is obvious that $\beta \in T(X, Y)$. Let $x \in X$. This implies that $x\beta\alpha^2 = x'\alpha^2 = (x'\alpha)\alpha = x\alpha$. Hence, $\alpha = \beta\alpha^2$ and so α is left regular. □

If we replace Y with X in Theorem 9, we have the following corollary.

Corollary 1. *Let* $\alpha \in T(X)$. *Then,* α *is left regular if and only if for each* $P \in \pi(\alpha)$, $X\alpha \cap P \neq \emptyset$.

Proof. By taking $X = Y$, we obtain $T(X, Y) = T(X, X) = T(X)$ and $X\alpha = Y\alpha$. By Theorem 9(3), we obtain that α is regular if and only if for each $P \in \pi(\alpha)$, $X\alpha \cap P = Y\alpha \cap P \neq \emptyset$. □

Theorem 10. *Let* $\alpha \in T(X, Y)$. *Then,* α *is right regular if and only if* $\alpha|_{X\alpha}$ *is injective.*

Proof. Assume that α is right regular. Then, there exists $\beta \in T(X, Y)$ such that $\alpha = \alpha^2\beta$. We will show that $\alpha|_{X\alpha}$ is injective. Let $x, y \in X\alpha$ be such that $x\alpha = y\alpha$. Thus, there exist $x', y' \in X$ such that $x = x'\alpha$ and $y = y'\alpha$. We obtain that

$$x = x'\alpha = x'\alpha^2\beta = (x'\alpha)\alpha\beta = x\alpha\beta = y\alpha\beta = y'\alpha^2\beta = y'\alpha = y.$$

Therefore, $\alpha|_{X\alpha}$ is injective.

Conversely, suppose $\alpha|_{X\alpha}$ is injective. Let $z \in X\alpha^2$. Then, there exists a unique $z' \in X\alpha$ such that $z'\alpha = z$. We choose $y \in Y$. Define $\beta : X \to X$ by

$$z\beta = \begin{cases} z' & \text{if } z \in X\alpha^2, \\ y & \text{otherwise.} \end{cases}$$

From $X\alpha \subseteq Y$, we obtain that $X\beta \subseteq X\alpha \cup \{y\} \subseteq Y$. Let $x \in X$. Note that $x\alpha^2 \in X\alpha^2$ and $(x\alpha^2)\beta = (x\alpha^2)'$ where $(x\alpha^2)'\alpha = x\alpha^2 = x\alpha\alpha$. Since $\alpha|_{X\alpha}$ is injective, we have $(x\alpha^2)' = x\alpha$. So $x\alpha^2\beta = (x\alpha^2)' = x\alpha$. Hence, $\alpha^2\beta = \alpha$ and so α is right regular. □

Corollary 2. *Let* $\alpha \in T(X)$. *Then,* α *is right regular if and only if* $\alpha|_{X\alpha}$ *is injective.*

From Theorems 4 and 10, we obtain the following corollary immediately.

Corollary 3. *For* $X \neq Y$, *every left magnifying element of a semigroup* $T(X, Y)$ *is right regular.*

Corollary 4. *Every right magnifying element of a semigroup* $T(X, Y)$ *is left regular.*

Proof. Let α be a right magnifying element. By Remark 1, we have $Y\alpha = Y$. It follows that $X\alpha \subseteq Y = Y\alpha = (Y\alpha)\alpha = Y\alpha^2 \subseteq X\alpha$. Hence, $Y\alpha^2 = X\alpha$ and so α is left regular. □

Example 4. *Let* $X = \mathbb{N}$ *and* $Y = 2\mathbb{N}$. *Define* $\alpha : X \to Y$ *by*

$$x\alpha = \begin{cases} x + 2 & \text{if } x \in 2\mathbb{N}, \\ x + 1 & \text{otherwise.} \end{cases}$$

Then, $\alpha \in T(X,Y)$. Clearly, α is not injective. We see that $X\alpha = 2\mathbb{N}$. This means that $\alpha|_{X\alpha} : X\alpha \to X\alpha$ is an injection. This means that α is right regular but it is not left magnifying.

Example 5. *Let $X = \mathbb{N}$ and $Y = 2\mathbb{N}$. Define $\alpha : X \to Y$ by*

$$x\alpha = \begin{cases} 4 & \text{if } x \in \{1,2,3,4\}, \\ x-2 & \text{if } x \in 2\mathbb{N} \setminus \{2,4\}, \\ x+1 & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \in T(X,Y)$. We see that $X\alpha = 2\mathbb{N} \setminus \{2\} = Y\alpha^2$ and $Y\alpha = 2\mathbb{N} \setminus \{2\} \neq Y$. Hence, α is a left regular element but it is not right magnifying.

Notice that for $|X| \leq 2$, we obtain that $T(X,Y)$ is left and right regular. Now, we consider the other case.

Theorem 11. *Let $X \neq Y$ be such that $|X| \geq 3$. Then, $T(X,Y)$ is a right regular semigroup if and only if $|Y| = 1$.*

Proof. If $|Y| = 1$, then $|T(X,Y)| = 1$ and $T(X,Y)$ is a right regular semigroup. Assume that $T(X,Y)$ is a right regular semigroup and suppose $|Y| \neq 1$. Let $a,b,c \in X$ be distinct elements and $a,b \in Y$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a & \text{if } x \in \{a,b\}, \\ b & \text{otherwise.} \end{cases}$$

Then, $\alpha \in T(X,Y)$. However, $\alpha|_{X\alpha}$ is not injective. Thus, α is not right regular, which is a contradiction. Hence, $|Y| = 1$. □

Theorem 12. *Let $X \neq Y$ be such that $|X| \geq 3$. Then, $T(X,Y)$ is a left regular semigroup if and only if $|Y| = 1$.*

Proof. If $|Y| = 1$, then $|T(X,Y)| = 1$ and $T(X,Y)$ is a left regular semigroup. Assume that $T(X,Y)$ is a left regular semigroup and $|Y| \neq 1$. Let $a,b,c \in X$ be distinct elements and $a,b \in Y$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a & \text{if } x \in \{a,b\}, \\ b & \text{otherwise.} \end{cases}$$

Then, $\alpha \in T(X,Y)$. Note that $a,b \notin b\alpha^{-1}$ and $Y\alpha \subseteq \{a,b\}$. So $Y\alpha \cap b\alpha^{-1} = \emptyset$. Therefore, α is not left regular. This is a contradiction. Hence, $|Y| = 1$. □

Corollary 5. *Every left regular element of a semigroup $T(X,Y)$ is regular.*

Proof. We first note that $F(X,Y) = \{\alpha \in T(X,Y) : X\alpha = Y\alpha\}$ is the largest regular subsemigroup of $T(X,Y)$. Let α be a left regular element of $T(X,Y)$. It follows from Theorem 9(2) that $X\alpha = Y\alpha^2 = (Y\alpha)\alpha \subseteq Y\alpha \subseteq X\alpha$. Thus, $X\alpha = Y\alpha$ and so $\alpha \in F(X,Y)$. Therefore, α is a regular element of $T(X,Y)$. □

Example 6. *Let $X = \mathbb{N}$ and $Y = 2\mathbb{N}$. Define $\alpha : X \to Y$ by*

$$x\alpha = \begin{cases} x+2 & \text{if } x \in 2\mathbb{N}, \\ x+3 & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \in T(X,Y)$. Consider $X\alpha = Y\alpha = 2\mathbb{N} \setminus \{2\}$ and $Y\alpha^2 = 2\mathbb{N} \setminus \{2,4\}$. Then, α is regular but it is not left regular.

Example 7. *Let $X = \mathbb{N}$ and $Y = 2\mathbb{N} + 1$. Define $\alpha : X \to Y$ by*

$$x\alpha = \begin{cases} x+1 & \text{if } x \in 2\mathbb{N}, \\ x+2 & \text{otherwise.} \end{cases}$$

It is verifiable that $\alpha \in T(X,Y)$. We obtain that $\alpha|_{X\alpha}$ is injective and $X\alpha \neq Y\alpha$. Thus, α is right regular but it is not regular.

Finally, we consider the set of all left regular elements $LReg(T(X,Y))$ and the set of all right regular elements $RReg(T(X,Y))$ of $T(X,Y)$. We begin with the following example.

Example 8. *Let $X = \{1,2,3,4,5\}$ and $Y = \{1,2,3,4\}$. We consider the mappings $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 3 & 3 & 4 & 3 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 1 & 2 & 1 \end{pmatrix}$. We note that $\alpha, \beta \in T(X,Y)$. Moreover, $\alpha|_{X\alpha}$ and $\beta|_{X\beta}$ are injective; that is $\alpha, \beta \in RReg(T(X,Y))$. Clearly, $\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 2 & 1 \end{pmatrix}$. This implies that $\alpha\beta|_{X\alpha\beta}$ is not injective and so $\alpha\beta \notin RReg(T(X,Y))$. In this case, we obtain that $RReg(T(X,Y))$ is not a semigroup.*

Theorem 13. *Let $|X| \geq 3$. Then, $RReg(T(X,Y))$ is a semigroup if and only if $|Y| \leq 2$.*

Proof. Suppose that $|Y| \geq 3$. Let $a,b,c \in Y$ be distict elements. Define $\alpha : X \to Y$ by

$$x\alpha = \begin{cases} a & \text{if } x = b, \\ b & \text{if } x = a, \\ c & \text{otherwise.} \end{cases}$$

We see that $X\alpha = \{a,b,c\}$ and $\alpha|_{X\alpha}$ is injective. Define $\beta : X \to Y$ by

$$x\beta = \begin{cases} a & \text{if } x = a, \\ c & \text{otherwise.} \end{cases}$$

Then, $X\beta = \{a,c\}$ and $\beta|_{X\beta}$ is injective. Since $b\alpha\beta = a$ and $c\alpha\beta = c$, we obtain $a,c \in X\alpha\beta$. From $a\alpha\beta = c = c\alpha\beta$, we conclude that $\alpha\beta|_{X\alpha\beta}$ is not injective. Therefore, $\alpha\beta \notin RReg(T(X,Y))$ and $RReg(T(X,Y))$ is not closed.

Assume that $|Y| \leq 2$. If $|Y| = 1$, then $T(X,Y)$ is a right regular semigroup. Therefore, $RReg(T(X,Y))$ is a semigroup. Suppose that $|Y| = 2$. Let $Y = \{a,b\}$ and $\alpha, \beta \in RReg(T(X,Y))$. We will show that $\alpha\beta|_{X\alpha\beta}$ is injective. Let $x,y \in X\alpha\beta$ be such that $x\alpha\beta = y\alpha\beta$. If α or β is a constant mapping, then $\alpha\beta$ is a constant mapping. Thus, $\alpha\beta|_{X\alpha\beta}$ is injective. Suppose that α and β are not constant mappings. Then, $X\alpha = Y = X\beta$. We observe that $x,y \in X\alpha\beta \subseteq X\beta = X\alpha$. If $x \neq y$, then $x\alpha \neq y\alpha$ since $\alpha|_{X\alpha}$ is injective. Note that $x\alpha, y\alpha \in Y = X\beta$. Then, $(x\alpha)\beta \neq (y\alpha)\beta$ since $\beta|_{X\beta}$ is injective. This is a contradition. Hence, $x = y$. So $\alpha\beta|_{X\alpha\beta}$ is injective. Therefore, $RReg(T(X,Y))$ is closed. □

Theorem 14. *Let $|X| \geq 3$. Then, $LReg(T(X,Y))$ is a semigroup if and only if $|Y| \leq 2$.*

Proof. Suppose that $|Y| \geq 3$. Let $a,b,c \in Y$ be distinct elements. Define $\alpha : X \to Y$ by

$$x\alpha = \begin{cases} b & \text{if } x = c, \\ c & \text{otherwise.} \end{cases}$$

And we define $\beta : X \to Y$ by

$$x\beta = \begin{cases} a & \text{if } x = b, \\ b & \text{otherwise.} \end{cases}$$

Then, $Y\alpha^2 = (Y\alpha)\alpha = \{b,c\}\alpha = \{b,c\} = X\alpha$ and $Y\beta^2 = (Y\beta)\beta = \{a,b\}\beta = \{a,b\} = X\beta$. Thus, $\alpha, \beta \in LReg(T(X,Y))$. It is easy to verify that

$$x\alpha\beta = \begin{cases} a & \text{if } x = c, \\ b & \text{otherwise} \end{cases}$$

such that $\pi(\alpha\beta) = \{\{c\}, X \setminus \{c\}\}$ and $Y\alpha\beta = \{a, b\}$. Clearly, $Y\alpha\beta \cap \{c\} = \emptyset$. Hence, $\alpha\beta \notin LReg(T(X,Y))$. So $\alpha\beta$ is not left regular.

Conversely, suppose $|Y| \leq 2$. If $|Y| = 1$, then $T(X,Y)$ is a left regular semigroup. We have $LReg(T(X,Y))$ is a semigroup. Assume that $Y = \{a, b\}$. Let $\alpha, \beta \in LReg(T(X,Y))$. If $|X\alpha| = 1$ or $|X\beta| = 1$, then $\alpha\beta$ is a constant mapping. Suppose that $|X\alpha| = |X\beta| = 2$. Then, $\pi(\alpha) = \{a\alpha^{-1}, b\alpha^{-1}\}$ and so it is a left regular element. From α being a left regular element, we obtain $Y\alpha \cap a\alpha^{-1} \neq \emptyset$ and $Y\alpha \cap b\alpha^{-1} \neq \emptyset$. This implies that $|a\alpha^{-1} \cap Y| = 1 = |b\alpha^{-1} \cap Y|$ and $X\alpha = Y\alpha = \{a, b\}$. Similarly, $|a\beta^{-1} \cap Y| = 1 = |b\beta^{-1} \cap Y|$ and $X\beta = Y\beta = \{a, b\}$. It is easy to verify that $\pi(\alpha\beta) = \pi(\alpha)$ and $Y\alpha\beta = \{a, b\} = Y\alpha$. Hence, $Y\alpha\beta \cap a\alpha^{-1} \neq \emptyset$ and $Y\alpha\beta \cap b\alpha^{-1} \neq \emptyset$. Therefore, $\alpha\beta$ is a left regular element. □

Theorem 15. *If Y is finite, then $RReg(T(X,Y)) = LReg(T(X,Y))$.*

Proof. Assume that Y is finite. Let α be a left regular element of $T(X,Y)$. Then, $Y\alpha^2 = X\alpha$. It follows that $X\alpha = Y\alpha^2 \subseteq X\alpha^2 \subseteq X\alpha$; that is, $X\alpha^2 = X\alpha$. From $(X\alpha)\alpha = X\alpha^2 = X\alpha$, we obtain $\alpha|_{X\alpha} : X\alpha \to X\alpha$ is surjective. Since $X\alpha \subseteq Y$ and Y is a finite set, we obtain $\alpha|_{X\alpha}$ is an injection. So α is right regular.

Assume that α is right regular. Thus, $\alpha|_{X\alpha} : X\alpha \to X\alpha$ is injective and also $\alpha|_{X\alpha} : X\alpha \to X\alpha$ is surjective since $X\alpha$ is finite. This means that $(X\alpha)\alpha = X\alpha$. We see that $X\alpha = (X\alpha)\alpha \subseteq Y\alpha \subseteq X\alpha$. Hence, $X\alpha = Y\alpha$ and so $Y\alpha^2 = X\alpha^2 = X\alpha$. Therefore, α is a left regular element of $T(X,Y)$. □

Next, the cardinality of right regular elements in the semigroup $T(X,Y)$ are investigated when X is finite.

Theorem 16. *Let $|X| = n$ and $|Y| = r$. Then,*

$$|LReg(T(X,Y))| = |RReg(T(X,Y))| = \sum_{k=1}^{r} k!\binom{r}{k}k^{n-k}$$

where $1 \leq k \leq r$.

Proof. By Theorem 15, we have $LReg(T(X,Y)) = RReg(T(X,Y))$. This implies that $|LReg(T(X,Y))| = |RReg(T(X,Y))|$. Let $1 \leq k \leq r$ and $B_k = \{\alpha \in RReg(T(X,Y)) : |X\alpha| = k\}$. From Y being finite, we have $\alpha|_{X\alpha} : X\alpha \to X\alpha$ is bijective for all $\alpha \in B_k$ by Theorem 10. Notice that the number of image sets in Y of cardinality k is equal to $\binom{r}{k}$. Since there are k^{n-k} ways of partitioning the remaining $n-k$ elements into k subsets, we obtain $|B_k| = \binom{r}{k}k!\,k^{n-k}$. Therefore,

$$|RReg(T(X,Y))| = \sum_{k=1}^{r} |B_k| = \sum_{k=1}^{r} k!\binom{r}{k}k^{n-k}.$$

□

4. Left Regular and Right Regular Elements in $S(X,Y)$

Theorem 17 ([18]). *Let $\alpha \in S(X,Y)$. Then, α is left regular if and only if $X\alpha = X\alpha^2$ and $Y\alpha = Y\alpha^2$.*

Theorem 18 ([18]). *Let $\alpha \in S(X,Y)$. Then, α is right regular if and only if $\pi(\alpha) = \pi(\alpha^2)$ and $\sigma(\alpha) = \sigma(\alpha^2)$ where $\sigma(\alpha) = \{y\alpha^{-1} : y \in X\alpha \cap Y\}$.*

Although the left and right regular elements of $S(X,Y)$ were characterized in [18], in this section we obtain the different results; see the following theorems.

Theorem 19. *Let $\alpha \in S(X,Y)$. Then, α is a left regular element if and only if for every $P \in \pi(\alpha)$, $P \cap X\alpha \neq \emptyset$ and for every $P \in \pi_Y(\alpha)$, $P \cap Y\alpha \neq \emptyset$.*

Proof. Assume that α is a left regular element. Thus, $\alpha = \beta\alpha^2$ for some $\beta \in S(X,Y)$. Let $P \in \pi(\alpha)$ and let $x \in P$. Then, $P = (x\alpha)\alpha^{-1}$ and $x\alpha = x\beta\alpha^2 = [(x\beta)\alpha]\alpha$. We see that $(x\beta)\alpha \in (x\alpha)\alpha^{-1} = P$ and $(x\beta)\alpha \in X\alpha$. Therefore, $P \cap X\alpha \neq \emptyset$. Let $P \in \pi_Y(\alpha)$ and let $y \in P \cap Y$. Then, $P = (y\alpha)\alpha^{-1}$, $y\beta \in Y$ and $y\alpha = y\beta\alpha^2 = [(y\beta)\alpha]\alpha$. We note that $(y\beta)\alpha \in Y\alpha$ and $(y\beta)\alpha \in (y\alpha)\alpha^{-1} = P$. Hence, $P \cap Y\alpha \neq \emptyset$.

Conversely, suppose the conditions hold. Let $x \in X$. Since $\pi(\alpha)$ is a partition of X, there is a unique $P_x \in \pi(\alpha)$ satisfying $x \in P_x$. If $P_x \cap Y \neq \emptyset$, then $P_x \in \pi_Y(\alpha)$ and $P_x \cap Y\alpha \neq \emptyset$ by our assumption. So, we choose $x' \in Y$ satisfying $x'\alpha \in P_x$. If $P_x \cap Y = \emptyset$, then since $P_x \cap X\alpha \neq \emptyset$, we choose $x' \in X$ satisfying $x'\alpha \in P_x$. Define $\beta : X \to X$ by $x\beta = x'$ for all $x \in X$. Then, β is well-defined and $x\beta\alpha^2 = x'\alpha^2 = (x'\alpha)\alpha = x\alpha$. Let $y \in Y$. Then, there is a unique $P_y \in \pi(\alpha)$ such that $y \in P_y$. Thus, $P_y \cap Y \neq \emptyset$. Therefore, $y\beta = y' \in Y$; that is, $Y\beta \subseteq Y$. So α is left regular. □

Theorem 20. *Let $\alpha \in S(X,Y)$. Then, α is a right regular element if and only if $\alpha|_{X\alpha}$ is injective and $(X\alpha \setminus Y)\alpha \subseteq X \setminus Y$.*

Proof. Assume that α is right regular. Thus, α is also a right regular element in $T(X)$. From Corollary 2, we have $\alpha|_{X\alpha}$ is injective. Next, we will show that $(X\alpha \setminus Y)\alpha \subseteq X \setminus Y$. Let $z \in X\alpha \setminus Y$. Then, $z = z'\alpha$ for some $z' \in X$. Thus, $z = z'\alpha^2\beta$ for some $\beta \in S(X,Y)$ since α is a right regular element in $S(X,Y)$. If $z'\alpha^2 \in Y$, then $z = (z'\alpha^2)\beta \in Y\beta \subseteq Y$, which is a contradiction. Hence, $z\alpha = z'\alpha^2 \notin Y$ and so $(X\alpha \setminus Y)\alpha \subseteq X \setminus Y$.

Assume that $\alpha|_{X\alpha}$ is injective and $(X\alpha \setminus Y)\alpha \subseteq X \setminus Y$. Let β be defined in the converse part of Theorem 10; we note that $\alpha = \alpha^2\beta$. It is enough to verify that $\beta \in S(X,Y)$. Let $x \in Y$. If $x \notin X\alpha^2$, then by the definition of β, we have $x\beta \in Y$. Assume that $x \in X\alpha^2$. There is a unique $x' \in X\alpha$ satisfying $x'\alpha = x$. If $x' \notin Y$, then $x' \in X\alpha \setminus Y$. By assumption, we have $x'\alpha \in X \setminus Y$ which is a contradiction. This means that $x\beta = x' \in Y$. Hence, $Y\beta \subseteq Y$ and so $\beta \in S(X,Y)$. □

Example 9. *Let $X = \mathbb{N}$ and $Y = 2\mathbb{N}$. Define $\alpha : X \to X$ by*

$$x\alpha = \begin{cases} 2 & \text{if } x \in Y, \\ 4 & \text{if } x = 1, \\ x-2 & \text{otherwise.} \end{cases}$$

Then, $X\alpha = \{2,4\} \cup \{2n-1 : n \in \mathbb{N}\}$ and $Y\alpha = \{2\} \subseteq Y$. So $\alpha \in S(X,Y)$. Moreover, we obtain $\pi(\alpha) = \{Y\} \cup \{\{2n-1\} : n \in \mathbb{N}\}$ and $\pi_Y(\alpha) = \{Y\}$. It is clear that $P \cap X\alpha \neq \emptyset$ for every $P \in \pi(\alpha)$ and $P \cap Y\alpha \neq \emptyset$ for all $P \in \pi_Y(\alpha)$. From Theorem 19, α is left regular. Note that $X\alpha \cap Y = \{2,4\} \neq Y\alpha$. By Theorem 3, α is also not regular.

Example 10. *Let α be defined in Example 2. Then, $Y\alpha \subseteq Y$ and also $X\alpha \cap Y = Y\alpha$. Thus, $\alpha \in S(X,Y)$ and α is regular. Note that $4\alpha^{-1} = \{1,2\}$ and $X\alpha \cap 4\alpha^{-1} = \emptyset$. Hence, α is not a left regular element of $S(X,Y)$.*

Example 11. *Let α be defined in Example 5. Then, $Y\alpha \subseteq Y$ and so $\alpha \in S(X,Y)$. Consider $Y \cap X\alpha = 2\mathbb{N} \setminus \{2\} = Y\alpha$ and $\alpha|_{X\alpha}$ is not injective. Hence, α is regular but not right regular in $S(X,Y)$.*

Example 12. *Recall α from Example 4. Then, $Y\alpha \subseteq Y$ and so $\alpha \in S(X,Y)$. We see that $\alpha|_{X\alpha}$ is injective and $(X\alpha \setminus Y)\alpha = \emptyset \subseteq X \setminus Y$. From Theorem 20, we obtain α is right regular. Consider $X\alpha \cap Y = 2\mathbb{N}$ and $Y\alpha = 2\mathbb{N} \setminus \{2\}$. Hence, $X\alpha \cap Y \neq Y\alpha$. From Theorem 3, we obtain α is not regular.*

Theorem 21. *The following statements are equivalent.*

(1) $S(X,Y) = LReg(S(X,Y))$.
(2) $S(X,Y) = RReg(S(X,Y))$.
(3) $|X| \leq 2$.

Proof. (1) $\Leftrightarrow$ (3). Assume that $S(X,Y) = LReg(S(X,Y))$. We will show that $|X| \leq 2$. Suppose that $|X| \geq 3$. Let $a, b, c \in X$ be distinct elements and $a \in Y$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} b & \text{if } x = c, \\ a & \text{if } x \neq c. \end{cases}$$

Claim that $\alpha \in S(X,Y)$. If $c \notin Y$, then $Y\alpha = \{a\} \subseteq Y$. Moreover, if $b, c \in Y$, then $Y\alpha = \{a,b\} \subseteq Y$. Thus, $\alpha \in S(X,Y)$ when $c \notin Y$ or $b, c \in Y$. Note that $\pi(\alpha) = \{\{c\}, X \setminus \{c\}\}$ and $X\alpha = \{a,b\}$. Clearly, $X\alpha \cap \{c\} = \emptyset$. Hence, α is not left regular. For the case $b \notin Y$ and $c \in Y$, we define $\beta : X \to X$ by

$$x\beta = \begin{cases} c & \text{if } x = b, \\ a & \text{if } x \neq b. \end{cases}$$

Then, $Y\beta = \{a\} \subseteq Y$, $\pi(\beta) = \{\{b\}, X \setminus \{b\}\}$ and $X\beta = \{a,c\}$. It follows that $\beta \in S(X,Y)$ but $\beta \notin LReg(S(X,Y))$. We conclude that $S(X,Y) \neq LReg(S(X,Y))$, which is a contradiction. So $|X| \leq 2$.

Conversely, assume that $|X| \leq 2$. Then, it is easy to verify that $S(X,Y)$ is a left regular semigroup. Hence, $S(X,Y) = LReg(S(X,Y))$.

(2) $\Leftrightarrow$ (3). Assume that $S(X,Y) = RReg(S(X,Y))$. We will show that $|X| \leq 2$. Suppose that $|X| \geq 3$. We consider α, β from condition (1) $\Leftrightarrow$ (3). Then $\alpha, \beta \in S(X,Y)$. Since $X\alpha = \{a,b\}$ and $\alpha|_{X\alpha}$ is not injective, we have α is not right regular. Similarly, $X\beta = \{a,c\}$ and $\beta|_{X\beta}$ is not injective. So β is not right regular, which is a contradition. Hence, $|X| \leq 2$.

Conversely, suppose that $|X| \leq 2$. Then, it is clear that $S(X,Y)$ is a right regular semigroup. Hence, $S(X,Y) = RReg(S(X,Y))$. □

Theorem 22. *The following statements are equivalent.*

(1) $LReg(S(X,Y))$ *is a semigroup.*
(2) $RReg(S(X,Y))$ *is a semigroup.*
(3) $|X| \leq 2$.

Proof. (1) $\Leftrightarrow$ (3). Suppose that $|X| \geq 3$. Let $a, b, c \in X$ be distinct elements and $a \in Y$. It is enough to consider only two cases.

Case 1: $\{b,c\} \subseteq Y$. Recall α, β from Theorem 14, we have $\alpha, \beta \in S(X,Y)$. Note that $\pi(\alpha) = \pi_Y(\alpha) = \{\{c\}, X \setminus \{c\}\}$ and $X\alpha = \{b,c\}$. Clearly, $Y\alpha \cap \{c\} \neq \emptyset$ and $Y\alpha \cap X \setminus \{c\} \neq \emptyset$. Thus, α is left regular; that is, $\alpha \in LReg(S(X,Y))$. Similarly, $\beta \in LReg(S(X,Y))$. Consider $\{c\} \in \pi(\alpha\beta)$ and $X\alpha\beta = \{a,b\}$. Therefore, $X\alpha\beta \cap \{c\} = \emptyset$ and so $\alpha\beta$ is not left regular; that is, $\alpha\beta \notin LReg(S(X,Y))$. Hence, $LReg(S(X,Y))$ is not a semigroup.

Case 2: $\{b,c\} \nsubseteq Y$. Assume that $c \notin Y$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} c & \text{if } x = c, \\ a & \text{otherwise.} \end{cases}$$

Then, $Y\alpha = \{a\} \subseteq Y$. So $\alpha \in S(X,Y)$. Note that $\pi(\alpha) = \{\{c\}, X \setminus \{c\}\}$, $\pi_Y(\alpha) = \{X \setminus \{c\}\}$ and $X\alpha = \{a,c\}$. Clearly, $X\alpha \cap \{c\} \neq \emptyset$, $X\alpha \cap (X \setminus \{c\}) \neq \emptyset$ and $Y\alpha \cap (X \setminus \{c\}) \neq \emptyset$. Therefore, α is a left regular element of $S(X,Y)$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} b & \text{if } x \in \{b,c\}, \\ a & \text{otherwise.} \end{cases}$$

If $b \in Y$, then $Y\beta = \{a, b\} \subseteq Y$. Thus, $\beta \in S(X, Y)$. If $b \notin Y$, then $Y\beta = \{a\} \subseteq Y$. So $\beta \in S(X, Y)$. We can show that $\beta \in LReg(S(X, Y))$. Then, we note that $\pi(\alpha\beta) = \{\{c\}, X \setminus \{c\}\}$ and $X\alpha\beta = \{a, b\}$. Clearly, $X\alpha\beta \cap \{c\} = \emptyset$. Hence, $\alpha\beta \notin LReg(S(X, Y))$ and so $LReg(S(X, Y))$ is not a semigroup.

Conversely, suppose $|X| \leq 2$. Then, we have $LReg(S(X, Y)) = S(X, Y)$ is a semigroup from Theorem 21(1).

(2) $\Leftrightarrow$ (3). Suppose that $|X| \geq 3$. Let $a, b, c \in X$ be distinct elements and $a \in Y$. Recall α and β from the proof of (1) $\Leftrightarrow$ (3). It is enough to show that α and β are right regular elements of $S(X, Y)$. Clearly, $\alpha|_{X\alpha}$ and $\beta|_{X\beta}$ are injective. Consider $(X\alpha \setminus Y)\alpha = \{c\} \subseteq X \setminus Y$ and $(X\beta \setminus Y)\beta = \begin{cases} \emptyset & \text{if } b \in Y, \\ \{b\} & \text{if } b \notin Y \end{cases} \subseteq X \setminus Y.$

Then, α and β are right regular elements of $S(X, Y)$; that is, $\alpha, \beta \in RReg(S(X, Y))$. Note that $\alpha\beta|_{X\alpha\beta}$ is not injective. We conclude that $\alpha\beta$ is not right regular. Thus, $\alpha\beta \notin RReg(S(X, Y))$ and so $RReg(S(X, Y))$ is not a semigroup.

Conversely, suppose $|X| \leq 2$. Then, we have $RReg(S(X, Y)) = S(X, Y)$ is a semigroup from Theorem 21(2). $\square$

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