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Mathematical Modeling of Mechanical Forces and Power Balance in Electromechanical Energy Converter

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Abstract: This article proposes a calculation method for mechanical (electromagnetic) forces arising in an electromechanical energy converter acting on current circuits in a magnetic field, or on capacitor plates in an electric one. Transformations were performed on the basis of the principle of possible displacements involving the apparatus of partial derivatives. It was found that the power converted into mechanical power is partially spent on changing the energy of the electromagnetic field, and the remaining power, determined by the co-energy, is converted into mechanical power. Expressions for the mechanical (electromagnetic) forces were obtained based on the power balance. The reliability of the obtained results was compared with the known results. Of these, one can observe the well-known 50/50 principle, which states that only part of the power associated with the movement of the circuits is converted into mechanical power, while the rest is intended for changing the energy of the magnetic field.

Keywords: mathematical modeling; differential equations; partial differential equations; electromechanical energy conversion; energy; co-energy; mechanical force; power balance; electromagnetic field

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1. Introduction

Electromechanical systems are widely used in such fields of science and technology as energy, electromechanics and electrical engineering.

An electromechanical system (EMS) is a system that converts electrical energy into mechanical energy, and vice versa.

A mathematical model is a description of an original system using mathematical symbols. The calculation and design of a mathematical model consists in determining the parameters and characteristics of individual elements, links, blocks and devices, as well as parts of the system and the entire system as a whole [1]. To achieve this, algorithms and techniques are compiled to calculate:

- Load (external) and performance characteristic dependencies on power and other parameters;
- Amplitude characteristics;
- Transfer functions;
- Outputs and consumed powers;
- Energy indicators and the gain and transmission coefficients of both individual elements and parts of the system, the system as a whole, etc.

The basis for the construction of mathematical models of electromechanical systems comprises the physical laws that determine the principles of the functioning of these systems. At the same time, depending on the task, it is necessary to identify the laws that affect the most significant characteristics of the research object, without overloading the model with additional parameters that are not of interest in the study, analysis, synthesis and design of electromechanical systems [2].

It should be noted that the approximate solution, using mathematical modeling, has been found. The main research methods are approximation and numerical analysis. In the development of computer technology and applied software, numerical analysis using "computer mathematics" has practically supplanted other classical methods of solving equations.

The final step in building a mathematical model is the preparation of an algorithm and a user program, taking into account the type, the operation features and the necessary application software of the computer on which mathematical modeling will be performed [3].

Electrical and electromechanical equipment and systems subject to research can be conditionally divided into information parts and power parts.

The conditionality of such division is due to the widespread introduction of semiconductor technology, the use of new materials, qualitative changes in the methods and means of control due to the intensive development and implementation of computer and microprocessor systems, and the development of new information technologies. Examples of power units can be seen in the research and development of interconnected electric drives of technological lines and complexes, and research and development in the field of mechatronics [4].

The principle of operation of any automatic control system is to detect deviation in the controlled values that characterize the operation of the machine, or the course of the process, and at the same time, act on the machine or process in such a way as to eliminate the deviations that have arisen.

The integral components of modern electromechanical systems are electric motors, converters, control devices, transmission and actuators. Most of these devices have typical circuit solutions and do not undergo changes in their design processes and EMC research.

However, the quality of design results and the speed of their receipt are determined by the level of the mathematical models, their accuracy and their adequacy for real devices.

The emergence of modern computers with high computing power has become a real breakthrough in the field of technology, and it has become possible to model all the described elements in the shortest possible time. Additionally, the development of EMC at the present stage proceeds both along the path of improving technical means and in the direction of finding new control algorithms. The achieved level and prospects for the development of automated electric drive and computer control systems are such that they make it possible to practically implement control algorithms of any complexity and configuration in real time. The problem consists only in determining the optimal control algorithm and the reliability of the process equipment.

In these cases, the following main areas are distinguished in the design and study of EMC:

- The autonomous design and study of EMC elements;
- Joint design of the power part of the automated electric drive (AED) and the technological unit (TA) with subsequent autonomous design of the control part of the AED and the EMC control device;
- Joint design of the power part of the AED and HE with subsequent joint design of the control devices of the AED and the system as a whole;
- The integrated joint design and study of all EMC elements.

The greatest efficiency of design solutions, obviously, can be achieved through the joint design of all elements of the system.

It follows from the above that there are no problems in building models of analog controllers.

The problem of automating the design and research of electromechanical systems is associated with achieving the automated construction of a rational EMC power circuit and the automated synthesis of a multilevel control system of a hierarchical structure. The solution to the first problem, as a rule, does not differ in the variety of options for design solutions, each of which is obtained by searching a limited set of rational elements of the power unit and designing devices for their interaction.

The many possible options for control algorithms at each control level, the many ways of implementing these algorithms and the simplicity of their restructuring and adjustment cause significant difficulties in solving the second problem. However, in this case, significant progress can be achieved in improving the quality of designed systems, and the field of using computer-aided design is most effective [5].

The calculation of mechanical forces and power balance in an electromechanical energy converter is a complex task that includes calculations of electromagnetic forces and electromagnetic moments, as well as calculations of mechanical force and power balance. Developing a unified theory that would form the basis of methods for practical activities related to the design, manufacture and operation of electromechanical energy converters is an important and necessary task in the design of electrical machines with various operating principles [6].

Therefore, the purpose of this article is to obtain an algorithm and solve a mathematical problem to compile a theoretical mathematical model, in an analytical form, for the mechanical force and power balance of an electrical system resulting from the conversion of electrical energy into mechanical, and vice versa. Energy conversion is carried out through the interaction of electromagnetic fields created by the currents of circuits, the value of which, along with the resistance of the circuit, is determined by the applied voltage and induced electromotive forces (EMF) or currents. This problem is proposed to be solved in a general form using the apparatus of partial derivatives, the correct application of which makes it possible to significantly simplify the search for strength and energy balance. The novelty of this solution consists in considering the division of electrical power, converted into mechanical power, spent on changing the electromagnetic field, and co-energy, which is converted into mechanical power and spent on the production of mechanical work.

2. Materials and Methods

In this paper, we consider the compilation of a mathematical model for mechanical force and power balance resulting from the conversion of electrical energy into mechanical, or vice versa. It is carried out through the interaction of electromagnetic fields created by the circuit currents i_k, the value of which, along with the resistance of the circuit R_k, is determined by the applied voltage and induced electromotive forces (EMF) or currents. This is a cumbersome task; therefore, in the vast majority of works devoted to this issue, systems with a minimum amount of freedom are considered, of which one is associated with mechanical movement, and the other with an electrical one, consisting of a single coil or capacitor [7,8]. Additionally, in our opinion, many difficulties in achieving results can be solved when effectively using the apparatus of partial derivatives, the correct use of which makes it possible to significantly simplify the search for strength and energy balance [8]. To demonstrate the proposed approach (technique), we first consider a case whereby the system is connected by a magnetic field, and then, by an electric field [9].

When starting to solve the problem, one must note that it is largely solved. This solution represents the law of conservation of incoming energy (power), which is based on the Poynting theorem. The Poynting theorem is a theorem describing the law of conservation of energy in an electromagnetic field [10]. The theorem was proven in 1884 by John Henry Poynting. It is represented by the following formula:

$$\frac{\partial u}{\partial t} + \nabla \cdot S = -J \cdot E, \tag{1a}$$

where u is energy density: $u = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right)$,

where ε_0 is an electrical constant;
μ_0 is a magnetic constant;
∇ is the Nabla operator;
S is the Poynting vector;
J is the current density;
B is the induction of the magnetic field; and
E is the electric field strength.

Pointing's theorem in its integral form is represented as follows:

$$\frac{\partial}{\partial t}\int_V u \cdot dV + \oint_{\partial V} S \cdot dA = -J\int_V J \cdot E \cdot dV, \tag{1b}$$

where ∂V is the volume bounding surface V.

Pointing's theorem can also be written in the form:

$$\nabla \cdot S + \varepsilon_0 E\frac{\partial \varepsilon}{\partial t} + \frac{B}{\mu_0} \cdot \frac{\partial B}{\partial t} + J \cdot E = 0, \tag{1c}$$

where $\varepsilon_0 E\frac{\partial \varepsilon}{\partial t}$ is the electric field energy density;
$\frac{B}{\mu_0} \cdot \frac{\partial B}{\partial t}$ is the magnetic field energy density; and
$J \cdot E$ is the Joule loss power per unit volume.

The electrical power is said to be supplied to the terminals of the circuits (minus the Joule losses), and it is spent on changing the energy of the magnetic or electric field and performing mechanical work. When places are changed, a similar formulation of the law can be given for a converter operating as a source of electrical energy. However, for practical usage, this provision must occur in a form whereby its components are analytical functions of independent variables characterizing the state of the system. Such a function is known to be the function of the energy stored in the communication field, which carries all the information about the state of the system [11,12].

We will perform this first for a case whereby the connection between the current loops is provided by the magnetic fields created by them. We will find the energy relationships for the case under consideration on the basis of Ohm's law, supplemented by Faraday's law of electromagnetic induction [13]. In accordance with this law, we obtain:

$$u_k + e_k = i_k R_k. \tag{1}$$

Here u_k, i_k, R_k are voltage at the circuit terminals, current and resistance, and $e_k = -\frac{d\psi_k}{dt}$ is an EMF induced in the circuit, caused by a change in flux linkage ψ_k. Upon multiplying the above equation by the loop current i_k, after simple transformations, we obtain:

$$u_k i_k = (u_k - i_k R_k) i_k = i_k d\psi_k/dt \tag{2}$$

The same equations will represent the remaining "n" circuits of the system.

Let us reveal the expression of the electromotive force (EMF) [14]. The flux linkages of the circuits are functions of the currents flowing in them, with which connections are carried out through their own L_{kk} and mutual L_{kr} inductance, depending on their position in space, which is determined using the coordinates $x_1 \ldots x_m$. In this case, the mutual inductances L_{kr} and L_{rk} are always equal. This is the condition under which both Newton's third law and the independence of the accumulated energy from the transition trajectory proceeding from the initial to the final position of the system are satisfied [15,16]. Taking into account the above remarks, we can state that:

$$\psi_k = \sum_{r=1}^{n} L_{kr} i_r, \tag{3}$$

where the inductance of the circuits L_{kr} represents functions of the spatial coordinates $x_1 \ldots x_m$, defining their mutual position. In accordance with the principle of possible displacements, we assume that it is in force under the condition $dx_j = 0$ at $j \neq \nu$. In this case, for the EMF of the circuit [17] we obtain:

$$-e_k = \sum_{r=1}^{n} \frac{\partial \psi_k}{\partial i_r}\frac{di_r}{dt} + \frac{\partial \psi_k}{\partial x_\nu}\frac{dx_\nu}{dt}. \quad (4)$$

Upon substituting (4) into Equation (2), extended to all "n" circuits of the system, we obtain:

$$\begin{array}{l} u_1 \cdot i_1 = \sum\limits_{r=1}^{n} i_1 \frac{\partial \psi_1}{\partial i_r}\frac{di_r}{dt} + i_1 \frac{\partial \psi_1}{\partial x_\nu}\frac{dx_\nu}{dt} \\ \ldots \\ u_n \cdot i_n = \sum\limits_{r=1}^{n} i_n \frac{\partial \psi_n}{\partial i_r}\frac{di_r}{dt} + i_n \frac{\partial \psi_n}{\partial x_\nu}\frac{dx_\nu}{dt} \end{array} \quad (5)$$

The total power supplied to the converter is determined by the sum of the above equations:

$$\sum_{k=1}^{n} u_k \cdot i_k = \sum_{k=1}^{n}\sum_{r=1}^{n} i_k \cdot \frac{\partial \psi_r}{\partial i_k}\frac{di_k}{dt} + \sum_{k=1}^{n} i_k \frac{\partial \psi_k}{\partial x} \cdot \frac{dx}{dt}. \quad (6)$$

Let us expand the first term on the right side of the resulting equation. For clarity, we will achieve this using the example of three contours (n = 3); the specific results obtained can then be easily generalized to an arbitrary number of contours. In this case, we obtain:

$$\begin{array}{l} \sum\limits_{k=1}^{3} u_k i_k = (i_1 \frac{\partial \psi_1}{\partial i_1} + i_2 \frac{\partial \psi_2}{\partial i_1} + i_3 \frac{\partial \psi_3}{\partial i_1})\frac{di_1}{dt} + \\ +(i_1 \frac{\partial \psi_1}{\partial i_2} + i_2 \frac{\partial \psi_2}{\partial i_2} + i_3 \frac{\partial \psi_3}{\partial i_2})\frac{di_2}{dt} + \\ +(i_1 \frac{\partial \psi_1}{\partial i_3} + i_2 \frac{\partial \psi_2}{\partial i_3} + i_3 \frac{\partial \psi_3}{\partial i_3})\frac{di_3}{dt} \end{array} \quad (7)$$

The sum of the terms in parentheses represents the flux linkage of the ψ_n nth contour. This can easily be seen if we take into account that $L_{kk} = \partial\psi_k/\partial i_k$. This is an expression of the self-inductance of the circuit, and $L_{kr} = \partial\psi_k/\partial i_r$ is an expression of its mutual inductance. In accordance with (3), we obtain:

$$\begin{array}{l} \psi_1 = L_{11}i_1 + L_{12}i_2 + L_{13}i_3 \\ \psi_2 = L_{21}i_1 + L_{22}i_2 + L_{23}i_3 \\ \psi_3 = L_{31}i_1 + L_{32}i_2 + L_{33}i_3 \end{array} \quad (8)$$

In turn, the flux linkage of the circuits ψ_k [18,19] can be found through the value of the energy stored in the magnetic field W_m [20]:

$$W_m = \frac{1}{2}\sum_{k=1}^{n} \psi_k i_k = \frac{1}{2}\sum_{k=1}^{n}\sum_{r=1}^{n} L_{kr} i_k i_r = \sum_{k=1}^{n}\int_0^{\psi_k} i_k d\psi_k \quad (9)$$

which, for the convenience of further transformations, for example, with n = 3, can be conveniently represented in the following form:

$$W_m = \frac{1}{2}(L_{11}{i_1}^2 + L_{22}i_2^2 + L_{33}i^2) + L_{12}i_1 i_2 + L_{13}i_1 i_3 + Li_2 i_3 \quad (10)$$

Using (10), we find that $\frac{\partial W_m}{\partial i_k} = \psi_k$, which will correspond to Equation (8). Thus, the considered term can be written in the following form:

$$\sum_{k=1}^{n}\sum_{r=1}^{n} i_n \frac{\partial \psi_r}{\partial i_n}\frac{di_n}{dt} = \sum_{k=1}^{n} \frac{\partial W_m}{\partial i_k}\frac{di_k}{dt} \quad (11)$$

Let us further consider the second term of Equation (6). Here, the partial derivative $\partial\psi_k/\partial x_\nu$ is taken by displacement x_ν when other independent variables remain constant. Therefore, we can transform it as follows:

$$\sum i_k \frac{\partial\psi_k}{\partial x_\nu} = \sum_{k=1}^{n} \frac{\partial(\psi_k i_k)}{\partial x_\nu} \tag{12}$$

This means it is possible to bring the current i_k under the sign of the derivative. This step will greatly simplify the achievement of the goal.

Work $i_k\psi_k$ can be found directly using Equation (2) (although other possibilities can be noted). Upon dividing the variables and integrating them by part, by extending the result to all 'n' contours, we obtain:

$$W_m = \sum_{k=1}^{n} \int_0^{\psi_k} i_k d\psi_k = \sum_{k=1}^{n} \left(i_k\psi_k - \int_0^{i_k} \psi_k di_k \right) \tag{13}$$

where the last term on the right side is called the co-energy. It has been established that the electrical power converted into mechanical power is partially spent on changing the energy of the electromagnetic field W_m. Additionally, the remaining power, determined by the co-energy W'_m, is converted into mechanical power [21]. Co-energy includes part of the energy of the magnetic field, which is expended during the performance of mechanical work. It follows that the energy W_m and the co-energy W'_m are related through the ratio:

$$W_m + W'_m = \sum_{k=1}^{n} i_k\psi_k. \tag{14}$$

Thus, taking into account (14), expression (12) is reduced to the form:

$$\sum_{k=1}^{n} \frac{\partial(i_k\psi_k)}{\partial x_\nu}\frac{dx_\nu}{dt} = (\frac{\partial W_m}{\partial x_\nu} + \frac{\partial W'_m}{\partial x_\nu})\frac{dx_\nu}{dt}. \tag{15}$$

Taking into account the obtained relations, the power balance in Equation (6) is reduced to the following form:

$$\sum_{k}^{n} u_k i_k = \sum_{k}^{n} \frac{\partial W_m}{\partial i_k}\frac{di_k}{dt} + \frac{\partial W_m}{\partial x_\nu}\frac{dx_\nu}{dt} + \frac{\partial W'_m}{\partial x_\nu}\frac{dx_\nu}{dt}. \tag{16}$$

Here, the sum of the first two terms on the right side, as is easy to establish, is the derivative of the stored energy dW_m/dt, and is disclosed for all independent variables: currents i_k and displacement X_ν. Taking into account the noted circumstances, let us rewrite Equation (16):

$$\sum_{k}^{n} u_k i_k = \frac{dW_m}{dt} + F_{em\nu}\frac{dx_\nu}{dt}. \tag{17}$$

3. Results and Discussion

The resulting Equation (17) is an equation for the conservation of power or energy per unit of time and is read in a known way: the electrical power supplied to the terminals of the converter (minus electrical losses) is spent on changing the stored energy Wm and changing the mechanical power produced by the force of electromagnetic nature as follows:

$$F_{em\nu} = \partial W'_m/\partial x_\nu. \tag{18}$$

Moreover, we note that the expression for force (18) was a product of the search for the power balance equation. In fact, from this, it can be seen that the change in the stored energy of the magnetic field (in the linear formulation of the problem, when the energy and co-energy are equal) consumes power equal to that converted into mechanical power.

In the case where instead of currents i_k, we accepted flux linkages ψ_k and moving x_ν as independent variables, the expression for strength $F_{em\nu}$ can be easily found if we take into account that the balance Equation (17) is an invariant of the independent variables. It retains its form when they are replaced. For this, it is enough to replace the derivative functions included in it (dW_m/dt and $\partial W'_m/\partial x_\nu$) to express new variables.

In our case, according to (1), we use the following expression for energy:

$$W_m = \sum_{k=1}^{n} \int_0^{\psi_k} i_k d\psi_k = \frac{1}{2} \sum_{k=1}^{n} \sum_{r=1}^{n} \Gamma_{kr}\psi_k\psi_r. \tag{19}$$

This will be obtained, provided that the relationship between currents ik and flux linkages ψ_k is defined as [22]:

$$i_k = \sum_{r=1}^{n} \Gamma_{kr}\psi_r. \tag{20}$$

For example, for $n = 3$, $i_1 = \Gamma_{11}\psi_1 + \Gamma_{12}\psi_2 + \Gamma_{13}\psi_3$, etc.

In (20), Γ_{kr} represents inverse inductances; k = r is the own inductance and $k \neq r$ is mutual inductance. They can be found by solving Equation (3) for currents i_k. Additionally, in this case, $\Gamma_{kr} = \Gamma_{rk}$. Therefore, when searching for energy, the same methods as have been mentioned above can be used.

Thus, Equation (17) can be written using new variables, taking into account that the energy derivative is:

$$\frac{dW_m}{dt} = \sum_{k=1}^{n} \frac{\partial W_m}{d\psi_k}\frac{d\psi_k}{dt} + \frac{\partial W_m}{\partial x_\nu}\frac{dx_\nu}{dt}$$

in the following form:

$$\sum_{k=1}^{n} u_k i_k = \sum_{k=1}^{n} \frac{\partial W_m}{\partial \psi_k}\frac{d\psi_k}{dt} + \frac{\partial W_m}{\partial x_\nu}\frac{dx_\nu}{dt} + F_{эM\nu}\frac{dx_\nu}{dt}. \tag{21}$$

Let us transform the resulting expression because:

$$\frac{\partial W_m}{\partial \psi_k} = i_k; \frac{d\psi_k}{dt} = u_k.$$

Then, we can immediately obtain the equation for mechanical force:

$$F_{em\nu} = -\frac{\partial W_m}{\partial x_\nu}. \tag{22}$$

It is interesting to give another derivation of the force expression (22). According to (2), we obtain the following expression for the electric power supplied to the terminals:

$$\sum_{k=1}^{n} u_k i_k = \sum_{k=1}^{n} i_k \frac{d\psi_k}{dt}. \tag{23}$$

Let us transform the above expression, taking into account that $i_k = \frac{\partial W_m}{\partial \psi_k}$, and add the sum to the right side (noting that it is equal to zero):

$$\frac{\partial W_m}{\partial x_j}\frac{dx_j}{dt} - \frac{\partial W_m}{\partial x_j}\frac{dx_j}{dt} = 0. \tag{24}$$

As a result, after some grouping, we obtain:

$$\sum_{k=1}^{w} u_k i_k = \sum_{k=1}^{n} \frac{\partial W_m}{\partial \psi_k}\frac{d\psi_k}{dt} + \frac{\partial W_m}{\partial x_\nu}\frac{dx_\nu}{dt} - \frac{\partial W_m}{\partial x_\nu}\frac{dx_\nu}{dt}. \tag{25}$$

Here, the first two terms on the right-hand side represent a derivative of the energy $\frac{dW_m}{dt}$ written in terms of independent variables, and the reduced expression is therefore reduced to the form:

$$\sum_{k=1}^{w} u_k i_k = \frac{dW_m}{dt} - \frac{\partial W_m}{\partial x_\nu}\frac{dx_\nu}{dt}. \tag{26}$$

Comparing the resulting equation with the initial balance level (17), it is easy to see that here, we obtain the same expression for force as that obtained earlier (21):

$$F_{em} = -\frac{\partial W_m}{\partial x_j}, \tag{27}$$

Using the relationship between energy and co-energy (14), the above expressions for force, (18) and (27), are reduced to the following well-known form:

$$\begin{array}{l} F_{em\nu} = \sum\limits_{k=1}^{n} i_k \frac{d\psi_k}{dx_\nu} - \frac{\partial W_m}{\partial x_\nu} \\ F_{em\nu} = \frac{\partial W_m}{\partial x_j} - \sum\limits_{k=1}^{n} \sum\limits_{k=1} \psi_k \frac{\partial i_k}{\partial x_\nu} \end{array}, \tag{28}$$

where the first expression is obtained when the current is taken as the independent variables i_k and moving x_ν, and the second as flux linkage ψ_k and moving x_ν.

Let us further consider an electromechanical converter, where the current circuits interact by means of an electric field. For the initial analysis in the case under consideration, it is convenient to take the known dependence between the current i_k charge (on the plates) of the capacitor q_k:

$$i_k = \frac{dq_k}{dt}, \tag{29}$$

which is similar to the expression of the induced EMF (Faraday's law). Upon multiplying the left and right parts of the above ratio by the magnitude of the voltage $u_k = u'_k - i_k R_k$, we obtain:

$$u_k i_k = (u'_k - i_k R_k) i_k = u_k \frac{dq_k}{dt}, \tag{30}$$

which is similar to Equation (2) if we replace the last voltage on the right side u_k with current i_k, and the charge q_k with flux linkage ψ_k. Therefore, it is possible to use the previously obtained expressions, changing the variables in them in accordance with the noted analogy [23]. As a result, we obtain:

$$\begin{array}{l} W_e = \sum\limits_{k=1}^{n} \int\limits_0^{q_k} u'_k dq'_k = \frac{1}{2} \sum\limits_{k=1}^{n} \sum\limits_{r=1}^{n} C_{kr} u'_k i_k \\ W'_e = \sum\limits_{k=1}^{n} \int\limits_0^{u_k} q'_k du'_k = \frac{1}{2} \sum\limits_{k=1}^{n} \sum\limits_{r=1}^{n} S_{kr} q'_k u_k \end{array}, \tag{31}$$

where C_{kr} and S_{kr} are containers and inverted containers. Moreover, the latter can be found through the value of C_{kr}, assuming that C_{kr} is known:

$$q'_k = \sum_{kr=1}^{n} C_{kr} u'_k.$$

Accordingly, the expressions for the power equation will also be written as follows:

$$\sum_{k=1}^{w} u_k i_k = \frac{dW_e}{dt} + F_{ei}\frac{dx_i}{dt}, \quad (32)$$

where the power F_{ei} is defined as:

$$F_{ei} = \frac{\partial W'_e}{\partial x_k} = \sum_{k=1}^{k=n} u_k \frac{\partial q_k}{\partial x_k} - \frac{\partial W_e}{\partial x_k} \text{ or } F_{ei} = -\frac{\partial W_e}{\partial x_k} = \frac{\partial W'_e}{\partial x_k} - \sum_{k=1}^{k=n} q_k \frac{\partial u_k}{\partial x_k}. \quad (33)$$

Similarly, there are forces acting along other possible displacements. At the same time, we note that this is usually solved in the reverse order. First, the force is found, and then, the balance equation is defined [24], which is not used in the analysis below; that is, it plays a decorative or demonstrative role [25].

In the power (energy) balance Equation (32), mechanical power is produced by the force:

$$f_{ek} = \frac{\partial W'_e}{\partial x_k}. \quad (34)$$

Its expression is the basis for obtaining other forms of the expression of forces represented by Equation (31).

It is easy to establish, even for all four expressions, that the forces are identical to those given in [7,8], but their derivation is completely different, just as the formulation of the problem adopted by the authors is different. It, in its essence, does not allow us to obtain an energy balance, which shows that only part of the power associated with movement is converted into mechanical power, while the other part is converted into a change in the electric or magnetic energy of the field.

As a result, we obtained a solution to the mathematical problem of compiling a theoretical mathematical model for mechanical force and power balance arising from the conversion of electrical energy into mechanical, and vice versa. Energy conversion is carried out through the interaction of electromagnetic fields created by the currents of the circuits, the value of which, along with the resistance of the circuit, is determined by the applied voltage and induced electromotive forces or currents. This cumbersome problem was solved in a general form using the apparatus of partial derivatives, the correct application of which makes it possible to significantly simplify the search for force and energy balance. The novelty of this solution lies in the fact that it takes into account the division of electrical power into that partly spent on changing the electromagnetic field and co-energy, which is converted into mechanical power and spent on the performance of mechanical work.

4. Conclusions

A mathematical model is proposed, in an analytical form, for the mechanical forces arising between the current circuits they create in magnetic and electric fields. A distinctive feature of the above approach is that the forces are based on the balance applied to the terminals of the power converter. This approach, which allows us to find a mathematical description in the form of a power balance equation, and an expression for forces, provides clarity in the given solution and reliability in the obtained results.

The mathematical expressions were derived from mechanical (electromagnetic) forces arising in electromechanical energy converters. Using the independence condition intended for variables describing the state of the EMEE of the converter, it is possible to significantly simplify the derivation of the energy balance equations and, on its basis, obtain expressions for the forces arising as a result of the interaction of currents with both magnetic and electric fields. At the same time, their derivation can be significantly simplified without going beyond the generally accepted assumptions.

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