mathematics

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Hamiltonians of the Generalized Nonlinear Schrödinger Equations

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Abstract: Some types of the generalized nonlinear Schrödinger equation of the second, fourth and sixth order are considered. The Cauchy problem for equations in the general case cannot be solved by the inverse scattering transform. The main objective of this paper is to find the conservation laws of the equations using their transformations. The algorithmic method for finding Hamiltonians of some equations is presented. This approach allows us to look for Hamiltonians without the derivative operator and it can be applied with the aid of programmes of symbolic calculations. The Hamiltonians of three types of the generalized nonlinear Schrödinger equation are found. Examples of Hamiltonians for some equations are presented.

Keywords: nonlinear Schrödinger equation; Hamiltonian; conservation law; optical soliton; conservative quantity

MSC: 37M10; 37K06

1. Introduction

It is well known that the conservation laws of a nonlinear evolution equation are among its most important properties [1–5]. The study of conservation laws allows us to understand the property of integrability of an equation because an infinite number of conservation laws leads to its integrability by the inverse scattering transform. The conservation law allows us to estimate the conservative quantity for the solution of the equation, which is especially important when conducting numerical modeling of the physical processes described by the equations [6–8]. There are several methods for finding conservation laws [9–14].

However, the construction of Hamiltonians for generalized nonlinear Schrodinger equations is a difficult problem since the Lagrangians generating these Hamiltonians are degenerate [8]. In this regard, the Bergman-Dirac formalism is used to construct the corresponding Hamiltonians [15–17], as a result of which a certain set of constraints is imposed that contributes to the Hamiltonian [8].

The essence of our method is that we use direct transformations of the system of equations corresponding to the original generalized nonlinear Schrödinger equation. Compared to other methods, our approach is not as formal as the approaches mentioned above, but it allows us to use the symbolic computation programs for whole classes of generalized nonlinear Schrödinger equations. To the best of our knowledge, this approach has not yet been used to construct Hamiltonians for generalized nonlinear Schrödinger equations.

In this paper, we present a simple algorithm for construction of Hamiltonians using transformations for some generalized nonlinear Schrödinger equations. The method presented can be used to construct Hamiltonians of a number of equations. The advantage of our method lies in its scope of application and in the fact that it allows us to practically construct Hamiltonians for whole classes of nonlinear evolutionary equations. We demonstrate the application of the method for the following types of nonlinear Schrödinger equations:

$$i\,q_t + a\,q_{xx} + \alpha\,F(|q|^2)\,q = 0, \tag{1}$$

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Citation: Kudryashov, N.A. Hamiltonians of the Generalized Nonlinear Schrödinger Equations. *Mathematics* **2023**, *11*, 2304. https://doi.org/10.3390/math11102304

Academic Editor: Xiangmin Jiao

Received: 21 April 2023
Revised: 11 May 2023
Accepted: 13 May 2023
Published: 15 May 2023

$$i\,q_t + a\,q_{xx} + i\,c\,q_{xxx} + b\,q_{xxxx} + \beta\,F(|q|^2)\,q = 0, \tag{2}$$

$$i\,q_t + a\,q_{xx} + i\,c\,q_{xxx} + b\,q_{xxxx} + i\,h\,q_{xxxxx} + d\,q_{xxxxxx} + \beta\,F(|q|^2)\,q = 0 \tag{3}$$

and

$$i\,q_t + a\,q_{xx} + i\,\beta\,\frac{\partial}{\partial x}\Big(F(|q|^2)\,q\Big) = 0. \tag{4}$$

In Equations (1)–(4) $q(x,t)$ is a complex valued function, x and t are the spatial and temporal coordinates; a, b, c, h, d, α and β are parameters of nonlinear differential equation (1). The function $F(|q|^2)$ of $|q|^2$ characterizes the nonlinear properties of the medium [2,3]. Expressions $q_{xx}, q_{xxx}, q_{xxxx}, q_{xxxxx}$ and q_{xxxxxx} correspond to the various types of dispersion of optical solitons.

The Cauchy problem for Equations (1)–(4) in the general case cannot be solved by the inverse scattering transform [18–20]. However, there are a number of nonlinear Schrödinger equations in the form (1) useful for description of the pulse propagation in optical fiber. Using the formula [18],

$$F(|q|^2) = |q|^2, \tag{5}$$

we have the famous nonlinear Schrödinger equation

$$i\,q_t + a\,q_{xx} + \alpha\,|q|^2\,q = 0, \tag{6}$$

where $q(x,t)$ is a complex function, a and α are the parameters of the mathematical model, q_{xx} corresponds to the the light diffraction and $|q|^2\,q$ is the term with Kerr medium.

Taking into account

$$F(|q|^2) = \ln(|q|^2), \tag{7}$$

we obtain from Equation (1) the logarithmic Schrödinger equation [8]

$$i\,q_t + a\,q_{xx} + \alpha\,\log(|q|^2)\,q = 0. \tag{8}$$

In the case of

$$F(|q|^2) = \alpha\,|q|^{-4n} + \beta\,|q|^{-2n} + \gamma\,|q|^{2n} + \delta\,|q|^{4n}, \tag{9}$$

where α, β, γ and δ are parameters of mathematical model, n is the value of refractive index for non-Kerr optical medium. We have the equation in the form [21]

$$i\,q_t + a\,q_{xx} + \Big(\alpha\,|q|^{-4n} + \beta\,|q|^{-2n} + \gamma\,|q|^{2n} + \delta\,|q|^{4n}\Big)\,q = 0. \tag{10}$$

This paper is organized as follows. In Section 2, we describe the method for finding the conservation laws including the Hamiltonians of the generalized nonlinear Schrödinger equations. In this section, we also present examples of some Hamiltonians corresponding to Equation (1). The generalization of the method to classes of Equations (2) and (3) is given in Sections 3 and 4. In Section 5, using our approach, we present the conservation laws for the generalized Kaup–Newell equation.

2. Method Applied

The proposed algorithm for constructing Hamiltonians of nonlinear Schrödinger equations is based on the construction of conservation laws. Apparently, our method does not have the degree of generality of the construction of Hamiltonians characteristic for the classical method of constructing Hamiltonians. However, for the whole class of functions, the method is universal.

In the first step, equations like Equations (1) and (2) are written as the system of equations. In the the second step, each of the equations is multiplied by q^* and by q and resulting expressions are added. As a result, we obtain the first conservation law that is further used for construction of the Hamiltonians. In the third step, we multiply each equation of the system by conjugate functions of derivatives and add the resulting expressions as well. In the fourth step, we multiply the first conservation law by the corresponding expressions and add it to the result obtained earlier. In the final step, we have to obtain expressions containing time derivatives, which we write down in the form of the conservation law from which the Hamiltonian follows.

Let us demonstrate the method for finding the Hamiltonian for Equation (1). Firstly, we write Equation (1) as the system of equations

$$i\, q_t + a\, q_{xx} + \alpha\, F(|q|^2)\, q = 0, \tag{11}$$

and

$$-i\, q_t^* + a\, q_{xx}^* + \alpha\, F(|q|^2)\, q^* = 0. \tag{12}$$

We multiply Equation (11) by q^* and Equation (12) by $-q$ and add the expressions. We have

$$i\, \frac{\partial}{\partial t}\, |q|^2 + a\, (q_{xx}\, q^* - q\, q_{xx}^*) = 0. \tag{13}$$

One can see that Equation (13) can be presented as the conservation law in the form

$$i\, \frac{\partial}{\partial t}\, (|q|^2) + a\, \frac{\partial}{\partial x} (q_x\, q^* - q\, q_x^*) = 0. \tag{14}$$

We get the following conservative quantity

$$P = \int\limits_{-\infty}^{\infty} |q|^2\, dx = Const. \tag{15}$$

After multiplying Equation (11) by q_x^* and Equation (12) by q_x and adding the resulting expressions, we obtain the conservation law as follows:

$$\begin{gathered} \frac{i}{2}\, \frac{\partial}{\partial t}\, (q_x^*\, q - q_x\, q^*) - \frac{i}{2}\, \frac{\partial}{\partial x}\, (q_t^*\, q - q_t\, q^*) + \\ a\, \frac{\partial}{\partial x} (|q|^2) + \alpha\, \frac{\partial}{\partial x}\, (Q(|q|^2)) = 0, \end{gathered} \tag{16}$$

where $Q(|q|^2)$ is the following integral

$$Q(|q|^2) = \int\limits_{0}^{|q|^2} F(\xi) d\xi. \tag{17}$$

From Equation (16), we obtain the conservative quantity in the form

$$M = \int\limits_{-\infty}^{\infty} (q_x^*\, q - q_x\, q^*)\, dx = Const. \tag{18}$$

In order to obtain the Hamiltonian corresponding to Equation (1), we multiply Equation (11) by the function q_{xx}^* and Equation (12) by $-q_{xx}$ and add these expressions, which yields

$$i\, (q_{xx}^*\, q_t + q_{xx}\, q_t^*) + \alpha\, F(|q|^2)\, (q_{xx}^*\, q - q_{xx}\, q^*) = 0. \tag{19}$$

Taking into account the following formula

$$i\, (q_{xx}^*\, q_t + q_{xx}\, q_t^*) = -i\, \frac{\partial}{\partial t} \left(|q_x|^2 \right) + i\, \frac{\partial}{\partial x} (q_x^*\, q_t + q_x\, q_t^*), \tag{20}$$

we rewrite Equation (19) as the following:

$$-i\,\frac{\partial}{\partial t}\Big(|q_x|^2\Big)+i\,\frac{\partial}{\partial x}(q_x^*\,q_t+q_x\,q_t^*)+\alpha\,F(|q|^2)\,(q_{xx}^*\,q-q_{xx}\,q^*). \tag{21}$$

Multiplying (13) by $-a\,F(|q|^2)$ and adding it to Equation (21) multiplied by a, we get the following expression

$$-i\,a\,\frac{\partial}{\partial t}\Big(|q_x|^2\Big)+i\,a\,\frac{\partial}{\partial x}(q_x^*\,q_t+q_x\,q_t^*)+i\,\alpha\,F(|q|^2)\,\frac{\partial}{\partial t}(|q|^2)=0. \tag{22}$$

Equation (22) allows us to obtain the Hamiltonians for many nonlinear Schrödinger equations. Equation (22) can be written as the conservation law in the form

$$\frac{\partial H}{\partial t}+\frac{\partial X}{\partial x}=0, \tag{23}$$

where H and X are expressed by the formulas

$$H=a\,|q_x|^2-\alpha\,Q(|q|^2),\qquad Q(|q|^2)=\int\limits_0^{|q|^2}F(\xi)d\xi \tag{24}$$

and

$$X=q_x^*\,q_t+q_x\,q_t^*. \tag{25}$$

Formula (24) gives the Hamiltonian for the nonlinear Schrödinger equation (1). One can note that the the conservative quantities P and M do not depend on the form of the function $F(|q|^2)$ but the Hamiltonian corresponding to Equation (1) does.

Let us present some examples.

Example 1. *The generalized nonlinear Schrödinger equation [2]*

$$i\,q_t+a\,q_{xx}+\alpha\,|q|^{2n}\,q=0, \tag{26}$$

where $q(x,t)$ is a complex function, a and α are the parameters, n is the value of refractive index. The term with the second derivative corresponds to the diffraction of the light. The nonlinear expression $|q|^{2n}\,q$ is responsible for the reflection of light in the optical fiber. Equation (26) has the function $F(|q|^2)$ as the following

$$F=|q|^{2n} \tag{27}$$

and the Hamiltonian in the form

$$H_1=\frac{\alpha\,|q|^{2n+2}}{n+1}-a\,|q_x|^2. \tag{28}$$

Example 2. *The logarithmic nonlinear Schrödinger equation [8]*

$$i\,q_t+a\,q_{xx}+\alpha\,\ln(|q|^2)\,q=0, \tag{29}$$

with the function F in the form

$$F_2=\ln(|q|^2) \tag{30}$$

has the following Hamiltonian

$$H_2=a\,|q_x|^2+\alpha\,|q|^2)-|q|^2\,\ln(|q|^2). \tag{31}$$

Example 3. *The equation for description of the pulse propagation in saturable medium [22–27]*

$$i\,q_t + a\,q_{xx} + \alpha |q|^2\,q - m\left(q - \frac{q}{(1 + a\,|q|^2)^p}\right) = 0. \tag{32}$$

Using the function $F(|q|^2)$ in the from

$$F(|q|^2) = \alpha\,|q|^2 - m\left(1 - \frac{1}{(1 + a\,|q|^2)^p}\right) \tag{33}$$

we obtain at $p \neq 1$ the following Hamiltonian

$$H_{3a} = a\,|q_x|^2 - \alpha\,\frac{|q|^4}{2} - m\,|q|^2 + \frac{(1 + b\,|q|^2)^{1-p}}{b\,(p-1)}. \tag{34}$$

In the case $p = 1$, we have the Hamiltonian corresponding to Equation (32) in the form

$$H_{3b} = a\,|q_x|^2 - \alpha\,\frac{|q|^4}{2} - m\,|q|^2 + m\,\ln(1 + a\,|q|^2). \tag{35}$$

Example 4. *The generalized nonlinear Schrödinger equation with anti-cubic nonlinearities [28–32]*

$$i\,q_t + a\,q_{xx} + \left(b_1\,|q|^{-4} + b_2\,|q|^2 + b_3\,|q|^4\right)q = 0. \tag{36}$$

The function $F(|q|^2)$ is determined by the formula

$$F(|q|^2) = b_1\,|q|^{-4} + b_2\,|q|^2 + b_3\,|q|^4 \tag{37}$$

and the Hamiltonian has the form

$$H_4 = a\,|q_x|^2 - b_1\,|q|^{-2} + \frac{b_2}{2}\,|q|^4 + \frac{b_3}{3}\,|q|^6. \tag{38}$$

Example 5. *The generalized nonlinear Schrödinger equation with four nonlinearities [33–36]*

$$i\,q_t + a\,q_{xx} + \left(\alpha\,|q|^{2n} + \beta\,|q|^{4n} + \gamma\,|q|^{6n} + \delta\,|q|^{8n}\right)q = 0. \tag{39}$$

The function $F(|q|^2)$ takes the form

$$F(|q|^2) = \alpha\,|q|^{2n} + \beta\,|q|^{4n} + \gamma\,|q|^{6n} + \delta\,|q|^{8n} \tag{40}$$

and the Hamiltonian is as follows:

$$\begin{aligned} H_5 = a\,|q_x|^2 - \Bigg(&\frac{\alpha}{n+1}\,|q|^{2n+2} + \frac{\beta}{2\,n+1}\,|q|^{4n+2} + \\ &\frac{\gamma}{3n+1}\,|q|^{6n+2} + \frac{\delta}{4n+1}\,|q|^{8n+2}\Bigg). \end{aligned} \tag{41}$$

It is clear that using formula (24) we can find a number of Hamiltonians which correspond to Equation (1).

3. Hamiltonian of the Generalized Nonlinear Schrödinger Equation of the Fourth Order

Let us demonstrate that we can use the same algorithm for finding the Hamiltonian of Equation (2). First of all, we write Equation (2) as the system of equations

$$i\,q_t + a\,q_{xx} + i\,c\,q_{xxx} + b\,q_{xxxx} + \beta\,F(|q|^2)\,q = 0, \tag{42}$$

and

$$-i\,q_t^* + a\,q_{xx}^* - i\,c\,q_{xxx} + b\,q_{xxxx}^* + \beta\,F(|q|^2)\,q^* = 0. \tag{43}$$

Multiplying Equation (42) by q^* and Equation (43) by $-q$ and adding the expressions obtained, we have the equation

$$\begin{aligned} i\,\frac{\partial}{\partial t}\,|q|^2 + a\,(q^*\,q_{xx} - q\,q_{xx}^*) + i\,c\,(q^*\,q_{xxx} + q\,q_{xxx}^*) + \\ b\,(q^*\,q_{xxxx} - q\,q_{xxxx}^*) = 0. \end{aligned} \tag{44}$$

Equation (44) can be written as the conservation law in the form

$$\begin{aligned} i\,\frac{\partial}{\partial t}\,|q|^2 + a\,\frac{\partial}{\partial x}\,(q^*\,q_x - q\,q_x^*) + i\,c\,\frac{\partial}{\partial x}(q^*\,q_{xx} + q\,q_{xx}^* - |q_x|^2) + \\ b\,\frac{\partial}{\partial x}\,(q^*\,q_{xxx} - q\,q_{xxx}^*) - b\,\frac{\partial}{\partial x}\,(q_x^*\,q_{xx} - q_x\,q_{xx}^*) = 0. \end{aligned} \tag{45}$$

As a result, we have the conservative quantity for the function q in the form of Equation (15). We also obtain the conservative quantity (18) using the approach presented in Section 2.

One can find the Hamiltonian corresponding to Equation (2) if we multiply Equation (42) by q_{xx}^* and Equation (43) by $-q_{xx}$ and add these expressions. We have

$$\begin{aligned} i\,(q_{xx}^*\,q_t + q_{xx}\,q_t^*) + b\,(q_{xx}^*\,q_{xxxx} - q_{xx}\,q_{xxxx}^*) + \\ i\,c\,(q_{xx}\,q_{xxx}^* + q_{xx}^*\,q_{xxx}) + \beta\,F(|q|^2)\,(q\,q_{xx}^* - q^*\,q_{xx}) = 0. \end{aligned} \tag{46}$$

After multiplying Equation (42) by q_{xxx}^* and Equation (43) by q_{xxx} and adding them, we obtain

$$\begin{aligned} i\,(q_{xxx}^*\,q_t - q_{xxx}\,q_t^*) + a\,(q_{xxx}^*\,q_{xx} + q_{xxx}\,q_{xx}^*) + \\ b\,(q_{xxx}^*\,q_{xxxx} + q_{xxx}\,q_{xxxx}^*) + \beta\,F(|q|^2)\,(q\,q_{xxx}^* + q^*\,q_{xxx}) = 0. \end{aligned} \tag{47}$$

Multiplying Equation (42) by q_{xxxx}^* and (43) by q_{xxxx} and adding the resulting expressions, we obtain the equation

$$\begin{aligned} i\,(q_{xxxx}^*\,q_t + q_{xxxx}\,q_t^*) + a\,(q_{xxxx}^*\,q_{xx} - q_{xxxx}\,q_{xx}^*) + \\ +i\,c\,(q_{xxxx}^*\,q_{xxx} + q_{xxxx}\,q_{xxx}^*) + \beta\,F(|q|^2)\,(q\,q_{xxxx}^* - q^*\,q_{xxxx}) = 0. \end{aligned} \tag{48}$$

Adding Equation (44) multiplied by $F(|q|^2)$, Equation (46) multiplied by a, Equation (47) multiplied by $-i\,c$ and Equation (48) multiplied by b, yields the equation

$$\begin{aligned} i\,a\,(q_{xx}^*\,q_t + q_{xx}\,q_t^*) + i\,b\,(q_{xxxx}^*\,q_t + q_{xxxx}\,q_t^*) + \\ c\,(q_{xxx}^*\,q_t - q_{xxx}\,q_t^*) + i\,\beta\,F(|q|^2)\,\frac{\partial}{\partial t}\,|q|^2 = 0. \end{aligned} \tag{49}$$

Using the formulas

$$q_{xx}^*\,q_t + q_{xx}\,q_t^* = \frac{\partial}{\partial x}(q_x\,q_t^* + q_t\,q_x^*) - \frac{\partial}{\partial t}\,|q_x|^2, \tag{50}$$

$$\begin{aligned} q_{xxx}^*\,q_t - q_{xxx}\,q_t^* = \frac{1}{2}\,\frac{\partial}{\partial t}(q_{xxx}^*\,q - q_{xxx}\,q^*) - \frac{1}{2}\,\frac{\partial}{\partial x}(q_{xxt}^*\,q - q_{xxt}\,q^*) - \\ -\frac{1}{2}\,\frac{\partial}{\partial x}(q_{xt}^*\,q_x - q_{xt}\,q_x^*) - \frac{1}{2}\,\frac{\partial}{\partial x}(q_t^*\,q_{xx} - q_t\,q_{xx}^*) \end{aligned} \tag{51}$$

and

$$\begin{aligned} q_{xxxx}^*\,q_t + q_{xxxx}\,q_t^* = \frac{\partial}{\partial x}(q_{xxx}\,q_t^* + q_t\,q_{xxx}^*) - \\ \frac{\partial}{\partial x}(q_{xx}\,q_{xt}^* + q_{xt}\,q_{xx}^*) + \frac{\partial}{\partial t}\,|q_{xx}|^2. \end{aligned} \tag{52}$$

we rewrite Equation (49) in the form of the following conservation law:

$$b\,\frac{\partial}{\partial t}\,|q_{xx}|^2-a\,\frac{\partial}{\partial t}\,|q_x|^2-\frac{i\,c}{2}\frac{\partial}{\partial t}(q_{xxx}^*\,q-q_{xxx}\,q^*)+\beta\,\frac{\partial}{\partial t}\,Q(|q|^2)+$$
$$+a\frac{\partial}{\partial x}(q_x\,q_t^*+q_t\,q_x^*)+b\,\frac{\partial}{\partial x}(q_{xxx}\,q_t^*+q_t\,q_{xxx}^*-q_{xx}\,q_{xt}^*-q_{xt}\,q_{xx}^*)+$$
$$\frac{i\,c}{2}\frac{\partial}{\partial x}(q\,q_{xxt}^*-q^*\,q_{xxt}-q_x\,q_{xt}^*+q_x^*\,q_{xt}+q_{xx}\,q_t^*-q_{xx}^*\,q_t)=0, \tag{53}$$

where $(Q(|q|^2))$ is determined by the formula

$$Q(|q|^2)=\int\limits_0^{|q|^2}F(\xi)d\xi \tag{54}$$

From the conservation law (53), we get the Hamiltonian of Equation (2) in the form

$$H_3=b\,|q_{xx}|^2-a\,|q_x|^2+\beta\,Q(|q|^2)-\frac{i\,c}{2}\,(q\,q_{xxx}^*-q^*\,q_{xxx}). \tag{55}$$

Let us again consider examples of Hamiltonians.

Example 6. *The generalized nonlinear Schrödinger equation with nonlinearities [37–42]*

$$i\,q_t+a\,q_{xx}+b\,q_{xxxx}+\left(\alpha\,|q|^{2n}+\beta\,|q|^{4n}+\gamma\,|q|^{6n}+\delta\,|q|^{8n}\right)q=0. \tag{56}$$

The function $F(|q|^2)$ takes the form

$$F(|q|^2)=\alpha\,|q|^{2n}+\beta\,|q|^{4n}+\gamma\,|q|^{6n}+\delta\,|q|^{8n} \tag{57}$$

and the Hamiltonian is as follows

$$H_4=b\,|q_{xx}|^2-a\,|q_x|^2-\left(\frac{\alpha}{n+1}\,|q|^{2n+2}+\frac{\beta}{2\,n+1}\,|q|^{4n+2}+\right.$$
$$\left.\frac{\gamma}{3n+1}\,|q|^{6n+2}+\frac{\delta}{4n+1}\,|q|^{8n+2}\right). \tag{58}$$

Some other examples of Hamiltonians (55) can be found for other functions $F(|q|^2)$ too.

4. Hamiltonian of the Generalized Nonlinear Schrödinger Equation of the Sixth Order

Using the presented approach we can find the Hamiltonian of Equation (3). We write the system of equations corresponding to Equation (3) in the form

$$i\,q_t+a\,q_{xx}+i\,c\,q_{xxx}+b\,q_{xxxx}+i\,h\,q_{xxxxx}+d\,q_{xxxxxx}+\beta\,F(|q|^2)\,q=0 \tag{59}$$

and

$$-i\,q_t^*+a\,q_{xx}^*-c\,q_{xxx}^*+bq_{xxxx}^*-i\,h\,q_{xxxxx}^*+dq_{xxxxxx}^*+\beta F(|q|^2)\,q^*=0. \tag{60}$$

Multiplying Equation (59) by q^* and Equation (60) by $-q^*$ and adding them, we obtain the equation

$$i\,\frac{\partial}{\partial t}\,|q|^2+a\,(q^*\,q_{xx}-q\,q_{xx}^*)+i\,c\,(q^*\,q_{xxx}+q\,q_{xxx}^*)+b\,(q^*\,q_{xxxx}-$$
$$q\,q_{xxxx}^*)+i\,h\,(q^*\,q_{xxxxx}+q\,q_{xxxxx}^*)+d\,(q^*\,q_{xxxxxx}-q\,q_{xxxxxx}^*)=0. \tag{61}$$

From Equation (61) follows the conservation Law

$$i\,\frac{\partial}{\partial t}\,|q|^2 + a\,\frac{\partial}{\partial x}\,(q^*\,q_x - q\,q_x^*) + i\,c\,\frac{\partial}{\partial x}(q^*\,q_{xx} + q\,q_{xx}^* - |q_x|^2) + b\,\frac{\partial}{\partial x}\,(q^*\,q_{xxx} - q\,q_{xxx}^* - q_x^*\,q_{xx} + q_x\,q_{xx}^*) + i\,h\,\frac{\partial}{\partial x}\,(q^*\,q_{xxxx} + q\,q_{xxxx}^* - q_x^*\,q_{xxx} - q_x\,q_{xxx}^* + |q_{xx}|^2) + d\,\frac{\partial}{\partial x}\,(q_{xxxxx}^*\,q - q^*\,q_{xxxxx} - q_{xxxx}^*\,q_x + q_x^*\,q_{xxxx} + q_{xxx}^*\,q_{xx} - q_{xx}^*\,q_{xxx}) = 0. \tag{62}$$

As a result, we have the conservative quantity for the function q in the form of Equation (15). One can obtain the conservative quantity (18) using the approach mentioned in Section 2.

Further, we multiply Equation (59) by q_{xx}^* and (60) by $-q_{xx}$ and add the resulting expressions together. We have the equation

$$i\,(q_{xx}^*\,q_t + q_{xx}\,q_t^*) + i\,c\,(q_{xx}^*\,q_{xxx} + q_{xx}\,q_{xxx}^*) + b\,(q_{xx}^*\,q_{xxxx} - q_{xx}\,q_{xxxx}^*) + {}+i\,h\,(q_{xx}^*\,q_{xxxxx} + q_{xx}\,q_{xxxxx}^*) + d\,(q_{xx}^*\,q_{xxxxxx} - q_{xx}\,q_{xxxxxx}^*) + \beta\,F\,(q_{xx}^*\,q - q^*\,q_{xx}) = 0 \tag{63}$$

By multiplying Equation (59) by q_{xxx}^* and (60) by q_{xxx} and adding them, we obtain the equation

$$i\,(q_{xxx}^*\,q_t - q_{xxx}\,q_t^*) + a\,(q_{xxx}^*\,q_{xx} + q_{xxx}\,q_{xx}^*) + b\,(q_{xxx}^*\,q_{xxxx} + q_{xxx}\,q_{xxxx}^*) + {}+i\,h\,(q_{xxx}^*\,q_{xxxxx} - q_{xxx}\,q_{xxxxx}^*) + d\,(q_{xxx}^*\,q_{xxxxxx} + q_{xxx}\,q_{xxxxxx}^*) + \beta\,F\,(q_{xxx}^*\,q + q^*\,q_{xxx}) = 0. \tag{64}$$

We also multiply Equation (59) by q_{xxxx}^* and (60) by $-q_{xxxx}$ and add them

$$i\,(q_{xxxx}^*\,q_t + q_{xxxx}\,q_t^*) + a\,(q_{xxxx}^*\,q_{xx} - q_{xxxx}\,q_{xx}^*) + ic\,(q_{xxxx}^*\,q_{xxx} + q_{xxxx}\,q_{xxx}^*) + i\,h\,(q_{xxxx}^*\,q_{xxxxx} + q_{xxxx}\,q_{xxxxx}^*) + d\,(q_{xxxx}^*\,q_{xxxxxx} - q_{xxxx}\,q_{xxxxxx}^*) + \beta\,F\,(q_{xxxx}^*\,q - q^*\,q_{xxxx}) = 0. \tag{65}$$

We multiply Equation (59) by q_{xxxxx}^* and (60) by q_{xxxxx} and add the resulting expressions together, which yields

$$i\,(q_{xxxxx}^*\,q_t - q_{xxxxx}\,q_t^*) + a\,(q_{xxxxx}^*\,q_{xx} + q_{xxxxx}\,q_{xx}^*) + b\,(q_{xxxxx}^*\,q_{xxxxx} + q_{xxxxx}\,q_{xxxx}^*) + i\,c\,(q_{xxxxx}^*\,q_{xxx} - q_{xxxxx}\,q_{xxx}^*) + d\,(q_{xxxxx}^*\,q_{xxxxxx} + q_{xxxxx}\,q_{xxxxxx}^*) + \beta\,F\,(q_{xxxxx}^*\,q + q^*\,q_{xxxxx}) = 0. \tag{66}$$

At last, by multiplying Equation (59) by q_{xxxxxx}^* and (60) by $-q_{xxxxxx}$ and adding, we get

$$i\,(q_{xxxxxx}^*\,q_t + q_{xxxxxx}\,q_t^*) + a\,(q_{xxxxxx}^*\,q_{xx} - q_{xxxxxx}\,q_{xx}^*) + ic\,(q_{xxxxxx}^*\,q_{xxx} + q_{xxxxxx}\,q_{xxx}^*) + i\,h\,(q_{xxxxxx}^*\,q_{xxxxx} + q_{xxxxxx}\,q_{xxxxx}^*) + b\,(q_{xxxxxx}^*\,q_{xxxx} - q_{xxxxxx}\,q_{xxxx}^*) + \beta\,F\,(q_{xxxxxx}^*\,q - q^*\,q_{xxxxxx}) = 0. \tag{67}$$

Taking into consideration the following direct calculations:

$$\beta\,F(|q|^2)\,*\,\text{Equation (61)} - i\,a\,*\,\text{Equation (63)} - c\,*\,\text{Equation (64)} - i\,b\,*\,\text{Equation (65)} - h\,*\,\text{Equation (66)} - i\,d\,*\,\text{Equation (67)} = 0, \tag{68}$$

we obtain the equation

$$
\begin{aligned}
a(q_t\, q^*_{xx} + q^*_t\, q_{xx}) - i\, c\,(q_t\, q^*_{xxx} - q^*_t\, q_{xxx}) + b\,(q_t\, q^*_{xxxx} + q^*_t\, q_{xxxx}) -\\
i\, h\,(q_t\, q^*_{xxxxx} - q^*_t\, q_{xxxxx}) + d\,(q_t\, q^*_{xxxxxx} + q^*_t\, q_{xxxxxx}) +\\
\beta\, F(|q|^2)\,\frac{\partial}{\partial t}(|q|^2) = 0.
\end{aligned}
\tag{69}
$$

Using Equations (50)–(52) and the formulas

$$
\begin{aligned}
q^*_{xxxxx}\, q_t - q_{xxxxx}\, q^*_t = \frac{1}{2}\frac{\partial}{\partial t}(q^*_{xxxxx}\, q - q_{xxxxx}\, q^*) - \frac{1}{2}\frac{\partial}{\partial x}(q^*_{xxxxt}\, q -\\
q_{xxxxt}\, q^*) + \frac{1}{2}\frac{\partial}{\partial x}(q^*_{xxxt}\, q_x - q_{xxxt}\, q^*_x) - \frac{1}{2}\frac{\partial}{\partial x}(q^*_{xxt}\, q_{xx} - q_{xxt}\, q^*_{xx}) +\\
\frac{1}{2}\frac{\partial}{\partial x}(q^*_{xt}\, q_{xxx} - q_{xt}\, q^*_{xxx}) - \frac{1}{2}\frac{\partial}{\partial x}(q^*_t\, q_{xxxx} - q_t\, q^*_{xxxx})
\end{aligned}
\tag{70}
$$

and

$$
\begin{aligned}
q^*_{xxxxxx}\, q_t + q_{xxxxxx}\, q^*_t = \frac{\partial}{\partial x}(q_{xxxxx}\, q^*_t + q_t\, q^*_{xxxxx}) -\\
\frac{\partial}{\partial x}(q_{xxxx}\, q^*_{xt} + q_{xt}\, q^*_{xxxx}) + \frac{\partial}{\partial x}(q_{xxx}\, q^*_{xxt} + q_{xxt}\, q^*_{xxx}) - \frac{\partial}{\partial t}\,|q_{xxx}|^2
\end{aligned}
\tag{71}
$$

we get the conservation law corresponding to Equation (3). From this conservation law, we have the Hamiltonian of Equation (3) in the form

$$
\begin{aligned}
H_5 = Q(|q|^2) - d\,|q_{xxx}|^2 + b\,|q_{xx}|^2 - a\,|q_x|^2 -\\
i\, c\,(q^*_{xxx}\, q - q_{xxx}\, q^*) - i\, h\,(q^*_{xxxxx}\, q - q_{xxxxx}\, q^*).
\end{aligned}
\tag{72}
$$

Using the explicit expression of $F(|q|^2)$ in Equation (54), we can obtain a number of Hamiltonians corresponding to Equation (2).

5. Conservation Laws of the Generalized Kaup–Newell Equation

A modification of the method can be used to construct the conservation laws of Equation (4). We present Equation (4) as the system of equations again

$$
i\, q_t + a\, q_{xx} + i\,\beta\,\frac{\partial}{\partial x}\Big(F(|q|^2)\, q\Big) = 0,
\tag{73}
$$

and

$$
-i\, q^*_t + a\, q^*_{xx} - i\,\beta\,\frac{\partial}{\partial x}\Big(F(|q|^2)\, q\Big) = 0.
\tag{74}
$$

Multiplying Equation (73) by q^* and Equation (74) by q and adding these expressions, we obtain the equation

$$
i\,\frac{\partial}{\partial t}(|q|^2) + a\,(q^*\, q_{xx} - q\, q^*_{xx}) + i\,\beta\,\frac{\partial}{\partial x}\Big(|q|^2\, F(|q|^2)\Big) = 0.
\tag{75}
$$

One can note that Equation (75) is written in the form of the conservation Law at $F = F(|q|^2)$.

At the next step, we multiply Equation (73) by q^*_x and Equation (74) by q_x. After adding the resulting expressions, we obtain

$$
\begin{aligned}
\frac{i}{2}\frac{\partial}{\partial x}(q^*\, q_t - q\, q^*_t) + a\,\frac{\partial}{\partial x}\Big(|q_x|^2\Big) - i\,\beta\,(q_x\, q^* - q\, q^*_x)\frac{\partial}{\partial x}\Big(F(|q|^2)\Big) -\\
\frac{i}{2}\frac{\partial}{\partial t}(q^*\, q_x - q\, q^*_x) = 0
\end{aligned}
\tag{76}
$$

By multiplying Equation (75) by $i\,\beta\,F(|q|^2)$ and Equation (76) by $-a$ and adding, we obtain the conservation law in the form

$$\begin{gathered}\frac{i\,a}{2}\,\frac{\partial}{\partial t}(q^*\,q_x - q\,q_x^*) - \beta\,F(|q|^2)\,\frac{\partial}{\partial t}(|q|^2) - \\ \frac{i\,a}{2}\,\frac{\partial}{\partial x}(q^*\,q_t - q\,q_t^*) - \beta^2\,\frac{\partial}{\partial x}\Big(|q|^2\,F(|q|^2)\Big) - a^2\,\frac{\partial}{\partial x}\,(|q_x|^2) + \\ i\,\beta\,a\,\frac{\partial}{\partial x}\Big(F(|q|^2)(q_x\,q^* - q\,q_x^*)\Big) = 0.\end{gathered} \tag{77}$$

The last equation can be presented in the form of conservation law

$$\frac{\partial H_6}{\partial t} + \frac{\partial X_2}{\partial x} = 0 \tag{78}$$

where H_6 and X_2 are determined by the formulas

$$H_6 = \frac{i\,a}{2}\,(q^*\,q_x - q\,q_x^*) - \beta\,Q(|q|^2), \qquad Q(|q|^2) = \int\limits_0^{|q|^2} F(\xi)\,d\xi \tag{79}$$

and

$$\begin{gathered}X_2 = \frac{i\,a}{2}\,(q^*\,q_t - q\,q_t^*) - \beta^2\,\Big(|q|^2\,F(|q|^2)\Big) - a^2\,(|q_x|^2) + \\ i\,\beta\,a\,\Big(F(|q|^2)(q_x\,q^* - q\,q_x^*)\Big) = 0.\end{gathered} \tag{80}$$

The conservative quantity T_2 does not give the Hamiltonian. This value corresponds to conservative density in the form

$$T_2 = \int\limits_{-\infty}^{\infty}\left[\frac{i\,a}{2}\,(q^*\,q_x - q\,q_x^*) - \beta\,Q(|q|^2)\right]dx = Const. \tag{81}$$

The conservation law (81) at $F = |q|^2$ was obtained in paper [14]; in this work, it has been indicated that this law does not correspond to either energy or momentum. It seems to be a hybrid of the Hamiltonian at $a = 0$ and the momentum in the case $\beta = 0$.

6. Conclusions

In this paper, we have considered the generalized nonlinear Schrödinger equations of the second, fourth and sixth order with integrable nonlinearity in the form (17). In contrast to the generally accepted approaches by means of the Euler–Lagrange operators used before for the construction of Hamiltonians of evolution differential equations, in this paper, we used direct transformations of the original system of equations. We have shown that this rather large class of nonlinear Schrödinger equations has three conservation laws characterizing the power, moment and energy of the wave. One of the obtained conservation laws corresponds to the Hamiltonian of the original equation. We present six examples of constructed Hamiltonians of the well-known nonlinear Schrödinger equations to demonstrate our method.

Funding: This research was supported by Russian Science Foundation Grant No. 23-41-00070, https://rscf.ru/en/project/23-41-00070/ (accessed on 12 May 2023).

Data Availability Statement: Not applicable.

Conflicts of Interest: The author also declares that there is no conflict of interest.

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