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Fractional Langevin Coupled System with Stieltjes Integral Conditions

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Abstract: This article outlines the necessary requirements for a coupled system of fractional order boundary value involving the Caputo fractional derivative, including its existence, uniqueness, and various forms of Ulam stability. We demonstrate the existence and uniqueness of the proposed coupled system by using the cone-type Leray–Schauder result and the Banach contraction principle. Based on the traditional method of nonlinear functional analysis, the stability is examined. An example is used to provide a clear illustration of our main results.

Keywords: caputo derivative; implicit coupled system; existence theory; green function; Ulam stability

MSC: 34A08; 34B15; 34B27; 34D20

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1. Introduction

Fractional differential equations ($\mathcal{FDE}s$, for short) have recently received a lot of attention from scholars working on a variety of problems. Numerous fields of engineering and science, including signal and image processing, polymer rheology, complex media electrodynamics, chemistry, aerodynamics, economics, biophysics, control theory, physics, blood flow phenomena, etc., use the aforementioned equations to mathematically model processes and systems. For details, see [1–7]. As a result, scholars are paying close attention to the topic of the aforementioned equations. The theory of boundary value problems for nonlinear $\mathcal{FDE}s$, however, is still in its development and needs more research in many areas. For detailed studies, see [8–11].

By utilizing a variety of fixed point techniques, several scholars have come to some surprising conclusions on the availability of solutions to boundary value problems for $\mathcal{FDE}s$. The study of coupled systems of differential equations is extremely important since these kinds of systems commonly arise in real situations with powerful applications. For more details, see [12–15].

Approximate solutions are frequently used in disciplines such as numerical analysis, optimization theory, and nonlinear analysis; thus, it is essential to understand how closely these solutions resemble the real solutions of the relevant system or systems. Other approaches might be used for this, but the Ulam–Hyers stability ($\mathcal{UHS}$, for short) procedure is simple and straightforward. In 1940, Ulam [16] first brought up the aforementioned stability, to which Hyers brilliantly countered in 1941 [17]. In 1978, Rassias developed the mathematical $\mathcal{UHS}$ approach by taking variables into account. Following this, some researchers developed the concepts of functionals, differentials, and integrals, and subsequently, some researchers developed the concept of $\mathcal{FDE}s$, see, for example [18–23].

Recent years have seen an increase in the study of the existence, uniqueness, and various forms of the $\mathcal{UHS}$ of solutions to nonlinear implicit $\mathcal{FDE}s$ with the Caputo fractional derivative. For more information, see [24–29]. We provide some related results in the list that follows:

- Ali et al. [30] identified four alternative types of Ulam stability, as well as the existence and uniqueness of a solution, for the implicit $\mathcal{FDE}$ given by

$$\begin{cases} \mathcal{D}^{\alpha}\mathsf{v}(\zeta) = \phi(\zeta, \mathsf{v}(\zeta), \mathcal{D}^{\alpha}\mathsf{v}(\zeta)), \quad \zeta \in \mathsf{J} = [0, T], \quad T > 0, \quad \alpha \in (1, 2], \\ \mathcal{D}^{\alpha-2}\mathsf{v}(0^+) = \gamma_1 \mathcal{D}^{\alpha-2}\mathsf{v}(T^-), \\ \mathcal{D}^{\alpha-1}\mathsf{v}(0^+) = \delta_1 \mathcal{D}^{\alpha-1}\mathsf{v}(T^-), \end{cases}$$

where $\delta_1, \gamma_1 \neq 1$. For $\mathcal{FDE}s$ that are coupled, researchers are currently concentrating their efforts on analysing various types of Ulam stability. Details can be found in [31–34].
- Ali et al. [35] examined several forms of stability for the implicit coupled system described below in the Ulam sense as well as existence theory:

$$\begin{cases} \mathcal{D}^{\alpha}\mathsf{v}(\zeta) = \phi_1(\zeta, \mathsf{v}(\zeta), \mathcal{D}^{\alpha}\mathsf{v}(\zeta)), \quad \zeta \in \mathsf{J} = [0, T], \quad T > 0, \quad \alpha \in (1, 2], \\ \mathcal{D}^{\beta}\mathsf{w}(\zeta) = \phi_2(\zeta, \mathsf{w}(\zeta), \mathcal{D}^{\beta}\mathsf{w}(\zeta)), \quad \zeta \in \mathsf{J} = [0, T], \quad T > 0, \quad \beta \in (1, 2], \\ \mathcal{D}^{\alpha-2}\mathsf{v}(0^+) = \gamma_1 \mathcal{D}^{\alpha-2}\mathsf{v}(T^-), \quad \mathcal{D}^{\alpha-1}\mathsf{v}(0^+) = \delta_1 \mathcal{D}^{\alpha-1}\mathsf{v}(T^-), \\ \mathcal{D}^{\beta-2}\mathsf{w}(0^+) = \gamma_2 \mathcal{D}^{\beta-2}\mathsf{w}(T^-), \quad \mathcal{D}^{\beta-1}\mathsf{w}(0^+) = \delta_2 \mathcal{D}^{\beta-1}\mathsf{w}(T^-). \end{cases}$$

- Zhang et al. [36] investigated the following mixed-derivative nonlinear implicit Langevin equation with Stieltjes integral conditions:

$$\begin{cases} {}^{c}\mathcal{D}^{\alpha}_{0,\zeta}(\mathcal{D} + \lambda)\mathsf{v}(\zeta) = \phi(\zeta, \mathsf{v}(\zeta)), \quad \zeta \in (0, 1], \\ \mathsf{v}(0) = 0, \quad {}^{c}\mathcal{D}^{\gamma_0}_{0,\zeta}\mathsf{v}(1) = \sum_{i=1}^{p} \int_0^1 {}^{c}\mathcal{D}^{\gamma_i}_{0,\zeta}\mathsf{v}(\zeta) d\mu_i(\zeta). \end{cases}$$

Inspired by the previous discussions, in this article, we explore the existence, uniqueness, and different types of Ulam stability for the following coupled system:

$$\begin{cases} \begin{cases} {}^{c}\mathcal{D}^{\alpha}_{0,\zeta}(\mathcal{D} + \lambda_1)\mathsf{v}(\zeta) = \phi_1(\zeta, \mathsf{v}(\zeta)), \quad \zeta \in (0, 1], \\ \mathsf{v}(0) = 0, \quad {}^{c}\mathcal{D}^{\gamma_0}_{0,\zeta}\mathsf{v}(1) = \sum_{i=1}^{p} \int_0^1 {}^{c}\mathcal{D}^{\gamma_i}_{0,\zeta}\mathsf{v}(\zeta) d\mu_i(\zeta), \end{cases} \\ \begin{cases} {}^{c}\mathcal{D}^{\beta}_{0,\zeta}(\mathcal{D} + \lambda_2)\mathsf{w}(\zeta) = \phi_2(\zeta, \mathsf{w}(\zeta)), \quad \zeta \in (0, 1], \\ \mathsf{w}(0) = 0, \quad {}^{c}\mathcal{D}^{\delta_0}_{0,\zeta}\mathsf{w}(1) = \sum_{i=1}^{q} \int_0^1 {}^{c}\mathcal{D}^{\delta_i}_{0,\zeta}\mathsf{w}(\zeta) d\mu_i(\zeta), \end{cases} \end{cases} \tag{1}$$

where $\mathsf{J} = (0, 1]$, ${}^{c}\mathcal{D}^{(\cdot)}_{0,\zeta}$ represents the classical Caputo derivative of order $(\cdot)$, with the lower bound zero, $0 < (\cdot) < 1$, $\lambda_1, \lambda_2 \in \mathbb{R}\backslash\{0\}$, $p, q \in \mathbb{N}$, $\gamma_i \in \mathbb{R}$ for all $i = 0, \ldots, p$, $0 \leqslant \gamma_1 < \gamma_2 < \cdots < \gamma_p < \beta$, $\gamma_0 \in [0, \alpha)$, $\delta_i \in \mathbb{R}$ for all $i = 0, \ldots, q$, $0 \leqslant \delta_1 < \delta_2 < \cdots < \delta_p < \alpha$, $\delta_0 \in [0, \beta)$, $\phi_1, \phi_2 : \mathsf{J} = [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous, and the integrals from the boundary condition are Riemann–Stieltjes integrals with μ_i $(i = 1, \ldots, p/q)$ functions of bounded variation.

We list the following as the important points of this paper:

1. We use pointwise Stieltjes integrals inspired by our previous paper [36] to model a coupled system for the first time in the literature.
2. In this article, we shall look into the existence, uniqueness, and several kinds of Ulam stability for the given coupled system.
3. We achieve better results by employing the Stieltjes integral conditions, even though we operated on the coupled system in the same way as in [37].

The arrangement of the paper is as follows. We provide a uniform structure for the suggested model in Section 2. For the existence and uniqueness of the solution of system (1), we employ several conditions and a few common fixed point theorems in Section 3. Ulam's stabilities are presented in Section 4. Finally, we present an example that illustrates our main results in Section 5.

2. Preliminary

Let the space $\mathsf{X} = \mathsf{C}(\mathsf{J},\mathbb{R})$ be a Banach space with the following defined norm $\|\mathsf{v}\|_{\mathsf{X}} = \max_{\zeta\in \mathsf{J}}\{|\mathsf{v}(\zeta)| : \zeta \in \mathsf{J}\}$. Similarly, the norm defined on the product space is $\|(\mathsf{v},\mathsf{w})\|_{\mathsf{X}\times\mathsf{X}} = \|\mathsf{v}\|_{\mathsf{X}} + \|\mathsf{w}\|_{\mathsf{X}}$. Obviously, $(\mathsf{X}\times\mathsf{X}, \|(\mathsf{v},\mathsf{w})\|_{\mathsf{X}\times\mathsf{X}})$ is a Banach space. Furthermore, the cone $\mathcal{C} \subset \mathsf{X}\times\mathsf{X}$ is defined as

$$\mathcal{C} = \{(\mathsf{v},\mathsf{w}) \in \mathsf{X}\times\mathsf{X} : \mathsf{v}(\zeta) \geq 0, \mathsf{w}(\zeta) \geq 0\}.$$

Consider the linear form of the first differential equation of (1) as follows:

$$\begin{cases} {}^{c}\mathcal{D}^{\alpha}_{0,\zeta}(\mathcal{D}+\lambda_1)\mathsf{v}(\zeta) = \phi_1(\zeta), \quad \zeta \in \mathsf{J}, \\ \mathsf{v}(0) = 0, \quad {}^{c}\mathcal{D}^{\gamma_0}_{0,\zeta}\mathsf{v}(1) = \sum_{i=1}^{p}\int_0^1 {}^{c}\mathcal{D}^{\gamma_i}_{0,\zeta}\mathsf{v}(\zeta)d\mu_i(\zeta). \end{cases} \tag{2}$$

We recall some definitions of fractional calculus from [3,38–40] as follows:

Definition 1. *For the function* v*, the fractional integral of order* α *from* 0 *to* ζ *is defined by*

$$I^{\alpha}_{0,\zeta}\mathsf{v}(\zeta) = \frac{1}{\Gamma(\alpha)}\int_0^{\zeta}(\zeta-s)^{\alpha-1}\mathsf{v}(s)ds, \quad \zeta > 0,\ \alpha > 0,$$

where $\Gamma(\cdot)$ *is the Gamma function.*

Definition 2. *For a function* v*, the Caputo derivative of fractional order* α *from* 0 *to* ζ *is defined as*

$${}^{c}\mathcal{D}^{\alpha}_{0,\zeta}\mathsf{v}(\zeta) = \frac{1}{\Gamma(n-\alpha)}\int_0^{\zeta}(\zeta-s)^{n-\alpha-1}\mathsf{v}^{(\mathsf{n})}(s)ds, \quad \text{where } n = [\alpha]+1.$$

Lemma 1. *The* $\mathcal{FDE}$ ${}^{c}\mathcal{D}^{\alpha}_{0,\zeta}\mathsf{v}(\zeta) = 0$ *with* $\alpha > 0$*, involving Caputo differential operator* ${}^{c}\mathcal{D}^{\alpha}_{0,\zeta}$*, has a solution in the following form:*

$$\mathsf{v}(\zeta) = c_0 + c_1\zeta + c_2\zeta^2 + \cdots + c_{m-1}\zeta^{m-1},$$

where $c_k \in \mathbb{R}, k = 0,1,\ldots,m-1$ *and* $m = [\alpha+1]$*.*

Theorem 1. *Suppose a Banach space* X *contains a cone* $\mathcal{C}$ *and* $\mathfrak{D} \subset \mathcal{C}$ *with* $0 \in \mathfrak{D}$ *being a substantially open set. Let the operator* $\mathcal{T} : \mathfrak{D} \to \mathfrak{D}$ *be completely continuous. Then, one of the following circumstances holds true:*

(i) *There is* $\mathsf{v} \in \partial\mathfrak{D}$ *and* $\lambda \in (0,1)$ *such that* $\mathsf{v} = \lambda\mathcal{T}\mathsf{v}$*;*
(ii) $\mathcal{T}$ *has a fixed point in* $\mathfrak{D}$*.*

Definition 3. *The system (1) is* $\mathcal{UHS}$*, if there is a constant* $C_{\alpha,\beta} = (C_\alpha, C_\beta) > 0$ *such that, for some* $\epsilon = (\epsilon_\alpha, \epsilon_\beta) > 0$ *and for each* $\zeta \in \mathsf{J}$ *and solution* $(\mathsf{v},\mathsf{w}) \in \mathsf{X}\times\mathsf{X}$ *of the following:*

$$\begin{cases} |{}^{c}\mathcal{D}^{\alpha}_{0,\zeta}\mathsf{v}(\zeta) - \phi_1(\zeta,\mathsf{v}(\zeta))| \leq \epsilon_\alpha, \\ |{}^{c}\mathcal{D}^{\beta}_{0,\zeta}\mathsf{w}(\zeta) - \phi_2(\zeta,\mathsf{w}(\zeta)| \leq \epsilon_\beta, \end{cases} \tag{3}$$

there is a unique solution $(\omega,\theta) \in \mathsf{X}\times\mathsf{X}$ *with*

$$|(\mathsf{v},\mathsf{w})(\zeta) - (\omega,\theta)(\zeta)| \leqslant C_{\alpha,\beta}\epsilon. \tag{4}$$

Definition 4. *The system* (1) *is the generalized* $\mathcal{UHS}$*, if there is* $\Theta_{\alpha,\beta} \in \mathsf{C}(\mathbb{R}^+, \mathbb{R}^+)$ *with* $\Theta_{\alpha,\beta}(0) = 0$*, such that, for each* $\zeta \in \mathsf{J}$ *and solution* $(\mathsf{v}, \mathsf{w}) \in \mathsf{X} \times \mathsf{X}$ *of* (3)*, there is a unique solution* $(\omega, \theta) \in \mathsf{X} \times \mathsf{X}$ *of* (1)*, which satisfies*

$$|(\mathsf{v}, \mathsf{w})(\zeta) - (\omega, \theta)(\zeta)| \leqslant \Theta_{\alpha,\beta}(\epsilon). \tag{5}$$

Definition 5. *The system* (1) *is Ulam–Hyers–Rassias stable (*$\mathcal{UHRS}$*, for short), with respect to* $\Phi_{\alpha,\beta} = (\Phi_\alpha, \Phi_\beta) \in \mathsf{X}$*, if there is a constant* $C_{\Phi_\alpha,\Phi_\beta} = (C_{\Phi_\alpha}, C_{\Phi_\beta}) > 0$ *such that, for some* $\epsilon = (\epsilon_\alpha, \epsilon_\beta) > 0$ *and for each* $\zeta \in \mathsf{J}$ *and solution* $(\mathsf{v}, \mathsf{w}) \in \mathsf{X} \times \mathsf{X}$ *of the following:*

$$\begin{cases} |{}^c\mathcal{D}^\alpha_{0,\zeta}\mathsf{v}(\zeta) - \phi_1(\zeta, \mathsf{v}(\zeta))| \leq \Phi_\alpha(\zeta)\epsilon_\alpha, \\ |{}^c\mathcal{D}^\beta_{0,\zeta}\mathsf{w}(\zeta) - \phi_2(\zeta, \mathsf{w}(\zeta)| \leq \Phi_\beta(\zeta)\epsilon_\beta, \end{cases} \tag{6}$$

there is a unique solution $(\omega, \theta) \in \mathsf{X} \times \mathsf{X}$ *with*

$$|(\mathsf{v}, \mathsf{w})(\zeta) - (\omega, \theta)(\zeta)| \leqslant C_{\Phi_\alpha,\Phi_\beta}\Phi_{\alpha,\beta}(\zeta)\epsilon. \tag{7}$$

Definition 6. *The system* (1) *is the generalized* $\mathcal{UHRS}$*, with respect to* $\Phi_{\alpha,\beta} = (\Phi_\alpha, \Phi_\beta) \in \mathsf{X}$*, if there is a constant* $C_{\Phi_\alpha,\Phi_\beta} = (C_{\Phi_\alpha}, C_{\Phi_\beta}) > 0$ *such that, for each* $\zeta \in \mathsf{J}$ *and solution* $(\mathsf{v}, \mathsf{w}) \in \mathsf{X} \times \mathsf{X}$ *of* (6)*, there is a unique solution* $(\omega, \theta) \in \mathsf{X} \times \mathsf{X}$ *of* (1) *that fulfils*

$$|(\mathsf{v}, \mathsf{w})(\zeta) - (\omega, \theta)(\zeta)| \leqslant C_{\Phi_\alpha,\Phi_\beta}\Phi_{\alpha,\beta}(\zeta). \tag{8}$$

Remark 1. *If there are* $\psi_1, \psi_2 \in \mathsf{X}$*, which depend upon* v *and* w*, satisfying the following* (i) *and* (ii)*, then we say that* $(\mathsf{v}, \mathsf{w}) \in \mathsf{X} \times \mathsf{X}$ *is a solution of* (1)*.*

(i) $\quad |\psi_1(\zeta)| \leqslant \epsilon_\alpha,\ |\psi_2(\zeta)| \leqslant \epsilon_\beta, \quad \zeta \in \mathsf{J};$

(ii)

$$\begin{cases} {}^c\mathcal{D}^\alpha_{0,\zeta}(\mathcal{D} + \lambda_1)\mathsf{v}(\zeta) = \phi_1(\zeta, \mathsf{v}(\zeta)) + \psi_1(\zeta), \quad \zeta \in \mathsf{J}, \\ {}^c\mathcal{D}^\beta_{0,\zeta}(\mathcal{D} + \lambda_2)\mathsf{w}(\zeta) = \phi_2(\zeta, \mathsf{w}(\zeta)) + \psi_2(\zeta), \quad \zeta \in \mathsf{J}. \end{cases}$$

3. Existence and Uniqueness

The criteria required to confirm the existence and uniqueness of solution to $\mathcal{FDE}$ (1) under examination will be built up in this section.

Lemma 2. *The function* $\mathsf{v} \in \mathsf{X}$ *is a solution of* (2) *if and only if*

$$\mathsf{v}(\zeta) = \int_0^1 \mathbb{G}_\alpha(\zeta, s)\phi_1(s)ds,$$

where

$$\begin{aligned} \mathbb{G}_\alpha(\zeta, s) &= \mathbb{G}_1(\zeta, s) + \mathbb{G}_2(\zeta, s), \\ \mathbb{G}_1(\zeta, s) &= \begin{cases} e^{-\lambda_1(\zeta - s)} I^\alpha_{0,\zeta}(1) + \dfrac{(1 - e^{-\lambda_1\zeta})e^{-\lambda_1(1-s)} I^{\alpha-\gamma_0}_{0,\zeta}(1)}{\Delta_\alpha}, & 0 \leqslant s \leqslant \zeta \leqslant 1, \\ \dfrac{(1 - e^{-\lambda_1\zeta})e^{-\lambda_1(1-s)} I^{\alpha-\gamma_0}_{0,\zeta}(1)}{\Delta_\alpha}, & 0 \leqslant \zeta \leqslant s \leqslant 1, \end{cases} \\ \mathbb{G}_2(\zeta, s) &= \frac{e^{-\lambda_1\zeta} - 1}{\Delta_\alpha} \sum_{i=1}^p \int_0^\zeta e^{-\lambda_1(\zeta - s)} I^{\alpha-\gamma_i}_{0,\zeta}(1) d\mu_i(\zeta) \end{aligned}$$

and

$$\Delta_\alpha = \sum_{i=1}^p \int_0^1 (-\lambda_1)^{\gamma_i} e^{-\lambda_1\zeta} d\mu_i(\zeta) - (-\lambda_1)^{\gamma_0} e^{-\lambda_1\zeta} \neq 0.$$

Proof. For the proof, see Lemma 4 in [36]. □

The solution of system (1) is therefore identical to the coupled system of integral equations provided by Lemma 2 for $\zeta \in \mathsf{J}$

$$\begin{cases} \mathsf{v}(\zeta) = \int_0^1 \mathbb{G}_\alpha(\zeta, s)\phi_1(s, \mathsf{v}(s))ds, \\ \mathsf{w}(\zeta) = \int_0^1 \mathbb{G}_\beta(\zeta, s)\phi_2(s, \mathsf{w}(s))ds, \end{cases} \tag{9}$$

where the Green function $\mathbb{G}_\beta(\zeta, s)$ is given as

$$\mathbb{G}_\beta(\zeta, s) = \mathbb{G}_3(\zeta, s) + \mathbb{G}_4(\zeta, s),$$

$$\mathbb{G}_3(\zeta, s) = \begin{cases} e^{-\lambda_2(\zeta - s)} I^{\beta}_{0,\zeta}(1) + \dfrac{(1 - e^{-\lambda_2 \zeta})e^{-\lambda_2(1-s)} I^{\beta-\delta_0}_{0,\zeta}(1)}{\Delta_\beta}, & 0 \leqslant s \leqslant \zeta \leqslant 1, \\ \dfrac{(1 - e^{-\lambda_2 \zeta})e^{-\lambda_2(1-s)} I^{\beta-\delta_0}_{0,\zeta}(1)}{\Delta_\beta}, & 0 \leqslant \zeta \leqslant s \leqslant 1, \end{cases}$$

$$\mathbb{G}_4(\zeta, s) = \frac{e^{-\lambda_2 \zeta} - 1}{\Delta_\beta} \sum_{i=1}^{q} \int_0^\zeta e^{-\lambda_2(\zeta - s)} I^{\beta-\delta_i}_{0,\zeta}(1) d\mu_i(\zeta),$$

where

$$\Delta_\beta = \sum_{i=1}^{q} \int_0^1 (-\lambda_2)^{\delta_i} e^{-\lambda_2 \zeta} d\mu_i(\zeta) - (-\lambda_2)^{\delta_0} e^{-\lambda_2 \zeta} \neq 0.$$

Lemma 3. *The Green function, $\mathbb{G}_{\alpha,\beta}(\zeta, s) = (\mathbb{G}_\alpha(\zeta, s), \mathbb{G}_\beta(\zeta, s))$ of system (1) has properties defined in the following:*

(1) $\mathbb{G}_{\alpha,\beta}(\zeta, s)$ *is continuous over* J;

(2) $\max_{\zeta \in \mathsf{J}} \int_0^1 |\mathbb{G}_\alpha(\zeta, s)| ds \leqslant \mathcal{Y}_\alpha$;

(3) $\max_{\zeta \in \mathsf{J}} \int_0^1 |\mathbb{G}_\beta(\zeta, s)| ds \leqslant \mathcal{Y}_\beta$,

where

$$\mathcal{Y}_\alpha = \frac{|1 - e^{-\lambda_1}|}{\lambda_1 \Gamma(\alpha + 1)} + \frac{|(1 - e^{-\lambda_1})^2|}{\lambda_1 \nabla_\alpha \Gamma(\alpha - \gamma_0 + 1)} + \sum_{i=1}^{p} \frac{|1 - e^{-\lambda_1}||1 - e^{-\lambda_1 \rho}|\rho^{\alpha - \gamma_i} \mathfrak{P}_i}{\lambda_1 \nabla_\alpha \Gamma(\alpha - \gamma_i + 1)},$$

$$\mathcal{Y}_\beta = \frac{|1 - e^{-\lambda_2}|}{\lambda_2 \Gamma(\beta + 1)} + \frac{|(1 - e^{-\lambda_2})^2|}{\lambda_2 \nabla_\beta \Gamma(\beta - \delta_0 + 1)} + \sum_{i=1}^{q} \frac{|1 - e^{-\lambda_2}||1 - e^{-\lambda_2 \rho}|\rho^{\beta - \delta_i} \mathfrak{P}_i}{\lambda_2 \nabla_\beta \Gamma(\beta - \delta_i + 1)},$$

$$\nabla_\alpha = \sum_{i=1}^{p} (-\lambda_1)^{\gamma_i} e^{-\lambda_1 \rho} \mathfrak{P}_i - (-\lambda_1)^{\gamma_0} e^{-\lambda_1} \text{ and}$$

$$\nabla_\beta = \sum_{i=1}^{q} (-\lambda_2)^{\delta_i} e^{-\lambda_2 \rho} \mathfrak{P}_i - (-\lambda_2)^{\delta_0} e^{-\lambda_2}.$$

Proof. It is very easy to prove **(1)**, **(2)**, and **(3)**; the reader may refer to [36]. □

If v, w, and $\zeta \in \mathsf{J}$ are solutions of system (1), then

$$\mathsf{v}(\zeta) = \int_0^\zeta e^{-\lambda_1(\zeta - s)} I^{\alpha}_{0,\zeta} \phi_1(s, \mathsf{v}(s)) ds + \frac{1 - e^{-\lambda_1 \zeta}}{\Delta_\alpha} \int_0^1 e^{-\lambda_1(1-s)} I^{\alpha-\gamma_0}_{0,\zeta} \phi_1(s, \mathsf{v}(s)) ds$$
$$- \frac{1 - e^{-\lambda_1 \zeta}}{\Delta_\alpha} \sum_{i=1}^{p} \int_0^1 \int_0^\zeta e^{-\lambda_1(\zeta - s)} I^{\alpha-\gamma_i}_{0,\zeta} \phi_1(s, \mathsf{v}(s)) ds d\mu_i(\zeta)$$

and

$$\mathsf{w}(\zeta) = \int_0^\zeta e^{-\lambda_2(\zeta-s)} I_{0,\zeta}^{\beta}\phi_2(s,\mathsf{w}(s))ds + \frac{1-e^{-\lambda_2\zeta}}{\Delta_\beta}\int_0^1 e^{-\lambda_2(1-s)} I_{0,\zeta}^{\beta-\delta_0}\phi_2(s,\mathsf{w}(s))ds$$
$$-\frac{1-e^{-\lambda_2\zeta}}{\Delta_\beta}\sum_{i=1}^{q}\int_0^1\int_0^\zeta e^{-\lambda_2(\zeta-s)} I_{0,\zeta}^{\beta-\delta_i}\phi_2(s,\mathsf{w}(s))dsd\mu_i(\zeta).$$

Our next step is to convert system (1) into a fixed point problem. Let the operator $\mathcal{T} : \mathsf{X}\times\mathsf{X}\to\mathsf{X}\times\mathsf{X}$ be defined as

$$\mathcal{T}(\mathsf{v},\mathsf{w})(\zeta) = \begin{pmatrix} \int_0^1 \mathbb{G}_\alpha(\zeta,s)\phi_1(s,\mathsf{v}(s))ds \\ \int_0^1 \mathbb{G}_\beta(\zeta,s)\phi_2(s,\mathsf{w}(s))ds \end{pmatrix} = \begin{pmatrix} \mathcal{T}_\alpha(\mathsf{v})(\zeta) \\ \mathcal{T}_\beta(\mathsf{w})(\zeta) \end{pmatrix}. \tag{10}$$

Hence, the fixed point of $\mathcal{T}$ and the solution of (1) are congruent, where

$$\mathcal{T}_\alpha(\mathsf{v})(\zeta) = \int_0^\zeta e^{-\lambda_1(\zeta-s)} I_{0,\zeta}^{\alpha}\phi_1(s,\mathsf{v}(s))ds + \frac{1-e^{-\lambda_1\zeta}}{\Delta_\alpha}\int_0^1 e^{-\lambda_1(1-s)} I_{0,\zeta}^{\alpha-\gamma_0}\phi_1(s,\mathsf{v}(s))ds$$
$$-\frac{1-e^{-\lambda_1\zeta}}{\Delta_\alpha}\sum_{i=1}^{p}\int_0^1\int_0^\zeta e^{-\lambda_1(\zeta-s)} I_{0,\zeta}^{\alpha-\gamma_i}\phi_1(s,\mathsf{v}(s))dsd\mu_i(\zeta)$$

and

$$\mathcal{T}_\beta(\mathsf{w})(\zeta) = \int_0^\zeta e^{-\lambda_2(\zeta-s)} I_{0,\zeta}^{\beta}\phi_2(s,\mathsf{w}(s))ds + \frac{1-e^{-\lambda_2\zeta}}{\Delta_\beta}\int_0^1 e^{-\lambda_2(1-s)} I_{0,\zeta}^{\beta-\delta_0}\phi_2(s,\mathsf{w}(s))ds$$
$$-\frac{1-e^{-\lambda_2\zeta}}{\Delta_\beta}\sum_{i=1}^{q}\int_0^1\int_0^\zeta e^{-\lambda_2(\zeta-s)} I_{0,\zeta}^{\beta-\delta_i}\phi_2(s,\mathsf{w}(s))dsd\mu_i(\zeta).$$

The following assumptions are imposed for further analysis:

($\mathbf{H_1}$) For $\zeta\in\mathsf{J}$ and $\mathsf{v}\in\mathsf{X}$, there are $\pi_1,\pi_2\in\mathsf{C}(\mathsf{J},\mathbb{R}^+)$, such that

$$|\phi_1(\zeta,\mathsf{v}(\zeta)| \leqslant \pi_1(\zeta) + \pi_2(\zeta)|\mathsf{v}(\zeta)|$$

with $\pi_1^* = \sup_{\zeta\in\mathsf{J}}\pi_1(\zeta)$ and $\pi_2^* = \sup_{\zeta\in\mathsf{J}}\pi_2(\zeta) < 1$. Similarly, for $\zeta\in\mathsf{J}$ and $\mathsf{w}\in\mathsf{X}$, there are $\pi_3,\pi_4\in\mathsf{C}(\mathsf{J},\mathbb{R}^+)$, such that

$$|\phi_2(\zeta,\mathsf{w}(\zeta)| \leqslant \pi_3(\zeta) + \pi_4(\zeta)|\mathsf{w}(\zeta)|$$

with $\pi_3^* = \sup_{\zeta\in\mathsf{J}}\pi_3(\zeta)$ and $\pi_4^* = \sup_{\zeta\in\mathsf{J}}\pi_4(\zeta) < 1$.

($\mathbf{H_2}$) For all $\mathsf{v},\bar{\mathsf{v}}\in\mathbb{R}$ and for each $\zeta\in\mathsf{J}$, there exists a constant $\mathbb{K}_{\phi_1} > 0$, such that

$$|\phi_1(\zeta,\mathsf{v}) - \phi_1(\zeta,\bar{\mathsf{v}})| \leqslant \mathbb{K}_{\phi_1}|\mathsf{v}-\bar{\mathsf{v}}|.$$

Similarly, for all $\mathsf{w},\bar{\mathsf{w}}\in\mathbb{R}$ and for each $\zeta\in\mathsf{J}$, there exists a constant $\mathbb{K}_{\phi_2} > 0$, such that

$$|\phi_2(\zeta,\mathsf{w}) - \phi_2(\zeta,\bar{\mathsf{w}})| \leqslant \mathbb{K}_{\phi_2}|\mathsf{w}-\bar{\mathsf{w}}|.$$

Theorem 2. *Let* ($\mathbf{H_1}$) *and* $\phi_1,\phi_2 : \mathsf{J}\times\mathbb{R}\to\mathbb{R}$ *hold. When that happens, the operator* $\mathcal{T} : \mathcal{C}\to\mathcal{C}$ *described in* (10) *is completely continuous.*

Proof. $\mathcal{T}$ is continuous for all $(\mathsf{v},\mathsf{w}) \in \mathcal{C}$ given the continuity of ϕ_1, ϕ_2, and $\mathbb{G}_{\alpha,\beta}(\zeta,s)$. Consider that $\mathcal{B} \subset \mathcal{C}$ is a bounded set. Thus, for every $\mathsf{v} \in \mathcal{B}$ we have

$$\begin{aligned} |\mathcal{T}_\alpha(\mathsf{v})(\zeta)| = \bigg| & \int_0^\zeta e^{-\lambda_1(\zeta-s)} I_{0,\zeta}^{\alpha}\phi_1(s,\mathsf{v}(s))ds + \frac{1-e^{-\lambda_1\zeta}}{\Delta_\alpha}\int_0^1 e^{-\lambda_1(1-s)} I_{0,\zeta}^{\alpha-\gamma_0}\phi_1(s,\mathsf{v}(s))ds \\ & - \frac{1-e^{-\lambda_1\zeta}}{\Delta_\alpha}\sum_{i=1}^{p}\int_0^1\int_0^\zeta e^{-\lambda_1(\zeta-s)} I_{0,\zeta}^{\alpha-\gamma_i}\phi_1(s,\mathsf{v}(s))dsd\mu_i(\zeta)\bigg|. \end{aligned} \tag{11}$$

Now, by $(\mathbf{H_1})$, we have

$$\begin{aligned} |\phi_1(\zeta,\mathsf{v}(\zeta)| &\leqslant a(\zeta) + b(\zeta)|\mathsf{v}(\zeta)| \\ &\leqslant \pi_1^* + \pi_2^*\|\mathsf{v}\|_{\mathsf{X}} = \gamma_\alpha. \end{aligned} \tag{12}$$

Now, by using **(2)** of Lemma 3 and (12) in (11), we get

$$\|\mathcal{T}_\alpha(\mathsf{v})\|_{\mathsf{X}} \leqslant \mathcal{Y}_\alpha\gamma_\alpha. \tag{13}$$

Similarly, we obtain

$$\|\mathcal{T}_\beta(\mathsf{w})\|_{\mathsf{X}} \leqslant \mathcal{Y}_\beta\gamma_\beta, \tag{14}$$

where

$$\gamma_\beta = \pi_3^* + \pi_4^*\|\mathsf{w}\|_{\mathsf{X}}.$$

Thus, from (13) and (14), we get

$$\|\mathcal{T}_\alpha(\mathsf{v})\|_{\mathsf{X}} + \|\mathcal{T}_\beta(\mathsf{w})\|_{\mathsf{X}} = \mathcal{Y}_\alpha\gamma_\alpha + \mathcal{Y}_\beta\gamma_\beta = \mathcal{M}_0,$$

which yields

$$\|\mathcal{T}(\mathsf{v},\mathsf{w})\|_{\mathsf{X}\times\mathsf{X}} \leqslant \mathcal{M}_0.$$

As a result, $\mathcal{T}$ is uniformly bounded. We can now demonstrate that the operator $\mathcal{T}$ is equi-continuous. To achieve this, suppose $\zeta_1 < \zeta_2 \in \mathsf{J}$ and $\mathsf{v} \in \mathcal{B}$, then

$$\begin{aligned} |\mathcal{T}_\alpha(\mathsf{v})(\zeta_1) - \mathcal{T}_\alpha(\mathsf{v})(\zeta_2)| = \bigg| & \int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)} I_{0,\zeta_1}^{\alpha}\phi_1(s,\mathsf{v}(s))ds + \frac{1-e^{-\lambda_1\zeta_1}}{\Delta_\alpha}\int_0^1 e^{-\lambda_1(1-s)} I_{0,\zeta_1}^{\alpha-\gamma_0}\phi_1(s,\mathsf{v}(s))ds \\ & - \frac{1-e^{-\lambda_1\zeta_1}}{\Delta_\alpha}\sum_{i=1}^{p}\int_0^1\int_0^{\zeta_1} e^{-\lambda_1(\zeta_1-s)} I_{0,\zeta_1}^{\alpha-\gamma_i}\phi_1(s,\mathsf{v}(s))dsd\mu_i(\zeta_1) \\ & - \int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)} I_{0,\zeta_2}^{\alpha}\phi_1(s,\mathsf{v}(s))ds - \frac{1-e^{-\lambda_1\zeta_2}}{\Delta_\alpha}\int_0^1 e^{-\lambda_1(1-s)} I_{0,\zeta_2}^{\alpha-\gamma_0}\phi_1(s,\mathsf{v}(s))ds \\ & + \frac{1-e^{-\lambda_1\zeta_2}}{\Delta_\alpha}\sum_{i=1}^{p}\int_0^1\int_0^{\zeta_2} e^{-\lambda_1(\zeta_2-s)} I_{0,\zeta_2}^{\alpha-\gamma_i}\phi_1(s,\mathsf{v}(s))dsd\mu_i(\zeta_2)\bigg| \\ \leqslant \gamma_\alpha \Bigg(& \frac{|1-e^{-\lambda_1(\zeta_1-\zeta_2)}|(\zeta_1-\zeta_2)^\alpha}{\lambda_1\Gamma(\alpha+1)} + \frac{|(1-e^{-\lambda_1(\zeta_1-\zeta_2)})||(1-e^{-\lambda_1})|(\zeta_1-\zeta_2)^{\alpha-\gamma_0}}{\lambda_1\nabla_\alpha\Gamma(\alpha-\gamma_0+1)} \\ & + \sum_{i=1}^{p}\frac{|(1-e^{-\lambda_1(\zeta_1-\zeta_2)})||(1-e^{-\lambda_1\rho})|\rho^{\alpha-\gamma_i}\mathfrak{P}_i}{\lambda_1\nabla_\alpha\Gamma(\alpha-\gamma_i+1)}\Bigg). \end{aligned} \tag{15}$$

In a same way, we get

$$|\mathcal{T}_\beta(\mathsf{w})(\zeta_1)-\mathcal{T}_\beta(\mathsf{w})(\zeta_2)| \leqslant \gamma_\beta\left(\frac{|1-e^{-\lambda_2(\zeta_1-\zeta_2)}|(\zeta_1-\zeta_2)^\beta}{\lambda_2\Gamma(\beta+1)}+\frac{|(1-e^{-\lambda_2(\zeta_1-\zeta_2)})||(1-e^{-\lambda_2})|(\zeta_1-\zeta_2)^{\beta-\delta_0}}{\lambda_2\nabla_\beta\Gamma(\beta-\delta_0+1)}\right.$$
$$\left.+\sum_{i=1}^{q}\frac{|(1-e^{-\lambda_2(\zeta_1-\zeta_2)})||(1-e^{-\lambda_2\rho})|\rho^{\beta-\delta_i}\mathfrak{P}_i}{\lambda_2\nabla_\beta\Gamma(\beta-\delta_i+1)}\right). \tag{16}$$

In the event that $\zeta_1 \to \zeta_2$, the right hand sides of (15) and (16) move in the direction of zero. The Arzela–Ascoli theorem is then applied to prove that $\mathcal{T}$ is uniformly equi-continuous. It is also quite easy to show that $\mathcal{T}(\mathcal{B}) \subset \mathcal{B}$. As a result, $\mathcal{T}$ is completely continuous. □

Theorem 3. *Consider hypothesis* $(\mathbf{H_2})$ *and*

$$\mathcal{Y}_\alpha\mathbb{K}_{\phi_1}+\mathcal{Y}_\beta\mathbb{K}_{\phi_2}<1. \tag{17}$$

Then, problem (1) *has a unique solution.*

Proof. Let $\mathsf{v},\bar{\mathsf{v}} \in \mathcal{C}$, we have

$$|\mathcal{T}_\alpha(\mathsf{v})(\zeta)-\mathcal{T}_\alpha(\bar{\mathsf{v}})(\zeta)| \leqslant \int_0^\zeta e^{-\lambda_1(\zeta-s)}I_{0,\zeta}^{\alpha}|\phi_1(s,\mathsf{v}(s))-\phi_1(s,\bar{\mathsf{v}}(s))|ds$$
$$+\frac{|1-e^{-\lambda_1\zeta}|}{|\Delta_\alpha|}\int_0^1 e^{-\lambda_1(1-s)}I_{0,\zeta}^{\alpha-\gamma_0}|\phi_1(s,\mathsf{v}(s))-\phi_1(s,\bar{\mathsf{v}}(s))|ds$$
$$+\frac{|1-e^{-\lambda_1\zeta}|}{|\Delta_\alpha|}\sum_{i=1}^{p}\int_0^1\int_0^\zeta e^{-\lambda_1(\zeta-s)}I_{0,\zeta}^{\alpha-\gamma_i}|\phi_1(s,\mathsf{v}(s))-\phi_1(s,\bar{\mathsf{v}}(s))|dsd\mu_i(\zeta). \tag{18}$$

Now, by $(\mathbf{H_2})$, we have

$$|\phi_1(\zeta,\mathsf{v}(\zeta))-\phi_1(\zeta,\bar{\mathsf{v}}(\zeta))| \leqslant \mathbb{K}_{\phi_1}|\mathsf{v}(\zeta)-\bar{\mathsf{v}}(\zeta)|. \tag{19}$$

Substituting (19) in (18) and taking a maximum over J, we obtain

$$\|\mathcal{T}_\alpha(\mathsf{v})-\mathcal{T}_\alpha(\bar{\mathsf{v}})\|_{\mathsf{X}} \leqslant \mathcal{Y}_\alpha\mathbb{K}_{\phi_1}\|\mathsf{v}-\bar{\mathsf{v}}\|_{\mathsf{X}}. \tag{20}$$

In the same way, we can obtain

$$\|\mathcal{T}_\beta(\mathsf{w})-\mathcal{T}_\beta(\bar{\mathsf{w}})\|_{\mathsf{X}} \leqslant \mathcal{Y}_\beta\mathbb{K}_{\phi_2}\|\mathsf{w}-\bar{\mathsf{w}}\|_{\mathsf{X}}. \tag{21}$$

Thus, from (20) and (21), we get

$$\|\mathcal{T}(\mathsf{v},\mathsf{w})-\mathcal{T}(\bar{\mathsf{v}},\bar{\mathsf{w}})\|_{\mathsf{X}\times\mathsf{X}} \leqslant \left(\mathcal{Y}_\alpha\mathbb{K}_{\phi_1}+\mathcal{Y}_\beta\mathbb{K}_{\phi_2}\right)\|(\mathsf{v},\mathsf{w})-(\bar{\mathsf{v}},\bar{\mathsf{w}})\|_{\mathsf{X}\times\mathsf{X}}.$$

Therefore, $\mathcal{T}$ is a contraction mapping. The Banach contraction principle states that $\mathcal{T}$ has a fixed point as a result. This leads to a unique solution for system (1). □

Theorem 4. *Consider the continuity of the functions* ϕ_1 *and* ϕ_2 *and presume* $(\mathbf{H_1})$ *and*

$$(\mathbf{H_3}): \quad \mathfrak{A}_1=\int_0^1\mathbb{G}_\alpha(1,s)\pi_1(s)ds,\ \mathfrak{B}_1=\int_0^1\mathbb{G}_\alpha(1,s)\pi_2(s)ds<1,$$
$$\mathfrak{A}_2=\int_0^1\mathbb{G}_\beta(1,s)\pi_3(s)ds,\ \mathfrak{B}_2=\int_0^1\mathbb{G}_\beta(1,s)\pi_4(s)ds<1$$

hold. Then, system (1) *has at least one solution.*

Proof. Let a set $\mathfrak{D}$ be defined as

$$\mathfrak{D} = \{(\mathsf{v}, \mathsf{w}) \in \mathsf{X} \times \mathsf{X} : ||(\mathsf{v}, \mathsf{w})\|_{\mathsf{X}\times\mathsf{X}} < \mathcal{R}_{\mathfrak{D}}\},$$

where $\max\{\frac{2\mathfrak{A}_1}{1-2\mathfrak{B}_1}, \frac{2\mathfrak{A}_2}{1-2\mathfrak{B}_2}\} < \mathcal{R}_{\mathfrak{D}}$. Additionally, the operator $\mathcal{T} : \overline{\mathfrak{D}} \to \mathcal{C}$ defined in (10) is a completely continuous operator. Considering $(\mathsf{v}, \mathsf{w}) \in \mathfrak{D}$, then by definition of $\mathfrak{D}$, we have $\|(\mathsf{v}, \mathsf{w})\|_{\mathsf{X}\times\mathsf{X}} < \mathcal{R}_{\mathfrak{D}}$;

$$\begin{aligned}
\|\mathcal{T}_\alpha(\mathsf{v})\|_{\mathsf{X}} &\leqslant \max_{\zeta\in\mathsf{J}} \int_0^1 |\mathbb{G}_\alpha(\zeta, s)||\phi_1(s, \mathsf{v}(s))|ds \\
&\leqslant \max_{\zeta\in\mathsf{J}} \int_0^1 |\mathbb{G}_\alpha(\zeta, s)|\pi_1(s)ds + \max_{\zeta\in\mathsf{J}} \int_0^1 |\mathbb{G}_\alpha(\zeta, s)|\pi_2(s)|\mathsf{v}(s)|ds \\
&\leqslant \int_0^1 \mathbb{G}_\alpha(1, s)\pi_1(s)ds + \mathcal{R}_{\mathfrak{D}} \int_0^1 \mathbb{G}_\alpha(1, s)\pi_2(s)ds \\
&= \mathfrak{A}_1 + \mathcal{R}_{\mathfrak{D}}\mathfrak{B}_1 \leqslant \frac{\mathcal{R}_{\mathfrak{D}}}{2}.
\end{aligned}$$

Additionally,

$$\|\mathcal{T}_\beta(\mathsf{v})\|_{\mathsf{X}} \leqslant \frac{\mathcal{R}_{\mathfrak{D}}}{2}.$$

Thus, we can write

$$\|\mathcal{T}(\mathsf{v}, \mathsf{w})\|_{\mathsf{X}\times\mathsf{X}} \leqslant \mathcal{R}_{\mathfrak{D}}.$$

Therefore, $\mathcal{T}(\mathsf{v}, \mathsf{w}) \in \overline{\mathfrak{D}}$. Thus, $\mathcal{T} : \overline{\mathfrak{D}} \to \overline{\mathfrak{D}}$ is completely continuous in view of Theorem 2. We now investigate an eigenvalue problem that is defined as

$$(\mathsf{v}, \mathsf{w}) = \lambda\mathcal{T}(\mathsf{v}, \mathsf{w}), \quad 0 < \lambda < 1. \tag{22}$$

In context of the (v, w) solution to (22), we therefore obtain

$$\begin{aligned}
\|\mathsf{v}\|_{\mathsf{X}} &= \|\lambda\mathcal{T}_\alpha(\mathsf{v})\| \\
&\leqslant \max_{\zeta\in\mathsf{J}} \int_0^1 |\mathbb{G}_\alpha(\zeta, s)||\phi_1(s, \mathsf{v}(s))|ds \\
&\leqslant \max_{\zeta\in\mathsf{J}} \int_0^1 |\mathbb{G}_\alpha(\zeta, s)|\pi_1(s)ds + \max_{\zeta\in\mathsf{J}} \int_0^1 |\mathbb{G}_\alpha(\zeta, s)|\pi_2(s)|\mathsf{v}(s)|ds \\
&\leqslant \int_0^1 \mathbb{G}_\alpha(1, s)\pi_1(s)ds + \mathcal{R}_{\mathfrak{D}} \int_0^1 \mathbb{G}_\alpha(1, s)\pi_2(s)ds \\
&= \mathfrak{A}_1 + \mathcal{R}_{\mathfrak{D}}\mathfrak{B}_1 \leqslant \frac{\mathcal{R}_{\mathfrak{D}}}{2}.
\end{aligned}$$

Similarly,

$$\|\mathsf{w}\|_{\mathsf{X}} \leqslant \frac{\mathcal{R}_{\mathfrak{D}}}{2}.$$

Thus,

$$\|(\mathsf{v}, \mathsf{w})\|_{\mathsf{X}\times\mathsf{X}} \leqslant \mathcal{R}_{\mathfrak{D}}. \tag{23}$$

The result of Equation (23) is $(\mathsf{v}, \mathsf{w}) \notin \partial\mathfrak{D}$. Therefore, Theorem 1 states that $\mathcal{T}$ has at least one fixed point that is located in $\overline{\mathfrak{D}}$. This demonstrates that system (1) has at least one solution. □

4. Ulam Stability Analysis

In this section, we will look at stability outcomes for problem (1) in the sense of Ulam stability.

Lemma 4. *Consider* $(\mathsf{v},\mathsf{w}) \in \mathsf{X} \times \mathsf{X}$ *as the solution of* (3)*. Then, for* $\zeta \in \mathsf{J}$*, we have:*

$$\begin{cases} |\mathsf{v}(\zeta) - \mathsf{y}(\zeta)| \leqslant \mathcal{Y}_\alpha \epsilon_\alpha, \\ |\mathsf{w}(\zeta) - \mathsf{z}(\zeta)| \leqslant \mathcal{Y}_\beta \epsilon_\beta. \end{cases} \tag{24}$$

Proof. In view of (ii) in Remark 1 and for $\zeta \in \mathsf{J}$, we have

$$\begin{cases} \begin{cases} {}^c\mathcal{D}^{\alpha}_{0,\zeta}(\mathcal{D}+\lambda_1)\mathsf{v}(\zeta) = \phi_1(\zeta,\mathsf{v}(\zeta)) + \psi_1(\zeta), \quad \zeta \in \mathsf{J}, \\ \mathsf{v}(0) = 0, \quad {}^c\mathcal{D}^{\gamma_0}_{0,\zeta}\mathsf{v}(1) = \sum_{i=1}^{p} \int_0^1 {}^c\mathcal{D}^{\gamma_i}_{0,\zeta}\mathsf{v}(\zeta) d\mu_i(\zeta), \end{cases} \\ \begin{cases} {}^c\mathcal{D}^{\beta}_{0,\zeta}(\mathcal{D}+\lambda_2)\mathsf{w}(\zeta) = \phi_2(\zeta,\mathsf{w}(\zeta)) + \psi_2(\zeta), \quad \zeta \in \mathsf{J}, \\ \mathsf{w}(0) = 0, \quad {}^c\mathcal{D}^{\delta_0}_{0,\zeta}\mathsf{w}(1) = \sum_{i=1}^{q} \int_0^1 {}^c\mathcal{D}^{\delta_i}_{0,\zeta}\mathsf{w}(\zeta) d\mu_i(\zeta), \end{cases} \end{cases} \tag{25}$$

and thus the solution of (25) will be in the following form

$$\begin{cases} \mathsf{v}(\zeta) = \int_0^1 \mathbb{G}_\alpha(\zeta,s)\phi_1(s,\mathsf{v}(s))ds + \int_0^1 \mathbb{G}_\alpha(\zeta,s)\psi_1(s)ds, \\ \mathsf{w}(\zeta) = \int_0^1 \mathbb{G}_\beta(\zeta,s)\phi_2(s,\mathsf{w}(s))ds + \int_0^1 \mathbb{G}_\beta(\zeta,s)\psi_2(s)ds. \end{cases} \tag{26}$$

For the first equation in (26), we have

$$\begin{aligned} \mathsf{v}(\zeta) = {}& \int_0^\zeta e^{-\lambda_1(\zeta-s)} I^{\alpha}_{0,\zeta}\phi_1(s,\mathsf{v}(s))ds + \frac{1-e^{-\lambda_1\zeta}}{\Delta_\alpha} \int_0^1 e^{-\lambda_1(1-s)} I^{\alpha-\gamma_0}_{0,\zeta}\phi_1(s,\mathsf{v}(s))ds \\ & - \frac{1-e^{-\lambda_1\zeta}}{\Delta_\alpha} \sum_{i=1}^{p} \int_0^1 \int_0^\zeta e^{-\lambda_1(\zeta-s)} I^{\alpha-\gamma_i}_{0,\zeta}\phi_1(s,\mathsf{v}(s))dsd\mu_i(\zeta) \\ & + \int_0^\zeta e^{-\lambda_1(\zeta-s)} I^{\alpha}_{0,\zeta}\psi_1(s)ds + \frac{1-e^{-\lambda_1\zeta}}{\Delta_\alpha} \int_0^1 e^{-\lambda_1(1-s)} I^{\alpha-\gamma_0}_{0,\zeta}\psi_1(s)ds \\ & - \frac{1-e^{-\lambda_1\zeta}}{\Delta_\alpha} \sum_{i=1}^{p} \int_0^1 \int_0^\zeta e^{-\lambda_1(\zeta-s)} I^{\alpha-\gamma_i}_{0,\zeta}\psi_1(s)dsd\mu_i(\zeta). \end{aligned} \tag{27}$$

For simplicity, let us denote the sum of terms free of ψ_1 by $\mathsf{y}(\zeta)$, then we have

$$\begin{aligned} \mathsf{y}(\zeta) = {}& \int_0^\zeta e^{-\lambda_1(\zeta-s)} I^{\alpha}_{0,\zeta}\phi_1(s,\mathsf{v}(s))ds + \frac{1-e^{-\lambda_1\zeta}}{\Delta_\alpha} \int_0^1 e^{-\lambda_1(1-s)} I^{\alpha-\gamma_0}_{0,\zeta}\phi_1(s,\mathsf{v}(s))ds \\ & - \frac{1-e^{-\lambda_1\zeta}}{\Delta_\alpha} \sum_{i=1}^{p} \int_0^1 \int_0^\zeta e^{-\lambda_1(\zeta-s)} I^{\alpha-\gamma_i}_{0,\zeta}\phi_1(s,\mathsf{v}(s))dsd\mu_i(\zeta). \end{aligned}$$

Thus, from above, we have

$$\begin{aligned} |\mathsf{v}(\zeta) - \mathsf{y}(\zeta)| \leqslant {}& \int_0^\zeta e^{-\lambda_1(\zeta-s)} I^{\alpha}_{0,\zeta}|\psi_1(s)|ds + \frac{|1-e^{-\lambda_1\zeta}|}{|\Delta_\alpha|} \int_0^1 e^{-\lambda_1(1-s)} I^{\alpha-\gamma_0}_{0,\zeta}|\psi_1(s)|ds \\ & + \frac{|1-e^{-\lambda_1\zeta}|}{|\Delta_\alpha|} \sum_{i=1}^{p} \int_0^1 \int_0^\zeta e^{-\lambda_1(\zeta-s)} I^{\alpha-\gamma_i}_{0,\zeta}|\psi_1(s)|dsd\mu_i(\zeta). \end{aligned}$$

Using **(2)** of Lemma 3 and (i) of Remark 1, we get

$$|\mathsf{v}(\zeta) - \mathsf{y}(\zeta)| \leqslant \mathcal{Y}_\alpha \epsilon_\alpha.$$

Similarly, for the second equation from (26), we have

$$|\mathsf{w}(\zeta) - \mathsf{z}(\zeta)| \leqslant \mathcal{Y}_\beta \epsilon_\beta.$$

As desired. □

Theorem 5. *Under hypothesis* $(\mathbf{H_2})$ *and if*

$$\mathcal{Y}_\alpha \mathbb{K}_{\phi_1} < 1,\ \mathcal{Y}_\beta \mathbb{K}_{\phi_2} < 1 \tag{28}$$

holds, then system (1) *is* $\mathcal{UHS}$.

Proof. Consider $(\mathsf{v}, \mathsf{w}) \in \mathsf{X} \times \mathsf{X}$ as the solution of (3) and $(\omega, \theta) \in \mathsf{X} \times \mathsf{X}$ as the unique solution to the system given by

$$\begin{cases} \begin{cases} {}^c\mathcal{D}^{\alpha}_{0,\zeta}(\mathcal{D} + \lambda_1)\omega(\zeta) = \phi_1(\zeta, \omega(\zeta)), \\ \omega(0) = 0, \quad {}^c\mathcal{D}^{\gamma_0}_{0,\zeta}\omega(1) = \sum_{i=1}^{p} \int_0^1 {}^c\mathcal{D}^{\gamma_i}_{0,\zeta}\omega(\zeta) d\mu_i(\zeta), \end{cases} \\ \begin{cases} {}^c\mathcal{D}^{\beta}_{0,\zeta}(\mathcal{D} + \lambda_2)\theta(\zeta) = \phi_2(\zeta, \theta(\zeta)), \\ \theta(0) = 0, \quad {}^c\mathcal{D}^{\delta_0}_{0,\zeta}\theta(1) = \sum_{i=1}^{q} \int_0^1 {}^c\mathcal{D}^{\delta_i}_{0,\zeta}\theta(\zeta) d\mu_i(\zeta); \end{cases} \end{cases} \tag{29}$$

then, for $\zeta \in \mathsf{J}$, the solution of (29) is

$$\begin{cases} \omega(\zeta) = \int_0^1 \mathbb{G}_\alpha(\zeta, s)\phi_1(s, \omega(s)) ds, \\ \theta(\zeta) = \int_0^1 \mathbb{G}_\beta(\zeta, s)\phi_2(s, \theta(s)) ds. \end{cases}$$

Consider

$$|\mathsf{v}(\zeta) - \omega(\zeta)| \leqslant |\mathsf{v}(\zeta) - \mathsf{y}(\zeta)| + |\mathsf{y}(\zeta) - \omega(\zeta)|. \tag{30}$$

By using Lemma 4 in (30), we have

$$\begin{aligned} |\mathsf{v}(\zeta) - \omega(\zeta)| \leqslant & \mathcal{Y}_\alpha \epsilon_\alpha + \int_0^\zeta e^{-\lambda_1(\zeta - s)} I^{\alpha}_{0,\zeta} |\phi_1(s, \mathsf{v}(s)) - \phi_1(s, \omega(s))| ds \\ & + \frac{|1 - e^{-\lambda_1 \zeta}|}{|\Delta_\alpha|} \int_0^1 e^{-\lambda_1(1-s)} I^{\alpha - \gamma_0}_{0,\zeta} |\phi_1(s, \mathsf{v}(s)) - \phi_1(s, \omega(s))| ds \\ & + \frac{|1 - e^{-\lambda_1 \zeta}|}{|\Delta_\alpha|} \sum_{i=1}^{p} \int_0^1 \int_0^\zeta e^{-\lambda_1(\zeta - s)} I^{\alpha - \gamma_i}_{0,\zeta} |\phi_1(s, \mathsf{v}(s)) - \phi_1(s, \omega(s))| ds d\mu_i(\zeta). \end{aligned} \tag{31}$$

Now, by $(\mathbf{H_2})$, we have

$$|\phi_1(\zeta, \mathsf{v}(\zeta)) - \phi_1(\zeta, \omega(\zeta))| \leqslant \mathbb{K}_{\phi_1} |\mathsf{v}(\zeta) - \omega(\zeta)|. \tag{32}$$

Using **(2)** of Lemma 3 and (32) in (31), we have

$$\begin{aligned} \|\mathsf{v} - \omega\|_{\mathsf{X}} &\leqslant \mathcal{Y}_\alpha \epsilon_\alpha + \mathcal{Y}_\alpha \mathbb{K}_{\phi_1} \|\mathsf{v} - \omega\|_{\mathsf{X}} \\ &\leqslant \frac{\mathcal{Y}_\alpha \epsilon_\alpha}{1 - \mathcal{Y}_\alpha \mathbb{K}_{\phi_1}}. \end{aligned}$$

Similarly, we have

$$\|\mathsf{w} - \theta\|_{\mathsf{X}} \leqslant \frac{\mathcal{Y}_\beta \epsilon_\beta}{1 - \mathcal{Y}_\beta \mathbb{K}_{\phi_2}}.$$

Thus, now we have

$$\|\mathsf{v}-\omega\|_{\mathsf{X}}+\|\mathsf{w}-\theta\|_{\mathsf{X}}\leqslant\frac{\mathcal{Y}_{\alpha}\epsilon_{\alpha}}{1-\mathcal{Y}_{\alpha}\mathbb{K}_{\phi_1}}+\frac{\mathcal{Y}_{\beta}\epsilon_{\beta}}{1-\mathcal{Y}_{\beta}\mathbb{K}_{\phi_2}}. \tag{33}$$

Now, by taking $max\{\epsilon_{\alpha},\epsilon_{\beta}\}=\epsilon$, then we can write above equation as

$$\|(\mathsf{v},\mathsf{w})-(\omega,\theta)\|_{\mathsf{X}\times\mathsf{X}}\leqslant C_{\alpha,\beta}\epsilon, \tag{34}$$

where

$$C_{\alpha,\beta}=\left[\frac{\mathcal{Y}_{\alpha}}{1-\mathcal{Y}_{\alpha}\mathbb{K}_{\phi_1}}+\frac{\mathcal{Y}_{\beta}}{1-\mathcal{Y}_{\beta}\mathbb{K}_{\phi_2}}\right].$$

Thus, problem (1) is $\mathcal{UHS}$. □

Remark 2. *By setting $C_{\alpha,\beta}(\epsilon)=C_{\alpha,\beta}\epsilon$, $C_{\alpha,\beta}(0)=0$ in (34) yields that the problem (1) is generalized $\mathcal{UHS}$.*

($\mathbf{H_4}$) Suppose $\Phi_{\alpha},\Phi_{\beta}\in\mathsf{X}$ are increasing functions. Then, there are $\Lambda_{\Phi_{\alpha}},\Lambda_{\Phi_{\beta}}>0$, such that, for each $\zeta\in\mathsf{J}$, the given inequalities

$$I^{\alpha}_{0,\zeta}\Phi_{\alpha}(\zeta)\leqslant\Lambda_{\Phi_{\alpha}}\Phi_{\alpha}(\zeta)$$

and

$$I^{\beta}_{0,\zeta}\Phi_{\beta}(\zeta)\leqslant\Lambda_{\Phi_{\beta}}\Phi_{\beta}(\zeta)$$

hold.

Remark 3. *According to supposition ($\mathbf{H_4}$) and (28) and by using Definitions 5 and 6, one can repeat the process of Lemma 4 and Theorem 5, and thus system (1) will be $\mathcal{UHRS}$ and generalized $\mathcal{UHRS}$.*

5. Example

We provide an example in this section to illustrate the major points.

Example 1. *Suppose the $\mathcal{FDE}$*

$$\begin{cases}\begin{cases}{}^{c}\mathcal{D}^{\frac{3}{4}}_{0,\zeta}(\mathcal{D}-1)\mathsf{v}(\zeta)=\dfrac{2+|v(\zeta)|}{12e^{\zeta+1}(1+|v(\zeta)|)}, & \zeta\in\mathsf{J},\\ \mathsf{v}(0)=0,\quad {}^{c}\mathcal{D}^{\gamma_0}_{0,\zeta}\mathsf{v}(1)=\displaystyle\sum_{i=1}^{2}\int_0^1{}^{c}\mathcal{D}^{\gamma_i}_{0,\zeta}\mathsf{v}(\zeta)d\mu_i(\zeta),\end{cases}\\ \begin{cases}{}^{c}\mathcal{D}^{\frac{1}{4}}_{0,\zeta}\left(\mathcal{D}-\dfrac{3}{2}\right)\mathsf{w}(\zeta)=\dfrac{1}{90}(\zeta\cos(\mathsf{w}(\zeta))-\mathsf{w}(\zeta)\sin(\zeta)), & \zeta\in\mathsf{J},\\ \mathsf{w}(0)=0,\quad {}^{c}\mathcal{D}^{\delta_0}_{0,\zeta}\mathsf{w}(1)=\displaystyle\sum_{i=1}^{3}\int_0^1{}^{c}\mathcal{D}^{\delta_i}_{0,\zeta}\mathsf{w}(\zeta)d\mu_i(\zeta),\end{cases}\end{cases} \tag{35}$$

where $\alpha=\frac{3}{4}$, $\beta=\frac{1}{4}$, $\lambda_1=-1$, $\lambda_2=-\frac{3}{2}$, $p=2$, $q=3$, $\rho=1$, $\mathfrak{P}_1=10$, $\mathfrak{P}_2=20$, $\mathfrak{P}_3=30$, $\gamma_0=\frac{1}{4}$, $\gamma_1=\frac{1}{2}$, $\gamma_2=\frac{7}{10}$, $\delta_0=\frac{1}{16}$, $\delta_1=\frac{1}{8}$, $\delta_2=\frac{3}{16}$, and $\delta_3=\frac{5}{16}$. Furthermore, we can easily find $\mathbb{K}_{\phi_1}=\frac{1}{12e}$ and $\mathbb{K}_{\phi_2}=\frac{1}{90}$. Therefore,

$$\mathcal{Y}_{\alpha}\mathbb{K}_{\phi_1}+\mathcal{Y}_{\beta}\mathbb{K}_{\phi_2}\approx-0.142154<1.$$

Thus, system (1) has a unique solution. Additionally, the requirement (28) is also satisfied. System (1) is therefore $\mathcal{UHS}$, generalized $\mathcal{UHS}$, $\mathcal{UHRS}$, and generalized $\mathcal{UHRS}$.

6. Conclusions

For the given coupled system, we have established the necessary conditions for existence, uniqueness, and other sorts of stability of the solutions of system (1) in the sense of Ulam stability. Banach and Leray–Schauder's cone-type fixed point theory has been used to achieve the necessary results. Additionally, we have added criteria to the offered solution for system (1), which are suitable for various sorts of Ulam stability. To strengthen the main theoretical conclusion, we included an example.

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References

1. Hilfer, R. *Applications of Fractional Calculus in Physics*; World Scientific: Singapore, 2000.
2. Lakshmikantham, V.; Leela, S.; Devi, J.V. *Theory of Fractional Dynamic Systems*; Cambridge Scientific Publishers: Cambridge, UK, 2009.
3. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equation*; Elsevier Science B.V.: Amsterdam, The Netherlands, 2006; Volume 204.
4. Liu, F.; Burrage, K. Novel techniques in parameter estimation for fractional dynamical models arising from biological systems. *Comput. Math. Appl.* **2011**, *62*, 822–833. [CrossRef]
5. Meral, F.C.; Royston, T.J.; Magin, R. Fractional calculus in viscoelasticity: An experimental study. *Commun. Nonlinear Sci. Numer. Simul.* **2010**, *15*, 939–945. [CrossRef]
6. Oldham, K.B. Fractional differential equations in electrochemistry. *Adv. Eng. Softw.* **2010**, *41*, 9–12. [CrossRef]
7. Podlubny, I. *Fractional Differential Equations*; Academic Press: San Diego, CA, USA, 1999.
8. Ahmad, B.; Nieto, J.J. Existence of solutions for nonlocal boundary value problems of higher-order nonlinear fractional differential equations. *Abstr. Appl. Anal.* **2009**, *2009*, 494720. [CrossRef]
9. Benchohra, M.; Hamani, S.; Ntouyas, S.K. Boundary value problems for differential equations with fractional order and nonlocal conditions. *Nonlinear Anal.* **2009**, *71*, 2391–2396. [CrossRef]
10. Shah, K.; Khalil, H.; Khan, R.A. Investigation of positive solution to a coupled system of impulsive boundary value problems for nonlinear fractional order differential equations. *Chaos Solitons Fractals* **2015**, *77*, 240–246. [CrossRef]
11. Shah, K.; Khan, R.A. Multiple positive solutions to a coupled systems of nonlinear fractional differential equations. *SpringerPlus* **2016**, *5*, 1116. [CrossRef]
12. Ahmad, B.; Nieto, J.J. Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions. *Comput. Math. Appl.* **2009**, *58*, 1838–1843. [CrossRef]
13. Bai, C.; Fang, J. The existence of a positive solution for a singular coupled system of nonlinear fractional differential equations. *Appl. Math. Comput.* **2004**, *150*, 611–621. [CrossRef]
14. Chen, Y.; An, H.L. Numerical solutions of coupled Burgers equations with time and space fractional derivatives. *Appl. Math. Comput.* **2008**, *200*, 87–95. [CrossRef]
15. Daftardar-Gejji, V. Positive solutions of a system of non-autonomous fractional differential equations. *J. Math. Anal. Appl.* **2005**, *302*, 56–64. [CrossRef]
16. Ulam, S.M. *A Collection of Mathematical Problems*; Interscience: New York, NY, USA, 1960.
17. Hyers, D.H. On the stability of the linear functional equation. *Proc. Natl. Acad. Sci. USA* **1941**, *27*, 222–224. [CrossRef] [PubMed]
18. Alam, M.; Shah, D. Hyers—Ulam stability of coupled implicit fractional integro-differential equations with Riemann—Liouville derivatives. *Chaos Solitons Fractals* **2021**, *150*, 111122. [CrossRef]
19. Alam, M.; Zada, A. Implementation of q-calculus on q-integro-differential equation involving anti-periodic boundary conditions with three criteria. *Chaos Solitons Fractals* **2022**, *154*, 111625. [CrossRef]
20. Luo, D.; Alam, M.; Zada, A.; Riaz, U.; Luo, Z. Existence and stability of implicit fractional differential equations with Stieltjes boundary conditions having Hadamard derivatives. *Complexity* **2021**, *2021*, 8824935. [CrossRef]
21. Shah, S.O.; Zada, A. Existence, uniqueness and stability of the solution to mixed integral dynamic systems with instantaneous and noninstantaneous impulses on time scales. *Appl. Math. Comput.* **2019**, *359*, 202–213. [CrossRef]

22. Zada, A.; Ali, W.; Park, C. Ulam's type stability of higher order nonlinear delay differential equations via integral inequality of Grönwall—Bellman—Bihari's type. *Appl. Math. Comput.* **2019**, *350*, 60–65.
23. Zada, A.; Shaleena, S.; Li, T. Stability analysis of higher-order nonlinear differential equations in β—Normed spaces. *Math. Methods Appl. Sci.* **2019**, *42*, 1151–1166. [CrossRef]
24. Alam, M.; Zada, A.; Popa, I.; Kheiryan, A.; Rezapour, S.; Kaabar, M.K.A. A fractional differential equation with multi-point strip boundary condition involving the Caputo fractional derivative and its Hyers—Ulam Stability. *Bound. Value Probl.* **2021**, *2021*, 73. [CrossRef]
25. Alam, M.; Zada, A.; Riaz, U. On a coupled impulsive fractional integrodifferential system with Hadamard derivatives. *Qual. Theory Dyn. Syst.* **2021**, *21*, 8. [CrossRef]
26. Wang, X.; Alam, M.; Zada, A. On coupled impulsive fractional integro-differential equations with Riemann—Liouville derivatives. *AIMS Math.* **2020**, *6*, 1561–1595. [CrossRef]
27. Wang, J.; Zada, A.; Waheed, H. Stability analysis of a coupled system of nonlinear implicit fractional anti-periodic boundary value problem. *Math. Methods Appl. Sci.* **2019**, *42*, 6706–6732. [CrossRef]
28. Zada, A.; Alam, M.; Khalid, K.H.; Iqbal, R.; Popa, I. Analysis of q-fractional implicit differential equation with nonlocal Riemann—Liouville and Erdélyi-Kober q-fractional integral conditions. *Qual. Theory Dyn. Syst.* **2022**, *21*, 1–39. [CrossRef]
29. Zada, A.; Alam, M.; Riaz, U. Analysis of q-fractional implicit boundary value problems having Stieltjes integral conditions. *Math. Methods Appl. Sci.* **2020**, *44*, 4381–4413. [CrossRef]
30. Ali, Z.; Zada, A.; Shah, K. Ulam stability solutions for the solutions of nonlinear implicit fractional order differential equations. *Hacet. J. Math. Stat.* **2019**, *48*, 1092–1109. [CrossRef]
31. Ali, A.; Samet, B.; Shah, K.; Khan, R.A. Existence and stability of solution to a toppled systems of differential equations of non-integer order. *Bound. Value Probl.* **2017**, *1*, 1–13. [CrossRef]
32. Khan, A.; Shah, K.; Li, Y.; Khan, T.S. Ulam type stability for a coupled systems of boundary value problems of nonlinear fractional differential equations. *J. Funct. Spaces* **2017**, *2017*, 3046013. [CrossRef]
33. Shah, K.; Tunç, C. Existence theory and stability analysis to a system of boundary value problem. *J. Taibah Univ. Sci.* **2017**, *11*, 1330–1342. [CrossRef]
34. Shah, K.; Wang, J.; Khalil, H. Existence and numerical solutions of a coupled system of integral BVP for fractional differential equations. *Adv. Differ. Equ.* **2018**, *2018*, 149. [CrossRef]
35. Ali, Z.; Zada, A.; Shah, K. On Ulam's stability for a coupled systems of nonlinear implicit fractional differential equations. *Bull. Malays. Math. Sci. Soc.* **2019**, *42*, 2681–2699. [CrossRef]
36. Zhang, B.; Majeed, R.; Alam, M. On Fractional Langevin Equations with Stieltjes Integral Conditions. *Mathematics* **2022**, *10*, 3877. [CrossRef]
37. Ali, Z.; Zada, A.; Shah, K. Ulam stability to a toppled systems of nonlinear implicit fractional order boundary value problem. *Bound. Value Probl.* **2018**, *2018*, 175. [CrossRef]
38. Ahmad, B.; Nieto, J.J. Riemann—Liouville fractional differential equations with fractional boundary conditions. *Fixed Point Theory* **2012**, *13*, 329–336.
39. Gafiychuk, V.; Datsko, B.; Meleshko, V. Analysis of the solutions of coupled nonlinear fractional reaction-diffusion equations. *Chaos Solitons Fractals* **2009**, *41*, 1095–1104. [CrossRef]
40. Rus, I.A. Ulam stabilities of ordinary differential equations in a Banach space. *Carpathian J. Math.* **2010**, *26*, 103–107.