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Ulam–Hyers Stability and Well-Posedness of Fixed Point Problems in C^*-Algebra Valued Bipolar b-Metric Spaces

Manoj Kumar [1], Pankaj Kumar [1], Ali Mutlu [2], Rajagopalan Ramaswamy [3,*], Ola A. Ashour Abdelnaby [3,4] and Stojan Radenović [5]

1 Department of Mathematics, Baba Mastnath University, Asthal Bohar, Rohtak 124021, Haryana, India
2 Department of Mathematics, Faculty of Science and Arts, Manisa Celal Bayar University, Martyr Prof. Dr. İlhan Varank Campus, 45140 Manisa, Türkiye
3 Department of Mathematics, College of Science and Humanities in Alkharj, Prince Sattam bin Abdulaziz University, AlKharj 11942, Saudi Arabia
4 Department of Mathematics, Cairo University, Cairo 12613, Egypt
5 Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Belgrade, Serbia
* Correspondence: r.gopalan@psau.edu.sa

Abstract: Here, we shall introduce the new notion of C^*-algebra valued bipolar b-metric spaces as a generalization of usual metric spaces, C^*-algebra valued metric space, b-metric spaces. In the above-mentioned spaces, we shall define $(\alpha_{\mathfrak{A}} - \psi_{\mathfrak{A}})$ contractions and prove some fixed point theorems for these contractions. Some existing results from the literature are also proved by using our main results. As an application Ulam–Hyers stability and well-posedness of fixed point problems are also discussed. Some examples are also given to illustrate our results.

Keywords: C^*-algebra valued bipolar b-metric space; covariant mapping; contravariant mapping; fixed points; Ulam–Hyers stability; well-posdeness

MSC: 47H10; 54H25

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1. Introduction

In 1922, Banach [1] gave a constructive method to obtain a fixed point for a self-map in metric spaces. Since then, the researchers have given many generalizations to the Banach contraction theorem and metric space to obtain new results in fixed point theory (see [2–5]). In 1990, Murphi [6] gave the C^*-algebra and operator theory. For further reference on operator theory and related topics, one can see [7,8].

In the spade of generalizations, in 2014, Ma et al. [9] granted C^*-algebra valued metric space and proved fixed point results thereon. Later [10,11] established fixed point results in the setting of C^*-algebra valued b-metric spaces.

In 2016, Mutlu et al. [12] generalized the notion of metric on two different sets and called that space as bipolar metric space. In continuation of this, Mutlu et al. [13] defined $\alpha - \psi$ contractive mappings and multivalued mappings, respectively, and proved fixed point theorems in the setting of bipolar metric spaces.

In 2022, Saha and Roy [14] introduced the new notion of bipolar p-metric spaces as an extension of predefined spaces like usual metric spaces, b-metric spaces, and p-metric spaces.

In the same year, Murthy et al. [15] proved some fixed point theorems via Meir–Keeler contraction in bipolar metric spaces.

Recently in 2022, Mani et al. [16] presented the concept of C^*-algebra valued bipolar metric space as a generalization of C^*-algebra valued (introduced by Ma et al., in 2014) [9] and bipolar metric space (introduced by Mutlu and Gürdal in 2016) [12] and established fixed point theorems on C^*-algebra valued bipolar metric space.

Inspired by the present article, we generalize the notion of C^*-algebra valued bipolar metric space and introduce a generalized C^*-algebra valued bipolar b - metric space and establish fixed point results. We supplement the derived results with nontrivial examples.

As a matter of fact, the remaining portion of the paper is organized as follows: Section 2 must go over some definitions and monographs that are considered necessary for our main results. In Section 3, we introduce generalized C^*-algebra valued bipolar b-metric space and establish fixed point results supported by examples. In Section 4, we addressed some results which are direct consequences of our key results. As an application of the finding of this study, Sections 5 and 6 discuss Ulam-stability Hyer's and the well-posedness of fixed point problems.

2. Preliminaries

The following are the basic notions from literature which are required for proving results.

Definition 1 ([10]). *Let $\psi : \mathfrak{A} \to \mathfrak{B}$ is a mapping in C^* algebra which is linear, then it is called positive if $\psi(\mathfrak{A}_+) \subseteq \mathfrak{B}_+$. Here, $\psi(\mathfrak{A}_h) \subseteq \mathfrak{B}_h$, and the map $\psi : \mathfrak{A}_h \to \mathfrak{B}_h$ is monotonically increasing in nature.*

Definition 2 ([10]). *Assume that $\mathfrak{A}$ and $\mathfrak{B}$ are two C^*-algebras, then $\psi : \mathfrak{A} \to \mathfrak{B}$ is said to be C^*-homomorphism if the following conditions are holds.*

1. $\psi(\mathfrak{p}\mathfrak{g} + \mathfrak{q}\mathfrak{f}) = \mathfrak{p}\psi(\mathfrak{g}) + \mathfrak{q}\psi(\mathfrak{f})$ *for all* $\mathfrak{p}, \mathfrak{q} \in \mathbb{C}$ *and* $\mathfrak{g}, \mathfrak{f} \in \mathfrak{A}$;
2. $\psi(\mathfrak{g}\mathfrak{f}) = \psi(\mathfrak{g})\psi(\mathfrak{f})$ *for all* $\mathfrak{g}, \mathfrak{f} \in \mathfrak{A}$;
3. $\psi(\mathfrak{g}^*) = \psi(\mathfrak{g})^*$ *for all* $\mathfrak{g} \in \mathfrak{A}$;
4. ψ *takes the unity of* $\mathfrak{A}$ *to the unity of* $\mathfrak{B}$.

Definition 3 ([10]). *Let $\Psi_{\mathfrak{A}}$ be the collection of all positive functions $\psi_{\mathfrak{A}} : \mathfrak{A}_+ \to \mathfrak{A}_+$ possessing the following properties:*

1. $\psi_{\mathfrak{A}}$ *having property of continuity and non decreasing;*
2. $\psi_{\mathfrak{A}}(\mathfrak{z}) = 0$ *if and only if* $\mathfrak{z} = 0$;
3. $\sum_{n=1}^{\infty} \psi_{\mathfrak{A}}^n(\mathfrak{z}) < \infty$ *and* $\lim n \to \infty \psi_{\mathfrak{A}}^n(\mathfrak{z}) = 0$ *for each* $\mathfrak{z} \succ 0$, *where* $\psi_{\mathfrak{A}}^n$ *is the* n^{th} *iteration of* $\psi_{\mathfrak{A}}$;
4. *The summation* $\sum_{k=1}^{\infty} \mathfrak{b}^k \psi_{\mathfrak{A}}^k(\mathfrak{z}) < \infty$, *for all* $\mathfrak{z} \succ 0$ *is monotonically non decreasing and having continuity at* 0.

Lemma 1 ([6–8]). *Let us take $\mathfrak{A}$ a unital C^*-algebra with unital $I_{\mathfrak{A}}$. Then:*

1. *Suppose* $\mathfrak{z} \in \mathfrak{A}_+$, *with* $||\mathfrak{z}|| < \frac{1}{2}$, *then* $I - \mathfrak{z}$ *has inverse and* $||\mathfrak{z}(I - \mathfrak{z})^{-1}|| < 1$;
2. *If* $\mathfrak{x} \in \mathfrak{A}$ *and* $\mathfrak{z}, \mathfrak{w} \in \mathfrak{A}_+$ *with* $\mathfrak{z} \preceq \mathfrak{w}$, *then* $\mathfrak{x}^*\mathfrak{z}\mathfrak{x}$ *and* $\mathfrak{x}^*\mathfrak{w}\mathfrak{x}$ *are positive elements and satisfying* $\mathfrak{x}^*\mathfrak{a}\mathfrak{x} \preceq \mathfrak{x}^*\mathfrak{b}\mathfrak{x}$;
3. *If* $0_A \preceq \mathfrak{z} \preceq \mathfrak{w}$, *then* $||\mathfrak{z}|| \leq ||\mathfrak{w}||$;
4. *Let* $\mathfrak{A}'$ *denote the set* $\{\mathfrak{a} \in \mathfrak{A} : \mathfrak{z}\mathfrak{w} = \mathfrak{w}\mathfrak{z} \forall \mathfrak{w} \in \mathfrak{A}\}$ *and let* $\mathfrak{w} \in \mathfrak{A}'$, *if* $\mathfrak{z}, \mathfrak{k} \in \mathfrak{A}$ *with* $\mathfrak{w} \succeq \mathfrak{k} \succeq 0_{\mathfrak{A}}$ *and* $I - \mathfrak{z} \in (\mathfrak{A}')_+$ *has an inverse, then* $(I_{\mathfrak{A}} - \mathfrak{z})^{-1}\mathfrak{w} \preceq (I_{\mathfrak{A}} - \mathfrak{z})^{-1}\mathfrak{k}$;
5. *If* $\mathfrak{z}, \mathfrak{w} \in \mathfrak{A}_+$ *and* $\mathfrak{z}\mathfrak{w} = \mathfrak{w}\mathfrak{z}$, *then* $\mathfrak{z}.\mathfrak{w} \succeq I_{\mathfrak{A}}$.

In 2021, Saha et al. [14] introduced the well-posedness of fixed point problem in bipolar b-metric space as defined below:

Definition 4 ([14]). *Let $(\Sigma, \mathfrak{N}, \sigma_b)$ be a bipolar-b metric space and $F : (\Sigma, \mathfrak{N}) \rightrightarrows (\Sigma, \mathfrak{N})$ is a covariant function. Then, the fixed point problem of F is said to be well-posed if*

1. $\mathfrak{F}$ *possesses a fixed point* $u \in \Sigma \cap \Pi$, *which is unique ;*
2. *For any sequence* (ϑ_n, ϱ_n) *in* (Σ, Π) *with* $\sigma_b(\vartheta_n, \mathfrak{F}\varrho_n) \to 0$ *and* $\sigma_b(\mathfrak{F}\vartheta_n, \varrho_n) \to 0$ *as* $n \to \infty$, *we have* $\sigma_b(\vartheta_n, u) \to 0$ *and* $\sigma_b(u, \varrho_n) \to 0$ *as* $n \to \infty$.

Definition 5 ([14]). *Let $(\Sigma, \mathfrak{N}, \sigma_b)$ be a bipolar-b metric space and $F : (\Sigma, \mathfrak{N}) \rightleftarrows (\Sigma, \mathfrak{N})$ is a contravariant function. Then, the fixed point problem of $\mathfrak{F}$ is said to be well-posed if*

1. *$\mathfrak{F}$ has fixed point $u \in \Sigma \cap \Pi$ which is unique;*
2. *For any sequence (ϑ_n, ϱ_n) in (Σ, Π) with $\sigma_b(\vartheta_n, \mathfrak{F}\varphi_n) \to 0$ and $\sigma_b(\mathfrak{F}\varrho_n, \varrho_n) \to 0$ as $n \to \infty$, we have $\sigma_b(\vartheta_n, u) \to 0$ and $\sigma_b(u, \varrho_n) \to 0$ as $n \to \infty$.*

Suppose that $\mathfrak{A}$ is a unital C^*-algebra with a unital $I_{\mathfrak{A}}$. Assume that $\mathfrak{A}_{\mathfrak{h}} = \{\mathfrak{z} \in \mathfrak{A} : \mathfrak{z} = \mathfrak{z}^*\}$. If $\mathfrak{z} = \mathfrak{z}^*$ for any element $\mathfrak{z} \in \mathfrak{A}$ then $\mathfrak{z}$ is called a positive element and $\sigma(\mathfrak{z}) \subset \mathbb{R}^+$ is the spectrum of $\mathfrak{z}$. Consider the partial order $\preceq$ on $\mathfrak{A}$ as $\mathfrak{z} \preceq \mathfrak{w}$ if $0_{\mathfrak{A}} \preceq \mathfrak{w} - \mathfrak{z}$, where $0_{\mathfrak{A}}$ is the zero element in $\mathfrak{A}$, and we denote the $\{\mathfrak{z} \in \mathfrak{A} : \mathfrak{z} \succeq 0_{\mathfrak{A}}\}$ by $\mathfrak{A}_+$ and $|\mathfrak{z}| = (\mathfrak{z}^*\mathfrak{z})^{\frac{1}{2}}$.

In 2015, Ma et al., gave C^*-algebra valued b-metric space as defined below:

Definition 6 ([11]). *Consider a unital C^*-algebra $\mathfrak{A}$ with a unital $I_{\mathfrak{A}}$, a set $\Sigma \neq \phi$, and $I \preceq b \in \mathfrak{A}_+$. A distance function $\sigma : \Sigma \times \Sigma \to \mathfrak{A}_+$ is such that*

1. *$\sigma(\vartheta, \varrho) = 0_{\mathfrak{A}}$ if and only if $\vartheta = \varrho$ for all $(\vartheta, \varrho) \in \Sigma \times \Sigma$;*
2. *$\sigma(\vartheta, \varrho) = \sigma(\varrho, \vartheta)$ for all $\vartheta, \varrho \in \Sigma$;*
3. *$\sigma(\vartheta, \varrho) \preceq b[\sigma(\vartheta, \mathfrak{h}) + \sigma(\mathfrak{h}, \varrho)]$ for all $\vartheta, \varrho, \mathfrak{h} \in \Sigma$.*

   Then $(\Sigma, \mathfrak{A}, \sigma)$ is known as C^*-algebra valued b-metric space.

In 2022, Mani et al. [16] gave the following:

Definition 7 ([16]). *Consider a unital C^*-algebra $\mathfrak{A}$ with a unital $I_{\mathfrak{A}}$, two sets $\Sigma, \Pi \neq \phi$, and $I \preceq b \in \mathfrak{A}_+$. A distance function $\sigma : \Sigma \times \Pi \to \mathfrak{A}_+$ with the following*

1. *$\sigma(\vartheta, \varrho) = 0_{\mathfrak{A}}$ if and only if $\vartheta = \varrho$ for all $(\vartheta, \varrho) \in \Sigma \times \Pi$;*
2. *$\sigma(\vartheta, \varrho) = \sigma(\varrho, \vartheta)$ for all $\vartheta, \varrho \in \Sigma \cap \Pi$;*
3. *$\sigma(\vartheta_1, \varrho_2) \preceq \sigma(\vartheta_1, \varrho_1) + \sigma(\vartheta_2, \varrho_1) + \sigma(\vartheta_2, \varrho_2)$ for all $(\vartheta_1, \varrho_1), (\vartheta_2, \varrho_2) \in \Sigma \times \Pi$.*

is called C^*-algebra valued bipolar metric and $(\Sigma, \Pi, \mathfrak{A}, \sigma)$ is called C^*-algebra valued bipolar metric space.

Definition 8 ([16]). *Suppose $(\Sigma, \Pi, \mathfrak{A}, \sigma)$ be a C^*-algebra valued bipolar metric space. Then*

1. *Elements of Σ are called left elements, of Π are right elements and of $\Sigma \cup \Pi$ are central elements.*
2. *A left sequence is a sequence $(\vartheta_n) \subseteq \Sigma$. If $||\sigma(\vartheta_n, \varrho)|| \to 0$ as $n \to \infty$ for some $\varrho \in \Pi$, then it is said to be right convergent to ϱ. Similarly, for the right convergent.*
3. *A bisequence is a sequence of the form $(\vartheta_n, \varrho_n) \in \Sigma \times \Pi$.*
4. *If in the bisequence (ϑ_n, ϱ_n)both the sequences (ϑ_n) and (ϱ_n) converge, then the bisequence (ϑ_n, ϱ_n) is said to be convergent. If they converge to the same point $u \in \mathfrak{M} \cap \mathfrak{N}$ then the bisequence (ϑ_n, ϱ_n) is called biconvergent.*
5. *A bisequence (ϑ_n, ϱ_n) on $(\mathfrak{M}, \mathfrak{N}, \mathfrak{A}, \sigma)$ is said to be Cauchy bisequence, if for each $\epsilon > 0$ there exists a positive integer $N \in \mathbb{N}$ such that $||\sigma(\vartheta_n, \varrho_m)|| < \epsilon$ for all $n, m \geq N$.*
6. *If every Cauchy bisequence is convergent then C^*-algebra valued bipolar metric space is said to be complete.*

Definition 9 ([12]). *Let $(\Sigma_1, \Pi_1, \sigma_1)$ and $(\Sigma_2, \Pi_2, \sigma_2)$ are the bipolar metric spaces and $\mathfrak{T} : \Sigma_1 \cup \Pi_1 \to \Sigma_2 \cup \Pi_2$ be a function:*

1. *If $\mathfrak{T}\Sigma_1 \subseteq \Sigma_2$ and $\mathfrak{T}\Pi_1 \subseteq \Pi_2$, then $\mathfrak{T}$ is known as covariant mapping and is represented as $\mathfrak{T} : (\Sigma_1, \Pi_1, \sigma_1) \rightrightarrows (\Sigma_2, \Pi_2, \sigma_2)$.*
2. *If $\mathfrak{T}\Sigma_1 \subseteq \Pi_2$ and $\mathfrak{T}\Pi \subseteq \Sigma_2$, then $\mathfrak{T}$ is termed as contravariant mapping and is represented by $\mathfrak{T} : (\Sigma_1, \Pi_1, \sigma_1) \rightleftarrows (\Sigma_2, \Pi_2, \sigma_2)$.*

Definition 10 ([16]). *Let $(\Sigma_1, \Pi_1, A, \sigma_1)$ and $(\Sigma_2, \Pi_2, A, \sigma_2)$ are C^*-algebra valued bipolar metric spaces.*

1. *A covariant map is called left continuous at a point $\vartheta_0 \in \Sigma_1$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $||\sigma_2(\mathfrak{T}\vartheta_0, \mathfrak{T}\varrho)|| < \epsilon$ whenever $||\sigma_1(\vartheta_0, \varrho)|| < \delta$.*
2. *A covariant map is called right continuous at a point $\varrho_0 \in \Pi_1$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $||\sigma_2(\mathfrak{T}\vartheta, \mathfrak{T}\varrho_0)|| < \epsilon$ whenever $||\sigma_1(\vartheta, \varrho_0)|| < \delta$.*

3. *A covariant map is called continuous if it is left continuous at each* $\vartheta_0 \in \Sigma_1$ *and right continuous at each* $\vartheta_0 \in \Pi_1$.
4. *A contravariant map is called continuous if it is continuous as a covariant map.*

3. Main Results

We start off the section by presenting a new notion C^*-algebra valued bipolar b-metric space as a generalization of C^*-algebra valued bipolar metric space and also prove some problems for fixed points in this space.

Definition 11. *Assume that* $\mathfrak{A}$ *is a unital* C^**-algebra with a unity* $I_{\mathfrak{A}}$, $b \succeq I_{\mathfrak{A}}$ *and* $\mathfrak{M}, \mathfrak{N}$ *are two non void sets. A mapping* $\sigma : \mathfrak{M} \times \mathfrak{N} \to \mathfrak{A}_+$ *is such that*

1. $\sigma(\vartheta, \varrho) = 0_{\mathfrak{A}}$ *if and only if* $\vartheta = \varrho$ *for all* $(\vartheta, \varrho) \in \mathfrak{M} \times \mathfrak{N}$;
2. $\sigma(\vartheta, \varrho) = \sigma(\varrho, \vartheta)$ *for all* $\vartheta, \varrho \in \mathfrak{M} \cap \mathfrak{N}$;
3. $\sigma(\vartheta_1, \varrho_2) \preceq b[\sigma(\vartheta_1, \varrho_1) + \sigma(\vartheta_2, \varrho_1) + \sigma(\vartheta_2, \varrho_2)]$ *for all* $(\vartheta_1, \varrho_1), (\vartheta_2, \varrho_2) \in \mathfrak{M} \times \mathfrak{N}$.

Then $(\mathfrak{M}, \mathfrak{N}, \mathfrak{A}, \sigma)$ *is called* C^**-algebra valued bipolar b-metric space.*

Remark 1. *The space is joint if* $\mathfrak{M} \cap \mathfrak{N} \neq \emptyset$ *otherwise disjoint.*

Example 1. *Consider* $\mathfrak{M} = (-\infty, 0], \mathfrak{N} = [0, \infty), \mathfrak{A} = M_2(\mathbb{R})$ *and* $\sigma : \mathfrak{M} \times \mathfrak{N} \to \mathfrak{A}$ *as* $\sigma(\vartheta, \varrho) = diag\{c_1(|\vartheta - \varrho|)^p, c_2(|\vartheta - \varrho|)^p\}$ *where* $p > 1$ *and* $c_1, c_2 > 0$.
Easily one can check that conditions 1 and 2 of Definition 11 are holds.
Using $|\vartheta_1 - \varrho_2|p \leq 2^{2p}(|\vartheta_1 - \varrho_1|^p + |\vartheta_2 - \varrho_1|^p + |\vartheta_2 - \varrho_2|^p)$ *one can also prove third condition where* $b = 2^{2p}I$. *So,* $(\mathfrak{M}, \mathfrak{N}, \mathfrak{A}, \sigma)$ *is a complete* C^**-algebra valued bipolar b-metric space.*
If we take $\vartheta_1 = \frac{-1}{2}, \vartheta_2 = 0, \varrho_1 = 0, \varrho_2 = \frac{1}{2}$, *then* $\sigma(\vartheta_1, \varrho_2) \succeq \sigma(\vartheta_1, \varrho_1) + \sigma(\vartheta_2, \varrho_1) + \sigma(\vartheta_2, \varrho_2)$ *for all* $(\vartheta_1, \varrho_1), (\vartheta_2, \varrho_2) \in \mathfrak{M} \times \mathfrak{N}$. *So, it is not* C^**-algebra valued bipolar metric space.*

Remark 2. *Taking* $b = I, \mathfrak{M} = \mathfrak{N}$, *one can obtain* C^**-algebra valued bipolar and* C^**-algebra valued metric spaces, respectively.*

Definition 12. *Let* $(\mathfrak{M}, \mathfrak{N}, \mathfrak{A}, \sigma)$ *be* C^**-algebra valued bipolar b-metric space and* $\alpha_{\mathfrak{A}} : \mathfrak{M} \times \mathfrak{N} \to \mathfrak{A}'_+$ *be a function. A covariant map* $J : (\mathfrak{M}, \mathfrak{N}) \rightrightarrows (\mathfrak{M}, \mathfrak{N})$ *is said to be* $\alpha_{\mathfrak{A}}$*-admissible if*

$$\alpha_{\mathfrak{A}}(\vartheta, \varrho) \succeq I_{\mathfrak{A}} \Rightarrow \alpha_{\mathfrak{A}}(J\vartheta, J\varrho) \succeq I_{\mathfrak{A}}, \tag{1}$$

for all $(\vartheta, \varrho) \in \mathfrak{M} \times \mathfrak{N}$.

Definition 13. *Let* $(\mathfrak{M}, \mathfrak{N}, \mathfrak{A}, \sigma)$ *be a* C^**-algebra valued bipolar b-metric space and* $\alpha_{\mathfrak{A}} : \mathfrak{M} \times \mathfrak{N} \to \mathfrak{A}'_+$ *be a function. A contravariant map* $J : (\mathfrak{M}, \mathfrak{N}) \rightleftarrows (\mathfrak{M}, \mathfrak{N})$ *is said to be* $\alpha_{\mathfrak{A}}$*-admissible if*

$$\alpha_{\mathfrak{A}}(\vartheta, \varrho) \succeq I_{\mathfrak{A}} \Rightarrow \alpha_{\mathfrak{A}}(J\varrho, J\vartheta) \succeq I_{\mathfrak{A}}, \tag{2}$$

for all $(\vartheta, \varrho) \in \mathfrak{M} \times \mathfrak{N}$.

Definition 14. *Let* $(\mathfrak{M}, \mathfrak{N}, \mathfrak{A}, \sigma)$ *be* C^**-algebra valued bipolar b-metric space and* $J : (\mathfrak{M}, \mathfrak{N}) \rightrightarrows (\mathfrak{M}, \mathfrak{N})$ *be a covariant mapping. If there exists two functions* $\alpha_{\mathfrak{A}} : \mathfrak{M} \times \mathfrak{N} \to \mathfrak{A}'_+$ *and* $\psi_{\mathfrak{A}} \in \Psi_{\mathfrak{A}}$ *such that*

$$\alpha_{\mathfrak{A}}(\vartheta, \varrho)\sigma(J\vartheta, J\varrho) \preceq \psi_{\mathfrak{A}}(\sigma(\vartheta, \varrho)), \tag{3}$$

for all $(\vartheta, \varrho) \in \mathfrak{M} \times \mathfrak{N}$.

Then, we say that J *is* $(\alpha_{\mathfrak{A}} - \psi_{\mathfrak{A}})$ *covariant contractive mapping.*

Definition 15. *Let $(\mathfrak{M}, \mathfrak{N}, \mathfrak{A}, \sigma)$ be a C^*-algebra valued bipolar b-metric space and $J : (\mathfrak{M}, \mathfrak{N}) \rightleftarrows (\mathfrak{M}, \mathfrak{N})$ be a contravariant mapping. If there exists two functions $\alpha_{\mathfrak{A}} : \mathfrak{M} \times \mathfrak{N} \to \mathfrak{A}'_+$ and $\psi_{\mathfrak{A}} \in \Psi_{\mathfrak{A}}$ such that*

$$\alpha_{\mathfrak{A}}(\vartheta, \varrho)\sigma(J\varrho, J\vartheta) \preceq \psi_{\mathfrak{A}}(\sigma(\vartheta, \varrho)), \tag{4}$$

for all $(\vartheta, \varrho) \in \mathfrak{M} \times \mathfrak{N}$.

Then we say that J is $(\alpha_{\mathfrak{A}} - \psi_{\mathfrak{A}})$ contravariant contractive mapping.

Theorem 1. *Let $(\mathfrak{M}, \mathfrak{N}, \mathfrak{A}, \sigma)$ be a joint C^*-algebra valued bipolar b-metric space and J be a $(\alpha_{\mathfrak{A}} - \psi_{\mathfrak{A}})$ covariant map satisfying Equation (3) with the following conditions:*

1. *J is $\alpha_{\mathfrak{A}}$-admissible;*
2. *There is $(\vartheta_0, \varrho_0) \in \mathfrak{M} \times \mathfrak{N}$ such that $\alpha_{\mathfrak{A}}(\vartheta_0, \varrho_0) \succeq I_{\mathfrak{A}}$ and $\alpha_{\mathfrak{A}}(\vartheta_0, J\varrho_0) \succeq I_{\mathfrak{A}}$;*
3. *J is continuous.*

Then J possesses a fixed point.

Proof. Suppose that $(\vartheta_0, \varrho_0) \in \mathfrak{M} \times \mathfrak{N}$ such that $\alpha_{\mathfrak{A}}(\vartheta_0, \varrho_0) \succeq I_{\mathfrak{A}}$ and $\alpha_{\mathfrak{A}}(\vartheta_0, J\varrho_0) \succeq I_{\mathfrak{A}}$. Now construct iteration sequences $\vartheta_n = J\vartheta_{n-1}$ and $\varrho_n = J\varrho_{n-1}$. Then, clearly (ϑ_n, ϱ_n) is a bisequence.

Now

$$\begin{aligned}
\alpha_{\mathfrak{A}}(\vartheta_0, y\varrho_0) \succeq I_{\mathfrak{A}} &\quad\Rightarrow\quad \alpha_{\mathfrak{A}}(J\vartheta_0, J\eta_0) \succeq I_{\mathfrak{A}}, \\
\alpha_{\mathfrak{A}}(\vartheta_0, \varrho_1) = \alpha_{\mathfrak{A}}(\vartheta_0, J\varrho_0) \succeq I_{\mathfrak{A}} &\quad\Rightarrow\quad \alpha_{\mathfrak{A}}(J\vartheta_0, J\varrho_1) = \alpha_{\mathfrak{A}}(\vartheta_1, \varrho_2) \succeq I_{\mathfrak{A}}, \\
\alpha_{\mathfrak{A}}(\vartheta_1, \varrho_1) = \alpha_{\mathfrak{A}}(J\vartheta_0, J\varrho_0) \succeq I_{\mathfrak{A}} &\quad\Rightarrow\quad \alpha_{\mathfrak{A}}(J\vartheta_1, J\varrho_1) = \alpha_{\mathfrak{A}}(\vartheta_2, \varrho_2) \succeq I_{\mathfrak{A}}, \\
\alpha_{\mathfrak{A}}(\vartheta_1, \varrho_2) = \alpha_{\mathfrak{A}}(\vartheta_1, J\varrho_1) \succeq I_{\mathfrak{A}} &\quad\Rightarrow\quad \alpha_{\mathfrak{A}}(J\vartheta_1, J\varrho_2) = \alpha_{\mathfrak{A}}(\vartheta_2, \varrho_3) \succeq I_{\mathfrak{A}}, \\
\alpha_{\mathfrak{A}}(\vartheta_2, \varrho_2) = \alpha_{\mathfrak{A}}(J\vartheta_1, J\varrho_1) \succeq I_{\mathfrak{A}} &\quad\Rightarrow\quad \alpha_{\mathfrak{A}}(J\vartheta_2, J\varrho_2) = \alpha_{\mathfrak{A}}(\vartheta_3, \varrho_3) \succeq I_{\mathfrak{A}}.
\end{aligned}$$

By continuing this process, we have

$$\alpha_{\mathfrak{A}}(\vartheta_{n+1}, \varrho_{n+1}) \succeq I_{\mathfrak{A}}, \alpha_{\mathfrak{A}}(\vartheta_n, \varrho_{n+1}) \succeq I_{\mathfrak{A}}, \forall n \in \mathbb{N} \tag{5}$$

Using Equations (3) and (5), for $\vartheta = \vartheta_n$ and $\varrho = \varrho_n$, we obtain that

$$\sigma(\vartheta_{n+1}, \varrho_{n+1}) = \sigma(J\vartheta_n, J\varrho_n) \preceq \alpha_{\mathfrak{A}}(\vartheta_n, \varrho_n)\sigma(J\vartheta_n, J\varrho_n) \preceq \psi_{\mathfrak{A}}(\sigma(\vartheta_n, \varrho_n))$$

Similarly, using Equations (3) and (5), for $\vartheta = \vartheta_{n-1}$ and $\varrho = \varrho_n$, we have

$$\sigma(\vartheta_n, \varrho_{n+1}) = \sigma(J\vartheta_{n-1}, J\varrho_n) \preceq \alpha_{\mathfrak{A}}(\vartheta_{n-1}, \vartheta_n)\sigma(J\vartheta_{n-1}, J\varrho_n) \preceq \psi_{\mathfrak{A}}(\sigma(\vartheta_{n-1}, \varrho_n))$$

By using mathematical induction, the above two inequalities imply that

$$\sigma(\vartheta_{n+1}, \varrho_{n+1}) \preceq \psi_{\mathfrak{A}}^{n+1}(\sigma(\vartheta_0, \varrho_0)), \sigma(\vartheta_n, \varrho_{n+1}) \preceq \psi_{\mathfrak{A}}^{n}(\sigma(\vartheta_0, \varrho_1)). \tag{6}$$

Now for $n, p \geq 1$, using Definition 11 and Equation (6), we have

$$\begin{aligned}
\sigma(\vartheta_n, \varrho_{n+p}) &\preceq b[\sigma(\vartheta_n, \varrho_{n+1}) + \sigma(\vartheta_{n+1}, \varrho_{n+1}) + \sigma(\vartheta_{n+1}, \varrho_{n+p})] \\
&\preceq b[\sigma(\vartheta_n, \varrho_{n+1}) + \sigma(\vartheta_{n+1}, \varrho_{n+1})] + b^2[\sigma(\vartheta_{n+1}, \varrho_{n+2}) + \sigma(\vartheta_{n+2}, \varrho_{n+2}) \\
&+ \sigma(\vartheta_{n+2}, \varrho_{n+p})] \\
&\preceq b[\sigma(\vartheta_n, \varrho_{n+1}) + \sigma(\vartheta_{n+1}, \varrho_{n+1})] + \cdots + b^p[\sigma(\vartheta_{n+p-1}, \varrho_{n+p}) + \sigma(\vartheta_{n+p}, \varrho_{n+p})] \\
&\preceq \sum_{k=1}^{p} b^k \sigma(\vartheta_{n+k-1}, \varrho_{n+k}) + \sum_{k=1}^{p} b^k \sigma(\vartheta_{n+k}, \varrho_{n+k}) \\
&\preceq \sum_{k=1}^{p} b^k \psi_A^{n+k-1}(\sigma(\vartheta_0, \varrho_1)) + \sum_{k=1}^{p} b^k \psi_A^{n+k+1}(\sigma(\vartheta_0, \varrho_0))
\end{aligned}$$

Taking $p \to \infty$ and using Definition 3, we have

$$\sigma(\vartheta_n, \varrho_{n+p}) \to 0. \tag{7}$$

Similarly, one can prove that

$$\sigma(\vartheta_{n+p}, \varrho_n) \to 0. \tag{8}$$

From Equations (7) and (8), it is clear that (ϑ_n, ϱ_n) is a Cauchy bisequence. As the space is complete, so (ϑ_n, ϱ_n) biconverges. It means, there exists $\varsigma \in \mathfrak{M} \cap \mathfrak{N}$ such that $\vartheta_n \to \varsigma$ and $\varrho_n \to \varsigma$ as $n \to \infty$. Since the map J is continuous, so $\vartheta_n \to \varsigma$ implies $J\vartheta_n \to J\varsigma$ and $\varrho_n \to \varsigma$ implies $J\varrho_n \to J\varsigma$.

By the uniqueness of limit, we obtain that $J\varsigma = \varsigma$. □

Theorem 2. *Let $(\mathfrak{M}, \mathfrak{N}, \mathfrak{A}, \sigma)$ be a joint C^*-algebra valued bipolar b-metric space, and J be a $(\alpha_{\mathfrak{A}} - \psi_{\mathfrak{A}})$ contravariant map satisfying Equation (4) with the following conditions:*

1. *J is $\alpha_{\mathfrak{A}}$-admissible;*
2. *There is $\vartheta_0 \in \mathfrak{M}$ with $\alpha_{\mathfrak{A}}(\vartheta_0, J\vartheta_0) \succeq I_{\mathfrak{A}}$;*
3. *J is continuous.*

Then J possesses a fixed point.

Proof. Assume that $\vartheta_0 \in \mathfrak{M}$ with $\alpha_{\mathfrak{A}}(\vartheta_0, J\vartheta_0) \succeq I_{\mathfrak{A}}$. Now construct iteration sequences $\vartheta_n = J\varrho_{n-1}$ and $\varrho_n = J\vartheta_n$. Then, clearly (ϑ_n, ϱ_n) is a bisequence.

Now

$$\begin{aligned}
\alpha_{\mathfrak{A}}(\vartheta_0, \varrho_0) = \alpha_{\mathfrak{A}}(\vartheta_0, J\vartheta_0) \succeq I_{\mathfrak{A}} &\Rightarrow \alpha_{\mathfrak{A}}(J\varrho_0, J\vartheta_0) \succeq I_{\mathfrak{A}}, \\
\alpha_{\mathfrak{A}}(\vartheta_1, \varrho_0) \succeq I_{\mathfrak{A}} &\Rightarrow \alpha_{\mathfrak{A}}(J\varrho_0, J\vartheta_1) = \alpha_{\mathfrak{A}}(\vartheta_1, \varrho_1) \succeq I_{\mathfrak{A}}, \\
\alpha_{\mathfrak{A}}(\vartheta_1, \varrho_1) \succeq I_{\mathfrak{A}} &\Rightarrow \alpha_{\mathfrak{A}}(J\varrho_1, J\vartheta_1) = \alpha_{\mathfrak{A}}(\vartheta_2, \varrho_1) \succeq I_{\mathfrak{A}}, \\
\alpha_{\mathfrak{A}}(\vartheta_2, \varrho_1) \succeq I_{\mathfrak{A}} &\Rightarrow \alpha_{\mathfrak{A}}(J\varrho_1, J\vartheta_2) = \alpha_{\mathfrak{A}}(\vartheta_2, \varrho_2) \succeq I_{\mathfrak{A}}.
\end{aligned}$$

By continuing this process, we have

$$\alpha_{\mathfrak{A}}(\vartheta_n, \varrho_n) \succeq I_{\mathfrak{A}}, \alpha_{\mathfrak{A}}(\vartheta_{n+1}, \varrho_n) \succeq I_{\mathfrak{A}}, \forall n \in \mathbb{N} \tag{9}$$

Using Equations (4) and (9), for $\vartheta = \vartheta_n$ and $\varrho = \varrho_{n-1}$, we obtain that

$$\sigma(\vartheta_n, \varrho_n) = \sigma(J\varrho_{n-1}, J\vartheta_n) \preceq \alpha_{\mathfrak{A}}(\vartheta_n, \varrho_{n-1}) d(T\varrho_{n-1}, T\vartheta_n) \preceq \psi_{\mathfrak{A}}(d(\vartheta_n, \varrho_{n-1}))$$

Similarly, using Equations (4) and (9), for $\vartheta = \vartheta_{n+1}$ and $\varrho = \varrho_n$, we have

$$\sigma(\vartheta_{n+1}, \varrho_n) = \sigma(J\varrho_n, J\vartheta_n) \preceq \alpha_{\mathfrak{A}}(\vartheta_n, \varrho_n)\sigma(J\varrho_n, J\vartheta_n) \preceq \psi_{\mathfrak{A}}(\sigma(\vartheta_n, \varrho_n))$$

By using mathematical induction, the above two inequalities imply that

$$\sigma(\vartheta_n, \varrho_n) \preceq \psi^n_{\mathfrak{A}}(\sigma(\vartheta_1, \varrho_0)), \sigma(\vartheta_{n+1}, \varrho_n) \preceq \psi^{n+1}_{\mathfrak{A}}(\sigma(\vartheta_0, \varrho_0)). \tag{10}$$

Now for $n, p \geq 1$, using Definition 11 and Equation (10), we have

$$\begin{aligned}
\sigma(\vartheta_n, \varrho_{n+p}) &\preceq b[\sigma(\vartheta_n, \varrho_n) + \sigma(\vartheta_{n+1}, \varrho_n) + \sigma(\vartheta_{n+1}, \varrho_{n+p})] \\
&\preceq b[\sigma(\vartheta_n, \varrho_n) + \sigma(\vartheta_{n+1}, \varrho_n)] + b^2[\sigma(\vartheta_{n+1}, \varrho_{n+1}) + \sigma(\vartheta_{n+2}, \varrho_{n+1}) + \sigma(\vartheta_{n+2}, \varrho_{n+p})] \\
&\preceq b[\sigma(\vartheta_n, \varrho_n) + \sigma(\vartheta_{n+1}, \varrho_n)] + \cdots + b^p[\sigma(\vartheta_{n+p}, \varrho_{n+p-1}) + \sigma(\vartheta_{n+p}, \varrho_{n+p})] \\
&\preceq \sum_{k=1}^{p} b^k \sigma(\vartheta_{n+k}, \varrho_{n+k-1}) + \sum_{k=1}^{p} b^k \sigma(\vartheta_{n+k}, \varrho_{n+k}) \\
&\preceq \sum_{k=1}^{p} b^k \psi_A^{k+1}(\sigma(\vartheta_0, \varrho_0)) + \sum_{k=1}^{p} b^k \psi_A^{k+1}(\sigma(\vartheta_1, \varrho_0))
\end{aligned}$$

Taking $p \to \infty$ and using Definition 3 , we have

$$\sigma(\vartheta_n, \varrho_{n+p}) \to 0. \tag{11}$$

Similarly, one can prove that

$$\sigma(\vartheta_{n+p}, \varrho_n) \to 0. \tag{12}$$

From Equations (11) and (12), one can easily check that (ϑ_n, ϱ_n) is a Cauchy bisequence. It is given that the space is complete, so (ϑ_n, ϱ_n) biconverges. Therefore, there exists $\varsigma \in \mathfrak{M} \cap \mathfrak{N}$ such that $\vartheta_n \to \varsigma$ and $\varrho_n \to \varsigma$ as $n \to \infty$. Since the map J is continuous, so $\vartheta_n \to \varsigma$ implies $\varrho_n = J\vartheta_n \to J\varsigma$ and $\varrho_n \to \varsigma$ implies $\vartheta_{n+1} = J\varrho_n \to J\varsigma$.
From this, we obtain that $J\varsigma = \varsigma$. □

At the end of Section 4, we will present two examples that prove that Theorems 1 and 2 are independent of each other.

(P) Suppose there is $z \in \mathfrak{M} \cap \mathfrak{N}$ with $\alpha_{\mathfrak{A}}(\vartheta, z) \succeq I_{\mathfrak{A}}$ and $\alpha_{\mathfrak{A}}(z, \varrho) \succeq I_{\mathfrak{A}}$ for all $(\vartheta, \varrho) \in \mathfrak{M} \times \mathfrak{N}$.

Theorem 3. *If in the conditions of Theorem 1 (or in Theorem 2), we add the condition (P) also, then the uniqueness of the fixed point occurs.*

Proof. Suppose, if possible, ς and φ are two distinct fixed point of J, then from the condition (P), there exists $w \in \mathfrak{M} \cap \mathfrak{N}$ such that

$$\alpha_{\mathfrak{A}}(\varsigma, w) \succeq I_{\mathfrak{A}}, \alpha_{\mathfrak{A}}(w, \varphi) \succeq I_{\mathfrak{A}}. \tag{13}$$

Since J is $\alpha_{\mathfrak{A}}$-admissible, using above, we have

$$\alpha_{\mathfrak{A}}(\varsigma, J^n w) \succeq I_{\mathfrak{A}}, \alpha_{\mathfrak{A}}(J^n w, \varphi) \succeq I_{\mathfrak{A}}, \forall n \in \mathbb{N}. \tag{14}$$

By Equations (3) and (14), we get

$$\begin{aligned}
\sigma(u, J^n w) &= \alpha_{\mathfrak{A}}(\varsigma, w)\sigma(J\varsigma, J(J^{n-1}w)) \\
&\preceq \psi_{\mathfrak{A}}(\sigma(\varsigma, J^{n-1}w)) \\
&\preceq \psi_{\mathfrak{A}}^n(\sigma(\varsigma, w)).
\end{aligned}$$

Similarly, $\sigma(J^n w, \varphi) \preceq \psi_{\mathfrak{A}}^n(\sigma(w, \varphi))$.

Letting $n \to \infty$ in the above inequalities and uniqueness of limit implies that $\varsigma = \varphi$. So, J has a unique fixed point.

The proof is similar for contravariant mappings. □

Theorem 4. *(Kannan Type) Let $(\mathfrak{M}, \mathfrak{N}, \mathfrak{A}, \sigma)$ be a joint C^*-algebra valued bipolar b-metric space and J be a $(\alpha_{\mathfrak{A}} - \psi_{\mathfrak{A}})$ contravariant map with*

$$\alpha_{\mathfrak{A}}(\vartheta, \varrho)\sigma(J\varrho, J\vartheta) \preceq \psi_{\mathfrak{A}}(\sigma(\vartheta, J\vartheta) + \sigma(J\vartheta, \vartheta)), \tag{15}$$

for all $(\vartheta, \varrho) \in \mathfrak{M} \times \mathfrak{N}$ *and*

1. J *is* $\alpha_{\mathfrak{A}}$*-admissible;*
2. *There is* $\vartheta_0 \in \mathfrak{M}$ *such that* $\alpha_{\mathfrak{A}}(\vartheta_0, J\vartheta_0) \succeq I_{\mathfrak{A}}$*;*
3. J *is continuous.*

Then J *possesses a fixed point. In addition, if* $\alpha_{\mathfrak{A}}(\varsigma, \varphi) \succeq I_{\mathfrak{A}}$ *where* ς *and* φ *are fixed points, then the uniqueness of the fixed point occurs.*

Proof. Let $\vartheta_0 \in \mathfrak{M}$ with $\alpha_{\mathfrak{A}}(\vartheta_0, J\vartheta_0) \succeq I_{\mathfrak{A}}$. Now construct iteration sequences $\vartheta_n = J\varrho_{n-1}$ and $\varrho_n = J\vartheta_n$. Then, clearly (ϑ_n, ϱ_n) is a bisequence.

From the proof of Theorem 2, we have

$$\alpha_{\mathfrak{A}}(\vartheta_n, \varrho_n) \succeq I_{\mathfrak{A}}\sigma(J\varrho, J\vartheta), \alpha_{\mathfrak{A}}(\vartheta_{n+1}, \varrho_n) \succeq I_{\mathfrak{A}}, \forall n \in \mathbb{N} \tag{16}$$

Using Equations (15) and (16), for $\vartheta = \vartheta_n$ and $\varrho = \varrho_{n-1}$, we obtain that

$$\sigma(\vartheta_n, \varrho_n) = \sigma(J\varrho_{n-1}, J\vartheta_n) \preceq \alpha_{\mathfrak{A}}(\vartheta_n, \varrho_{n-1})\sigma(J\varrho_{n-1}, J\vartheta_n) \preceq \psi_{\mathfrak{A}}(\sigma(\vartheta_n, \varrho_n) + \sigma(\vartheta_n, \varrho_{n-1})),$$

Using Definition 2, we have

$$\begin{aligned} \sigma(\vartheta_n, \varrho_n) &\preceq \psi_{\mathfrak{A}}(\sigma(\vartheta_n, \varrho_n)) + \psi_{\mathfrak{A}}(\sigma(\vartheta_n, \varrho_{n-1})) \\ (I - \psi_{\mathfrak{A}})\sigma(\vartheta_n, \varrho_n) &\preceq \psi_{\mathfrak{A}}(\sigma(\vartheta_n, \varrho_{n-1})) \\ \sigma(\vartheta_n, \varrho_n) &\preceq (I - \psi_{\mathfrak{A}})^{-1}\psi_{\mathfrak{A}}(\sigma(\vartheta_n, \varrho_{n-1})) \end{aligned}$$

Now assume that $\phi_{\mathfrak{A}} = (I - \psi_{\mathfrak{A}})^{-1}\psi_{\mathfrak{A}} = \sum_{n=0}^{\infty} \psi_{\mathfrak{A}}^n < \infty$.
So, the above becomes

$$\sigma(\vartheta_n, \varrho_n) \preceq \phi_{\mathfrak{A}}^n(\sigma(\vartheta_1, \varrho_0)). \tag{17}$$

Similarly, using Equations (15) and (16), for $\vartheta = \vartheta_{n+1}$ and $\varrho = \varrho_n$, we have

$$\sigma(\vartheta_{n+1}, \varrho_n) = \sigma(J\varrho_n, J\vartheta_n) \preceq \phi_{\mathfrak{A}}^{n+1}(\sigma(\vartheta_0, \varrho_0)). \tag{18}$$

From Equations (16) and (17) and Theorem 2, (ϑ_n, ϱ_n) is a Cauchy bisequence. As the space is complete, so (ϑ_n, ϱ_n) biconverges. It means, there is $\varsigma \in \mathfrak{M} \cap \mathfrak{N}$ with $\vartheta_n \to \varsigma$ and $\varrho_n \to \varsigma$ as $n \to \infty$. It is given that map J is continuous, so $\vartheta_n \to \varsigma$ implies $\varrho_n = J\vartheta_n \to J\varsigma$ and $\varrho_n \to \varsigma$ implies $\vartheta_{n+1} = J\varrho_n \to J\varsigma$.

From this, we obtain that $J\varsigma = \varsigma$.

Uniqueness:

Let, if possible, ς and φ are two distinct fixed points of J. Then

$$\begin{aligned} 0_{\mathfrak{A}} &\preceq \sigma(\varsigma, \varphi) = \sigma(J\varsigma, J\varphi) \\ &\preceq \alpha_{\mathfrak{A}}(\varsigma, \varphi)\sigma(J\varsigma, J\varphi) \\ &\preceq \psi_{\mathfrak{A}}(\sigma(J\varsigma, \varsigma) + \sigma(\varphi, J\varphi)) \\ &\preceq 0. \end{aligned}$$

This implies $\sigma(\varsigma, \varphi) = 0$, so, $\varsigma = \varphi$. □

Theorem 5. *(Banach–Kannan Type) Let* $(\mathfrak{M}, \mathfrak{N}, \mathfrak{A}, \sigma)$ *be a joint* C^**-algebra valued bipolar b-metric space and* J *be a* $(\alpha_{\mathfrak{A}} - \psi_{\mathfrak{A}})$ *contravariant map with*

$$\alpha_{\mathfrak{A}}(\vartheta, \varrho)\sigma(J\varrho, J\vartheta) \preceq \psi_{\mathfrak{A}}(\sigma(\vartheta, \varrho) + \sigma(\vartheta, J\vartheta) + \sigma(J\varrho, \varrho)), \tag{19}$$

for all $(\vartheta, \varrho) \in \mathfrak{M} \times \mathfrak{N}$, $(I - \psi_{\mathfrak{A}})^{-1}\psi_{\mathfrak{A}} \preceq \frac{1}{2}I_{\mathfrak{A}}$ *and*

1. J *is* $\alpha_{\mathfrak{A}}$*-admissible;*
2. *There is* $\vartheta_0 \in \mathfrak{M}$ *with* $\alpha_{\mathfrak{A}}(\vartheta_0, J\vartheta_0) \succeq I_{\mathfrak{A}}$*;*
3. J *is continuous.*

Then J possesses a fixed point.

Proof. Let $\vartheta_0 \in \mathfrak{M}$ such that $\alpha_{\mathfrak{A}}(\vartheta_0, J\vartheta_0) \succeq I_{\mathfrak{A}}$. Now construct iteration sequences $\vartheta_n = J\varrho_{n-1}$ and $\varrho_n = J\vartheta_n$. Then, clearly (ϑ_n, ϱ_n) is a bisequence.

From the proof of Theorem 2, we obtain that

$$\alpha_{\mathfrak{A}}(\vartheta_n, \varrho_n) \succeq I_{\mathfrak{A}}\sigma(J\varrho, J\vartheta), \alpha_{\mathfrak{A}}(\vartheta_{n+1}, \varrho_n) \succeq I_{\mathfrak{A}}, \forall n \in \mathbb{N} \tag{20}$$

Using Equations (19) and (20), for $\vartheta = \vartheta_n$ and $\varrho = \varrho_{n-1}$, we obtain that

$$\sigma(\vartheta_n, \varrho_n) = \sigma(J\varrho_{n-1}, J\vartheta_n) \preceq \alpha_{\mathfrak{A}}(\vartheta_n, \varrho_{n-1})\sigma(J\varrho_{n-1}, J\vartheta_n) \preceq \psi_{\mathfrak{A}}(\sigma(\vartheta_n, \varrho_n) + 2\sigma(\vartheta_n, \varrho_{n-1})),$$

Using Definition 2, we have

$$\begin{aligned} \sigma(\vartheta_n, \varrho_n) &\preceq \psi_{\mathfrak{A}}(\sigma(\vartheta_n, \varrho_n)) + 2\psi_{\mathfrak{A}}(\sigma(\vartheta_n, \varrho_{n-1})) \\ (I - \psi_{\mathfrak{A}})\sigma(\vartheta_n, \varrho_n) &\preceq \psi_{\mathfrak{A}}(\sigma(\vartheta_n, \varrho_{n-1})) \\ \sigma(\vartheta_n, \varrho_n) &\preceq (I - \psi_{\mathfrak{A}})^{-1} 2 I_{\mathfrak{A}} \psi_{\mathfrak{A}}(\sigma(\vartheta_n, \varrho_{n-1})) \end{aligned}$$

Putting $\phi_{\mathfrak{A}}(\frac{I_{\mathfrak{A}}}{2}) = (I - \psi_{\mathfrak{A}})^{-1}\psi_{\mathfrak{A}}$.
So, we

$$\sigma(\vartheta_n, \varrho_n) \preceq \phi^n_{\mathfrak{A}}(\sigma(\vartheta_1, \varrho_0)). \tag{21}$$

Similarly, using Equations (19) and (20), for $\vartheta = \vartheta_{n+1}$ and $\varrho = \varrho_n$, we have

$$\sigma(\vartheta_{n+1}, \varrho_n) = \sigma(J\varrho_n, J\vartheta_n) \preceq \phi^{n+1}_A(\sigma(\vartheta_0, \varrho_0)). \tag{22}$$

From Equations (21) and (22) and Theorem 2, we conclude that (ϑ_n, ϱ_n) is a Cauchy bisequence. As the space is complete, so (ϑ_n, ϱ_n) biconverges. So, there is $\varsigma \in \mathfrak{M} \cap \mathfrak{N}$ with $\vartheta_n \to \varsigma$ and $\varrho_n \to \varsigma$ as $n \to \infty$. According to the hypothesis, the map J is continuous, so $\vartheta_n \to \varsigma$ implies $\varrho_n = J\vartheta_n \to J\varsigma$ and $\varrho_n \to \varsigma$ implies $\vartheta_{n+1} = J\varrho_n \to J\varsigma$.

From this, we obtain that $J\varsigma = \varsigma$. □

4. Consequences

Definition 16. *Let $(\mathfrak{M}, \mathfrak{N}, \mathfrak{A}, \sigma)$ be C^*-algebra valued bipolar b-metric space and $J : (\mathfrak{M}, \mathfrak{N}) \rightrightarrows (\mathfrak{M}, \mathfrak{N})$ be a covariant map. J is said to be $\psi_{\mathfrak{A}}$-type covariant contractive map if there is a function $\psi_{\mathfrak{A}} \in \Psi_{\mathfrak{A}}$ such that*

$$\sigma(J\vartheta, J\varrho) \preceq \psi_{\mathfrak{A}}(\sigma(\vartheta, \varrho)), \tag{23}$$

for all $(\vartheta, \varrho) \in \mathfrak{M} \times \mathfrak{N}$.

Corollary 1. *Let $(\mathfrak{M}, \mathfrak{N}, A, \sigma)$ be a joint complete C^*-algebra valued bipolar b-metric space and $J : (\mathfrak{M}, \mathfrak{N}) \rightrightarrows (\mathfrak{M}, \mathfrak{N})$ be a continuous $\psi_{\mathfrak{A}}$-type covariant contractive mapping, then J possesses a fixed point.*

Proof. Proof follows directly by taking $\alpha_{\mathfrak{A}}(\vartheta, \varrho) = I_{\mathfrak{A}}$ in Theorem 1. □

Definition 17. *Let $(\mathfrak{M}, \mathfrak{N}, \mathfrak{A}, \sigma)$ be C^*-algebra valued bipolar b-metric space and $J : (\mathfrak{M}, \mathfrak{N}) \rightleftarrows (\mathfrak{M}, \mathfrak{N})$ be a contravariant map. J is said to be $\psi_{\mathfrak{A}}$-type contravariant contractive map if there is a function $\psi_{\mathfrak{A}} \in \Psi_{\mathfrak{A}}$ such that*

$$\sigma(J\varrho, J\vartheta) \preceq \psi_A(\sigma(\vartheta, \varrho)), \tag{24}$$

for all $(\vartheta, \varrho) \in \mathfrak{M} \times \mathfrak{N}$.

Corollary 2. *Let $(\mathfrak{M},\mathfrak{N},\mathfrak{A},\sigma)$ be a joint complete C^*-algebra valued bipolar b-metric space and $J:(\mathfrak{M},\mathfrak{N}) \rightleftarrows (\mathfrak{M},\mathfrak{N})$ be a continuous $\psi_{\mathfrak{A}}$-type contravariant contractive map, then J possesses a fixed point.*

Proof. The proof is a consequence of Theorem 2 for $\alpha_{\mathfrak{A}}(\vartheta,\varrho) = I_{\mathfrak{A}}$. □

Corollary 3. *Let $(\mathfrak{M},\mathfrak{N},\mathfrak{A},\sigma)$ be a joint C^*-algebra valued bipolar b-metric space and J be a continuous contravariant map with*

$$\sigma(J\varrho, J\vartheta) \preceq \psi_A(\sigma(\vartheta, J\vartheta) + \sigma(J\varrho,\varrho)), \tag{25}$$

where $\psi_{\mathfrak{A}} \in \Psi_{\mathfrak{A}}$, then J possesses a unique fixed point.

Proof. Taking $\alpha_{\mathfrak{A}}(\vartheta,\varrho) = I_{\mathfrak{A}}$ in Theorem 3, proof follows. □

Corollary 4. *Let $(\mathfrak{M},\mathfrak{N},\mathfrak{A},\sigma)$ be a joint C^*-algebra valued bipolar b-metric space and J be a continuous contravariant mapping such that*

$$\sigma(J\varrho, J\vartheta) \preceq \psi_A(\sigma(\vartheta,\varrho) + \sigma(\vartheta, J\vartheta) + \sigma(J\varrho,\varrho)), \tag{26}$$

where $\psi_{\mathfrak{A}} \in \Psi_{\mathfrak{A}}$, then J has a fixed point.

Proof. Putting $\alpha_{\mathfrak{A}}(\vartheta,\varrho) = I_{\mathfrak{A}}$ in Theorem 4 , result follows. □

Corollary 5. *Let $(\mathfrak{M},\mathfrak{N},\mathfrak{A},\sigma)$ be a joint C^*-algebra valued bipolar b-metric space and $J:(\mathfrak{M},\mathfrak{N}) \rightrightarrows (\mathfrak{M},\mathfrak{N})$ be a covariant mapping such that*

$$\sigma(J\vartheta, J\varrho) \preceq Q^*\sigma(\vartheta,\varrho)Q, \tag{27}$$

for all $(\vartheta,\varrho) \in \mathfrak{M}\times\mathfrak{N}$ and $Q \in \mathfrak{A}$ with $||Q|| < 1$.

Then J possesses a unique fixed point.

Proof. Taking $\psi_{\mathfrak{A}} = Q^*tQ$ in Corollary 1. □

Corollary 6. *Let $(\mathfrak{M},\mathfrak{N},\mathfrak{A},\sigma)$ be a joint complete C^*-algebra valued bipolar b-metric space and $J:(\mathfrak{M},\mathfrak{N}) \rightleftarrows (\mathfrak{M},\mathfrak{N})$ be a contravariant mapping such that*

$$\sigma(J\varrho, J\vartheta) \preceq Q^*\sigma(\vartheta,\varrho)Q, \tag{28}$$

for all $(\vartheta,\varrho) \in \mathfrak{M}\times\mathfrak{N}$ and $Q \in \mathfrak{A}$ with $||Q|| < 1$.

Then J possesses a unique fixed point.

Proof. Making $\psi_{\mathfrak{A}}(\vartheta,\varrho) = Q^*tQ$ in Corollary 2. □

Corollary 7. *Let $(\mathfrak{M},\mathfrak{N},\mathfrak{A},\sigma)$ be a joint complete C^*-algebra valued bipolar b-metric space and $J:(\mathfrak{M},\mathfrak{N}) \rightleftarrows (\mathfrak{M},\mathfrak{N})$ be a contravariant mapping such that*

$$\sigma(J\varrho, J\vartheta) \preceq Q[\sigma(\vartheta, J\vartheta) + \sigma(J\varrho,\varrho)], \tag{29}$$

for all $(\vartheta,\varrho) \in \mathfrak{M}\times\mathfrak{N}$ and $Q \in \mathfrak{A}$ with $||Q|| < \frac{1}{2}$.

Then J has a unique fixed point.

Proof. Making $\psi_{\mathfrak{A}}(\vartheta,\varrho) = Q(t)$ in Corollary 5, the proof follows. □

Example 2. *Let $\mathfrak{M} = (-\infty,0], \mathfrak{N} = [0,\infty), \mathfrak{A} = M_2(\mathbb{C})$. Define $\sigma : (-\infty,0]\times[0,\infty) \to M_2(\mathbb{C})$ as $\sigma(\vartheta,\varrho) = \begin{bmatrix} 2|\vartheta-\varrho|^2 & 0 \\ 0 & 3|\vartheta-\varrho|^2 \end{bmatrix}$. Clearly, $(\mathfrak{M},\mathfrak{N},\mathfrak{A},\sigma)$ is a joint complete C^*-algebra valued bipolar b-metric space, where $b = 2^{2p}I_A$.*

Define $J\vartheta = \frac{\vartheta}{2}$, *As*, $J([0,\infty)) \subseteq [0,\infty)$ *and* $J((-\infty,0]) \subseteq (-\infty,0]$, *J is covariant map.*

$\alpha_{\mathfrak{A}}(\vartheta,\varrho) = I_{\mathfrak{A}}$ *and* $\psi(a) = \frac{a}{2}$.
Now, the left-hand side of Equation (3) becomes

$$\begin{aligned} \alpha_{\mathfrak{A}}(\vartheta,\varrho)\sigma(J\vartheta,J\varrho) &= \begin{bmatrix} 2|\frac{\vartheta}{2}-\frac{\varrho}{2}|^2 & 0 \\ 0 & 3|\frac{\vartheta}{2}-\frac{\varrho}{2}|^2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}|\vartheta-\varrho|^2 & 0 \\ 0 & \frac{3}{4}|\vartheta-\varrho|^2 \end{bmatrix} \end{aligned} \tag{30}$$

Now, the right-hand side of Equation (3) using Equation (30) implies

$$\begin{aligned} \psi_{\mathfrak{A}}(\sigma(\vartheta,\varrho)) &= \frac{1}{2}\begin{bmatrix} 2|\vartheta-\varrho|^2 & 0 \\ 0 & 3|\vartheta-\varrho|^2 \end{bmatrix} \\ \begin{bmatrix} \frac{1}{2}|\vartheta-\varrho|^2 & 0 \\ 0 & \frac{3}{4}|\vartheta-\varrho|^2 \end{bmatrix} &\preceq \begin{bmatrix} |\vartheta-\varrho|^2 & 0 \\ 0 & \frac{3}{2}|\vartheta-\varrho|^2 \end{bmatrix} \end{aligned}$$

Therefore, Equation (3) holds. Since J is a continuous and $\alpha_{\mathfrak{A}}$*-admissible. So, all the assumptions of Theorems 1 and 3 occurred.*

Hence, J possesses a unique fixed point.

Clearly, 0 *is the fixed point of J, which is unique in nature.*

Thus, Theorems 1 and 3 are verified.

However, as $J([0,\infty)) = [0,\infty) \not\subset (-\infty,0]$ *and* $J((-\infty,0]) = (-\infty,0] \not\subset [0,\infty)$, *So, J is not a contravariant map.*

Hence, Theorem 2 cannot apply here.

Example 3. *Let* $\mathfrak{M} = (-\infty,0], \mathfrak{N} = [0,\infty), \mathfrak{A} = M_2(\mathbb{C})$. *Define* $\sigma : (-\infty,0] \times [0,\infty) \to M_2(\mathbb{C})$ *as* $\sigma(\vartheta,\varrho) = \begin{bmatrix} 3|\vartheta-\varrho|^2 & 0 \\ 0 & 4|\vartheta-\varrho|^2 \end{bmatrix}$. *Clearly,* $(\mathfrak{M},\mathfrak{N},\mathfrak{A},\sigma)$ *is a joint complete* C^**-algebra valued bipolar b-metric space, where* $b = 2^{2p}I_{\mathfrak{A}}$.

Suppose $J\vartheta = \frac{-\vartheta}{7}, \alpha_{\mathfrak{A}}(\vartheta,\varrho) = I_{\mathfrak{A}}$ *and* $\psi(a) = \frac{a}{3}$.
Now, the left-hand side of Equation (3) becomes

$$\begin{aligned} \alpha_{\mathfrak{A}}(\vartheta,\varrho)\sigma(J\varrho,J\vartheta) &= \begin{bmatrix} 3|\frac{-\varrho}{7}-\frac{-\vartheta}{7}|^2 & 0 \\ 0 & 4|\frac{-\varrho}{7}-\frac{-\vartheta}{7}|^2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{7^2}|\vartheta-\varrho|^2 & 0 \\ 0 & \frac{4}{7^2}|\vartheta-\varrho|^2 \end{bmatrix} \end{aligned} \tag{31}$$

Now, the right-hand side of Equation (4) using Equation (31) implies

$$\begin{aligned} \psi_{\mathfrak{A}}(\sigma(\vartheta,\varrho)) &= \frac{1}{3}\begin{bmatrix} 3|\vartheta-\varrho|^2 & 0 \\ 0 & 4|\vartheta-\varrho|^2 \end{bmatrix} \\ \begin{bmatrix} \frac{3}{7^2}|\vartheta-\varrho|^2 & 0 \\ 0 & \frac{4}{7^2}|\vartheta-\varrho|^2 \end{bmatrix} &\preceq \begin{bmatrix} |\vartheta-\varrho|^2 & 0 \\ 0 & \frac{4}{3}|\vartheta-\varrho|^2 \end{bmatrix} \end{aligned}$$

So, Equation (4) holds. Since J is a continuous and $\alpha_{\mathfrak{A}}$*-admissible. So, all the assumptions of Theorems 2 and 3 occurred.*

Hence, J possesses a unique fixed point.

Clearly, 0 *is the fixed point of J, which is unique.*

Thus, Theorems 2 and 3 are verified.

However, as $J([0,\infty)) = (-\infty.0] \nsubseteq [0,\infty)$ *and* $J((-\infty,0]) = [0,\infty) \nsubseteq (-\infty,0]$, *so J is not a covariant map. Hence Theorem 1 cannot be applied here.*

5. Well-Posedness of Fixed Point Problem

Theorem 6. *Let $(\mathfrak{M},\mathfrak{N},\mathfrak{A},\sigma)$ be a joint complete C^*-algebra valued bipolar b-metric space and $J:(\mathfrak{M},\mathfrak{N})\rightrightarrows(\mathfrak{M},\mathfrak{N})$ be a $(\alpha_{\mathfrak{A}}-\psi_{\mathfrak{A}})$ covariant contractive mapping such that condition (P) is hold and $(I-\psi_{\mathfrak{A}}(b)\psi^2_{\mathfrak{A}})^{-1}$ exists. If $\alpha_{\mathfrak{A}}(\vartheta,\varsigma),\alpha_{\mathfrak{A}}(\varsigma,\varrho)\succeq I_{\mathfrak{A}}$ for all $(\vartheta,\varrho)\in\mathfrak{M}\times\mathfrak{N}$, where $J\varsigma=\varsigma$. Then the fixed point problem of J is well-posed.*

Proof. From the Theorems 1 and 3, J has a unique fixed point say ς. Now, let us assume that (ϑ_n,ϱ_n) is a bisequence in $\mathfrak{M}\times\mathfrak{N}$ such that $\sigma(\vartheta_n,J\varrho_n)\to 0$ and $\sigma(J\vartheta_n,\varrho_n)\to 0$ as $n\to\infty$. Then

$$\begin{aligned}\sigma(\vartheta_n,\varsigma) &\preceq b[\sigma(\vartheta_n,J\varrho_n)+\sigma(\varsigma,J\varrho_n)+\sigma(\varsigma,\varsigma)]\\ &\preceq b[\sigma(\vartheta_n,J\varrho_n)+\alpha_{\mathfrak{A}}(\varsigma,\varrho_n)\sigma(J\varsigma,J\varrho_n)]\\ &\preceq b[\sigma(\vartheta_n,J\varrho_n)+\psi_{\mathfrak{A}}(\sigma(\varsigma,\varrho_n))].\end{aligned}\tag{32}$$

Similarly,

$$\begin{aligned}\sigma(\varsigma,\varrho_n) &\preceq b[\sigma(\varsigma,\varsigma)+\sigma(J\vartheta_n,\varsigma)+\sigma(J\vartheta_n,\varrho_n)]\\ &\preceq b[\sigma(J\vartheta_n,\varrho_n)+\alpha_A(\varrho_n,\varsigma)\sigma(J\vartheta_n,J\varsigma)]\\ &\preceq b[\sigma(J\vartheta_n,\varrho_n)+\psi_A(\sigma(\vartheta_n,\varsigma))].\end{aligned}\tag{33}$$

Now, by using Equation (33) in (32), we get $\sigma(\vartheta_n,\varsigma)\preceq(I-\psi_{\mathfrak{A}}(b)\psi^2_{\mathfrak{A}})^{-1}[b\sigma(\vartheta_n,J\varrho_n)+\psi_{\mathfrak{A}}(b\sigma(J\vartheta_n,\varrho_n))]$

Taking $n\to\infty$, we get
$\sigma(\vartheta_n,\varsigma)\to 0$.

In a similar way, one can prove easily that $\sigma(\varsigma,\varrho_n)\to 0$. Thus, the fixed point problem of J is well-posed. □

Theorem 7. *Let $(\mathfrak{M},\mathfrak{N},\mathfrak{A},\sigma)$ be a joint complete C^*-algebra valued bipolar b-metric space and $J:(\mathfrak{M},\mathfrak{N})\rightleftarrows(\mathfrak{M},\mathfrak{N})$ be a $(\alpha_{\mathfrak{A}}-\psi_{\mathfrak{A}})$ contravariant contractive mapping such that condition (P) is hold and $(I-b(\psi_{\mathfrak{A}}))^{-1}$ exists. If $\alpha_{\mathfrak{A}}(\vartheta,\varsigma),\alpha_{\mathfrak{A}}(\varsigma,\varrho)\succeq I_{\mathfrak{A}}$ for all $(\vartheta,\varrho)\in\mathfrak{M}\times\mathfrak{N}$, where $J\varsigma=\varsigma$. Then the fixed point problem of J is well-posed.*

Proof. From the proof of Theorems 2 and 3, J possesses a unique fixed point, say ς. Now, let us suppose that (ϑ_n,ϱ_n) is a bisequence in $\mathfrak{M}\times\mathfrak{N}$ such that $\sigma(\vartheta_n,J\vartheta_n)\to 0$ and $\sigma(J\varrho_n,\varrho_n)\to 0$ as $n\to\infty$. Then

$$\begin{aligned}\sigma(\vartheta_n,\varsigma) &\preceq b[\sigma(\vartheta_n,J\vartheta_n)+\sigma(\varsigma,J\vartheta_n)+\sigma(\varsigma,\varsigma)]\\ &\preceq b[\sigma(\vartheta_n,J\vartheta_n)+\alpha_{\mathfrak{A}}(\vartheta_n,\varsigma)\sigma(J\varsigma,J\vartheta_n)]\\ &\preceq b[\sigma(\vartheta_n,J\vartheta_n)+\psi_{\mathfrak{A}}(\sigma(\varsigma,\vartheta_n))]\\ \sigma(\vartheta_n,\varsigma) &\preceq (I-b(\psi_{\mathfrak{A}}))^{-1}b(\sigma(\vartheta_n,J\vartheta_n))\end{aligned}\tag{34}$$

Taking $n\to\infty$, we get
$\sigma(\vartheta_n,\varsigma)\to 0$.

In a similar way, one can prove easily that $\sigma(\varsigma,\varrho_n)\to 0$. Thus, the fixed point problem of J is well-posed. □

6. Ulam–Hyers Stability

Let $(\mathfrak{M},\mathfrak{N},\mathfrak{A},\sigma)$ be a joint complete C^*-algebra valued bipolar b-metric space and $J:(\mathfrak{M},\mathfrak{N})\rightleftarrows(\mathfrak{M},\mathfrak{N})$ be a mapping. Let us consider the fixed point equation

$$J\varsigma=\varsigma,\varsigma\in\mathfrak{M}\cap\mathfrak{N},\tag{35}$$

and for some $\varepsilon>0$

$$||\sigma(\omega,J\omega)||<\varepsilon\ for\ \omega\in\mathfrak{M}\ or\ ||\sigma(J\omega,\omega)||<\varepsilon\ for\ \omega\in\mathfrak{N}.\tag{36}$$

Any point $\omega \in \mathfrak{M} \cup \mathfrak{N}$, which is a solution of the Equation (36), is called an ε-solution of the mapping J. We say that the fixed point problem (35) is Ulam–Hyers stable in C^*-algebra valued bipolar b-metric space if there exists a function $\mathfrak{Z} : [0,\infty) \to [0,\infty)$ with $\mathfrak{Z}(t) > 0$ when $t > 0$, such that for each $\varepsilon > 0$ and an ε-solution $\omega \in \mathfrak{M} \cup \mathfrak{N}$, there exists a solution ς of the fixed point Equation (35) such that

$$||\sigma(\varsigma,\omega)|| < \mathfrak{Z}(\varepsilon) \text{ or } ||\sigma(\omega,\varsigma)|| < \mathfrak{Z}(\varepsilon). \tag{37}$$

Theorem 8. *Let $(\mathfrak{M},\mathfrak{N},\mathfrak{A},\sigma)$ be a joint complete C^*-algebra valued bipolar b-metric space and $J : (\mathfrak{M},\mathfrak{N}) \rightleftarrows (\mathfrak{M},\mathfrak{N})$ be a contravariant mapping satisfying Equation (4) and if $J\vartheta = \vartheta$ then $\alpha_{\mathfrak{A}}(\vartheta,\varrho) \succeq I_{\mathfrak{A}}$ for all $\varrho \in \mathfrak{N}$ and $\alpha_{\mathfrak{A}}(\varrho,\vartheta) \succeq I_{\mathfrak{A}}$ for all $\varrho \in \mathfrak{M}$. Suppose that $(I-b)^{-1}$ exists (where $I \preceq b$) . Moreover, the onto function $\pi : [0,\infty) \to [0,\infty)$ such that $\pi(t) = \lambda t (\lambda > 0)$ is strictly increasing, then the fixed point Equation (35) of J is Ulam–Hyers stable.*

Proof. Let $\varepsilon > 0$ be arbitrary and ω is a ε-solution with $\omega \in \mathfrak{M}$ that is $||\sigma(\omega, J\omega)|| < \varepsilon$. From Theorems 2 and 3, we obtain that J has a unique fixed point $\varsigma \in \mathfrak{M} \cap \mathfrak{N}$. Now, as J satisfies Equation (4), by using the fact that $\psi_{\mathfrak{A}}(t) \preceq t$, we have

$$\begin{aligned}
\sigma(\omega,\varsigma) &\preceq b[\sigma(\omega,J\omega) + \sigma(\varsigma,J\omega) + \sigma(\varsigma,\varsigma)] \\
&= b[\sigma(\omega,J\omega) + \sigma(J\varsigma,J\omega)] \\
&\preceq b[\sigma(\omega,J\omega) + \alpha_{\mathfrak{A}}(\omega,\varsigma)\sigma(J\varsigma,J\omega)] \\
&\preceq b[\sigma(\omega,J\omega) + \psi_{\mathfrak{A}}(\sigma(\omega,\varsigma))] \\
(1-b)\sigma(\omega,\varsigma) &\preceq b\sigma(\omega,J\omega) \\
\sigma(\omega,\varsigma) &\preceq \frac{b}{1-b}\sigma(\omega,J\omega) \\
||\sigma(\omega,\varsigma)|| &\leq ||\frac{b}{1-b}||(||\sigma(\omega,J\omega)||) \\
||\sigma(\omega,\varsigma)|| &\leq ||\frac{b}{1-b}||\varepsilon
\end{aligned}$$

Applying π, we get

$$\pi(||\sigma(\omega,\varsigma)||) = \lambda||\sigma(\omega,\varsigma)|| < \lambda||\frac{b}{1-b}||\varepsilon$$

This implies that

$$||\sigma(\omega,\varsigma)|| < \pi^{-1}(\lambda||\frac{b}{1-b}||\varepsilon).$$

where $\pi^{-1}(t) = \mathfrak{Z}(t)$.

Therefore, Equation (37) is satisfied. Similarly, we can prove that if ω be an ε-solution with $\omega \in \mathfrak{N}$ that is $||\sigma(J\omega,\omega)|| < \mathfrak{Z}(\varepsilon)$. So, the fixed point Equation (35) of J is Ulam–Hyers stable. □

Remark 3. *In the above theorem for each ε and ε-solution, if we take $\mathfrak{Z}(t) = \pi^{-1}(t)$ (as defined above), then there exists a solution of Equation (35) such that Equation (37) is satisfied.*

7. Conclusions

We put forward a novel idea of C^*-algebra valued bipolar b-metric spaces in this document. We derived fixed point results in the aforementioned spaces by using $(\alpha_{\mathfrak{A}} - \psi_{\mathfrak{A}})$ contractions. Our outcomes generalized some previously published insights. The obtained results have been utilized to establish Ulam–Hyers' stability and the well-posedness of fixed point problems. Some examples are also given to demonstrate our research results. It will be an open problem to generalize our outcomes to other contractive conditions in the context of C^*-algebra valued bipolar b-metric spaces.

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