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An Optimal Inequality for the Normal Scalar Curvature in Metallic Riemannian Space Forms

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Abstract: In this paper, we prove the DDVV conjecture for a slant submanifold in metallic Riemannian space forms with the semi-symmetric metric connection. The equality case of the derived inequality is discussed, and some special cases of the inequality are given.

Keywords: metallic structure; Riemannian manifolds; scalar curvature; Wintgen inequality

MSC: 53B05; 53B20; 53C25; 53C40

1. Introduction

In 1979, P. Wintgen [1] proved a basic inequality for the surface M^2 in the Euclidean 4-space $\mathbb{E}^4$, often referred to as the Wintgen inequality and involves both intrinsic and extrinsic invariants. He proved that the intrinsic Gaussian curvature G and the extrinsic normal curvature $K^{\perp}$ of M^2 in $\mathbb{E}^4$ satisfy

$$G + |K^{\perp}| \leq \|H\|^2$$

where $\|H\|^2$ is the squared norm of the mean curvature H. Additionally, the surface M^2 is called the Wintgen ideal surface if it satisfies the equality case, i.e., the equality holds iff the ellipse of the surface's curvature in $\mathbb{E}^4$ is a circle.

The aforementioned inequality was researched and extended independently in [2] and in [3] for surfaces of arbitrary co-dimension n in the real space form $\tilde{M}^{n+2}(c)$, $n \geq 2$ as

$$K + |K^{\perp}| \leq ||H||^2 + c.$$

Furthermore, B.-Y. Chen extended the Wintgen inequality in [4,5] to the surfaces in pseudo-Euclidean 4-spaces $\mathbb{E}^4_2$ with the neutral metric.

In 1999, researchers proposed in [6] a conjecture of the Wintgen inequality for general Riemannian submanifolds in real space forms, later known as the DDVV conjecture. They revealed that, for a submanifold M^n of real space form $\tilde{M}^{n+m}(c)$, the following hold

$$\rho + \rho^{\perp} \leq ||H||^2 + c,$$

ρ denotes the normalised scalar curvature and $\rho^{\perp}$ stands for the normalised normal scalar curvature of M. This inequality is also known as the *generalized Wintgen inequality or the normal scalar curvature conjecture* and was proved independently by Ge and Tang [7], and Lu [8]. Recently generalized Wintgen inequalities have been established for submanifolds in Golden Riemannian manifolds [9], complex space forms [10], Sasakian space form [11], (κ, μ)-space forms [12], etc. For further literature about the DDVV inequality, one can refer to [13] and the references therein.

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Since the first presentation of the structure of golden type on a Riemannian manifold in [14] in the year 2008, a great deal of work has been produced by numerous scholars using various theories. According to C. E. Hretcanu and M. Crasmareanu [15], the metallic structure on a Riemannian manifold generalises the structure of the golden type. Recent publications include a thorough analysis of Norden and metallic pseudo-Riemannian manifolds in [16] and several findings on curvature for generalised metallic pseudo-Riemannian structures in [17]. We should also mention that [18–21] have conducted a thorough examination into different submanifolds in metallic Riemannian manifolds. Metallic warped product manifolds were explored in 2018 by A. M. Blaga and C. E. Hretcanu [22], who also came up with some intriguing findings. The same authors [21,23] also looked at warped product submanifolds in metallic Riemannian manifolds.

On the other hand, Friedmann and Schouten in [24] first proposed the notion of a semi-symmetric linear connection (briefly SSLC) on a differentiable manifold. The concept of a metric connection with torsion on a Riemannian manifold was later proposed by Hayden [25]. According to [26], the author demonstrated that a Riemannian manifold is conformally flat iff it admits a semi-symmetric metric connection (shortly SSMC) whose curvature tensor disappears. In [27,28], Chen-like inequalities for submanifolds of real, complex, and Sasakian space forms endowed with SSMC were found.

Furthermore, a different algebraic strategy was used in [29] to derive certain optimal inequalities for submanifolds of a Riemannian manifold with an SSMC and quasi-constant curvature.

In this short note, the generalized Wintgen inequalities for slant submanifolds in the context of metallic Riemannian space forms with SSMC have been established, drawing inspiration from the aforementioned findings. The equality case is discussed and some special cases of the derived inequality are given.

2. Preliminaries

Let $\tilde{M}^m$ be a Riemannian manifold with the linear connection $\tilde{\nabla}$ and a torsion tensor T such that

$$T(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y], \ \ \forall X,Y \in \Gamma(T\tilde{M}).$$

Let ϕ be any 1-form satisfying

$$T(X,Y) = \phi(Y)X - \phi(X)Y;$$

then, $\tilde{\nabla}$ is a semi-symmetric connection on $\tilde{M}$. In addition, for the Riemannian metric g, if $\tilde{\nabla} g = 0$ on $\tilde{M}$, then $\tilde{\nabla}$ will be referred as a semi-symmetric metric connection (in short, SSMC).

Let $\tilde{\nabla}'$ denote the Levi–Civita connection and B symbolize a vector field satisfying $g(B,X) = \phi(X)$; the SSMC $\tilde{\nabla}$ on $\tilde{M}$ can be viewed as [26]

$$\tilde{\nabla}_X Y = \tilde{\nabla}'_X Y + \phi(Y)X - g(X,Y)B, \qquad \forall X,Y \in \Gamma(T\tilde{M}).$$

Let M be an isometric immersed submanifold of $\tilde{M}$ and $\tilde{\nabla}$ and $\tilde{\nabla}'$ are SSMC and Levi–Civita connections of $\tilde{M}$, and let ∇ and ∇' denote the induced SSMC and Levi–Civita connection of M. Then, the Gauss formulas for $\tilde{\nabla}$ and $\tilde{\nabla}'$ are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \ \ \tilde{\nabla}'_X Y = \nabla'_X Y + h'(X,Y),$$

h' denotes the second fundamental form of M in $\tilde{M}$ and h stands for $(0,2)$-tensor on M. Let A be the shape operator and ξ be a normal vector field to M. Then, we have

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla^{\perp}_X \xi,$$

where $\nabla^{\perp}$ is the normal connection on M. Furthermore, denote the curvature tensors of $\tilde{M}$ (respectively, M) by $\tilde{R}$ and $\tilde{R}'$ (respectively, R and R') with respect to $\tilde{\nabla}$ and $\tilde{\nabla}'$ (respectively, ∇ and ∇'). Then, we have [30]

$$\tilde{R}'(X,Y,Z,W) = R'(X,Y,Z,W) - g(h'(X,W),h'(Y,Z)) + g(h'(X,Z),h'(Y,W)). \quad (1)$$

In view of SSMC $\tilde{\nabla}$, one can express [31]

$$\tilde{R}(X,Y,Z,W) = \tilde{R}'(X,Y,Z,W) - \alpha(Y,Z)g(X,W) + \alpha(X,Z)g(Y,W) - \alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z), \ \ \forall X,Y,Z,W \in \Gamma(TM), \quad (2)$$

α being $(0,2)$-tensor such that

$$\alpha(X,Y) = (\tilde{\nabla}'_X \phi)Y - \phi(X)\phi(Y) + \frac{1}{2}\phi(B)g(X,Y). \quad (3)$$

Take into account the local orthonormal tangent and normal frames $\{e_1,\ldots,e_n\}$ and $\{e_{n+1},\ldots,e_m\}$ of TM^n and $T^{\perp}M^{m-n}$ of M in $\tilde{M}$, respectively.

The mean curvature vector and squared norm of h of M are, respectively, given by

$$H = \sum_{i=1}^{n} \frac{1}{n} h(e_i,e_i), \ \ ||h||^2 = \sum_{1\leq i,j\leq n} g\big(h(e_i,e_j),h(e_i,e_j)\big).$$

The scalar curvature τ and the normalised scalar curvature ρ are defined as

$$\tau = \sum_{1\leq i<j\leq n} R(e_i,e_j,e_j,e_i), \ \ \rho = \frac{2\tau}{n(n-1)} = \frac{2}{n(n-1)} \sum_{1\leq i<j\leq n} K(e_i \wedge e_j), \quad (4)$$

with K being the sectional curvature function on M. Moreover, the scalar normal curvature K_N and the normalized scalar normal curvature ρ_N are given by [32]

$$K_N = \sum_{1\leq r<s\leq m-n} \sum_{1\leq i<j\leq n} [\sum_{k=1}^{n} (h^r_{jk}h^s_{ik} - h^r_{ik}h^s_{jk})]^2, \ \ \rho_N = \frac{2}{n(n-1)}\sqrt{K_N}. \quad (5)$$

(Refs. [14,33]) Consider that $(\tilde{M}^m, g)$ is a Riemannian manifold and F is a $(1,1)$-tensor field on $\tilde{M}^m$. F is said to be a polynomial structure if $P(F) = 0$, where

$$P(Y) := Y^n + a_n Y^{n-1} + \cdots + a_2 Y + a_1 I,$$

for identity transformation I on $\Gamma(T\tilde{M}^m)$ and real numbers $a_1,\ldots,a_n$.

A $(1,1)$ tensor field φ is called a metallic structure on $\tilde{M}^m$ if [15]

$$\varphi^2 = p\varphi + qI, \quad (6)$$

for $p,q \in \mathbb{N}^*$ (set of positive integers), with I being the identity transformation on $T\tilde{M}$.

The Riemannian metric g is called φ-compatible if

$$g(X,\varphi Y) = g(\varphi X, Y), \qquad \forall X,Y \in \Gamma(T\tilde{M}^m). \quad (7)$$

A metallic Riemannian manifold is a smooth manifold $\tilde{M}^m$ with a metallic structure φ and a φ-compatible Riemannian metric g.

From (6) and (7), we find

$$g(\varphi X, \varphi Y) = pg(X,\varphi Y) + qg(X,Y).$$

Particularly, if $p = q = 1$, then $(\tilde{M}, \varphi, g)$ is simply referred as a golden Riemannian manifold [14,34].

If a $(1,1)$-tensor field F on a Riemannian manifold $(\tilde{M}^m, g)$ satisfies the conditions of $F^2 = I$ and $F \neq \pm I$, it is an almost product structure according to [35].

On a Riemannian manifold $\tilde{M}$, one obtains two F with metallic structure φ [15]:

$$F_1 = \frac{2}{2\sigma_{p,q} - p}\varphi - \frac{p}{2\sigma_{p,q} - p}I, \;\; F_2 = -\frac{2}{2\sigma_{p,q} - p}\varphi + \frac{p}{2\sigma_{p,q} - p}I, \tag{8}$$

in which the members of the family of metallic means are illustrated by $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$. Besides this, two metallic structures are identified by F on $\tilde{M}$:

$$\varphi_1 = \frac{p}{2}I + \frac{2\sigma_{p,q} - p}{2}F, \quad \varphi_2 = \frac{p}{2}I - \frac{2\sigma_{p,q} - p}{2}F. \tag{9}$$

A metallic (or golden) Riemannian manifold $\tilde{M}$ is called the *locally metallic (or locally golden) Riemannian manifold* if φ is parallel with respect to the Levi–Civita connection $\tilde{\nabla}$, i.e., $\tilde{\nabla}\varphi = 0$ on $\tilde{M}$ [18]. In a similar way, if an almost product structure F on a Riemannian manifold $\tilde{M}$ satisfies $\tilde{\nabla}F = 0$, then F is said to be locally product structure on $\tilde{M}$ [36].

If P and Q denote the tangential and normal components of φ, one can write

$$\varphi X = PX + QX, \qquad \forall X \in \Gamma(T\tilde{M}).$$

Let M^n be an isometrically immersed submanifold in a metallic Riemannian manifold $(\tilde{M}^m, g, \varphi)$, X be a nonzero vector tangent to M at $x \in M$, and the angle between φX and $T_x M$ be denoted by $\theta(X)$.

Submanifold M of $(\tilde{M}^m, g, \varphi)$ is said to be *slant* if $\theta(X)$ is constant. Invariant and anti-invariant submanifolds are the particular class of slant submanifolds with a $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively.

Next, we recall the following:

Lemma 1 (Refs. [18,20])**.** *Let M be a submanifold of a metallic Riemannian manifold $(\tilde{M}, \varphi, g)$. If M is slant with slant angle θ, then*

$$g(TX, TY) = \cos^2\theta[pg(X, TY) + qg(X, Y)]$$
$$g(NX, NY) = \sin^2\theta[pg(X, TY) + qg(X, Y)],$$

$\forall X, Y \in \Gamma(TM)$. *When I stands for the identity transformation on TM, thus we also obtain*

$$T^2 = \cos^2\theta(pT + qI), \quad \nabla T^2 = p\cos^2\theta(\nabla T).$$

Let M_1 and M_2 be two Riemannian manifolds with constant sectional curvatures c_1 and c_2, respectively. Then the product Riemannian manifold $(\tilde{M} = M_1 \times M_2, F)$ with locally product structure F is a locally Riemannian product manifold and its curvature tensor of $\tilde{M} = M_1(c_1) \times M_2(c_2)$ is given by [37]

$$
\begin{aligned}
\tilde{R}'(X,Y)Z = {} & \frac{1}{4}(c_1+c_2)[g(Y,Z)X-g(X,Z)Y] \\
& +\frac{1}{4}(c_1+c_2)\Big\{\frac{4}{(2\sigma_{p,q}-p)^2}[g(\varphi Y,Z)\varphi X-g(\varphi X,Z)\varphi Y] \\
& +\frac{p^2}{(2\sigma_{p,q}-p)^2}[g(Y,Z)X-g(X,Z)Y] \\
& +\frac{2p}{(2\sigma_{p,q}-p)^2}[g(\varphi X,Z)Y+g(X,Z)\varphi Y] \\
& -\frac{2p}{(2\sigma_{p,q}-p)^2}[g(\varphi Y,Z)X+g(Y,Z)\varphi X]\Big\} \\
& \pm\frac{1}{2}(c_1-c_2)\Big\{\frac{1}{2\sigma_{p,q}-p}[g(Y,Z)\varphi X-g(X,Z)\varphi Y] \\
& +\frac{1}{2\sigma_{p,q}-p}[g(\varphi Y,Z)X-g(\varphi X,Z)Y] \\
& +\frac{p}{2\sigma_{p,q}-p}[g(X,Z)Y-g(Y,Z)X]\Big\}. \qquad (10)
\end{aligned}
$$

Additionally, if $\tilde{M}$ is equipped with SSMC, then the curvature tensor of $\tilde{M}$ with the help of (2) and (10) is given by

$$
\begin{aligned}
\tilde{R}(X,Y,Z,W) = {} & \frac{1}{4}(c_1+c_2)[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)] \\
& +\frac{1}{4}(c_1+c_2)\Big\{\frac{4}{(2\sigma_{p,q}-p)^2}[g(\varphi Y,Z)g(\varphi X,W) \\
& -g(\varphi X,Z)g(\varphi Y,W)] \\
& +\frac{p^2}{(2\sigma_{p,q}-p)^2}[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)] \\
& +\frac{2p}{(2\sigma_{p,q}-p)^2}[g(\varphi X,Z)g(Y,W)+g(X,Z)g(\varphi Y,W) \\
& -g(\varphi Y,Z)g(X,W)-g(Y,Z)g(\varphi X,W)]\Big\} \\
& \pm\frac{1}{2}(c_1-c_2)\Big\{\frac{1}{2\sigma_{p,q}-p}[g(Y,Z)g(\varphi X,W)-g(X,Z)g(\varphi Y,W)] \\
& +\frac{1}{2\sigma_{p,q}-p}[g(\varphi Y,Z)g(X,W)-g(\varphi X,Z)g(Y,W)] \\
& +\frac{p}{2\sigma_{p,q}-p}[g(X,Z)g(Y,W)-g(Y,Z)g(X,W)]\Big\} \\
& -\alpha(Y,Z)g(X,W)+\alpha(X,Z)g(Y,W) \\
& -\alpha(X,W)g(Y,Z)+\alpha(Y,W)g(X,Z). \qquad (11)
\end{aligned}
$$

3. Generalized Wintgen Inequality for θ-Slant Submanifolds

Here, we establish the generalized Wintgen inequality for the θ-slant submanifold M^n of locally metallic space $(\tilde{M}=M_1(c_1)\times M_2(c_2),g,\varphi)$.

Theorem 1. *Let M^n be a θ-slant submanifold in a locally metallic space form $(\tilde{M}^m = M_1(c_1) \times M_2(c_2), g, \varphi)$ equipped with the SSMC. Then,*

$$\rho_N \leq ||H||^2 - 2\rho + (c_1 + c_2)\frac{p}{p^2 + 4q}\Big\{p^2 + 2q + \frac{2}{n(n-1)}[tr^2\varphi - (p \cdot trT + nq)\cos^2\theta] - \frac{2p}{n}tr\varphi\Big\} + \frac{1}{n}\frac{1}{\sqrt{p^2 + 4q}}(c_1 - c_2)(2tr\varphi - np) - 2(n-1)tr(\alpha). \tag{12}$$

Furthermore, the equality case in (12) holds identically iff, for the orthonormal frame $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\}$, the shape operators A satisfy

$$A_{n+1} = \begin{pmatrix} a & d & 0 & \ldots & 0 & 0 \\ d & a & 0 & \ldots & 0 & 0 \\ 0 & 0 & a & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a & 0 \\ 0 & 0 & 0 & \ldots & 0 & a \end{pmatrix}, \tag{13}$$

$$A_{n+2} = \begin{pmatrix} b+d & 0 & 0 & \ldots & 0 & 0 \\ 0 & b-d & 0 & \ldots & 0 & 0 \\ 0 & 0 & b & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & b & 0 \\ 0 & 0 & 0 & \ldots & 0 & b \end{pmatrix}, \tag{14}$$

$$A_{n+3} = \begin{pmatrix} c & 0 & 0 & \ldots & 0 & 0 \\ 0 & c & 0 & \ldots & 0 & 0 \\ 0 & 0 & c & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & c & 0 \\ 0 & 0 & 0 & \ldots & 0 & c \end{pmatrix}, \quad A_{n+4} = \cdots = A_m = 0, \tag{15}$$

a, b, c, and d are smooth functions on M.

Proof. From (11) and (1), we obtain

$$\sum_{1 \leq i < j \leq n} R(e_i, e_j, e_j, e_i) = \frac{1}{4}\frac{(n-1)}{\sqrt{p^2 + 4q}}(c_1 - c_2)(4tr\varphi - 2np) + \frac{1}{4}(c_1 + c_2)\frac{n(n-1)}{p^2 + 4q}\Big\{2p^2 + 4q + \frac{4}{n(n-1)}[tr^2\varphi - (p \cdot trT + nq)\cos^2\theta] - \frac{4p}{n}tr\varphi\Big\} + \sum_{\alpha=n+1}^{m}\sum_{1 \leq i < j \leq n}\Big[h_{ii}^{\alpha}h_{jj}^{\alpha} - (h_{ij}^{\alpha})^2\Big] - 2(n-1)tr(\alpha). \tag{16}$$

One can also observe that

$$2\tau = \sum_{1 \leq i < j \leq n} R(e_i, e_j, e_j, e_i), \tag{17}$$

which gives

$$
\begin{aligned}
2\tau = &\frac{1}{4}\frac{(n-1)}{\sqrt{p^2+4q}}(c_1-c_2)(4tr\varphi-2np) \\
&+\frac{1}{4}(c_1+c_2)\frac{n(n-1)}{p^2+4q}\Big\{2p^2+4q \\
&+\frac{4}{n(n-1)}[tr^2\varphi-(p\cdot trT+nq)\cos^2\theta]-\frac{4p}{n}tr\varphi\Big\} \\
&+\sum_{\alpha=n+1}^{m}\sum_{1\le i<j\le n}\Big[h^{\alpha}_{ii}h^{\alpha}_{jj}-(h^{\alpha}_{ij})^2\Big]-2(n-1)tr(\alpha).
\end{aligned} \tag{18}
$$

We also find that

$$
\begin{aligned}
n^2||H||^2 = &\sum_{\alpha=n+1}^{m}\Big(\sum_{i=1}^{n}h^{\alpha}_{ii}\Big)^2=\frac{1}{n-1}\sum_{\alpha=n+1}^{m}\sum_{1\le i<j\le n}(h^{\alpha}_{ii}-h^{\alpha}_{jj})^2 \\
&+\frac{2n}{n-1}\sum_{\alpha=n+1}^{m}\sum_{1\le i<j\le n}h^{\alpha}_{ii}h^{\alpha}_{jj}.
\end{aligned} \tag{19}
$$

Then, from [8], clearly, we know that

$$
\begin{aligned}
&\sum_{\alpha=n+1}^{m}\sum_{1\le i<j\le n}(h^{\alpha}_{ii}-h^{\alpha}_{jj})^2+2n\sum_{\alpha=n+1}^{m}\sum_{1\le i<j\le n}(h^{\alpha}_{ij})^2 \\
&\ge 2n\Big\{\sum_{n+1\le\alpha<\beta\le m-n}\sum_{1\le i<j\le n}[\sum_{k=1}^{n}(h^{\alpha}_{jk}h^{\beta}_{ik}-h^{\alpha}_{ik}h^{\beta}_{jk})]^2\Big\}^{\frac{1}{2}}.
\end{aligned} \tag{20}
$$

From (19) and (20) with the help of (5), we reach

$$
n^2||H||^2-n^2\rho_N\ge\frac{2n}{n-1}\sum_{\alpha=n+1}^{m-n}\sum_{1\le i<j\le n}[h^{\alpha}_{ii}h^{\alpha}_{jj}-(h^{\alpha}_{ij})^2]. \tag{21}
$$

As a result, using (5), (18), and (21), we derive

$$
\begin{aligned}
\rho_N-||H||^2\le &\frac{1}{2}(c_1+c_2)\frac{p}{p^2+4q}\Big\{2p^2+4q \\
&+\frac{4}{n(n-1)}[tr^2\varphi-(p\cdot trT+nq)\cos^2\theta]-\frac{4p}{n}tr\varphi\Big\} \\
&+\frac{1}{2n}\frac{1}{\sqrt{p^2+4q}}(c_1-c_2)(4tr\varphi-2np)-2\rho-2(n-1)tr(\alpha),
\end{aligned} \tag{22}
$$

demonstrating the necessary inequality.

Lastly, by examining the equality scenario in (12), We come to the conclusion using a ratiocination similar to that in [[7], Corollary 1.2] that the equality sign holds in (12) at a point $p \in M$ if and only if the shape operators have the forms (13)–(15) with respect to any appropriate tangent and normal orthonormal bases. □

4. Consequences of Theorem 1

We have two ways of its applications:

1. Firstly, we could consider the particular classes of θ-slant submanifolds i.e., either invariant or anti-invariant. In this case, we have the following result as a particular case of Theorem 1

Corollary 1. *Let M^n be an immersed submanifold of $(\tilde{M} = M_1(c_1) \times M_2(c_2), g, \varphi)$ be a locally metallic space form equipped with SSMC. Then, inequality (12) takes the following forms:*

(i) *If M is invariant, then*

$$\rho_N \leq ||H||^2 - 2\rho + (c_1 + c_2)\frac{p}{p^2 + 4q}\Big\{p^2 + 2q + \frac{2}{n(n-1)}[tr^2\varphi - (p \cdot trT + nq)] - \frac{2p}{n}tr\varphi\Big\} + \frac{1}{n}\frac{1}{\sqrt{p^2 + 4q}}(c_1 - c_2)(2tr\varphi - np) - 2(n-1)tr(\alpha). \tag{23}$$

(ii) *If M is anti-invariant, then*

$$\rho_N \leq ||H||^2 - 2\rho + (c_1 + c_2)\frac{p}{p^2 + 4q}\Big[p^2 + 2q + \frac{2}{n(n-1)}tr\varphi^2 - \frac{2p}{n}tr\varphi\Big] - \frac{p}{\sqrt{p^2 + 4q}}(c_1 - c_2) + \frac{1}{n}\frac{1}{\sqrt{p^2 + 4q}}(c_1 - c_2)(2tr\varphi - np) - 2(n-1)tr(\alpha). \tag{24}$$

Moreover, the equality case in (23) and (24) holds identically iff, for the orthonormal frame $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\}$, the shape operators A satisfy

$$A_{n+1} = \begin{pmatrix} a & d & 0 & \ldots & 0 & 0 \\ d & a & 0 & \ldots & 0 & 0 \\ 0 & 0 & a & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a & 0 \\ 0 & 0 & 0 & \ldots & 0 & a \end{pmatrix}, \tag{25}$$

$$A_{n+2} = \begin{pmatrix} b+d & 0 & 0 & \ldots & 0 & 0 \\ 0 & b-d & 0 & \ldots & 0 & 0 \\ 0 & 0 & b & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & b & 0 \\ 0 & 0 & 0 & \ldots & 0 & b \end{pmatrix}, \tag{26}$$

$$A_{n+3} = \begin{pmatrix} c & 0 & 0 & \ldots & 0 & 0 \\ 0 & c & 0 & \ldots & 0 & 0 \\ 0 & 0 & c & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & c & 0 \\ 0 & 0 & 0 & \ldots & 0 & c \end{pmatrix}, \quad A_{n+4} = \cdots = A_m = 0, \tag{27}$$

a, b, c, and d are smooth functions on M.

2. Secondly, we can consider the particular classes of metallic product spaces, such as a golden structure or so-called silver, copper, nickel, and bronze structures by taking the particular values of p and q. For example, the inequality (12) for the locally golden space form $(\tilde{M} = M_1(c_1) \times M_2(c_2), g, \varphi)$ will be

Corollary 2. *For a submanifold M^n of locally golden space form $(\tilde{M} = M_1(c_1) \times M_2(c_2), g, \varphi)$ endowed with SSMC, we have* *(i)* *If M is θ-slant, then*

$$\rho_N \leq ||H||^2 - 2\rho + \frac{1}{5}(c_1 + c_2)\Big\{3 + \frac{2}{n(n-1)}[tr^2\varphi - (trT + n)\cos^2\theta] - \frac{2}{n}tr\varphi\Big\} + \frac{1}{\sqrt{5}n}(c_1 - c_2)(2tr\varphi - n) - 2(n-1)tr(\alpha). \quad (28)$$

(ii) *If M is invariant, then*

$$\rho_N \leq ||H||^2 - 2\rho + \frac{1}{5}(c_1 + c_2)\Big\{3 + \frac{2}{n(n-1)}(tr^2\varphi - trT - n) - \frac{2}{n}tr\varphi\Big\} + \frac{1}{\sqrt{5}n}(c_1 - c_2)(2tr\varphi - n) - 2(n-1)tr(\alpha). \quad (29)$$

(iii) *If M is anti-invariant, then*

$$\rho_N \leq ||H||^2 - 2\rho + \frac{1}{5}(c_1 + c_2)\Big[3 + \frac{2}{n(n-1)}tr^2\varphi - \frac{2}{n}tr\varphi\Big] + \frac{1}{\sqrt{5}n}(c_1 - c_2)(2tr\varphi - n) - 2(n-1)tr(\alpha). \quad (30)$$

Moreover, the equality sign holds in (28)–(30) at a point $p \in M$ if and only if the shape operators have the forms (13)–(15) with respect to any appropriate tangent and normal orthonormal bases.

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References

1. Wintgen, P. Sur l'inegalite de Chen-Willmore. *CR Acad. Sci. Paris Ser. A-B* **1979**, *288*, A993–A995.
2. Guadalupe, I.V.; Rodriguez, L. Normal curvature of surfaces in space forms. *Pacific J. Math.* **1983**, *106*, 95–103. [CrossRef]
3. Rouxel, B. *Sur une Famille des A-Surfaces d'un Espace Euclidien E^4*; Osterreischer Mathematiker Kongress: Insbruck, Austria, 1981.
4. Chen, B.-Y. A Wintgen type inequality for surfaces in 4 D neutral pseudo-Riemannian space forms and its applications to minimal immersions. *JMI Int. J. Math. Sci.* **2010**, *1*, 1–12.
5. Chen, B.-Y. Wintgen ideal surfaces in four-dimensional neutral indefinite space form $R^4_2(c)$. *Results Math.* **2012**, *61*, 329–345. [CrossRef]
6. Smet, P.J.D.; Dillen, F.; Verstraelen, L.; Vrancken, L. A pointwise inequality in submanifold theory. *Arch. Math.* **1999**, *35*, 115–128.
7. Ge, J.Q.; Tang, Z.Z. A proof of the DDVV conjecture and its equality case. *Pacific J. Math.* **2008**, *237*, 87–95. [CrossRef]
8. Lu, Z. Normal scalar curvature conjecture and its applications. *J. Funct. Anal.* **2011**, *261*, 1284–1308. [CrossRef]
9. Choudhary, M.A.; Bahadir, O.; Alsulami, H. Generalized Wintgen inequality for some submanifolds in Golden Riemannian space forms. *Balk. J. Geom. Its Appl.* **2020**, *25*, 1–11.
10. Mihai, I. On the generalized Wintgen inequality for Lagrangian submanifolds in complex space forms. *Nonlinear Anal.* **2014**, *95*, 714–720. [CrossRef]
11. Mihai, I. On the generalized Wintgen inequality for Legendrian submanifolds in Sasakian space forms. *Tohoku Math. J.* **2017**, *69*, 43–53. [CrossRef]
12. Lee, C.W.; Lee, J.W.; Vilcu, G.-E. Generalized wintgen inequality for submanifolds in generalized (κ, μ)-space forms. *Quaest. Math.* **2022**, *45*, 497–511. [CrossRef]
13. Chen, B.Y. Recent developments in Wintgen inequality and Wintgen ideal submanifold. *Int. Electron. Geom.* **2021**, *14*, 6–45. [CrossRef]
14. Crasmareanu, M.; Hretcanu, C. Golden differential geometry. *Chaos Solitons Fractals* **2008**, *38*, 1229–1238. [CrossRef]
15. Hretcanu, C.; Crasmareanu, M. Metallic structures on Riemannian manifolds. *Rev. Union Mat. Argent.* **2013**, *54*, 15–27.

16. Blaga, A.M.; Nannicini, A. On curvature tensors of Norden and metallic pseudo-Riemannian manifold. *Complex Manifolds* **2019**, *6*, 150–159. [CrossRef]
17. Blaga, A.M.; Nannicini, A. On the geometry of generalized metallic pseudo-Riemannian structures. *Riv. Mat. Della Univ. Parma* **2020**, *11*, 69–87.
18. Blaga, A.M.; Hretcanu, C.E. Invariant, anti-invariant and slant submanifolds of a metallic Riemannian manifold. *Novi Sad J. Math.* **2018**, *48*, 57–82. [CrossRef]
19. Hretcanu, C.E.; Blaga, A.M. Hemi-slant submanifolds in metallic Riemannian manifolds. *Carpathian J. Math.* **2019**, *35*, 59–68. [CrossRef]
20. Hretcanu, C.E.; Blaga, A.M. Slant and semi-slant submanifolds in metallic Riemannian manifolds. *J. Funct. Spaces* **2018**, *2018*, 2864263. [CrossRef]
21. Hretcanu, C.E.; Blaga, A.M. Warped product pointwise bi-slant submanifolds in metallic Riemannian manifolds. *arXiv* **2020**, arXiv:2002.10909v2.
22. Blaga, A.M.; Hretcanu, C.E. Remarks on metallic warped product manifolds. *Facta Univ.* **2018**, *33*, 269–277. [CrossRef]
23. Hretcanu, C.E.; Blaga, A.M. Warped product submanifolds in metallic and Golden Riemannian manifolds. *Tamkang J. Math.* **2020**, *51*, 161–186. [CrossRef]
24. Friedmann, A.; Schouten, A. Uber die Geometrie der halbsymmetrischen Ubertragungen. *Math. Z.* **1924**, *21*, 211–223. [CrossRef]
25. Hayden, H.A. Subspaces of a space with torsion. *Proc. London Math. Soc.* **1932**, *34*, 27–50. [CrossRef]
26. Yano, K. On semi-symmetric metric connection. *Rev. Roum. Math. Pure Appl.* **1970**, *15*, 1579–1586.
27. Mihai, A.; Ozgur, C. Chen inequalities for submanifolds of real space forms with a semi-symmetric metric connection. *Taiwan. J. Math.* **2010**, *14*, 1465–1477. [CrossRef]
28. Mihai, A.; Ozgur, C. Chen inequalities for submanifolds of complex space forms and Sasakian space forms endowed with semi-symmetric metric connection. *Rocky Mt. J. Math.* **2011**, *41*, 1653–1673. [CrossRef]
29. Zhang, P.; Zhang, L.; Song, W. Inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature with a semi-symmetric metric connection. *Taiwan. J. Math.* **2014**, *18*, 1841–1862. [CrossRef]
30. Nakao, Z. Submanifolds of a Riemannian manifold with semi-symmetric metric connections. *Proc. Amer. Math. Soc.* **1976**, *54*, 261–266. [CrossRef]
31. Imai, T. Notes on semi-symmetric metric connection. *Tensor* **1972**, *24*, 293–296.
32. Mihai, I.; Al-Solamy, F.R.; Shahid, M.H. On Ricci curvature of a quaternion CR-submanifold in a quaternion space form. *Rad. Mat.* **2003**, *12*, 91–98.
33. Goldberg, S.I.; Yano, K. Polynomial structures on manifolds. *Kodai Math. Sem. Rep.* **1970**, *22*, 199–218. [CrossRef]
34. Hretcanu, C.; Crasmareanu, M. Applications of the golden ratio on Riemannian manifolds. *Turkish J. Math.* **2009**, *33*, 179–191. [CrossRef]
35. Bahadir, O.; Uddin, S. Slant submanifolds of golden Riemannian manifolds. *J. Math. Ext.* **2019**, *13*, 1–10.
36. Pitis, G. On some submanifolds of a locally product manifold. *Kodai Math. J.* **1986**, *9*, 327–333. [CrossRef]
37. Choudhary, M.A.; Blaga, A.M. Generalized Wintgen inequality for slant submanifolds in metallic Riemannian space forms. *J. Geom.* **2021**, *112*, 26. [CrossRef]

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