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Bound for an Approximation of Invariant Density of Diffusions via Density Formula in Malliavin Calculus

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Abstract: The *Kolmogorov* and *total variation* distance between the laws of random variables have upper bounds represented by the L^1-norm of densities when random variables have densities. In this paper, we derive an upper bound, in terms of densities such as the Kolmogorov and total variation distance, for several probabilistic distances (e.g., Kolmogorov distance, total variation distance, *Wasserstein* distance, *Forter–Mourier* distance, etc.) between the laws of F and G in the case where a random variable F follows the invariant measure that admits a density and a differentiable random variable G, in the sense of Malliavin calculus, and also allows a density function.

Keywords: Malliavin calculus; invariant measure; density function; Stein's bound; fourth moment theorem; probabilistic distance; Scheffe's theorem

MSC: 60H07; 60F17; 60F25;

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1. Introduction

Let $B = \{B(h), h \in \mathfrak{H}\}$, where $\mathfrak{H}$ is a real separable Hilbert space, be an isonormal Gaussian process defined on some probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ (see Definition 1). The authors in [1] discovered a celebrated central limit theorem, called the "*fourth moment theorem*", for a sequence of random variables belonging to a fixed Wiener chaos associated with B (see Section 2 for the definition of Wiener chaos).

Theorem 1 (Fourth moment theorem)**.** *Let $\{F_n, n \geq 1\}$ be a sequence of random variables belonging to the $q(\geq 2)$th Wiener chaos with $\mathbb{E}[F_n^2] = 1$ for all $n \geq 1$. Then $F_n \xrightarrow{\mathcal{L}} Z$ if and only if $\mathbb{E}[F_n^4] \to 3 = \mathbb{E}[Z^4]$, where Z is a standard Gaussian random variable and the notation $\xrightarrow{\mathcal{L}}$ denotes the convergence in distribution.*

After that, the authors in [2] obtained a quantitative bound of the distances between the laws of F and Z by developing the techniques based on the combination between Malliavin calculus (see, e.g., [3–7]) and Stein's method for normal approximation (see, e.g., [8–10]). These distances can be defined in several ways. More precisely, the distance between the laws of F and Z is given by

$$d(F, Z) \leq C_d \sqrt{\mathbb{E}[(1 - \langle DF, -DL^{-1}F \rangle_{\mathfrak{H}})^2]}. \quad (1)$$

where D and L^{-1} denote the Malliavin derivative and the pseudo-inverse of the Ornstein–Uhlhenbeck generator, respectively (see Definitions 2 and 5), and the constant C_d in (1) only depends on the distance d considered. In the particular case where F is an element in the qth Wiener chaos of B with $\mathbb{E}[F^2] = 1$, the upper bound (1) for Kolmogorov distance ($C_d = 1$) is given by

$$d_{Kol}(F, Z) \leq \sqrt{\frac{q-1}{3q}(\mathbb{E}[F^4] - 3)}. \tag{2}$$

where $\mathbb{E}[F^4] - 3$ is the fourth cumulant of F.

The application of the Stein's method related to Malliavin calculus has been extended from the normal distribution to the cases of Gamma and Pearson distributions (see e.g., [11,12]). Furthermore, the authors in [13] extend the upper bound (1) to a more general class of probability distribution. For a differentiable random variable in the sense of the Malliavin calculus, they obtain the upper bound of distance between its law and a law of a random variable with a density that is continuous, bounded, and strictly positive in the interval (l, u) $(-\infty \leq l < u \leq \infty)$ with finite variance. Their approach is based on the construction of an ergodic diffusion that has a density p as an invariant measure. The diffusion with the invariant density p has the form

$$dX_t = b(X_t)dt + \sqrt{a(X_t)}dW_t, \tag{3}$$

where W is a standard Brownian motion. Then, they consider the generator of the diffusion process X and use the integration by parts (see Definition 3 for the integration by parts formula) to find an upper bound for the distance between the law of a differentiable random variable G and the law of a random variable F with density p_F. This bound contains D and L^{-1} as in the bound (1). Precisely, for a suitable class of functions $\mathcal{F}$,

$$\begin{aligned}
&\sup_{f\in\mathcal{F}} |\mathbb{E}[f(G) - f(F)]| \\
\leq\ & C\mathbb{E}\left[\left|\frac{1}{2}a(G) + \mathbb{E}\left[\langle -DL^{-1}(b(G) - \mathbb{E}[b(G)]), DG\rangle_{\mathfrak{H}} \middle| G\right]\right|\right] \\
&+C|\mathbb{E}[b(G)]|.
\end{aligned} \tag{4}$$

If a random variable G admits a density with respect to the Lebesgue measure, the Kolmogorov (i.e., $\mathcal{F} = \{\mathbf{1}_{(l,z)}; z \in (l, u)\}$) and *total variation* distance ($\mathcal{F} = \{\mathbf{1}_B; B \in \mathcal{B}(\mathbb{R})\}$) can be bounded by

$$\sup_{f\in\mathcal{F}} |\mathbb{E}[f(G) - f(F)]| \leq \int_{-\infty}^{\infty} |p_G(x) - p_F(x)|dx. \tag{5}$$

We note that Scheffe's theorem implies that the pointwise convergence of densities is stronger than convergence in distribution. In this paper, we assume that the law of G admits a density with respect to the Lebesgue measure. This assumption on G is satisfied for all distributions considered throughout examples in the paper [13]. Using the bound of (4) and the diffusion coefficient in (3) given by

$$a(x) = \frac{-2\int_l^x (y-m)p_F(y)dy}{p_F(x)},$$

we derive a bound of general distances in the left-hand side of (4), being expressed in terms of the density functions of two random variables F and G as in the case of Kolmogorov and total variation distances. In addition, we deal with the computation of the conditional expectation in (4). When G is general, it is difficult to find an explicit computation of this expectation. The random variables in all examples covered in [13] are just functions of a Gaussian vector. In this case, it is possible to compute the explicit expectation. If the law of these random variables admits a density with respect to the Lebesgue measure, like all examples considered in [13], we can find the formula from which we can easily compute this expectation.

The rest of the paper is organized as follows. Section 2 reviews some basic notations and the results of Malliavin calculus. In Section 3, we describe the construction of a diffusion process with an invariant density p and derive an upper bound between the laws of F and G in terms of densities. In Section 4, we introduce a method that can directly compute the conditional expectation in (4). Finally, as an application of our main results, in Section 5, we obtain an upper bound of an example considered in [13]. Throughout this paper, c (or C) stands for an absolute constant with possibly different values in different places.

2. Preliminaries

In this section, we briefly review some basic facts about Malliavin calculus for Gaussian processes. For a more detailed explanation, see [6,7]. Fix a real separable Hilbert space $\mathfrak{H}$, with inner product $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$.

Definition 1. *We say that a stochastic process $B = \{B(h), h \in \mathfrak{H}\}$ defined on $(\Omega, \mathfrak{F}, P)$ is an isonormal Gaussian process if B is a centered Gaussian family of random variables such that $\mathbb{E}[B(g)B(h)] = \langle g, h \rangle_{\mathfrak{H}}$ for every $g, h \in \mathfrak{H}$.*

For the rest of this paper, we assume that $\mathfrak{F}$ is the σ-field generated by X. To simplify the notation, we write $L^2(\Omega)$ instead of $L^2(\Omega, \mathfrak{F}, P)$. For each $q \geq 1$, we write $\mathcal{H}_q$ to denote the closed linear subspace of $\mathbb{L}^2(\Omega)$ generated by the random variables $H_q(B(h)), h \in \mathfrak{H}$, $\|h\|_{\mathfrak{H}} = 1$, where the space H_q is the qth Hermite polynomial. The space $\mathcal{H}_q$ is called the qth Wiener chaos of B. Let $\mathcal{S}$ denote the class of smooth and cylindrical random variables F of the form

$$F = f(B(\varphi_1), \cdots, B(\varphi_m)), \ \ m \geq 1, \tag{6}$$

where $f : \mathbb{R}^m \to \mathbb{R}$ is a $\mathcal{C}^\infty$-function such that its partial derivatives have at most polynomial growth, and $\varphi_i \in \mathfrak{H}, i = 1, \cdots, m$. Then, the space $\mathcal{S}$ is dense in $L^q(\Omega)$ for every $q \geq 1$.

Definition 2. *For a given integer $p \geq 1$ and $F \in \mathcal{S}$, the pth Malliavin derivative of F with respect to B is the element of $L^2(\Omega; \mathfrak{H}^{\odot p})$, where the space $\mathfrak{H}^{\odot p}$ denotes the symmetric tensor product of $\mathfrak{H}$, defined by*

$$D^p F = \sum_{i_1,\ldots,i_p=1}^{m} \frac{\partial^p f}{\partial x_1, \ldots, \partial x_p}(B(\varphi_1), \ldots, B(\varphi_n))\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_p}. \tag{7}$$

For a fixed $p \in [1, \infty)$ and an integer $k \geq 1$, we denote by $\mathbb{D}^{k,p}$ the closure of its associated smooth random variable class of $\mathcal{S}$ with respect to the norm

$$\|F\|_{k,p}^p = \mathbb{E}[|F|^p] + \sum_{\ell=1}^{k} \mathbb{E}[\|D^\ell F\|^p_{\mathfrak{H}^{\otimes \ell}}].$$

For a given integer $p \geq 1$, we denote by $\delta^p : L^2(\Omega; \mathfrak{H}^{\otimes p}) \to L^2(\Omega)$ the adjoint of the operator $D^p : \mathbb{D}^{k,2} \to L^2(\Omega; \mathfrak{H}^{\odot q})$, called the *multiple divergence operator* of order p. The domain of δ^p, denoted by $\mathrm{Dom}(\delta^p)$, is the subset of $\mathbb{L}^2(\Omega; \mathfrak{H}^{\otimes p})$ composed of those elements u such that

$$|\mathbb{E}[\langle D^p F, u\rangle_{\mathfrak{H}^{\otimes p}}]| \leq C(\mathbb{E}[|F|^2]^{1/2} \ \text{ for all } F \in \mathbb{D}^{p,2}.$$

Definition 3. *If $u \in \mathrm{Dom}(\delta^p)$, then $\delta^p(u)$ is the element of $L^2(\Omega)$ defined by the duality relationship*

$$\mathbb{E}[F\delta^p(u)] = \mathbb{E}[\langle D^p F, u\rangle_{\mathfrak{H}^{\otimes p}}] \ \textit{ for every } F \in \mathbb{D}^{p,2}. \tag{8}$$

The above formula (8) is called an integration by parts formula. For a given integer $q \geq 1$ and $f \in \mathcal{H}^{\odot q}$, the qth multiple integral of f is defined by $I_q(f) = \delta^q(f)$. Let $h \in \mathfrak{H}$ with $\|h\|_{\mathfrak{H}} = 1$. Then, for any integer $q \geq 1$, we have $I_q(h^{\otimes q}) = q! H_q(B(h))$. From this, the linear mapping $I_q : \mathfrak{H}^{\odot q} \to \mathcal{H}_q$ by $I_q(h^{\otimes q}) = q! H_q(B(h))$ has an isometric property. It is well known that any square integrable random variable $F \in L^2(\Omega)$ can be expanded into a series of multiple integrals:

$$F = \mathbb{E}[F] + \sum_{q=1}^{\infty} I_q(f_q),$$

where the series converges in L^2, and the functions $f_q \in \mathfrak{H}^{\odot q}$, $q \geq 1$, are uniquely determined by F. Moreover, if $F \in \mathbb{D}^{m,2}$, then $f_q = \frac{1}{q!}\mathbb{E}[D^q F]$ for all $q \leq m$.

Definition 4. *For a given $F \in L^2(\Omega)$, we say that F belongs to $Dom(L)$ if*

$$\sum_{q=1}^{\infty} q^2 \mathbb{E}[J_q(F)^2] < \infty,$$

where J_q is the projection operator from $L^2(\Omega)$ into $\mathcal{H}_q$, that is, $J_q(F) = Proj(F|\mathcal{H}_q)$, $q = 0, 1, 2 \ldots$. For such an F, the operator L is defined through the projection operator J_q, $q = 0, 1, 2 \ldots$, as $LF = -\sum_{q=1}^{\infty} q J_q F$.

It is not difficult to see that the operator L coincides with the infinitesimal generator of the Ornstein–Uhlhenbeck semigroup $\{P_t, t \geq 0\}$. The following gives a crucial relationship between the operator D, δ, and L: Let $F \in L^2(\Omega)$. Then, we have $F \in Dom(L)$ if and only if $F \in \mathbb{D}^{1,2}$ and $DF \in Dom(\delta)$. In this case, $\delta(DF) = -LF$, that is, for $F \in L^2(\Omega)$, the statement $F \in \mathrm{Dom}(L)$ is equivalent to $F \in \mathrm{Dom}(\delta D)$.

Definition 5. *For any $F \in L^2(\Omega)$, we define the operator L^{-1}, called the pseudo-inverse of L, as $L^{-1}F = \sum_{q=1}^{\infty} \frac{1}{q} J_q(F)$.*

Note that L^{-1} is an operator with values in $\mathbb{D}^{2,2}$ and $LL^{-1}F = F - \mathbb{E}[F]$ for all $F \in L^2(\Omega)$.

3. Diffusion Process with Invariant Measures

In this section, we explain how a diffusion process is constructed to have an invariant measure μ that admits a density function, say p, with respect to the Lebesgue measure (see [13,14] for more information). Let μ be a probability measure on $I = (l, u)$ $(-\infty \leq l < u \leq \infty)$ with a continuous, bounded, and strictly positive density function p. We take a function $b : I \to \mathbb{R}$ that is continuous such that $e \in (l, u)$ exists for which $b(x) > 0$ for $x \in (l, e)$ and $b(x) < 0$ for $x \in (e, u)$ are satisfied. Moreover, the function bp is bounded on I and

$$\int_l^u b(x)p(x)dx = 0. \tag{9}$$

For $x \in I$, let us set

$$a(x) = \frac{2\int_l^x b(y)p_F(y)dy}{p(x)}. \tag{10}$$

Then, the stochastic differential equation (sde)

$$dX_t = b(X_t)dt + \sqrt{a(X_t)}dB_t \tag{11}$$

has a unique ergodic Markovian weak solution with the invariant measure μ.

The authors prove in [15] that the convergence of the elements of a Markov chaos to a Pearson distribution can be still bounded with just the first four moments by using the new concept of a *chaos grade*. Pearson diffusions are examples of the Markov triple and Itô diffusion given by the sde

$$dX_t = -(X_t - m)dt + \sqrt{a(X_t)}dB_t, \tag{12}$$

where m is the expectation of μ, and

$$a(x) = \frac{-2\int_l^x (y-m)p(y)dy}{p(x)} \quad \text{for} \quad x \in (l,u). \tag{13}$$

Let us define

$$\tilde{h}_f(y) = \frac{2\int_l^y (f(u) - \mathbb{E}[f(F)])p(u)du}{a(y)p_F(y)},$$

where F is a random variable having its law of μ. For $f \in \mathcal{C}_0(I)$, where $\mathcal{C}_0(I) = \{f : I \to \mathbb{R} | f \text{ is continuous on } I \text{ vanishing at the boundary of } I\}$, we define

$$h_f(x) = \int_0^x \tilde{h}_f(y)dy.$$

Then, h_f satisfies that

$$f - \mathbb{E}[f(F)] = b(x)h_f'(x) + \frac{1}{2}a(x)h_f''(x).$$

In [13], the authors derive the Stein's bound between the probability measure μ and the law of an arbitrary random variable G. This bound extends the results in [2,12] in the case where μ is a standard Gaussian and Gamma distribution, respectively.

Theorem 2 (Kusuoka and Tudor (2012) [13])**.** *Let F be a random variable having the target law μ with a probability distribution associated to the diffusion given by sde (11). Let G be an I-valued random variable in $\mathbb{D}^{1,2}$ with $b(G) \in \mathbb{L}^2(\Omega)$. Then, for every $f : I \to \mathbb{R}$ such that $\tilde{h}_f$ and $\tilde{h}_f'$ are bounded, the following holds:*

$$\begin{aligned} & \big|\mathbb{E}\big[f(G) - f(F)\big]\big| \\ \le \;& \|\tilde{h}_f'\|_\infty \mathbb{E}\left[\left|\frac{1}{2}a(G) + \big\langle -DL^{-1}(b(G) - \mathbb{E}[b(G)]), DG\big\rangle_{\mathfrak{H}}\right|\right] \\ & + \|\tilde{h}_f\|_\infty |\mathbb{E}[b(G)]|, \end{aligned} \tag{14}$$

and

$$\begin{aligned} & \big|\mathbb{E}\big[f(G) - f(F)\big]\big| \\ \le \;& \|\tilde{h}_f'\|_\infty \mathbb{E}\left[\left|\mathbb{E}\left[\frac{1}{2}a(G) + \big\langle -DL^{-1}(b(G) - \mathbb{E}[b(G)]), DG\big\rangle_{\mathfrak{H}}\Big| G\right]\right|\right] \\ & + \|\tilde{h}_f\|_\infty |\mathbb{E}[b(G)]|. \end{aligned} \tag{15}$$

When the laws of F and G admit densities p_F and P_G (with respect to Lebesgue measure), respectively, we derive an upper bound (14) in terms of the densities of F and G by using Theorem 2.

Theorem 3. *Let F be a random variable having the law μ with the density p_F associated to the diffusion given by sde (11). Let G be a random variable in $\mathbb{D}^{1,2}$ with $b(G) \in \mathbb{L}^2(\Omega)$. Suppose that the law of G has the density p_G with respect to the Lebesgue measure. Then, for every $f : I \to \mathbb{R}$ such that $\tilde{h}_f$ and $\tilde{h}_f'$ are bounded, we find that*

$$
\begin{aligned}
&\left|\mathbb{E}[f(G)-f(F)]\right| \\
\leq\ & \|h_f'\|_\infty \mathbb{E}\left[\left|\int_G^\infty b(y)\left(\frac{p_F(y)}{p_F(G)}-\frac{p_G(y)}{p_G(G)}\right)dy\right|\right] \\
&+\left(\|h_f'\|_\infty \mathbb{E}\left[\frac{\int_G^\infty p_G(y)dy}{p_G(G)}\right]+\|h_f\|_\infty\right)\left|\mathbb{E}[b(G)]\right|. \qquad (16)
\end{aligned}
$$

Proof. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^1$-function having a bounded derivative φ' with a compact support. Using the integration by parts yields

$$
\begin{aligned}
&\mathbb{E}\Big[\varphi'(G)\mathbb{E}\Big[\big\langle -DL^{-1}(b(G)-\mathbb{E}[b(G)]), DG\big\rangle_{\mathfrak{H}}\big|G\Big]\Big] \\
=\ & \mathbb{E}\Big[\big\langle -DL^{-1}(b(G)-\mathbb{E}[b(G)]), D\varphi(G)\big\rangle_{\mathfrak{H}}\Big] \\
=\ & \mathbb{E}[\varphi(G)(b(G)-\mathbb{E}[b(G)])] \\
=\ & -\int_{-\infty}^{\infty}\varphi(x)\frac{d}{dx}\left(\int_x^\infty (b(y)-\mathbb{E}[b(G)])p_G(y)dy\right)dx \\
=\ & -\varphi'(x)\int_x^\infty (b(y)-\mathbb{E}[b(G)])p_G(y)dy\Big|_{-\infty}^{\infty} \\
&+\int_{-\infty}^{\infty}\varphi'(x)\int_x^\infty (b(y)-\mathbb{E}[b(G)])p_G(y)dydx \\
=\ & \mathbb{E}\left[\varphi'(G)\frac{\int_G^\infty (b(y)-\mathbb{E}[b(G)])p_G(y)dy}{p_G(G)}\right]. \qquad (17)
\end{aligned}
$$

The above equality (17) obviously shows that

$$
\begin{aligned}
&\mathbb{E}\Big[\big\langle -DL^{-1}(b(G)-\mathbb{E}[b(G)]), DG\big\rangle_{\mathfrak{H}}\big|G\Big] \\
=\ & \frac{\int_G^\infty (b(y)-\mathbb{E}[b(G)])p_G(y)dy}{p_G(G)}. \qquad (18)
\end{aligned}
$$

Using the relations (10) and (17), the first expectation in the right-hand side of (15) can be written as

$$
\begin{aligned}
&\mathbb{E}\left[\left|\frac{1}{2}a(G)+\mathbb{E}\Big[\big\langle -DL^{-1}(b(G)-\mathbb{E}[b(G)]), DG\big\rangle_{\mathfrak{H}}\Big|G\Big]\right|\right] \\
=\ & \mathbb{E}\left[\left|\frac{\int_{-\infty}^G b(y)p_F(y)dy}{p_F(G)}+\frac{\int_G^\infty (b(y)-\mathbb{E}[b(G)])p_G(y)dy}{p_G(G)}\right|\right] \qquad (19)
\end{aligned}
$$

Since

$$
\frac{\int_l^u b(y)p_F(y)dy}{p_F(G)}=0,
$$

we have that

$$
\frac{\int_{-\infty}^G b(y)p_F(y)dy}{p_F(G)}=-\frac{\int_G^\infty b(y)p_F(y)dy}{p_F(G)},
$$

This implies that (19) can be written as

$$\begin{aligned}
& \mathbb{E}\left[\left|\mathbb{E}\left[\frac{1}{2}a(G)+\left\langle -DL^{-1}(b(G)-\mathbb{E}[b(G)]),DG\right\rangle_{\mathfrak{H}}\Big|G\right]\right|\right] \\
\le\ & \mathbb{E}\left[\left|\frac{\int_G^\infty b(y)p_F(y)dy}{p_F(G)}-\frac{\int_G^\infty b(y)p_G(y)dy}{p_G(G)}\right|\right] \\
& +|\mathbb{E}[b(G)]|\mathbb{E}\left[\frac{\int_G^\infty p_G(y)dy}{p_G(G)}\right]. \qquad (20)
\end{aligned}$$

Combining (15) and (20) completes the proof of this theorem. □

Remark 1. *In Theorem 2 of [13], the authors prove that if a random variable $G \in \mathbb{D}^{1,2}$ has the invariant measure μ, then $\mathbb{E}[b(G)] = 0$ and*

$$\mathbb{E}\left[\frac{1}{2}a(G)+\left\langle -DL^{-1}b(G),DG\right\rangle_{\mathfrak{H}}\Big|G\right]=0. \qquad (21)$$

Furthermore, if μ admits the density p_F, it is obvious from (19) that (21) holds.

Remark 2. *We think it would be interesting to give numerical examples from the computational validity in Theorem 3. In this respect, although not a numerical example, we give a simple example to deduce an upper bound for between the laws of two centered Gaussain random variables.*

Proposition 1. *Let F and G be two centered Gaussian random variables with variances $\sigma_1^2 > 0$ and $\sigma_2^2 > 0$. Then,*

$$d_{\mathcal{F}}(F,G) \le \sup_{f\in\mathcal{F}} \|h_f'\|_\infty |\sigma_F^2-\sigma_G^2|, \qquad (22)$$

where $\mathcal{F}$ is the class of functions to be chosen depending on the type of the distance d.

Proof. Obviously, the random variable F has the law μ with the density

$$p_F(x)=\frac{1}{\sqrt{2\pi}\sigma_F}\exp\left(-\frac{x^2}{2\sigma_F^2}\right)$$

associated to the diffusion given by sde with $b(x) = -x$ and $a(x) = 2\sigma_F^2$. Since $\mathbb{E}[b(G)] = 0$, the second sum in (16) is vanished. Hence, from Theorem 3, it follows that

$$\begin{aligned}
& \big|\mathbb{E}\big[f(G)-f(F)\big]\big| \\
\le\ & \|h_f'\|_\infty \mathbb{E}\left[\left|e^{\frac{G^2}{2\sigma_F^2}}\int_G^\infty ye^{-\frac{y^2}{2\sigma_F^2}}dy-e^{\frac{G^2}{2\sigma_G^2}}\int_G^\infty ye^{-\frac{y^2}{2\sigma_G^2}}dy\right|\right] \\
=\ & \|h_f'\|_\infty \mathbb{E}\left[\left|\sigma_F^2 e^{\frac{G^2}{2\sigma_F^2}}\int_{-\frac{G^2}{2\sigma_F^2}}^\infty e^{-u}du-\sigma_G^2 e^{\frac{G^2}{2\sigma_G^2}}\int_{-\frac{G^2}{2\sigma_G^2}}^\infty e^{-u}du\right|\right] \\
=\ & \|h_f'\|_\infty |\sigma_F^2-\sigma_G^2|. \qquad (23)
\end{aligned}$$

Since the distance $d_{\mathcal{F}}(F,G)$ between two distributions F and G is given by

$$d_{\mathcal{F}}(F,G)=\sup_{f\in\mathcal{F}}\big|\mathbb{E}\big[f(G)-f(F)\big]\big|,$$

the proof of this proposition is completed. □

Depending on the choice of $\mathcal{F}$, several types of distances can be defined (see Section 5.2). Comparing the upper bound in Proposition 3.6.1 of [6] obtained from an elementary application of Stein's method with the upper bound in (22) is very interesting. This shows that our study is differentiated from the existing ones.

4. Computation of $\mathbf{E}\left[\langle -\mathbf{DL^{-1}}(\mathbf{b}(\mathbf{G})-\mathbf{E}[\mathbf{b}(\mathbf{G})]),\mathbf{DG}\rangle_{\mathfrak{H}}|\mathbf{G}\right]$

When G is general, it is difficult to find an explicit computation of the right-hand side of (15). In particular, when $\big\langle -DL^{-1}(b(G)-\mathbb{E}[b(G)]),DG\big\rangle_{\mathfrak{H}}$ is not measurable with respect to the σ-field generated by G, there are cases where it is impossible to compute the expectation. The next proposition in [4] contains an explicit example.

Proposition 2. *Let $DG=\Psi_G(B)$, where B is an isonormal Gaussian process and $\Psi_G:\mathbb{R}^{\mathfrak{H}}\to\mathfrak{H}$ is a uniquely defined measurable function a.e. Then, we have*

$$\begin{aligned}&\big\langle -DL^{-1}(G-\mathbb{E}[G]),DG\big\rangle_{\mathfrak{H}}\\ =\;& \int_0^\infty e^{-t}\langle\Psi_G(B),\mathbb{E}'\left[\Psi_G(e^{-t}B+\sqrt{1-e^{-2t}}B')\right]\rangle_{\mathfrak{H}}dt,\end{aligned} \tag{24}$$

so that

$$\begin{aligned}&\mathbb{E}\big[\big\langle -DL^{-1}(G-\mathbb{E}[G]),DG\big\rangle_{\mathfrak{H}}|G\big]\\ =\;& \int_0^\infty e^{-t}\mathbf{E}\big[\langle\Psi_G(B),\Psi_G(e^{-t}B+\sqrt{1-e^{-2t}}B')\rangle_{\mathfrak{H}}|G\big]dt.\end{aligned} \tag{25}$$

Here, B and B' are defined on the product space $(\Omega\times\Omega',\mathcal{F}\otimes\mathcal{F}',\mathbb{P}\otimes\mathbb{P}')$ such that B' stands for an independent copy of B. $\mathbf{E}$ and $\mathbb{E}'$ denote the expectation with respect to $\mathbb{P}\otimes\mathbb{P}'$ and $\mathbb{P}'$, respectively.

If $G=h(N)-\mathbb{E}[h(N)]$, where $h:\mathbb{R}^d\to\mathbb{R}$ is a $\mathcal{C}^1$-function with bounded derivative and $N=(N_1,\dots N_d)$ is a d-dimensional Gaussian random variable with zero mean and covariance $\langle h_i,h_j\rangle_{\mathfrak{H}}=\mathbb{E}[N_iN_j]=(C_{i,j})$, $i,j=1,\dots,d$, where $\{h_i, i=1,\dots,n\}$ stands for the canonical basis of $\mathfrak{H}$. By using Proposition 2, the following useful formula can be proved:

$$\begin{aligned}&\big\langle -DL^{-1}(G-\mathbb{E}[G]),DG\big\rangle_{\mathfrak{H}}\\ =\;& \int_0^\infty e^{-x}\mathbb{E}'\left[\sum_{i,j=1}^d C_{i,j}\frac{\partial h}{\partial x_i}(N)\frac{\partial h}{\partial x_j}\Big(e^{-x}N+\sqrt{1-e^{-2x}}N'\Big)\right]dx.\end{aligned} \tag{26}$$

In order to show the significance of the bound (15), the authors in [13] consider the several random variables G. Here, among these random variables, we consider random variables with the uniform and Laplace distribution. The random variable defined by

$$G=e^{-\frac{1}{2}(B(f)+B(g))},$$

where $B(f)$ and $B(g)$ are independent standard Gaussian random variables, has the uniform distribution $\mathcal{U}([0,1])$. The authors in [13] compute the right-hand side of (26) to prove that

$$\mathbb{E}[\big\langle -DL^{-1}(G-\mathbb{E}[G]),DG\big\rangle_{\mathfrak{H}}|G]=G(1-G). \tag{27}$$

Computing in this way is tedious and lengthy. To overcome this situation, we can use Equation (18) to prove that (27) holds. Since G has the uniform distribution $\mathcal{U}([0,1])$, we have

$$\begin{aligned}\mathbb{E}\Big[\big\langle -DL^{-1}(G-\frac{1}{2}),DG\big\rangle_{\mathfrak{H}}|G\Big] &= \frac{\int_G^\infty(y-\frac{1}{2})\mathbf{1}_{[0,1]}(y)dy}{\mathbf{1}_{[0,1]}(G)}\\ &= G(1-G).\end{aligned} \tag{28}$$

In the case where G has a Laplace distribution, the authors in [13] consider two random variables:

$$G_1 = \frac{1}{2}\big(B(h_1)^2 + B(h_2)^2 - B(h_3)^2 - B(h_4)^2\big), \tag{29}$$

$$G_2 = B(h_1)B(h_2) + B(h_3)B(h_4). \tag{30}$$

where h_i, $i = 1, \ldots, 4$, are orthonormal functions in $L^2([0,T])$. It can be easily seen that G_i, $i = 1,2$, has the Laplace distribution with parameter 1. In the paper [13], the authors prove, using Theorem 2 in [13], that for $i = 1,2$,

$$\mathbb{E}\Big[\big\langle -DL^{-1}G_i, DG_i\big\rangle_{\mathfrak{H}}|G_i\Big] = 1 + |G_i|. \tag{31}$$

The authors argue that these identities are difficult to be proven directly. Here, we introduce a method that can directly prove these identities (31) by using the formula given in (18). Since G_i, $i = 1,2$, has a Laplace distribution with parameter 1, we find that for $i = 1,2$,

$$\mathbb{E}\Big[\big\langle -DL^{-1}(G_i - \frac{1}{2}), DG_i\big\rangle_{\mathfrak{H}}|G_i\Big] = \frac{\frac{1}{2}\int_{G_i}^{\infty} ye^{-|y|}dy}{\frac{1}{2}e^{-|G_i|}}. \tag{32}$$

An elementary computation yields that for $G_i \geq 0$ a.s,

$$\frac{\frac{1}{2}\int_{G_i}^{\infty} ye^{-|y|}dy}{\frac{1}{2}e^{-|G_i|}} = \frac{e^{-G_i}(1+G_i)}{e^{-G_i}} = 1 + G_i, \tag{33}$$

and for $G_i < 0$ a.s.

$$\begin{aligned}\frac{\frac{1}{2}\int_{G_i}^{\infty} ye^{-|y|}dy}{\frac{1}{2}e^{-|G_i|}} &= \frac{\frac{1}{2}\int_{G_i}^{0} ye^{y}dy + \frac{1}{2}\int_{0}^{\infty} ye^{-y}dy}{\frac{1}{2}e^{G_i}} \\ &= \frac{e^{G_i}(1-G_i)}{e^{G_i}} = 1 - G_i.\end{aligned} \tag{34}$$

Combining (33) and (34) proves that the identity (31) holds.

5. Example

In this section, we illustrate the upper bound of probabilistic distances in Theorem 3 through an example considered in [13]. We denote the Wiener integral of $h \in L^2([0,T])$ by $W(h)$. Let $\{h_i, i = 1,2,\ldots\}$ be a sequence of orthonormal bases of $L^2([0,T])$ and $\{G_N, N = 1,2\ldots\}$ a sequence of random variables defined by

$$G_N = e^{-\frac{1}{\sqrt{2N}}\sum_{i=1}^{N}(W(h_i)^2 - 1)}. \tag{35}$$

Let F be a random variable having log normal distribution with mean $m = 0$ and variance $\sigma^2 = 1$. Then, the density of F is given by

$$p_F(x) = \frac{1}{\sqrt{2\pi}x}\exp\left(-\frac{1}{2}(\log x)^2\right)\mathbf{1}_{(0,\infty)}(x). \tag{36}$$

Next, we compute the density of the random variable G_N given by (35). We first compute the cumulative distribution function of G_N. Let us set $X_N = \sum_{i=1}^{N} W(h_i)^2$. Then, the random variable $X_N = N - \sqrt{2N}\log G_N$ has a Gamma distribution with parameters $\alpha = \frac{N}{2}$ and $\beta = \frac{1}{2}$, that is,

$$P_{X_N}(x) = \frac{1}{2^{\frac{N}{2}}\Gamma(\frac{N}{2})} x^{\frac{N}{2}-1}e^{-\frac{x}{2}}\mathbf{1}_{(0,\infty)}(x). \tag{37}$$

Using (37), we find that for $x \geq 0$,

$$\begin{aligned} \mathbb{P}(G_N \leq x) &= \mathbb{P}\left(-\frac{1}{\sqrt{2N}} \sum_{i=1}^{N} (W(h_i)^2 - 1) \leq \log x \right) \\ &= \mathbb{P}(X_N \geq N - \sqrt{2N} \log x) \\ &= \int_{N-\sqrt{2N}\log x}^{\infty} p_{X_N}(y) dy. \end{aligned} \tag{38}$$

Differentiating Equation (38) proves that

$$p_{G_N}(x) = \frac{\sqrt{2N}}{x} p_{X_N}(N - \sqrt{2N} \log x). \tag{39}$$

From (39), it follows that

$$\begin{aligned} p_{G_N}(x) &= \frac{\sqrt{2N}}{2^{\frac{N}{2}} \Gamma(\frac{N}{2}) x} (N - \sqrt{2N} \log x)^{\frac{N}{2}-1} e^{-\frac{1}{2}(N-\sqrt{2N}\log x)} \mathbf{1}_{(0,\infty)}(x) \\ &= \frac{\sqrt{2N}}{2^{\frac{N}{2}} \Gamma(\frac{N}{2}) x} \exp\left\{ \left(\frac{N}{2} - 1\right) \log(N - \sqrt{2N} \log x) \right. \\ &\qquad \left. - \frac{1}{2}(N - \sqrt{2N} \log x) \right\} \mathbf{1}_{(0,\infty)}(x). \end{aligned} \tag{40}$$

5.1. Scheffe's Theorem

First, we prove that G_N converges in distribution to F by using Scheffe's theorem and then find a convergence rate of the Kolmogorov and total variation distance. The right-hand side of (40) can be written as

$$\begin{aligned} p_{G_N}(x) &= \frac{\sqrt{2N}}{2^{\frac{N}{2}} \Gamma(\frac{N}{2}) x} \exp\left\{ \left(\frac{N}{2} - 1\right) \log N - \frac{N}{2} \right\} \\ &\quad \times \exp\left\{ \left(\frac{N}{2} - 1\right) \log\left(1 - \sqrt{\frac{2}{N}} \log x\right) \right. \\ &\qquad \left. + \sqrt{\frac{N}{2}} \log x \right\} \mathbf{1}_{(0,\infty)}(x). \end{aligned} \tag{41}$$

For any fixed $x \in (0, \infty)$, we have, from (36) and (41), that

$$\begin{aligned} p_{G_N}(x) - p_F(x) &= \left[\frac{\sqrt{2N}}{2^{\frac{N}{2}} \Gamma(\frac{N}{2})} \exp\left\{ \left(\frac{N}{2} - 1\right) \log N - \frac{N}{2} \right\} \right. \\ &\qquad \left. - \frac{1}{\sqrt{2\pi}} \right] \frac{1}{x} e^{-\frac{1}{2}(\log x)^2} \\ &\quad + \frac{\sqrt{2N}}{2^{\frac{N}{2}} \Gamma(\frac{N}{2})} \exp\left\{ \left(\frac{N}{2} - 1\right) \log N - \frac{N}{2} \right\} \\ &\quad \times \frac{1}{x} \left[\exp\left\{ \left(\frac{N}{2} - 1\right) \log\left(1 - \sqrt{\frac{2}{N}} \log x\right) \right. \right. \\ &\qquad \left. \left. + \sqrt{\frac{N}{2}} \log x \right\} - e^{-\frac{1}{2}(\log x)^2} \right] \\ &= A_{1,N} + A_{2,N}. \end{aligned} \tag{42}$$

To estimate the first term $A_{1,N}$ in (42), we can use the following specific version of the Stirling formula of the Γ function, incorporating upper and lower bounds (see [16]):

Lemma 1. *Let $S(x) = x^{x-\frac{1}{2}}e^{-x}$. Then for all $x > 0$,*

$$\sqrt{2\pi}S(x) \leq \Gamma(x) \leq \sqrt{2\pi}S(x)e^{\frac{1}{12x}}. \tag{43}$$

The term $|A_{1,N}|$ in (42) can be written as

$$|A_{1,N}| = \frac{1}{\sqrt{2\pi}}|1 - A_{11,N} \times A_{12,N}|\frac{1}{x}e^{-\frac{1}{2}(\log x)^2}, \tag{44}$$

where

$$\begin{aligned} A_{11,N} &= \frac{\sqrt{2\pi}\sqrt{\frac{2}{N}}(\frac{N}{2})^{\frac{N}{2}}e^{-\frac{N}{2}}}{\Gamma(\frac{N}{2})}, \\ A_{12,N} &= \frac{\sqrt{2N}e^{(\frac{N}{2}-1)\log N-\frac{N}{2}}}{2^{\frac{N}{2}}\sqrt{\frac{2}{N}}(\frac{N}{2})^{\frac{N}{2}}e^{-\frac{N}{2}}}. \end{aligned}$$

Obviously,

$$A_{12,N} = \frac{\sqrt{2N}2^{\frac{N}{2}-1}(\frac{N}{2})^{\frac{N}{2}-1}}{2^{\frac{N}{2}}\sqrt{\frac{2}{N}}(\frac{N}{2})^{\frac{N}{2}}} = 1. \tag{45}$$

Hence, form (43) and (44),

$$\begin{aligned} |A_{1,N}| &= \frac{1}{\sqrt{2\pi}}\left|\frac{\Gamma(\frac{N}{2}) - \sqrt{2\pi}(\frac{N}{2})^{\frac{N}{2}-\frac{1}{2}}e^{-\frac{N}{2}}}{\Gamma(\frac{N}{2})}\right| \\ &\leq \frac{1}{\sqrt{2\pi}}(1 - e^{\frac{1}{12N}}) \\ &= \frac{1}{12\sqrt{2\pi}N} + o(\frac{1}{N}). \end{aligned} \tag{46}$$

Using the Taylor expansion of $\log\left(1 - \sqrt{\frac{2}{N}}\log x\right)$

$$\log\left(1 - \sqrt{\frac{2}{N}}\log x\right) = -\sqrt{\frac{2}{N}}\log x - \frac{2}{2N}(\log x)^2 + o_x(N^{-1}),$$

we write $A_{2,N}$ as

$$\begin{aligned} A_{2,N} &= \frac{\sqrt{2N}}{2^{\frac{N}{2}}\Gamma(\frac{N}{2})}\exp\left\{\left(\frac{N}{2} - 1\right)\log N - \frac{N}{2}\right\} \\ &\quad \times\frac{1}{x}\left[\exp\left\{-\frac{1}{2}(\log x)^2 + o(1)\right\} - e^{-\frac{1}{2}(\log x)^2}\right]. \end{aligned} \tag{47}$$

Since

$$\lim_{N\to\infty}\frac{\sqrt{2N}}{2^{\frac{N}{2}}\Gamma(\frac{N}{2})}\exp\left\{\left(\frac{N}{2} - 1\right)\log N - \frac{N}{2}\right\} = \frac{1}{\sqrt{2\pi}},$$

we will have that $\lim_{N\to\infty} A_{2,N} = 0$, and hence, from (42),

$$\lim_{N\to\infty} p_{G_N}(x) = p_F(x) \text{ for all } x \in (0,1).$$

This convergence implies, from Scheffe's theorem, that as $N \to \infty$,

$$\int_0^\infty |p_{G_N}(x) - p_F(x)|dx \to 0.$$

An upper bound for the Kolmogorov and total variation distance is given in (5). Hence, G_N converges in distribution to F. Next, we find the rate of convergence for an upper bound for these distances by using the bound (5). By using the change of variables $\log x = z$, we find, from (36) and (40), that

$$\begin{aligned} d(G,F) &\le \int_0^\infty \big|p_F(x) - p_G(x)\big|dx \\ &= \int_{-\infty}^\infty \left| \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} - \frac{\sqrt{2N}}{2^{\frac{N}{2}}\Gamma(\frac{N}{2})} e^{(\frac{N}{2}-1)\log N - \frac{N}{2}} \right. \\ &\qquad \left. \times\, e^{(\frac{N}{2}-1)\log(1-\sqrt{\frac{2}{N}}z)+\sqrt{\frac{N}{2}}z} \right| dz. \end{aligned} \tag{48}$$

Using the Taylor expansion of $\log(1 - \sqrt{\frac{2}{N}}z)$, the right-hand side of (48) can be represented as

$$\begin{aligned} d(G,F) &\le \frac{1}{2}\int_{-\infty}^\infty \left| \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} - \frac{\sqrt{2N}}{2^{\frac{N}{2}}\Gamma(\frac{N}{2})} e^{(\frac{N}{2}-1)\log N - \frac{N}{2}} \right. \\ &\qquad \left. \times\, e^{-\frac{z^2}{2}+\sqrt{\frac{2}{N}}z+o_z(N^{-\frac{1}{2}})} \right| dz \\ &\le \frac{1}{2}\left| \frac{1}{\sqrt{2\pi}} - \frac{\sqrt{2N}}{2^{\frac{N}{2}}\Gamma(\frac{N}{2})} e^{(\frac{N}{2}-1)\log N - \frac{N}{2}} \right| \int_{-\infty}^\infty e^{-\frac{z^2}{2}} dz \\ &\qquad + \frac{\sqrt{2N}}{2^{\frac{N}{2}+1}\Gamma(\frac{N}{2})} e^{(\frac{N}{2}-1)\log N - \frac{N}{2}} \int_{-\infty}^\infty e^{-\frac{z^2}{2}} \\ &\qquad \times \left| 1 - e^{\sqrt{\frac{2}{N}}z+o_z(N^{-\frac{1}{2}})} \right| dz \\ &= B_{1,N} + B_{2,N}. \end{aligned} \tag{49}$$

From (46), it follows that

$$B_{1,N} \le \frac{C}{\sqrt{N}}. \tag{50}$$

Obviously,

$$\begin{aligned} B_{2,N} &\le C\int_{-\infty}^\infty e^{-\frac{z^2}{2}} \left| 1 - e^{\sqrt{\frac{2}{N}}z+\frac{z^2}{N}++o_z(N^{-\frac{1}{2}})} \right| dz \\ &\le \frac{C}{\sqrt{N}}. \end{aligned} \tag{51}$$

From (50) and (51), we prove that the rate of convergence of the Kolmogorov and total variation distance between the laws of F and G_N is of order $\frac{1}{\sqrt{N}}$.

5.2. General Distance

In this section, we consider general distances between the laws of F and G_N defined by

$$d_{\mathcal{F}}(G_N, F) = \sup_{f\in\mathcal{F}} \big|\mathbb{E}[f(G_N)] - \mathbb{E}[f(F)]\big|, \tag{52}$$

where $\mathcal{F}$ is a class of functions defined on $\mathbb{R}$. Depending on the choice of $\mathcal{F}$, several types of distances can be defined. In addition to the Kolmogorov distance and total variation distance, the following distances can be obtained: for example, if $\mathcal{F} = \{f : \|f\|_L \leq 1\}$, where $\|\cdot\|_L$ denotes the Lipschitz seminorm defined by

$$\|f\|_L = \sup\left\{\frac{f(x) - f(y)|}{|x-y|} : x \neq y\right\}.$$

then the distance in (52) is called *Wasserstein*. If $\mathcal{F} = \{f : \|f\|_L + \|f\|_\infty \leq 1\}$, the Fortet-Mourier will be obtained. The rate of convergence of this distance can be found by using the bound given in Theorem 3. The drift coefficient of the associated diffusion is given by

$$a(x) = \frac{2e^{m+\frac{\sigma^2}{2}}}{p_F(x)}\left[\Phi\left(\frac{\log x - m}{\sigma}\right) - \Phi\left(\frac{\log x - m}{\sigma} - \sigma\right)\right], \tag{53}$$

where the function Φ denotes the distribution function of the standard Gaussian distribution. Let us set $\bar{G}_N = G_N - \mathbb{E}[G_N]$. From (18) and (39), it follows that

$$\begin{aligned}
&\mathbb{E}\Big[\big\langle -DL^{-1}\bar{G}_N, DG\big\rangle_{\mathfrak{H}}\big|\bar{G}_N\Big] \\
=&\quad \frac{\int_{G_N}^\infty (y-m)p_{G_N}(y)dy}{p_{G_N}(G_N)} \\
=&\quad \frac{G_N\int_{G_N}^\infty (y-m)\frac{\sqrt{2N}}{y}p_{X_N}\big(N - \sqrt{2N}\log y\big)dy}{\sqrt{2N}p_{X_N}\big(N-\sqrt{2N}\log G_N\big)} \\
=&\quad \frac{G_N\int_{-\infty}^{X_N}(e^{-\frac{1}{\sqrt{2N}}(x-N)} - m)p_{X_N}(x)dx}{\sqrt{2N}p_{X_N}(X_N)},
\end{aligned} \tag{54}$$

where m is the expectation of G_N given by

$$m = e^{\sqrt{\frac{N}{2}}}\left(1+\sqrt{\frac{2}{N}}\right)^{-\frac{N}{2}}.$$

The right-hand side of (54) can be written as

$$\begin{aligned}
&\mathbb{E}\Big[\big\langle -DL^{-1}\bar{G}_N, DG\big\rangle_{\mathfrak{H}}\big|\bar{G}_N\Big] \\
=&\quad \frac{e^{\sqrt{\frac{N}{2}}}G_N\int_{-\infty}^{X_N}\big[e^{-\frac{x}{\sqrt{2N}}} - (1+\sqrt{\frac{2}{N}})^{-\frac{N}{2}}\big]p_{X_N}(x)dx}{\sqrt{2N}p_{X_N}(X_N)} \\
=&\quad \frac{e^{\sqrt{\frac{N}{2}}}G_N X_N^{1-\frac{N}{2}}e^{-\frac{X_N}{2}}}{\sqrt{2N}} \\
&\quad \times\int_0^{X_N}\big[e^{-\frac{x}{\sqrt{2N}}} - (1+\sqrt{\frac{2}{N}})^{-\frac{N}{2}}\big]x^{\frac{N}{2}-1}e^{-\frac{x}{2}}dx \\
=&\quad \frac{e^{\sqrt{\frac{N}{2}}}G_N X_N^{1-\frac{N}{2}}e^{-\frac{X_N}{2}}}{\sqrt{2N}}\Bigg\{\int_0^{X_N}x^{\frac{N}{2}-1}e^{-\frac{1}{2}(\sqrt{\frac{2}{N}}+1)x}dx \\
&\quad -\int_0^{X_N}(1+\sqrt{\frac{2}{N}})^{-\frac{N}{2}}x^{\frac{N}{2}-1}e^{-\frac{x}{2}}dx\Bigg\}.
\end{aligned} \tag{55}$$

Using the change of variables $(\sqrt{\frac{2}{N}}+1)x = y$, we express the right-hand side of (55) as

$$\mathbb{E}\Big[\big\langle -DL^{-1}\bar{G}_N, DG\big\rangle_{\mathfrak{H}}\big|\bar{G}_N\Big] = \frac{e^{\sqrt{\frac{N}{2}}}G_N X_N^{1-\frac{N}{2}}e^{\frac{X_N}{2}}}{\sqrt{2N}}(1+\sqrt{\frac{2}{N}})^{-\frac{N}{2}} \times \int_{X_N}^{(\sqrt{\frac{2}{N}}+1)X_N} x^{\frac{N}{2}-1}e^{-\frac{x}{2}}dx. \quad (56)$$

By using the expansion

$$(1+\sqrt{\frac{2}{N}})^{-\frac{N}{2}} = e^{-\sqrt{\frac{N}{2}}+\frac{1}{2}-\frac{1}{3\sqrt{2N}}+o(N^{-\frac{1}{2}})},$$

the right-hand side of (56) can be expressed as

$$\mathbb{E}\Big[\big\langle -DL^{-1}\bar{G}_N, DG\big\rangle_{\mathfrak{H}}\big|\bar{G}_N\Big] = \frac{e^{\frac{1}{2}-\frac{1}{3\sqrt{2N}}+o(N^{-\frac{1}{2}})}G_N X_N^{1-\frac{N}{2}}e^{\frac{X_N}{2}}}{\sqrt{2N}} \times \int_{X_N}^{(\sqrt{\frac{2}{N}}+1)X_N} x^{\frac{N}{2}-1}e^{-\frac{x}{2}}dx. \quad (57)$$

The change of variables $\frac{x-X_N}{\sqrt{\frac{2}{N}}X_N} = z$ shows that (57) is

$$\mathbb{E}\Big[\big\langle -DL^{-1}\bar{G}_N, DG\big\rangle_{\mathfrak{H}}\big|\bar{G}_N\Big] = \frac{e^{\frac{1}{2}-\frac{1}{3\sqrt{2N}}+o(N^{-\frac{1}{2}})}G_N X_N}{N} \times \int_0^1 \Big(1+\sqrt{\frac{2}{N}}z\Big)^{\frac{N}{2}-1} e^{-\frac{X_N z}{\sqrt{2N}}}dz. \quad (58)$$

The Taylor expansion of $\log\Big(1+\sqrt{\frac{2}{N}}z\Big)$, $0 \le z \le 1$, is given by

$$\log\Big(1+\sqrt{\frac{2}{N}}z\Big) = \sqrt{\frac{2}{N}}z - \frac{1}{N}z^2 + o(N^{-1}). \quad (59)$$

Applying this expansion (59) to a function $\Big(1+\sqrt{\frac{2}{N}}z\Big)^{\frac{N}{2}-1}$, we have

$$\Big(1+\sqrt{\frac{2}{N}}z\Big)^{\frac{N}{2}-1} = e^{(\frac{N}{2}-1)\Big(\sqrt{\frac{2}{N}}z-\frac{1}{N}z^2+o(N^{-1})\Big)} = e^{\sqrt{\frac{N}{2}}z-\frac{z^2}{2}+No(N^{-1})}e^{-\sqrt{\frac{2}{N}}z+o(N^{-\frac{1}{2}})}. \quad (60)$$

Substituting (60) into the integrand in (58) yields that

$$\mathbb{E}\Big[\big\langle -DL^{-1}\bar{G}_N, DG\big\rangle_{\mathfrak{H}}\big|\bar{G}_N\Big] = \frac{e^{\frac{1}{2}-\frac{1}{3\sqrt{2N}}+o(N^{-\frac{1}{2}})}G_N X_N}{N} \times \int_0^1 e^{-\frac{z^2}{2}-\frac{X_N z}{\sqrt{2N}}+\sqrt{\frac{N}{2}}z+o(N^{-\frac{1}{2}})}dz. \quad (61)$$

From (36) and (53), the drift coefficient of diffusion is given by

$$\begin{aligned}\frac{1}{2}a(G_N) &= \frac{e^{\frac{1}{2}}}{p_F(G_N)}\int_{\log G_N-1}^{\log G_N}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \\ &= \sqrt{2\pi}e^{\frac{1}{2}}G_N e^{\frac{1}{2}\left(\frac{1}{\sqrt{2N}}(X_N-N)\right)^2}\int_{-\frac{1}{\sqrt{2N}}(X_N-N)-1}^{-\frac{1}{\sqrt{2N}}(X_N-N)}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \\ &= \frac{e^{\frac{1}{2}}G_N e^{\frac{1}{4N}(X_N-N)^2}}{\sqrt{2N}}\int_{X_N}^{X_N+\sqrt{2N}}e^{-\frac{(y-N)^2}{4N}}dz \\ &= \frac{e^{\frac{1}{2}}G_N}{\sqrt{2N}}\int_{X_N}^{X_N+\sqrt{2N}}e^{-\frac{(y-X_N)^2}{4N}-\frac{(X_N-N)(y-X_N)}{2N}}dy. \end{aligned} \tag{62}$$

The use of the change of variables $\frac{(y-X_N)}{\sqrt{2N}}=z$ makes the right-hand side of (62) equal to

$$\frac{1}{2}a(G_N) = e^{\frac{1}{2}}G_N\int_0^1 e^{-\frac{z^2}{2}-\frac{X_N z}{\sqrt{2N}}+\sqrt{\frac{N}{2}}z}dz. \tag{63}$$

From (61) and (63), we write $\frac{1}{2}a(G_N)-g_{\bar{G}_N}(\bar{G}_N)=D_{1,N}+D_{2,N}+D_{3,N}$, where

$$\begin{aligned} D_{1,N} &= e^{\frac{1}{2}}\left(1-e^{-\frac{1}{3\sqrt{2N}}+o(N^{-\frac{1}{2}})}\right)G_N \\ &\quad\times\int_0^1 e^{-\frac{z^2}{2}-\frac{X_N z}{\sqrt{2N}}+\sqrt{\frac{N}{2}}z}dz, \\ D_{2,N} &= e^{\frac{1}{2}-\frac{1}{3\sqrt{2N}}+o(N^{-\frac{1}{2}})}G_N\left(1--\frac{X_N}{N}\right) \\ &\quad\times\int_0^1 e^{-\frac{z^2}{2}-\frac{X_N z}{\sqrt{2N}}+\sqrt{\frac{N}{2}}z}dz, \\ D_{3,N} &= e^{\frac{1}{2}-\frac{1}{3\sqrt{2N}}+o(N^{-\frac{1}{2}})}G_N\frac{X_N}{N} \\ &\quad\times\int_0^1 e^{-\frac{z^2}{2}-\frac{X_N z}{\sqrt{2N}}+\sqrt{\frac{N}{2}}z}(1-e^{o(N^{-\frac{1}{2}})})dz. \end{aligned}$$

Lemma 2. *For every $x>0$, we have*

$$\mathbb{E}[G_N^x]=e^{\frac{x^2}{2}+o_x(N^{-\beta})}. \tag{64}$$

where $0<\beta<\frac{1}{2}$.

Proof. We write $G_N=e^{\sqrt{\frac{N}{2}}}\times e^{-\frac{X_N}{\sqrt{2N}}}$, where $X_N\sim\Gamma(\frac{N}{2},\frac{1}{2})$. Hence,

$$\begin{aligned}\mathbb{E}[G_N^x] &= e^{x\sqrt{\frac{N}{2}}}\mathbb{E}\left[e^{-\frac{x}{\sqrt{2N}}X_N}\right] \\ &= e^{x\sqrt{\frac{N}{2}}}\left(1+\frac{2x}{\sqrt{2N}}\right)^{-\frac{N}{2}}. \end{aligned} \tag{65}$$

Since

$$\log\left(1+\frac{2x}{\sqrt{2N}}\right)=\frac{2x}{\sqrt{2N}}-\frac{2x^2}{2N}+o_x(N^{-\alpha}),\ \ \alpha<\frac{3}{2},$$

we have

$$\begin{aligned} \left(1+\frac{2x}{\sqrt{2N}}\right)^{-\frac{N}{2}} &= e^{-\frac{N}{2}\log\left(1+\frac{2x}{\sqrt{2N}}\right)} \\ &= e^{-x\sqrt{\frac{N}{2}}+\frac{x^2}{2}+o_x(N^{-\beta})},\ 0<\beta<\frac{1}{2}. \end{aligned} \tag{66}$$

Substituting (66) into (65) proves this lemma. □

Next, we estimate $\mathbb{E}[|D_{k,N}|]$, $k=1,2,3$. The Cauchy–Schwartz inequality and Lemma 2 give the estimate

$$\begin{aligned} \mathbb{E}[|D_{1,N}|] &\le e^{\frac{1}{2}}|1-e^{-\frac{1}{3\sqrt{2N}}+o(N^{-\frac{1}{2}})}|\sqrt{\mathbb{E}[G_N^2]} \\ &\quad\times\left(\int_0^1 e^{-z^2}\mathbb{E}\left[G_N^{2z}\right]dz\right)^{\frac{1}{2}} \\ &\le \frac{e}{3\sqrt{2N}}(1+o(1))e^{o(1)} \le \frac{c}{\sqrt{N}}. \end{aligned} \tag{67}$$

By Hölder inequality and Lemma 2, we have

$$\begin{aligned} \mathbb{E}[|D_{2,N}|] &\le e^{\frac{1}{2}-\frac{1}{3\sqrt{2N}}+o(N^{-\frac{1}{2}})}(\mathbb{E}[G_N^3])^{\frac{1}{3}}\frac{(\mathbb{E}[|N-X_N|^3])^{\frac{1}{3}}}{N} \\ &\quad\times\left(\int_0^1 e^{-\frac{3z^2}{2}}\mathbb{E}[G_N^{3z}]dz\right)^{\frac{1}{3}} \\ &\le e^{\frac{1}{2}-\frac{1}{3\sqrt{2N}}+o(N^{-\frac{1}{2}})}e^{\frac{3}{2}+o(N^{-\beta})}\left(\mathbb{E}\left[\left(\frac{N-X_N}{\sqrt{2N}}\right)^4\right]\right)^{\frac{1}{4}}\sqrt{\frac{2}{N}} \\ &\quad\times\left(\int_0^1 e^{3z^2+o_z(N^{-\beta})}dz\right)^{\frac{1}{3}} \\ &\le e^{\frac{1}{2}-\frac{1}{3\sqrt{2N}}+o(N^{-\frac{1}{2}})}e^{\frac{3}{2}+o(N^{-\beta})}\left(3+\frac{12}{N}\right)^{\frac{1}{4}}\sqrt{\frac{2}{N}}e^{1+o(N^{-\beta})} \\ &\le \frac{c}{\sqrt{N}}. \end{aligned} \tag{68}$$

Similarly,

$$\begin{aligned} \mathbb{E}[|D_{3,N}|] &\le e^{\frac{1}{2}-\frac{1}{3\sqrt{2N}}+o(N^{-\frac{1}{2}})}(\mathbb{E}[G_N^3])^{\frac{1}{3}}\frac{(\mathbb{E}[|X_N|^3])^{\frac{1}{3}}}{N} \\ &\quad\times\left(\int_0^1 e^{-\frac{3z^2}{2}}\mathbb{E}[G_N^{3z}]dz\right)^{\frac{1}{3}}|1-e^{o(N^{-\frac{1}{2}})}| \\ &\le e^{\frac{1}{2}-\frac{1}{3\sqrt{2N}}+o(N^{-\frac{1}{2}})}e^{\frac{3}{2}+o(N^{-\beta})}(1+o(1)) \\ &\quad\times e^{1+o(N^{-\beta})}|1-e^{o(N^{-\frac{1}{2}})}| \\ &\le \frac{c}{\sqrt{N}}. \end{aligned} \tag{69}$$

Combining the bounds in (67), (68) and (69), we obtain

$$\begin{aligned} \left|\mathbb{E}[f(G_N) - f(F)]\right| &\leq \|h_f'\|_\infty \mathbb{E}\left[\left|\frac{1}{2}a(G_N) - g_{\bar{G}_N}(\bar{G}_N)\right|\right] \\ &\leq \frac{c}{\sqrt{N}}. \end{aligned} \tag{70}$$

Therefore, we find that the rate of convergence of the general distance is of order $\frac{1}{\sqrt{N}}$.

6. Conclusions and Future Works

When a random variable F follows the invariant measure that admits a density and a differentiable random variable G in the sense of Malliavin allows a density function, this paper derives an upper bound on several probabilistic distances (e.g., Kolmogorov distance, total variation distance, Wasserstein distance, and Forter–Mourier distance, etc.) between the laws of F and G in terms of two densities. Among these distances, it is well known that the upper bound of the Kolmogorov and total variation distance can be easily expressed in terms of densities. The significant feature of our works is to show that the bounds of distances other than the two distances mentioned above can be expressed in some form of two density functions. An insight into the main result of this study is that it is possible by applying our results to express an upper bound for the distance of two distributions in terms of two density functions even when it is difficult to express the distance as a density function of two distributions.

Future works will be carried out in two directions: (1) Using the results worked in this paper, we plan to conduct a study on the upper bound that is more rigorous than the results obtained in the papers [15,17]. (2) In the case when G is a random variable belonging to a fixed Wiener chaos, we will prove the fourth moment theorem by using the bound obtained in this paper.

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