mathematics

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On 2-Rainbow Domination of Generalized Petersen Graphs $P(ck, k)$

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Abstract: We obtain new results on the 2-rainbow domination number of generalized Petersen graphs $P(ck, k)$. Exact values are established for all infinite families where the general lower bound $\frac{4}{5}ck$ is attained. In all other cases lower and upper bounds with small gaps are given.

Keywords: rainbow domination; 2-rainbow domination number; generalized Petersen graphs

MSC: 05C69; 05C35

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Citation: Brezovnik, S.; Rupnik Poklukar, D.; Žerovnik, J. On 2-Rainbow Domination of Generalized Petersen Graphs $P(ck, k)$. *Mathematics* **2023**, *11*, 2271. https://doi.org/10.3390/math11102271

Academic Editor: Bo Zhou

Received: 8 April 2023
Revised: 5 May 2023
Accepted: 10 May 2023
Published: 12 May 2023

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1. Introduction

Considering domination in graphs as a model for applications where vertices are providing a service that should be accessible in the neighborhood of each vertex in the network, rainbow domination provides in a sense a more complex version of domination. Inspired by such problems, Brešar, Henning and Rall [1] initiated the study of rainbow domination. The original motivation for introducing rainbow domination is due to its essential relation with domination in Cartesian products of graphs and possible outcomes for the famous Vizing conjecture, see [2,3].

Initial results on rainbow domination were established by Hartnell et al. [4]. Some exact values for 2-rainbow domination can be found in [1,5,6] and for Cartesian products in [1,4,7,8], while some results on bounds of the 2-rainbow domination number are presented in [6,9]. Bounds of the 3-rainbow domination number were derived in [10,11] and bounds for general k-rainbow domination in [12,13].

Not surprisingly, rainbow domination problems are intractable in general. Computational complexity was considered in [5], where the NP-completeness of the 2-rainbow domination problem for bipartite and chordal graphs was proven. In [14], the complexity result was extended to k-rainbow domination for $k \geq 2$. On the positive side, linear time algorithms are known that provide: the 2-rainbow domination number for trees [1], and the k-rainbow number for trees [14] and for block graphs [15]. Some approximation results with open questions appear in [16].

The class of generalized Petersen graphs has drawn considerable attention when studying rainbow domination. Quite a few papers have investigated the 2-rainbow domination number in this class of graphs, see [17–26]. In this paper, we continue the study of rainbow domination of generalized Petersen graphs. In particular we generalize the result on 2-rainbow domination $P(5k, k)$ to $P(ck, k)$ for abritrary $c \geq 3$.

The rest of the paper is organized as follows. The Preliminaries section recalls some basic notions relevant to the present work. It is followed by a section with a brief overview of related previous work including some technical details that are used later. In Section 4, our results are summarized. Cases with small k ($k = 1, 2, 3$) are considered in Section 5. Then, Section 6 provides analysis of the cases where the general lower bound is attained. For the general case, an improved lower bound is provided in Section 7 and upper bounds are obtained by constructions in Section 8 and improved in Section 9. Section 10 brings some concluding remarks.

2. Preliminaries

2.1. Generalized Petersen Graphs

For convenience, throughout the paper, all subscripts will be taken modulo n. For $n \geq 3$ and k, $1 \leq k < \frac{1}{2}n$, the *generalized Petersen graph* $P(n,k)$ is a graph on $2n$ vertices with $V(P(n,k)) = \{v_i, u_i \mid 0 \leq i \leq n-1\}$ and $E(P(n,k)) = \{u_i u_{i+1}, u_i v_i, v_i v_{i+k} \mid 0 \leq i \leq n-1\}$. This standard notation was introduced by Watkins [27] (see Figure 1, left). Clearly, the set of vertices $U = \{u_i \mid 0 \leq i \leq n-1\}$ induces a cycle that is called the *outer cycle* and, when $n = ck$, the set of vertices $V = \{v_i \mid 0 \leq i \leq n-1\}$ induces k cycles called the *inner cycles*.

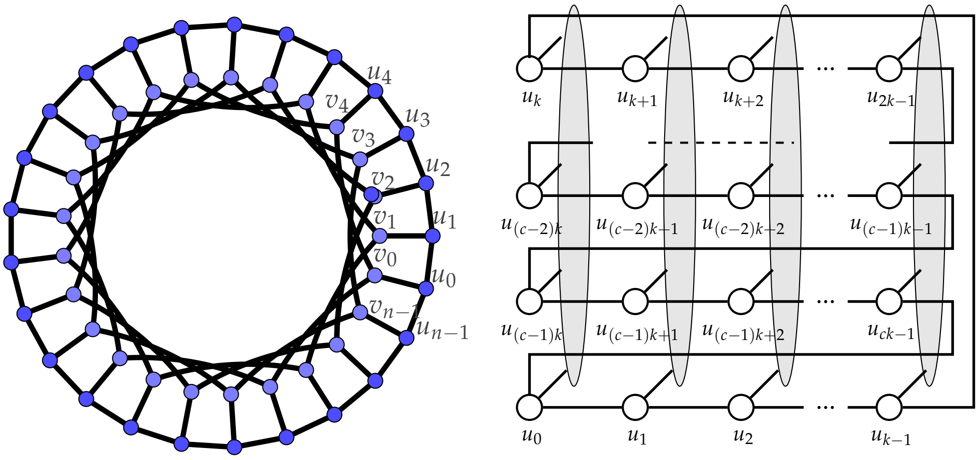

Figure 1. Drawing of generalized Petersen graph $P(n,k)$ (**left**) and an alternative drawing of $P(ck,k)$ (**right**).

Here we consider generalized Petersen graphs $P(ck,k)$, $c \geq 3$, $k \geq 1$. The structure of these graphs makes it possible to perform some special constructions. In particular, $P(ck,k)$ has one long (outer) cycle and k (inner) cycles of length c (see Figure 1, right). Analogously as in [26] we introduce the next notations. For $i = 1, 2, \ldots, c$ we define

$$V_i = \{v_{(i-1)k}, v_{(i-1)k+1}, v_{(i-1)k+2}, \ldots, v_{ik-1}\},$$
$$U_i = \{u_{(i-1)k}, u_{(i-1)k+1}, u_{(i-1)k+2}, \ldots, u_{ik-1}\},$$
$$V = \bigcup_{i=1}^{c} V_i,\ U = \bigcup_{i=1}^{c} U_i,\ \ V(P(ck,k)) = V \cup U.$$

2.2. Rainbow Domination

Let G be a graph and t a positive integer. We want to assign a subset of the color set $\{1, 2, \ldots, t\}$ to every vertex of graph G such that every vertex that has assigned the empty set has all t colors in the neighborhood. We refer to the defined assignment as a t-rainbow dominating function, abbreviated as tRD function or $tRDF$, of the graph G. The weight of the assignment g, which represents a $tRDF$ of G, is computed as $w(g) = \sum_{v \in V(G)} w(g(v))$, where $w(g(v))$ denotes the count of colors assigned to vertex v. We also use the term *tRD-colored*, or simply colored, to describe a graph G that has been colored by g. A vertex is said to be tRD-dominated if it is allocated a non-empty set of colors or if all t colors are assigned to the vertices in its locality. A vertex is considered colored if $g(v) \neq \emptyset$ and uncolored otherwise. The t-rainbow domination number $\gamma_{rt}(G)$ corresponds to the minimal weight among all tRD functions of G.

3. Related Previous Work

In this section, we recall some recent results that are closely connected to the present work. In most cases, the results or methods are used in the proofs. We first recall the next bounds for the 2-rainbow domination number of a generalized Petersen graph.

Proposition 1 ([5,24]). *Let $P(n,k)$ be a generalized Petersen graph, where $\gcd(n,k) = 1$. Then,*

$$\left\lceil \frac{4n}{5} \right\rceil \leq \gamma_{r2}(P(n,k)) \leq n.$$

Moreover, by using Theorem 2 from [12] the lower bound can be stated for general n and k.

Proposition 2. *Let $P(n,k)$ be a generalized Petersen graph. Then,*

$$\gamma_{r2}(P(n,k)) \geq \left\lceil \frac{4n}{5} \right\rceil.$$

An important tool relates the rainbow domination numbers of a graph and its h-lift. We omit the details, because we only need the fact (Proposition 3) that some Petersen graphs are covers of some other Petersen graphs in order to use the next theorem, recalled from [26]. (For more information on graph covers and h-lifts we refer to [28,29].)

Theorem 1 ([26]). *Let graph H be an h-lift of graph G. Then $\gamma_{rt}(H) \leq h\gamma_{rt}(G)$.*

Proposition 3 ([26]). *Let $k \geq 1$, $c_0 \geq 3$, and $h \geq 2$. Petersen graph $P((hc_0)k, k)$ is an h-lift of $P(c_0 k, k)$.*

Reference [26] also provides the exact values of the 2-rainbow domination number for some infinite subfamilies of Petersen graphs $P(5k,k)$ and bounds with gap at most 2 for all other infinite subfamilies of Petersen graphs $P(5k,k)$.

Theorem 2 ([26]). *Let $k > 3$. Then*

$$\gamma_{r2}(P(5k,k)) = \begin{cases} 4k, & k \equiv 2,8 \bmod 10 \\ 4k+1, & k \equiv 5,9 \bmod 10 \end{cases} \tag{1}$$

$$4k+1 \leq \gamma_{r2}(P(5k,k)) \leq \begin{cases} 4k+2, & k \equiv 1,6,7 \bmod 10 \\ 4k+3, & k \equiv 0,3,4 \bmod 10 \end{cases} \tag{2}$$

For establishing the lower bound of the main theorem, we will need a generalization of the next lemma that provided a lower bound of the 2-rainbow domination number of Petersen graphs $P(5k,k)$. Generalization is not trivial (see the proof of Lemma 2), so it is worth recalling the original lemma here.

Lemma 1 ([26]). *Assume $\gamma_{r2}(P(5k,k)) = 4k$. Let $C_i = \{v_i, v_{k+i}, v_{2k+i}, v_{3k+i}, v_{4k+i}\}$ and $V_i = \{u_i, u_{k+i}, u_{2k+i}, u_{3k+i}, u_{4k+i}\}$, for any $i \in \{0,1,\ldots,k-1\}$.Then, for a 2RD coloring of weight $4k$, we have*

(1) Exactly one vertex of C_i receives color 1 and exactly one vertex receives color 2;
(2) The two vertices on the cycle $v_i v_{k+i} v_{2k+i} v_{3k+i} v_{4k+i}$ that receive colors are not adjacent;
(3) Exactly one vertex of V_i receives color 1 and exactly one vertex receives color 2;
(4) Assume (wlog) $f(v_{k+i}) = 1$ and $f(v_{3k+i}) = 2$. Then $f(u_{4k+i}) = 1$ and $f(u_{0+i}) = 2$.

Recall two constructions from [30]. The first construction gives the Petersen graph $P((c-1)k,k)$ from the Petersen graph $P(ck,k)$ by deleting some vertices and adding some edges.

Construction 1 ([30]).

- *Start with* $P(ck,k)$.
- *Delete vertices* $V_c = \{v_{(c-1)k}, v_{(c-1)k+1}, v_{(c-1)k+2}, \dots v_{ck-1}\}$ *and* $U_c = \{u_{(c-1)k}, u_{(c-1)k+1}, u_{(c-1)k+2}, \dots u_{ck-1}\}$ *and delete all edges incident to these vertices.*
- *Add edges* $\{v_{(c-2)k}v_0, v_{(c-2)k+1}v_1, v_{(c-2)k+2}v_2, \dots v_{(c-1)k-1}v_{k-1}\}\}$ *on the inner cycles and edge* $\{u_{(c-1)k-1}u_0\}$ *on the outer cycle.*

Proposition 4 ([31]). *Construction 1 on* $P(ck,k)$ *results in the graph* $P((c-1)k,k)$.

The second construction transforms $P(ck,k)$ to $P(c(k-1),k-1)$.

Construction 2 ([30]).

- *Start with* $P(ck,k)$. *Choose* $K \in \{0,1,\dots,k-1\}$. *Delete the vertices* $Out_K = \{u_{jk+K} \mid j = 0,1,2,\dots,c-1\}$ *and vertices of the corresponding inner cycle* $Inn_K = \{v_{jk+K} \mid j = 0,1,2,\dots,c-1\}$ *and delete all edges incident to these vertices.*
- *Add edges* $u_{jk+K-1}u_{jk+K+1}$ *for* $j = 0,1,2,\dots,c-1$.

Proposition 5 ([31]). *Construction 2 on* $P(ck,k)$ *results in a graph that is isomorphic to* $P(c(k-1),k-1)$.

It is well known that the coloring on the outer cycle determines the coloring of the inner cycles (in particular, when f is a $2RD$ function of $P(ck,k)$ with $w(f) = 4hk$ where $c = 5h$.). Therefore, we follow [30] and provide the colorings as tables providing $f(u)$ for $u \in U$. The convention is given in Table 1, showing how a 2RD coloring of $P(5k,k)$ is outlined in five rows. An example, a 2RD coloring of $P(55,11)$, is given in Table 2. Note that the last columns duplicate the information, while the entries in the columns are shifted up by one (compare columns 0 and 11 in Table 2).

Table 1. A 2RD coloring of U_i for $P(5k,k)$.

$f(u_0)$	$f(u_1)$	...	$f(u_i)$	...	$f(u_{k-1})$	$f(u_k)$	$f(u_{k+1})$	...
$f(u_k)$	$f(u_{k+1})$	...	$f(u_{k+i})$	...	$f(u_{2k-1})$	$f(u_{2k})$	$f(u_{2k+1})$	...
$f(u_{2k})$	$f(u_{2k+1})$	...	$f(u_{2k+i})$	...	$f(u_{3k-1})$	$f(u_{3k})$	$f(u_{3k+1})$	...
$f(u_{3k})$	$f(u_{3k+1})$	...	$f(u_{3k+i})$	...	$f(u_{4k-1})$	$f(u_{4k})$	$f(u_{4k+1})$	...
$f(u_{4k})$	$f(u_{4k+1})$	...	$f(u_{4k+i})$	...	$f(u_{5k-1})$	$f(u_{5k}) = f(u_0)$	$f(u_1)$	...

Table 2. A 2RD coloring of $P(55,11)$.

1	0	2	0	0	2	0	1	0	0	1	2	0	0	...
2	0	0	2	0	1	0	0	1	0	2	0	2	0	...
0	2	0	1	0	0	1	0	2	0	**2**	0	1	0	...
0	1	0	0	1	0	2	0	0	2	**1**	0	0	1	...
0	0	1	0	2	0	0	2	0	1	0	1	0	2	...
0	1	2	3	4	5	6	7	8	9	10	11	12	13	...

4. Summary of Our Results

The main results are summarized in the next two theorems. The first theorem provides general bounds. These bounds are improved in special cases by the second theorem, that also provides characterization of the cases in which the general lower bound is attained.

Theorem 3. *For $c \geq 3$ it holds*

$$\frac{4}{5}ck \leq \gamma_{r2}(P(ck,k)) \leq \frac{4}{5}(c+1)(k+1)+1\,. \tag{3}$$

Proof. For $k > 3$, the bounds follow from Proposition 2 and Proposition 14. The small examples (k = 1, 2, 3) are elaborated in Propositions 6–8. □

Theorem 4. *Let $k > 3$. Then if $c \equiv 0 \mod 5$ and $k \equiv 2,8 \mod 10$,*

$$\gamma_{r2}(P(ck,k)) = \frac{4}{5}ck\,. \tag{4}$$

Otherwise, if either $c \not\equiv 0 \mod 5$ or $k \not\equiv 2,8 \mod 10$, we have

$$\frac{4}{5}ck < \gamma_{r2}(P(ck,k)) \leq \frac{4}{5}ck + \alpha(k) + \beta(c) + \gamma(k,c)\,, \tag{5}$$

where

$$\alpha(k) = \begin{cases} 0 & k \equiv 2,8 \bmod 10 \\ \frac{1}{5}c & k \equiv 5,9 \bmod 10 \\ \frac{2}{5}c & k \equiv 1,6,7 \bmod 10 \\ \frac{3}{5}c & k \equiv 0,3,4 \bmod 10 \end{cases} \quad \text{and} \quad \beta(c) = \begin{cases} 0 & c \equiv 0 \bmod 5 \\ \frac{2}{5}k & c \equiv 4 \bmod 5 \\ \frac{3}{5}k & c \equiv 1,2 \bmod 5 \\ \frac{4}{5}k & c \equiv 3 \bmod 5 \end{cases}$$

and $\gamma(k,c)$ is a constant,

$$\gamma(k,c) = \begin{cases} -\frac{6}{5} & k \equiv 7 \bmod 10 \wedge c \equiv 2 \bmod 5 \\ -1 & k \equiv 1,6 \bmod 10 \wedge c \equiv 2 \bmod 5 \\ -\frac{4}{5} & (k \equiv 1,6 \bmod 10 \wedge c \equiv 1 \bmod 5) \vee (k \equiv 3 \bmod 10 \wedge c \equiv 1,2 \bmod 5) \\ -\frac{3}{5} & k \equiv 0 \bmod 10 \wedge c \equiv 1 \bmod 5 \\ -\frac{2}{5} & (k \equiv 1,3,6,9 \bmod 10 \wedge c \equiv 3 \bmod 5) \vee \\ & (k \equiv 0 \bmod 10 \wedge c \equiv 4 \bmod 5) \vee (k \equiv 5 \bmod 10 \wedge c \equiv 2 \bmod 5) \\ -\frac{1}{5} & (k \equiv 1,3,6,7,9 \bmod 10 \wedge c \equiv 4 \bmod 5) \vee \\ & (k \equiv 4,5,7 \bmod 10 \wedge c \equiv 1 \bmod 5) \vee (k \equiv 0,9 \bmod 10 \wedge c \equiv 2 \bmod 5) \\ 0 & (k \equiv 4 \bmod 10 \wedge c \equiv 2 \bmod 5) \vee \\ & (k \equiv 0,1,3,4,5,6,7,9 \bmod 10 \wedge c \equiv 0 \bmod 5) \\ \frac{1}{5} & (k \equiv 0 \bmod 10 \wedge c \equiv 3 \bmod 5) \vee (k \equiv 9 \bmod 10 \wedge c \equiv 1 \bmod 5) \vee \\ & (k \equiv 4,5,8 \bmod 10 \wedge c \equiv 4 \bmod 5) \\ \frac{2}{5} & (k \equiv 8 \bmod 10 \wedge c \equiv 2,3 \bmod 5) \vee (k \equiv 4,5,7 \bmod 10 \wedge c \equiv 3 \bmod 5) \\ \frac{3}{5} & k \equiv 2 \bmod 10 \wedge c \equiv 2,3 \bmod 5 \\ \frac{4}{5} & (k \equiv 2 \bmod 10 \wedge c \equiv 4 \bmod 5) \vee (k \equiv 8 \bmod 10 \wedge c \equiv 1 \bmod 5) \\ \frac{6}{5} & k \equiv 2 \bmod 10 \wedge c \equiv 1 \bmod 5 \end{cases}$$

Proof. The results summarize Proposition 9, Propositions 11–13, and Proposition 19. □

5. 2RD Coloring of $P(c,1)$, $P(2c,2)$, and $P(3c,3)$

In this section, we consider the cases with small k.

First, observe that $P(c,1) = C_c \Box K_2$, the Cartesian product of cycle C_c and K_2. (The Cartesian product is one of the standard graph products [32].) It is well known that $\gamma_{rk}(G) = \gamma(G \Box K_k)$ [1] and $\gamma(C_n \Box C_4) = n$ [33]. Hence we have

$$\gamma_{r2}(C_n \Box K_2) = \gamma((C_n \Box K_2) \Box K_2) = \gamma(C_n \Box (K_2 \Box K_2)) = \gamma(C_n \Box C_4) = n\,.$$

For clarity, we write this fact as a proposition.

Proposition 6. $\gamma_{r2}(P(c,1)) = \gamma_{r2}(C_c \square K_2) = c$.

The graph $P(2c,2)$ has an outer cycle of length $2c$ and two inner cycles of length c. If $2c$ is a multiple of 8, then a 2RDF of weight $2c$ is obtained easily (see Figure 2, left). More precisely, a 2RDF is given by

$$f(u_i) = \begin{cases} 1, & i \equiv 1,4 \mod 8 \\ 2, & i \equiv 5,0 \mod 8 \\ 0, & i \equiv 2,3,6,7 \mod 8 \end{cases} \tag{6}$$

and

$$f(v_i) = \begin{cases} 1, & i \equiv 6,7 \mod 8 \\ 2, & i \equiv 2,3 \mod 8 \\ 0, & i \equiv 0,1,4,5 \mod 8 \end{cases} \tag{7}$$

If $2c \equiv 0 \mod 8$, then is straighforward to check that f is 2RDF for $P(2c,2)$ of weight $2c$.

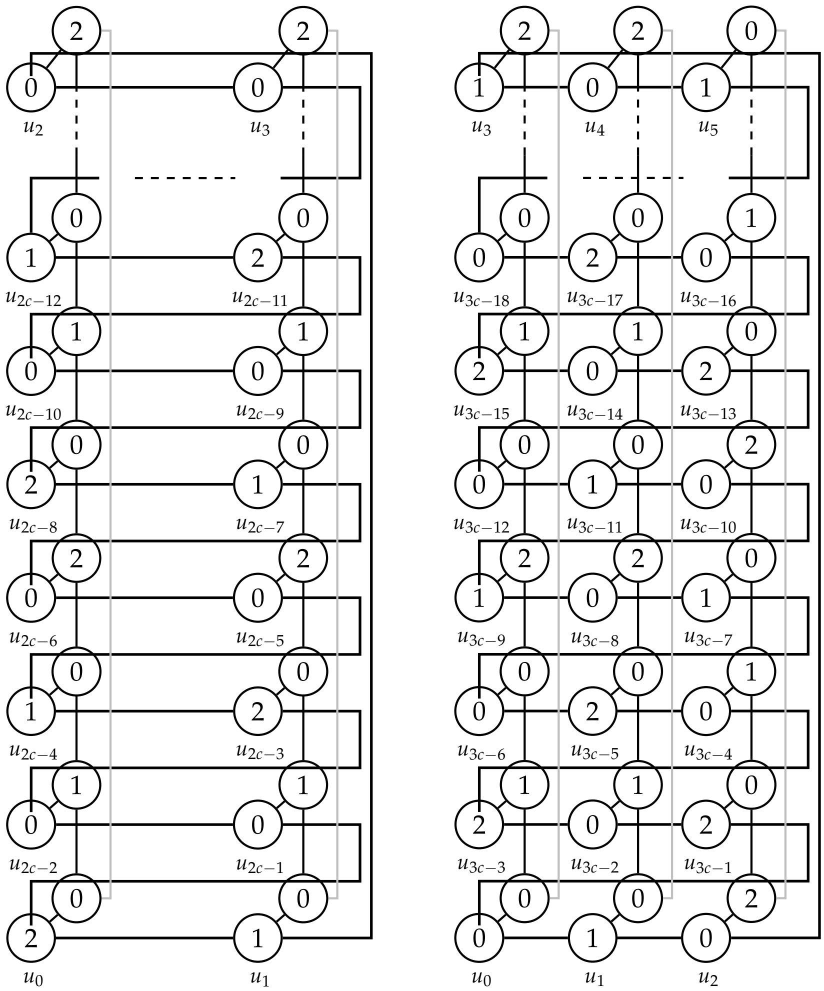

Figure 2. Colorings of $P(2c,2)$ in case $2c \equiv 0$ mod 8 (**left**) and $P(3c,3)$ in case $3c \equiv 0$ mod 12 (**right**).

Proposition 7. *For* $c \geq 3$, $\gamma_{r2}(P(2c,2)) \leq 2c + 4 \leq \frac{12}{5}(c+1)$.

Proof. If $c \equiv 0$ mod 4, then $\gamma_{r2}(P(2c,2)) \leq 2c$ by construction above. If $c \not\equiv 0$ mod 4, then define $\tilde{c} = 4\lceil \frac{c}{4} \rceil$. Recall the definition of f by Formulas (6) and (7). Note that in total $2\tilde{c}$ colors are used by f and $2c$ colors are used when f is restricted to $i = 0, 1, \ldots 2c-1$. However, f may not be a 2RDF because the vertices u_0, v_0, u_{2c-1}, and v_{2c-1} may not be properly dominated. To obtain a 2RDF $\tilde{f}$ we use f on the set of indices $i = -2, -1, 0, 1, \ldots 2c-1, 2c, 2c+1$, where, formally, the negative indices are defined as $-1 = 2\tilde{c} - 1$ and $-2 = 2\tilde{c} - 2$. It is straightforward to see that $\tilde{f}$ defined with $\tilde{f}(u_0) = f(u_0) \cup f(u_{-1})$, $\tilde{f}(v_0) = f(v_0) \cup f(v_{-2})$, $\tilde{f}(v_1) = f(v_1) \cup f(v_{-1})$, $\tilde{f}(u_{2c-1}) = f(u_{2c-1}) \cup f(u_{2c})$, $\tilde{f}(v_{2c-1}) = f(v_{2c-1}) \cup f(v_{2c+1})$, $\tilde{f}(v_{2c-2}) = f(v_{2c-2}) \cup f(v_{2c})$, and otherwise $\tilde{f}(x) = f(x)$, is a 2RDF. See Figure 3 (right) for the case $c \equiv 2 \mod 4$. The total number of colors used by $\tilde{f}$ is $2c + 4$.

Observe that, if $c \geq 5$, then $2c + 4 \leq \frac{12}{5}(c + 1)$, as needed.

Finally, for $c = 4$ we have a 2RDF of weight $2c < \frac{12}{5}(c + 1)$ and, for $c = 3$, observe that the number of colors needed is $2c + 2 = 8$ and, therefore, $\gamma_{r2}(P(6, 2)) = 8 = 2c + 2 = 2(c + 1) \leq \frac{12}{5}(c + 1)$. □

Remark 1. *Note that the case $c = 3$ generalizes to all $c = 4i + 3$ (see Figure 3, left), so we have $\gamma_{r2}(P(2c, 2)) \leq 2c + 2$ when $c = 3 \mod 4$.*

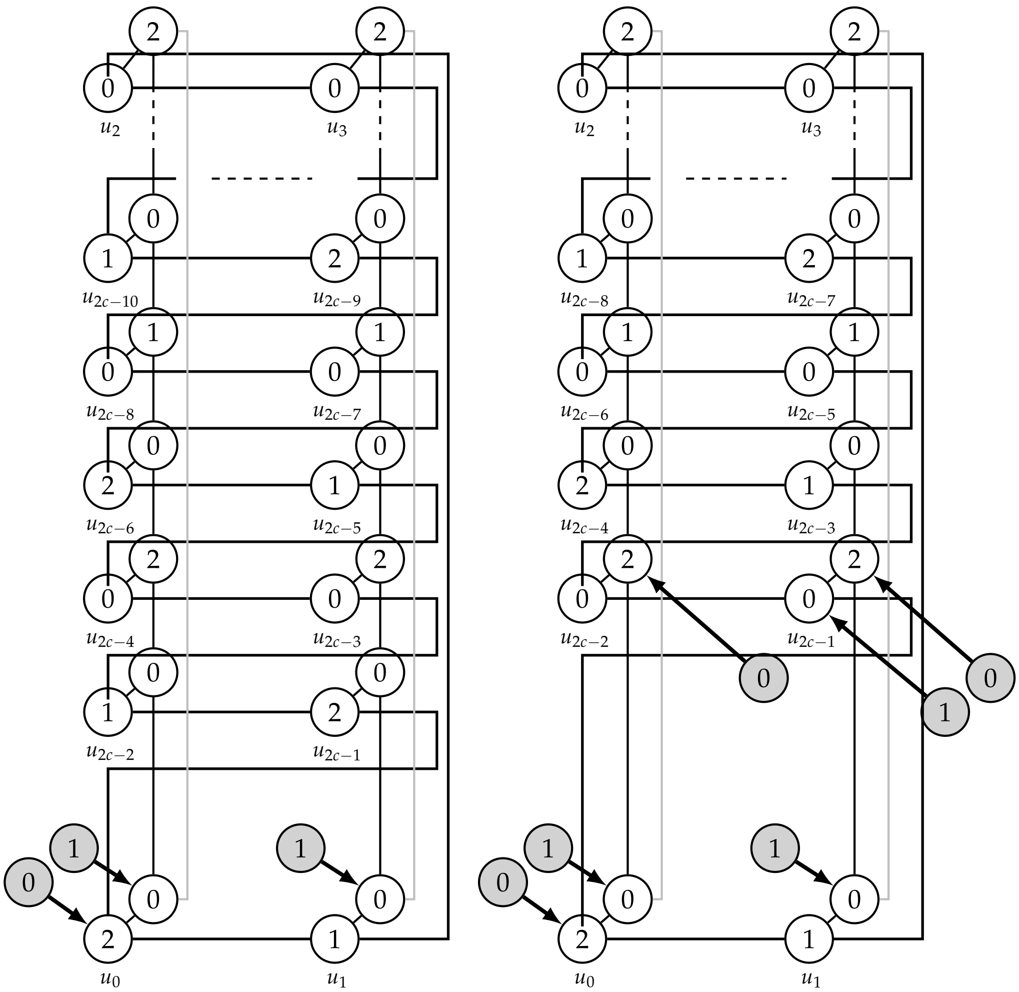

Figure 3. Colorings of $P(2c, 2)$ for $c \equiv 3 \mod 4$ (**left**) and $c \equiv 2 \mod 4$ (**right**). One or two rows are deleted, and colors are added to obtain a 2RDF.

The graph $P(3c, 3)$ has an outer cycle of length $3c$ and three inner cycles of length c. If $3c$ is a multiple of 12, then a 2RDF of weight $3c$ is obtained easily (see Figure 2, right). More precisely, a 2RDF is given by

$$f(u_i) = \begin{cases} 1, & i \equiv 1, 3, 5 \mod 12 \\ 2, & i \equiv 7, 9, 11 \mod 12 \\ 0, & i \equiv 2, 4, 6, 8, 10, 0 \mod 12 \end{cases} \quad (8)$$

and

$$f(v_i) = \begin{cases} 1, & i \equiv 8, 9, 10 \mod 12 \\ 2, & i \equiv 2, 3, 4 \mod 12 \\ 0, & i \equiv 1, 5, 6, 7, 11, 0 \mod 12 \end{cases} \tag{9}$$

Proposition 8. *For $c \geq 3$, $\gamma_{r2}(P(3c, 3)) \leq 3c + 6 \leq \frac{16}{5}(c+1)$.*

Proof. If $3c \equiv 0 \mod 12$, then $\gamma_{r2}(P(3c, 3)) \leq 3c$; a 2RDF is defined by (8) and (9). If $3c \not\equiv 0 \mod 12$, then the bound is obtained by analogous reasoning as in the proof of Proposition 7. Details are omitted. □

6. The Case with Exact Answer

Proposition 9. *Let $c \equiv 0 \mod 5$ and $k \equiv 2, 8 \mod 10$. Then*

$$\gamma_{r2}(P(ck, k)) = \frac{4}{5}ck.$$

Proof. As $c \equiv 0 \mod 5$, we can write $c = 5h$. The proof follows directly from Propositions 3 and 10, and Theorems 1 and 2. □

In addition, the desired 2RD function for the cases when $c \equiv 0 \mod 5$ and $k \equiv 2, 8 \mod 10$ are generated by the function $\mathbb{F}$, which is defined in the following way.

$$\mathbb{F}(u_j) = \begin{cases} 1, & j \equiv 0, 3 \mod 10 \\ 2, & j \equiv 5, 8 \mod 10 \\ 0, & j \equiv 1, 2, 4, 6, 7, 9 \mod 10 \end{cases}. \tag{10}$$

Moreover, we define the values of $\mathbb{F}$ on the inner cycles in the following way.

$$\mathbb{F}(v_j) = \begin{cases} 1, & j \equiv 6, 7 \mod 10 \\ 2, & j \equiv 1, 2 \mod 10 \\ 0, & j \equiv 0, 3, 4, 5, 8, 9 \mod 10 \end{cases}. \tag{11}$$

7. Towards an Improved Lower Bound

In this section, we prove a lemma which in turn implies that the lower bound, $\gamma_{r2}(P(ck, k)) \geq \frac{4}{5}ck$, can be improved in all cases not covered by Proposition 9. More precisely, if either $c \equiv 0 \mod 5$ or $k \equiv 2, 8 \mod 10$, then $\gamma_{r2}(P(ck, k)) > \frac{4}{5}ck$ (see Lemma 3).

We start with some basic observations. Proposition 2 immediately implies the next assertion that is heavily used later in this paper.

Proposition 10. *Let $P(ck, k)$ be a generalized Petersen graph, where $c = 5h$. Then,*

$$4hk \leq \gamma_{r2}(P(ck, k)) \leq 5hk\,.$$

It is easy to see that, to achieve the lower bound, the following conditions must be fulfilled.

Observation 1. *Let $c = 5h$ and f be a 2RD function of $G = P(ck, k)$ with $w(f) = 4hk$. Then:*

(i) For each vertex v of G, $|f(v)| = 0$ or 1.
(ii) The set of the colored vertices is independent.
(iii) If $|f(v)| = 0$ then exactly two neighbors of v are colored (with distinct colors).

Proof. A color at vertex v fulfills the demand of v and half of the demand of its three neighbors, in total $\frac{5}{2}$. If a vertex is assigned both colors, then it also fulfills the demand of its neighbors; hence, total demand 4 is covered by two colors. Total demand of G equals $|V(G)| = 2n = \frac{4n}{5} \times \frac{5}{2}$. So this is possible if $\frac{4n}{5}$ vertices are colored by one color and no demand is wasted. In other words, this means that each colored vertex must have three uncolored neighbors and each uncolored vertex must have exactly two colored neighbors, colored with distinct colors. □

Note that Observation 1(i) also means that any $2RD$ function of $P(ck,k)$ with $w(f) = 4hk$ is a singleton $2RD$ function.

The next technical lemma is a generalization of Lemma 1.

Lemma 2. *Let $G = P(ck,k)$, where $c = 5h$. Let $C_i = \{v_i, v_{k+i}, v_{2k+i}, \ldots, v_{(c-2)k+i}, v_{(c-1)k+i}\}$ and $V_i = \{u_i, u_{k+i}, u_{2k+i}, \ldots, u_{(c-2)k+i}, u_{(c-1)k+i}\}$, for any $i \in \{0, \ldots, k-1\}$. Then, for any 2RD coloring of G of weight $4hk$, we have*

(1) Exactly h vertices of C_i receive color 1 and exactly h vertices receive color 2;
(2) Exactly h vertices of V_i receive color 1 and exactly h vertices receive color 2;
(3) Assume (wlog) $f(v_{k+i}) = 1, f(v_{4k+i}) = 2, f(v_{6k+i}) = 1, f(v_{9k+i}) = 2, \ldots, f(v_{(c-4)k+i}) = 1$, and $f(v_{(c-1)k+i}) = 2$. Then $f(u_{2k+i}) = 2, f(u_{3k+i}) = 1, f(u_{7k+i}) = 2, f(u_{8k+i}) = 1, \ldots, f(u_{(c-3)k+i}) = 2$, and $f(u_{(c-2)k+i}) = 1$.

Proof. (1) and (2). Consider an arbitrary 2RD coloring of G of weight $4hk$. We first claim that the number of colored vertices on C_i is exactly $2h$. Recall that each vertex of C_i has exactly one neighbor outside C_i.

Case (a). Let us first suppose that the number of colors of the vertices from C_i is at most $2h-1$. Then there exists a path $(x_1, x_2, \ldots, x_5) \in C_i$ such that at most one of the vertices x_j is colored and it is colored with exactly one color, by Observation 1(i). Since the coloring is proper, the only possibility is that x_3 is colored and all of the other vertices on P_5 have no color. Let $(y_1, y_2, \ldots, y_5)$ be the corresponding set of vertices $y_j \in V_i$, where $x_j y_j \in E(G)$. It follows from Observation 1(ii) and from the definition of a proper coloring that y_3 is the only vertex without color and each of the remaining vertices y_j must be colored. Now let us consider the set of vertices $(z_1, z_2, \ldots, z_5) \in V_{i+1}$, where $y_j z_j \in E(G)$. By definition of a proper coloring, one of the neighboring vertices of a vertex y_3 that lies in V_{i+1} or V_{i-1} is colored. W.l.o.g. assume that z_3 is colored. Moreover, by Observation 1(ii) the vertices $z_j, j \neq 3$ should have no color. Finally, let us consider the path $(w_1, w_2, \ldots, w_5) \in C_{i+1}$, where $z_j w_j \in E(G)$. Since z_3 is colored, by Observation 1(ii) w_3 should have no color. Additionally, by Observation 1(iii) there is exactly one of the vertices w_2 and w_4 that is colored; w.l.o.g. let w_2 be such a vertex. However, then w_4 is not colored and has two non-colored neighbors (w_3 and z_4), which is a contradiction to Observation 1(iii).

Case (b). Suppose now that the number of colors of the vertices from C_i is at least $2h+1$. Then there exists a path $(x'_1, x'_2, \ldots, x'_5) \in C_i$ such that at least three among the vertices x'_j are colored (note that, by Observation 1(i), each of them is colored with exactly one color). By Observation 1(ii), the only possibility is that x'_1, x'_3, and x'_5 are colored (and the vertices x'_2 and x'_4 are non-colored). We will use the same idea as before to define the sets $(y'_1, y'_2, \ldots, y'_5)$, $(z'_1, z'_2, \ldots, z'_5)$, and $(w'_1, w'_2, \ldots, w'_5)$, namely choose $y'_j \in V_i$ such that $x'_j y'_j \in E(G)$. Similarly, $z'_j \in V_{i+1}$ and $w'_j \in C_{i+1}$ are determined by $y'_j z'_j \in E(G)$ and $z'_j w'_j \in E(G)$. Then, by Observation 1(ii) and (iii), all the vertices $y'_1, y'_2, \ldots, y'_5$ are not colored. Since x'_3 is colored with exactly one color, by Observation 1(ii) there is exactly one neighbor of y'_3 that lies in V_{i+1} or V_{i-1} and is colored with exactly one color. W.l.o.g. assume that z'_3 is such a vertex. Then by Observation 1(ii) w'_3 is not colored. Since the vertices x'_2 and y'_2 are not colored, z'_2 must be colored, by Observation 1(iii). By the same reasoning, because x'_4 and y'_4 are not colored, z'_4 has to be colored. Therefore, by Observation 1(ii), the vertices w'_2 and w'_4 are non-colored vertices. However, then w'_3 is not colored and has two non-colored neighbors (and the one of the neighbors is colored with exactly one color), which is a contradiction to the fact that the coloring is proper.

At last, we can suppose that there are exactly $2h$ colored vertices of C_i and the number of colors of one color class exceeds the number of the colors of the other color class. Then, there exists a path P on 5 vertices in C_i where at least two of the vertices are col-

ored with the same color. By the same arguments as before, we conclude that those two vertices are the only colored vertices on P. Since the coloring is proper and since Observation 1(i) holds, there are no three consecutive vertices on P that are not colored. Moreover by Observation 1(ii) those two colored vertices are not adjacent. The first possibility then is that those two vertices are at distance 2. However, this is a contradiction to fact (iii) from Observation 1. The second possibility is that those two vertices are at distance 3. We denote those two colored vertices by c_1 and c_2. Moreover, we denote by $(c_1, c_2, \ldots, c_{2h}), c_i \in C_i, \forall i \in [2h]$ the sequence of the colored vertices of C_i, traversing along cycle C_i. Moreover, let $(d(c_1, c_2), d(c_2, c_3), \ldots, d(c_{2h-1}, c_{2h}), d(c_{2h}, c_1))$ be the sequence of the distances between the colored vertices of C_i. One can easily check that $(d(c_1, c_2), d(c_2, c_3), \ldots, d(c_{2h-1}, c_{2h}), d(c_{2h}, c_1)) = (3, 2, 3, 2, \ldots, 2, 3, 2)$ (note that the distance $d(c_{2h}, c_1) = 2$, since $c = 5h$). Now, by applying Observation 1(iii) every pair of colored vertices, that are at distance 2, should be colored with different colors. Therefore there are exactly h vertices of one color and h vertices of the other color, which is a contradiction to the assumption that the number of colors of one color class exceeds the number of the colors of the other color class.
By the previous arguments, we conclude that C_i must include exactly h vertices of color 1 and exactly h vertices of color 2 (which confirms statement (1)).
Case (c). The coloring of any C_i discussed above implies that, in each V_i, at least h vertices must be colored with color 1 and at least h vertices must be colored with color 2. On the other hand, observe that the union of V_i induces the outer cycle of length $5hk$ and that, by the same arguments as above, we know that the total number of colored vertices on the outer cycle must be $2hk$ and both colors must be used evenly. Hence, in each V_i, exactly h vertices must be colored with color 1 and exactly h vertices must be colored with color 2.

(3) Moreover, the second part of statement (2) determines the two possible specific orders of colors for vertices in C_i. Again, we denote by $(c_1, c_2, \ldots, c_{2h}), c_i \in C_i, \forall i \in [2h]$ the sequence of the colored vertices of C_i, traversing along the cycle C_i. First, assume that w.l.o.g. $f(c_1) = 1$, $f(c_2) = 1$, $f(c_3) = 2$, $f(c_4) = 2$, $\ldots$, $f(c_{2h-1}) = 2$ and $f(c_{2h}) = 2$ (note that such a coloring is proper only for $c \equiv 0 \mod 10$, and, therefore, $f(c_{2h-1}) = 2$ and $f(c_{2h}) = 2$ holds). In such case it is an exercise to see that the colors of the vertices from the sets V_j and $C_j, \forall j \in \{1, 2, \ldots, i-2, i, i+1, \ldots, k\}$ are uniquely determined by the colors of C_i and that for an arbitrarily chosen $h = 2m, m \in \mathbb{Z}^+$ the obtained coloring is not proper. Secondly, let us assume that w.l.o.g. $f(v_{k+i}) = 1, f(v_{4k+i}) = 2, f(v_{6k+i}) = 1, f(v_{9k+i}) = 2, \ldots, f(v_{(c-4)k+i}) = 1$, and $f(v_{(c-1)k+i}) = 2$. Then v_{5k+i} is colored with v_{4k+i} and v_{6k+i}, while v_{2k+i} must have a neighbor of color 2 outside C_i and v_{3k+i} must have a neighbor of color 1 outside C_i. By the same reasoning one can observe that $f(u_{7k+i}) = 2, f(u_{8k+i}) = 1, \ldots, f(u_{(c-3)k+i}) = 2$ and $f(u_{(c-2)k+i}) = 1$ (which confirms statement (3)). □

Lemma 3. *If* $k \not\equiv 2, 8 \mod 10$ *and* $c = 5h$ *then* $\gamma_{r2}(P(ck, k)) > 4hk$.

Proof. Beginning with any column, there are exactly two possible extensions to potentially infinite pattern with the property that the minimal possible number of colors is used. As observed in Proposition 9 and Lemma 2, $k \equiv 2, 8 \mod 10$ are the only possibilities in which the extensions match when the two ends of pattern are identified. □

Lemma 4. *Let* $c \geq 3$. *If* $c \not\equiv 0 \mod 5$ *then* $\gamma_{r2}(P(ck, k)) > \frac{4}{5}ck$.

Proof. (Sketch.) The next general argument for $c > 5$ heavily relies on the proof of Lemma 2. Recall that it was shown in the proof of Lemma 2 that the sufficient condition for achieving the exact lower bound on the 2-rainbow domination number of Petersen graph $P(ck, k)$ is that, for every path P of length 5 that lies in $C_i = (v_1, v_2, \ldots, v_m)$, exactly two of the vertices, lying on P, are colored. In general it was proved that the two colored vertices on P may be colored with the same color and at distance 3, or they are at distance 2 and colored differently. W.l.o.g. let v_1 be colored. Then we have two possibilities, that v_3 is colored or v_4 is colored. By applying properties from Observation 1 one can observe that the set of colored vertices of C_i is now in both cases uniquely determined. Moreover, one can see that, for every $c \not\equiv 0 \mod 5$, the path $(v_{m-1}, v_m, v_1, v_2, v_3)$ has only one or exactly three colored vertices, which is a contradiction and, therefore, the lower bound can not be achieved.

Cases $c = 4$ and $c = 3$ easily follow from the facts that, when $\gamma_{r2}(P(n,k)) = \frac{4}{5}n$, each vertex is colored by at most one color and the set of colored vertices must be independent, see Observation 1.

Let $c = 3$ and assume $\gamma_{r2}(P(n,k)) = \frac{4}{5}n$. Consider an inner cycle C and observe that exactly one vertex of C must be colored. If this were not true, then at least one uncolored vertex on the cycle would have two uncolored vertices; thus, the neighbor on the outer cycle must provide two colors. However, then, to fulfill the demands of vertices of C, at least $2 = c - 1$ vertices must be colored in the neighborhood $N(C)$. As this holds for all inner cycles, the total number of colored vertices is at least $kc = n > \frac{4}{5}n$. Contradiction.

If $c = 4$, then exactly two vertices are colored on each inner cycle C by the same argument as in the previous case. Consequently, exactly one half of the vertices on inner cycles are colored and, consequently, exactly one half of vertices on the outer cycle receive one color from their inner neighbor. To fulfill the remaining total demand $\frac{3}{4}n$ on the outer cycle, at least $\frac{3}{8}n$ vertices must be colored (with one color each) on the outer cycle. Hence, the number of colored vertices is at least $\frac{3}{8}n + \frac{1}{2}n > \frac{4}{5}n$. □

8. Upper Bounds by Constructions

In this section, we give constructions of 2RDF that provide upper bounds for γ_{r2}. The constructions are based on the basic construction defined by Formulas (10) and (11). In each case, the basic construction is locally altered in order to get a proper 2RD function.

Proposition 11. *If $k \equiv 5, 9 \mod 10$ and $c \equiv 0 \mod 5$, then $\gamma_{r2}(P(ck,k)) \leq \frac{4}{5}ck + \frac{c}{5}$.*

Proof. Let $c = 5h$. We need to provide constructions of colorings, showing that if $k \equiv 5, 9 \bmod 10$ then $\gamma_{r2}(P(ck,k)) \leq \frac{4}{5}ck + \frac{c}{5}$. By Theorem 2, $\gamma_{r2}(P(5k,k)) = 4k + 1$. Since $P(5hk,k)$ is h-lift of $P(5k,k)$, by Lemma 1 we have $\gamma_{r2}(P(5hk,k)) \leq h\gamma_{r2}(P(5k,k)) = h(4k+1) = \frac{4}{5}ck + \frac{c}{5}$, as needed. □

Proposition 12. *Assume $k > 3$. If $k \equiv 1, 6, 7 \mod 10$ and $c \equiv 0 \mod 5$, then $\gamma_{r2}(P(ck,k)) \leq \frac{4}{5}ck + \frac{2c}{5}$.*

Proof. It is sufficient to provide constructions of colorings showing that if $k \equiv 1, 6, 7 \bmod 10$ then $\gamma_{r2}(P(ck,k)) \leq \frac{4}{5}ck + \frac{2c}{5}$. Since $P(5hk,k)$ is h-lift of $P(5k,k)$, by applying Lemma 1 we get $\gamma_{r2}(P(5hk,k)) \leq h\gamma_{r2}(P(5k,k))$ and, since Theorem 2 holds, the proof is completed. □

Proposition 13. *Assume $k > 3$. If $k \equiv 0, 3, 4 \bmod 10$ and $c \equiv 0 \mod 5$, then $\gamma_{r2}(P(ck,k)) \leq \frac{4}{5}ck + \frac{3c}{5}$.*

Proof. The proof is analogous to the proofs of Propositions 11 and 12. Details are omitted. □

This concludes the analysis of the cases with $c \equiv 0 \mod 5$. We postpone elaboration of other cases to the next section and use general ideas to obtain a general upper bound. This general bound will be later improved in special cases by more detailed arguments.

Proposition 14.

$$\gamma_{r2}(P(ck,k)) \leq \left\lceil \frac{4}{5}(c+\alpha)(k+\beta) \right\rceil + 1$$

where

$$\alpha = \begin{cases} 0 & c \equiv 0 \bmod 5 \\ \frac{1}{2} & c \equiv 4 \bmod 5 \\ 1 & c \equiv 1,2,3 \bmod 5 \end{cases} \quad \text{and} \quad \beta = \begin{cases} 0 & k \equiv 2,8 \bmod 10 \\ \frac{1}{2} & k \equiv 1,7 \bmod 10 \\ 1 & k \equiv 0,3,4,5,6,9 \bmod 10 \end{cases}$$

Proof. First recall that the case $c \equiv 0 \bmod 5$ follows from previous considerations.

For arbitrary $c \geq 3$ and k, define $\tilde{c} \geq c$ with $\tilde{c} \equiv 0 \bmod 5$ and $\tilde{k} \geq k$ with $\tilde{k} \equiv 8 \bmod 10$. Use the 2RDF for $P(\tilde{c}\tilde{k}, \tilde{k})$ defined by Formulas (10) and (11).

Delete the last $\tilde{c} - c$ consecutive rows or, more precisely, successively apply Construction 1 $(\tilde{c} - c)$ times. By Proposition 4, the construction defines $P(c\tilde{k}, \tilde{k})$.

Next, delete the last $(\tilde{k} - k)$ consecutive columns or, more precisely, successively apply Construction 2 $(\tilde{k} - k)$ times. By Proposition 5, such construction results in the graph G that is isomorphic to $P(ck, k)$.

Now we are going to define a 2RDF of $G \simeq P(ck, k)$. For clarity, we will be using vertex labels inherited from $P(\tilde{c}\tilde{k}, \tilde{k})$. Define the function g as follows. First, let $g(v) = \mathbb{F}(v)$ for all vertices v of G. Then, in some cases add some colors to some vertices by the rules below. (the rules differ depending on the number of rows (columns) that were deleted):

(a) If $\tilde{c} \neq c$ then alter $g(v_i) = g(v_i) \cup \mathbb{F}(v_{(\tilde{c}-1)\tilde{k}+i})$, for $i = 0, 1, 2, \ldots, k-1$.
If $\tilde{c} - c \geq 2$, i.e., if at least two rows were deleted, then also let $g(v_{(c-1)\tilde{k}+i}) = g(v_{(c-1)\tilde{k}+i}) \cup \mathbb{F}(v_{c\tilde{k}+i})$, for $i = 0, 1, 2, \ldots, k-1$.

(b) If $\tilde{k} \neq k$, let $g(u_0) = g(u_0) \cup \mathbb{F}(u_{\tilde{c}\tilde{k}-1})$ and $g(u_{i\tilde{k}}) = g(u_{i\tilde{k}-1}) \cup \mathbb{F}(u_{ik}u_{i\tilde{k}-1})$ for $i = 1, 2, \ldots, c-1$.
If $\tilde{k} - k \geq 2$, then also let $g(u_{i\tilde{k}+k-1}) = g(u_{i\tilde{k}+k-1}) \cup \mathbb{F}(u_{i\tilde{k}+\tilde{k}})$ for $i = 0, 1, 2, \ldots, c-1$.

(c) Finally, if $\tilde{k} = k$, and $\tilde{c} \neq c$, set $g(u_{c\tilde{k}-1}) = g(u_{c\tilde{k}-1}) \cup \mathbb{F}(u_{c\tilde{k}})$.

We claim that the rules assure that g is a 2RDF of G. The argument is as follows. If $\tilde{c} \neq c$ then, on each of the inner cycles, $\tilde{c} - c$ vertices were deleted and rule (a) assures that g is a 2RDF on the inner cycles. Application of Construction 1 also cuts the outer cycle. As $g(u_0) = \mathbb{F}(u_0) = 1$, only vertex $u_{c\tilde{k}-1}$ may need the color of deleted vertex $u_{c\tilde{k}}$, that is provided by rule (c). If $\tilde{k} \neq k$, then $\tilde{k} - k$ inner cycles were deleted and the outer cycle is broken in each row. The missing colors on the outer cycle are provided by rule (b) and, clearly, there are no missing colors on the inner cycles.

We claim that $w(g) \leq \left\lceil \frac{4}{5}(c+\alpha)(k+\beta) \right\rceil + 1$. First, observe that, in any column, there are exactly $\frac{2}{5}\tilde{c}$ colors and, in the rows, there are roughly $\frac{2}{5}\tilde{k}$ or, more precisely, in i consecutive rows, there are at most $\lceil \frac{2i}{5}\tilde{k} \rceil$ colors. First, observe that, after deletion of $\tilde{k} - k$ columns, the total weight of $\mathbb{F}$ on the undeleted vertices is $k \times \frac{4}{5}\tilde{c}$. By rule (b), a column receives additional $\frac{2}{5}\tilde{k}$ colors if $\tilde{k} - k = 1$ and $2 \times \frac{2}{5}\tilde{k}$ are given to two columns when $\tilde{k} - k \geq 2$. Observe that the weight of rows does not increase.

Now consider the effect of rule (a). If $\tilde{c} - c = 1$, then one row is deleted and rule (a) adds at most $\frac{2}{5}\tilde{c}$ colors to the total weight of g. If $\tilde{c} - c \geq$, then rule (a) adds at most $2 \times \frac{2}{5}\tilde{c}$ colors to the total weight of g.

Finally, note that rule (c) adds at most 1 to the total weight of g.

Hence, the total number of colors used is at most $\left\lceil \frac{4}{5}(c+\alpha)(k+\beta) \right\rceil + 1$, as claimed. □

In the continuation, we consider in more detail the cases where $c \not\equiv 0 \bmod 5$. First, we provide a general construction that gives a general upper bound which will be improved by explicit constructions in the next section.

9. Better Upper Bounds by Constructions

We start with $k \equiv 1 \bmod 10$ and consider different c.

Proposition 15. *Let $k \equiv 1 \bmod 10$ and $c \equiv 4 \bmod 5$. Then $\gamma_{r2}(P(ck,k)) \leq \frac{4ck}{5} + \frac{2c}{5} + \frac{2k-1}{5}$.*

Proof. Observe that the Petersen graph $P(ck,k)$ is obtained by application of Construction 1 starting from $P(Ck,k)$ where $C = c+1$. The 2RDF of $P(Ck,k)$ therefore can be constructed from the 2RD function of $P(55,11)$, see Table 2. Note that by deletion of all the vertices of V_C and U_C the vertex $u_{(C-1)k-1}$ has already been colored and there remain exactly $\frac{4(c+1)k}{5} + \frac{2(c+1)}{5} - (5 + 4(h-1))$ colored vertices, where $h = \frac{k-1}{10}$. Since some of the non-colored vertices of V_1 now obtain the colors of the corresponding vertices from V_C, the number of the colored vertices of Petersen graph $P(ck,k)$ is therefore equal

$$\frac{4(c+1)k}{5} + \frac{2(c+1)}{5} - (5 + 4(h-1)) = \frac{4ck}{5} + \frac{2c}{5} + \frac{2k-1}{5},$$

which completes the proof. □

Proposition 16. *Let* $k \equiv 1 \mod 10$ *and* $c \equiv 3 \mod 5$. *Then* $\gamma_{r2}(P(ck,k)) \leq \frac{4ck}{5} + \frac{2c}{5} + \frac{2(2k-1)}{5}$.

Proof. The Petersen graph $P(ck,k)$ now can be obtained by two applications of Construction 1 starting from $P(Ck,k)$ where $C = c+2$ and the 2RDF of $P(Ck,k)$ is again formed from the 2RD function of $P(55,11)$. By deletion of all the vertices of V_C, V_{C-1}, U_C, and U_{C-1} the vertex $u_{(C-2)k-1}$ has already been colored and there are exactly $\frac{4(c+2)k}{5} + \frac{2(c+2)}{5} - 2(5 + 4(h-1))$ colored vertices in $P(ck,k)$, $h = \frac{k-1}{10}$. Note that some of the non-colored vertices of V_1 now obtain the colors of the corresponding vertices from V_C and some of the non-colored vertices of V_{C-2} obtain the colors of the corresponding vertices from V_{C-1}. The number of the colored vertices of Petersen graph $P(ck,k)$ is therefore equal

$$\frac{4(c+2)k}{5} + \frac{2(c+2)}{5} - 2(5 + 4(h-1)) = \frac{4ck}{5} + \frac{2c}{5} + \frac{2(2k-1)}{5}.$$

□

Proposition 17. *Let* $k \equiv 1 \mod 10$ *and* $c \equiv 2 \mod 5$. *Then* $\gamma_{r2}(P(ck,k)) \leq \frac{4ck}{5} + \frac{2c}{5} + \frac{3k-5}{5}$.

Proof. A desired 2RDF for Petersen graph $P(ck,k)$ can be obtained by three applications of Construction 1 to $P(Ck,k)$ where $C = c+3$ and adapting 2RDF of $P(55,11)$. By deletion of all the vertices of V_C, V_{C-1}, V_{C-2} U_C, U_{C-1}, and U_{C-2} the vertex $u_{(C-3)k-1}$ has already been colored. Now, there are exactly $\frac{4(c+3)k}{5} + \frac{2(c+3)}{5} - 4(5 + 4(h-1)) - (2 + 2(h-1))$ colored vertices in $P(ck,k)$, where $h = \frac{k-1}{10}$ (since some of the non-colored vertices of V_1 now obtain the colors of the corresponding vertices from V_C and some of the non-colored vertices of V_{C-3} obtain the colors of the corresponding vertices from V_{C-2}). Therefore, the number of the colored vertices of Petersen graph $P(ck,k)$ is equal

$$\frac{4(c+3)k}{5} + \frac{2(c+3)}{5} - 4(5 + 4(h-1)) - (2 + 2(h-1)) = \frac{4ck}{5} + \frac{2c}{5} + \frac{3k-5}{5}.$$

□

Proposition 18. *Let* $k \equiv 1 \mod 10$ *and* $c \equiv 1 \mod 5$. *Then* $\gamma_{r2}(P(ck,k)) \leq \frac{4ck}{5} + \frac{2c}{5} + \frac{3k-4}{5}$.

Proof. Analogously as in proofs of Propositions 15–17, 2RD functions for Petersen graph $P(ck,k)$ can be obtained by four applications of Construction 1 to $P(Ck,k)$ where $C = c+4$ and adapting 2RDF of $P(55,11)$. By deletion of all the vertices of $V_C, V_{C-1}, V_{C-2}, V_{C-3}, U_C, U_{C-1}, U_{C-2}$, and U_{C-3} the vertex $u_{(C-4)k-1}$ has already been dominated by the vertices $v_{(C-4)k-1}$ and $u_{(C-4)k-2}$. In this case, there remain exactly $\frac{4(c+4)k}{5} + \frac{2(c+4)}{5} - 5(5 + 4(h -$

$1)) - (4+4(h-1)) - (2+2(h-1))$ colored vertices, $h = \frac{k-1}{10}$. Therefore, the number of the colored vertices of Petersen graph $P(ck, k)$ is equal

$$\frac{4(c+4)k}{5} + \frac{2(c+4)}{5} - 5(5+4(h-1)) - (4+4(h-1)) - (2+2(h-1)) = \frac{4ck}{5} + \frac{2c}{5} + \frac{3k-4}{5}.$$

□

This concludes the analysis of cases with $k \equiv 1 \mod 10$ and all c. It can be summarized as follows.

Proposition 19. *Let* $k \equiv 1 \mod 10$. *Then*

$$\frac{4}{5}ck \leq \gamma_{r2}(P(ck,k)) \leq \begin{cases} \frac{4}{5}ck + \frac{2c}{5} + \frac{3k-4}{5}, & c \equiv 1 \mod 5 \\ \frac{4}{5}ck + \frac{2c}{5} + \frac{3k-5}{5}, & c \equiv 2 \mod 5 \\ \frac{4}{5}ck + \frac{2c}{5} + \frac{2(2k-1)}{5}, & c \equiv 3 \mod 5 \\ \frac{4}{5}ck + \frac{2c}{5} + \frac{2k-1}{5}, & c \equiv 4 \mod 5 \end{cases} \quad (12)$$

Proof. The bounds $\frac{4}{5}ck \leq \gamma_{r2}(P(ck,k)) \leq \frac{4}{5}ck + \frac{2c}{5} + \frac{2k-1}{5}$ hold by Proposition 15, the bounds $\frac{4}{5}ck \leq \gamma_{r2}(P(ck,k)) \leq \frac{4}{5}ck + \frac{2c}{5} + \frac{2(2k-1)}{5}$ hold by Proposition 16, the bounds $\frac{4}{5}ck \leq \gamma_{r2}(P(ck,k)) \leq \frac{4}{5}ck + \frac{2c}{5} + \frac{3k-5}{5}$ hold by Proposition 17, and the bounds $\frac{4}{5}ck \leq \gamma_{r2}(P(ck,k)) \leq \frac{4}{5}ck + \frac{2c}{5} + \frac{3k-4}{5}$ hold by Proposition 18. □

It remains to consider $k \not\equiv 1 \mod 10$. The upper bounds can be found by analogous constructions, followed by suitable update of the colorings. Therefore, the details of this part are omitted.

10. Conclusions and Ideas for Future Work

We have answered some questions regarding rainbow domination of generalized Petersen graphs $P(ck, k)$. In particular, we have:

- Characterized all $P(ck, k)$ for which the lower bound $\frac{4}{5}n$ is obtained. In these cases, the 2-rainbow domination number is known.
- For all other cases, we provide lower and upper bounds with small gaps. In these cases, it remains open to find exact values, at least for some subfamilies.

We wish to note that our construction giving the exact values of $\gamma_{r2}(P(ck,k))$ assigns at most one color to each vertex. Thus, the corresponding 2RDF are also *singleton rainbow domination functions* as defined in [26]. We claim that other constructions here that are used for the upper bounds can be easily adapted to provide singleton 2RDFs. The idea is very simple: in each case, the union of generic values was used for simplicity. In fact, it was needed only for vertices that were originally not colored.

Hence, our main result, Theorem 4, holds for the singleton rainbow domination number as well. Two questions may therefore be asked:

- Can the bounds of Theorem 4 be improved for the rainbow domination number?
- Can the bounds of Theorem 4 be improved for the singleton rainbow domination number?

Generalized Petersen graphs $P(ck, k)$ are 3-regular. Hence the singleton rainbow domination makes sense only for $r = 1, 2,$ and 3. Therefore, this paper somehow closes the series of results on singleton rainbow domination of generalized Petersen graphs, because analogous analysis has already been carried out for "normal" domination ($r = 1$) [34] and for 3-rainbow domination [35]. The study of r-rainbow domination of generalized Petersen graphs $P(n, k)$ and in particular $P(ck, k)$ remains an interesting avenue of future research.

Author Contributions: Conceptualization, J.Ž.; Writing—original draft, S.B.; Writing—review & editing, S.B., D.R.P. and J.Ž.; Visualization, D.R.P. All authors have read and agreed to the published version of the manuscript.

Funding: Supported in part by ARRS, the Research Agency of Slovenia, grants J2-2512, P2-0248, and P1-0297.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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