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Analysis of Control Problems for Stationary Magnetohydrodynamics Equations Under the Mixed Boundary Conditions for a Magnetic Field

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Abstract: The optimal control problems for stationary magnetohydrodynamic equations under the inhomogeneous mixed boundary conditions for a magnetic field and the Dirichlet condition for velocity are considered. The role of controls in the control problems under study is played by normal and tangential components of the magnetic field given on different parts of the boundary and by the exterior current density. Quadratic tracking-type functionals for velocity, magnetic field or pressure are taken as cost functionals. The global solvability of the control problems under consideration is proved, an optimality system is derived and, based on its analysis, a mathematical apparatus for studying the local uniqueness and stability of the optimal solutions is developed. On the basis of the developed apparatus, the local uniqueness of solutions of control problems for specific cost functionals is proved, and stability estimates of optimal solutions are established.

Keywords: magnetohydrodynamics; mixed boundary conditions; optimal control; optimality system; uniqueness; stability estimates

MSC: 35Q35; 76D03; 76D55

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1. Introduction

Great attention has recently been paid to optimal control problems for the models of magnetohydrodynamics (MHD) for a viscous conducting incompressible fluid. There are a number of papers devoted to the theoretical study of such problems. Among these, we mention the papers [1–8], devoted to studying the control problems for a stationary MHD system, and papers [9–12], where the authors study control problems in a nonstationary case. In these papers, solvability of the control problems is proved, and optimality systems that describe the necessary conditions of the extremum are constructed and studied. In [5], the uniqueness and stability of the solutions of control problems are studied for some particular cases.

Along with optimal control problems, inverse or identification problems for the MHD models as well as for other hydrodynamic models play a key role in applications. In these problems, some parameters (constant or functional) that are included into a boundary value problem under study are unknown and are required to be determined together with the solution by using additional information about the state of the system. It is significant that the identification problems can be reduced to appropriate control problems by choosing a suitable tracking-type cost functional. As a result, control and identification problems can be analyzed using a common approach based on the theory of smooth-convex extremum problems in Hilbert or Banach spaces [13,14] (see also [15]).

In this paper, we study control problems for the stationary MHD model in which magnetohydrodynamic equations are considered under mixed boundary conditions for a magnetic field and under the Dirichlet boundary condition for velocity (see Section 2).

The role of controls in control problems under study is played by normal and tangential components of the magnetic field given on the different parts of the boundary and by exterior current density given within the flow region. Our goal is to construct a theory for studying local uniqueness and the stability of optimal solutions. Reaching of this goal will be based on generalization of the approach developed in [5,16] for studying the stability of optimal solutions for the stationary Navier–Stokes and MHD systems related to small perturbations of the cost functionals to be minimized. In this approach, there is no requirement to determine the second derivative of the cost functional under minimization, since it is based on the analysis of fundamental properties of the optimality system for the control problem under study and the use of special estimates for the difference of solutions of the original and perturbed control problems. The method is rather simple, natural and applicable for the models of hydrodynamics, heat convection, mass transfer and other hydrodynamic models based on the Navier–Stokes system [15]. The use of this approach allows us to obtain stability estimates for optimal solutions with respect to small perturbations of the cost functional in an explicit and sufficiently easy-to-interpret form.

One of the features of this work consists in proving the solvability of optimal control problems under minimum requirements on normal and tangential components of the magnetic field given on the different parts of boundary and playing the role of controls in control problems. This result, in contrast to [5], allows the use of simple L^2 norms for the normal and tangential components of the magnetic field instead of the standard $H^{1/2}$ norms of boundary controls as Tikhonov regularizers while studying control problems for the MHD system. This regularization is needed to prove the local uniqueness and stability of the control problem solutions. The latter is the main goal of the study.

The structure of this paper is as follows. In Section 2, the boundary value problem is formulated for stationary MHD equations under inhomogeneous mixed boundary conditions for a magnetic field, and some notations that will be used throughout the paper are introduced along with the presentation of some additional facts that are necessary for studying optimal control problems. Presented in Section 3 are two lemmas on the existence of lifting of the velocity and of the magnetic component of the solution of the boundary value problem under consideration. In addition, the unique solvability of a generalized linear analogue of the original MHD boundary value problem is proved. In Section 4, two optimal control problems are formulated, and their solvability is proved, the optimality system describing the first-order necessary optimality conditions is derived and, on the basis of its analysis, the additional properties of optimal solutions are established. Finally, in Section 5, we will prove the local uniqueness and stability of the solutions for the control problems for "magnetic field-tracking" or "velocity-tracking" cost functionals.

2. Statement of the Boundary Value Problem and Notation

While studying the flows of electrically conducting fluids in real-life devices, the necessity often arises in flows of conducting fluid modeling in domains with boundaries consisting of parts with different electrophysical properties. Mathematical modeling of conducting flows in such types of domains gives rise to the study of boundary value problems for MHD equations under the mixed boundary conditions for a magnetic field.

Let Ω be a bounded domain of space $\mathbb{R}^3$ with boundary $\Sigma = \partial\Omega$ consisting of two parts Σ_τ and Σ_ν. In this paper, we study control problems for the stationary magnetohydrodynamic equations of a viscous incompressible fluid

$$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \kappa\,\mathrm{rot}\,\mathbf{H}\times\mathbf{H} = \mathbf{f},\quad \mathrm{div}\,\mathbf{u} = 0, \tag{1}$$

$$\nu_1\mathrm{rot}\,\mathbf{H} - \rho_0^{-1}\mathbf{E} + \kappa\mathbf{H}\times\mathbf{u} = \nu_1\mathbf{j},\quad \mathrm{div}\,\mathbf{H} = 0,\quad \mathrm{rot}\,\mathbf{E} = \mathbf{0}, \tag{2}$$

considered in domain Ω under the following inhomogeneous boundary conditions:

$$\mathbf{u}|_\Sigma = \mathbf{g},\ \mathbf{H}\cdot\mathbf{n}|_{\Sigma_\tau} = q,\ \mathbf{H}\times\mathbf{n}|_{\Sigma_\nu} = \mathbf{q},\ \ \mathbf{E}\times\mathbf{n}|_{\Sigma_\tau} = \mathbf{k}. \tag{3}$$

Here $\mathbf{u}$ is the velocity vector, $\mathbf{H}$ and $\mathbf{E}$ are magnetic and electric fields, respectively, $p = P/\rho_0$, where P is the pressure, $\rho_0 = \text{const}$ is a fluid density, $\varkappa = \mu/\rho_0$, $\nu_1 = 1/\rho_0\sigma = \varkappa\nu_m$, ν and ν_m are constant kinematic and magnetic viscosity coefficients, σ is a constant electrical conductivity, μ is a constant magnetic permeability, $\mathbf{n}$ is the outer normal to $\partial\Omega$, and $\mathbf{j}$ is the exterior current density. Further, we will refer to problem (1)–(3) for given $\mathbf{f}, \mathbf{j}, \mathbf{g}, q, \mathbf{q}$ and $\mathbf{k}$ as Problem 1. It should be noted that all the quantities used in (1)–(3) are dimensional and, moreover, their physical dimensions are defined in terms of SI units.

We emphasize that Equations (1) and (2) are considered under mixed boundary conditions with respect to electromagnetic field $(\mathbf{H}, \mathbf{E})$ in (3). These conditions generalize two types of previously used boundary conditions. The first type is described by relations $\mathbf{H} \cdot \mathbf{n} = 0$ and $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$, corresponding to a perfectly conducting boundary (see, e.g., [2–5,17–20]). The second type is described by the condition $\mathbf{H} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ (see [8,21,22]) corresponding to a perfectly insulating boundary.

For the first time, global solvability of the homogeneous analogue of Problem 1 was proved in [23]. In this work, mathematical tools from [24–26] were essentially used. Global solvability of the inhomogeneous mixed Problem 1 was proved in [27]. In [28], the results obtained in [27] were generalized for the model of heat conducting magnetohydrodynamics. It should be noted that papers [29–31], magnetohydrodynamic equations are studied under mixed boundary conditions with respect to velocity and under the standard boundary conditions of the first type for an electromagnetic field. In [32], the author proves the existence of a very weak solution of the MHD boundary value problem using the Dirichlet boundary condition for a magnetic field. In [33], the MHD boundary value problem is studied in the case where a pressure and zero tangential velocity component are specified on an entire boundary. A local solvability of boundary value and boundary control problems for the model of magnetomicropolar flow is proved in paper [6], where the optimality systems are also derived and analyzed for the control problem.

Papers [4,28] and a number of others study the solvability of boundary value problems for stationary and nonstationary magnetohydrodynamics–Boussinesq systems. In [34], the authors study the properties of solutions (and, in particular, solvability) of the heat conducting magnetic hydrodynamic equations with the buoyancy effects due to temperature differences in the flow, Joule and viscous heating effects. In [35,36], the authors study the solvability of boundary value problems respectively for a steady or nonsteady MHD–Boussinesq system considered under mixed boundary conditions for temperature, magnetic field and velocity, in the general case when the thermal conductivity viscosity coefficient, electrical conductivity, magnetic permeability and specific heat of the fluid depend on the temperature.

We will assume below that the domain Ω and the boundary Σ partitioning into parts Σ_τ and Σ_ν satisfy

(i) Ω is a bounded domain in the space $\mathbb{R}^3$, and its boundary $\Sigma = \partial\Omega$ consists of $m+1$ disjoint closed C^2 – surfaces $\Sigma_0, \Sigma_1, \ldots, \Sigma_m$, each of which has a finite area where Σ_0 is the outer boundary of Ω.

(ii) Sets Σ_τ and Σ_ν are not empty, open and the following conditions take place:

$$\Sigma = \overline{\Sigma}_\tau \cup \overline{\Sigma}_\nu, \;\; \overline{\Sigma}_\tau \cap \overline{\Sigma}_\nu = \varnothing.$$

The conditions in (ii) mean that each of the parts Σ_τ and Σ_ν consists of a finite number of connected components of the boundary Σ.

It is assumed that, in the general case, Ω is a multi-connected domain and by $n \geq 1$, we denote the number of handles of Σ. The case $n = 0$ corresponds to a simply connected domain.

The numbers n and m are respectively called the first and second Betti numbers (see, e.g., [15] (p. 277) and [37]). Typical examples of domain Ω are shown in Figures 1 and 2, where a simply connected domain Ω with a disjoint boundary $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ (with Betti numbers $m = 2$ and $n = 0$) and multi-connected toroidal domain Ω with a disjoint boundary Σ (with Betti numbers $m = 2$ and $n = 1$), respectively, are presented.

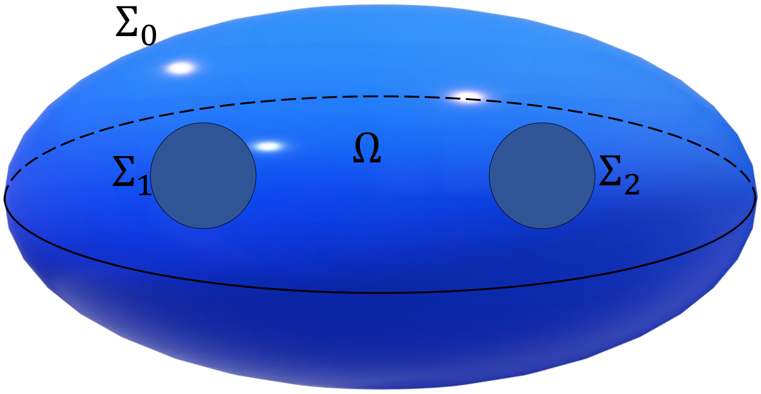

Figure 1. Example of simply connected domain with $n = 0$, $m = 2$.

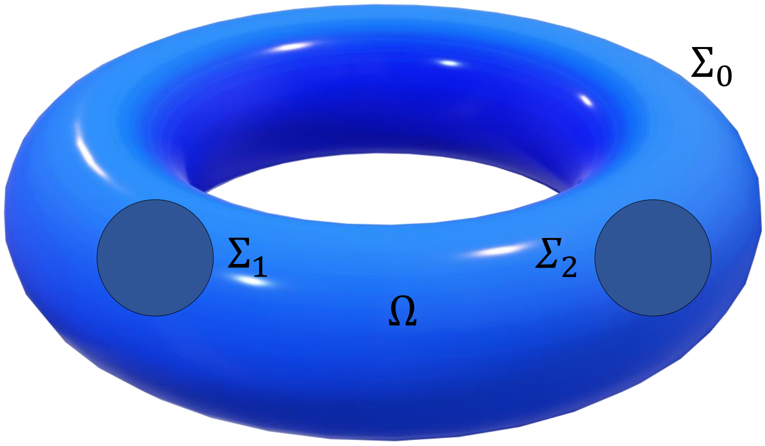

Figure 2. Example of multi-connected domain with $n = 1$, $m = 2$.

Similarly to [37], we denote, by $\hat{m} + 1$, the number of connected components of the part Σ_ν of the boundary Σ, by Σ_i, $i = 1, \ldots, m_0 \le m$ we denote the internal connected components of the boundary Σ contained in Σ_ν. It is clear that $\hat{m} = m_0$ if $\Sigma_0 \subset \Sigma_\nu$ and $\hat{m} = m_0 - 1$ if $\Sigma_0 \subset \Sigma_\tau$. Similarly, by n_1 (or n_2), we denote the number of handles of the part Σ_ν (or Σ_τ). Clearly, $n_1 + n_2 = n$ (where n is the number of handles of Σ).

Below we will use the Sobolev spaces $H^s(D)$, $s \in \mathbb{R}$, $H^0(D) \equiv L^2(D)$, where D denotes Ω or the boundary Σ or a part $\Sigma_0 \subset \Sigma$. The corresponding spaces of vector functions are denoted by $H^s(D)^3$ and $L^2(D)^3$. The inner products and norms in the spaces $H^s(D)$ and $H^s(D)^3$ are denoted by $(\cdot,\cdot)_{s,D}$ and $\|\cdot\|_{s,D}$. The inner products and norms in $L^2(\Omega)$ and $L^2(\Omega)^3$ are denoted by $(\cdot,\cdot)$ and $\|\cdot\|_\Omega$. By $\|\cdot\|_{1,\Omega}$ and $|\cdot|_{1,\Omega}$ we denote norm and seminorm in $H^1(\Omega)$ or in $H^1(\Omega)^3$. For arbitrary Hilbert space H by H^* we denote the dual space of H. By $H^s_T(\Sigma_0)$ we will denote the subspace in $H^s(\Sigma_0)^3$ consisting of tangential on $\Sigma_0 \subseteq \Sigma$ vector functions. Set $H^{-s}(\Sigma_\tau) = H^s(\Sigma_\tau)^*$, $H^{-s}_T(\Sigma_\nu) = H^s_T(\Sigma_\nu)^*$, $H^{-1/2}_T(\Sigma) = H^{1/2}_T(\Sigma)^*$ for $s \ge 0$.

Let $\mathcal{D}(\Omega)$ be the space of infinitely differentiable compactly supported functions in Ω, $H^1_0(\Omega)$ be the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$, $V = \{\mathbf{v} \in H^1_0(\Omega)^3 : \operatorname{div}\mathbf{v} = 0\}$, $H^{-1}(\Omega)^3 = (H^1_0(\Omega)^3)^*$, $L^2_0(\Omega) = \{p \in L^2(\Omega) : (p,1) = 0\}$, $H^1(\Omega,\Sigma_\tau) = \{\varphi \in H^1(\Omega) : \varphi|_{\Sigma_\tau} = 0\}$, $C_{\Sigma_\tau 0}(\overline{\Omega})^3 := \{\mathbf{v} \in C^0(\overline{\Omega})^3 : \mathbf{v}\cdot\mathbf{n}|_{\Sigma_\tau} = 0,\ \mathbf{v}\times\mathbf{n}|_{\Sigma_\nu} = \mathbf{0}\}$. In addition to the spaces defined above, we will use the spaces $H(\operatorname{div},\Omega) = \{\mathbf{v} \in L^2(\Omega)^3 : \operatorname{div}\mathbf{v} \in L^2(\Omega)\}$, $H(\operatorname{curl},\Omega) = \{\mathbf{v} \in L^2(\Omega)^3 : \operatorname{curl}\mathbf{v} \in L^2(\Omega)^3\}$, $H^0(\operatorname{curl},\Omega) = \{\mathbf{v} \in H(\operatorname{curl},\Omega) : \operatorname{curl}\mathbf{v} = \mathbf{0}\}$ and the space $H_{DC}(\Omega) = H(\operatorname{div},\Omega) \cap H(\operatorname{curl},\Omega)$, endowed with Hilbert norm defined by

$$\|\mathbf{u}\|^2_{DC} := l^{-2}\|\mathbf{u}\|^2_\Omega + \|\operatorname{div}\mathbf{u}\|^2_\Omega + \|\operatorname{curl}\mathbf{u}\|^2_\Omega. \qquad (4)$$

Here l is a dimensional factor of the dimension $[l] = L_0$, and its value is equal to 1, L_0 denotes the SI dimension of the length (for more detail, see Section 4).

Any vector $\mathbf{v}$ defined on the boundary Σ (or on a part $\Sigma_0 \subset \Sigma$) can be decomposed to the sum of two vectors—the normal and tangential components $\mathbf{v}_n$ and $\mathbf{v}_T$: $\mathbf{v} = \mathbf{v}_n + \mathbf{v}_T$. If $\mathbf{v} \in L^2(\Sigma)^3$, the components $\mathbf{v}_n$ and $\mathbf{v}_T$ are given by formulas $\mathbf{v}_n = (\mathbf{v} \cdot \mathbf{n})\mathbf{n} \equiv v_n \mathbf{n}$ and $\mathbf{v}_T = \mathbf{v} - \mathbf{v}_n \equiv (\mathbf{n} \times \mathbf{v}) \times \mathbf{n}$. Here $v_n = \mathbf{v} \cdot \mathbf{n}$ decribes the normal component of field $\mathbf{v}$, $\mathbf{v} \times \mathbf{n}$ describes the tangential vector orthogonal to the normal $\mathbf{n}$ and to $\mathbf{v}_T$. It is clear that $\mathbf{v}_T = \mathbf{0}$ on Σ if $\mathbf{v} \times \mathbf{n}|_\Sigma = \mathbf{0}$. By $\gamma_n|_{\Sigma_0}$ (or $\gamma_\tau|_{\Sigma_0}$) we denote the operator defined on $H_{DC}(\Omega)$, which is placed in correspondence to every function $\mathbf{h} \in H_{DC}(\Omega)$ with the normal trace $\gamma_n|_{\Sigma_0}\mathbf{h} = \mathbf{h} \cdot \mathbf{n}|_{\Sigma_0}$ (or tangential trace $\gamma_\tau|_{\Sigma_0}\mathbf{h} = \mathbf{h} \times \mathbf{n}|_{\Sigma_0}$).

The following Green's formulae (see, e.g., [38]) will be used below:

$$\int_\Omega \mathbf{v} \cdot \operatorname{grad} \varphi \, dx + \int_\Omega \operatorname{div} \mathbf{v} \, \varphi \, dx = \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} \, \varphi \, d\sigma \quad \forall \mathbf{v} \in H^1(\Omega)^3, \ \varphi \in H^1(\Omega), \tag{5}$$

$$\int_\Omega (\mathbf{v} \cdot \operatorname{curl} \mathbf{w} - \mathbf{w} \cdot \operatorname{curl} \mathbf{v}) dx = \int_{\partial\Omega} (\mathbf{v} \times \mathbf{n}) \cdot \mathbf{w}_T d\sigma \quad \forall \mathbf{v}, \mathbf{w} \in H^1(\Omega)^3. \tag{6}$$

If $\varphi \in H^1(\Omega, \Sigma_\tau)$ or $\mathbf{w} \in C_{\Sigma_\tau 0}(\overline{\Omega})^3 \cap H^1(\Omega)^3$ the right-hand sides of (5) or (6) become $\int_{\Sigma_\nu} \mathbf{v} \cdot \mathbf{n} \, \varphi d\sigma$ or $\int_{\Sigma_\tau} (\mathbf{v} \times \mathbf{n}) \cdot \mathbf{w}_T d\sigma$. Based on (5), (6) we say, following [25], that the function $\mathbf{v} \in H_{DC}(\Omega)$ weakly satisfies condition $\mathbf{v} \cdot \mathbf{n} = 0$ on Σ_ν if

$$\int_\Omega (\mathbf{v} \cdot \operatorname{grad} \varphi + \operatorname{div} \mathbf{v} \, \varphi) dx = 0 \quad \forall \varphi \in H^1(\Omega, \Sigma_\tau).$$

Similarly, we say that $\mathbf{v} \times \mathbf{n} = \mathbf{0}$ weakly on Σ_τ if

$$\int_\Omega (\mathbf{v} \cdot \operatorname{curl} \mathbf{w} - \mathbf{w} \cdot \operatorname{curl} \mathbf{v}) dx = 0 \; \forall \mathbf{w} \in C_{\Sigma_\tau 0}(\overline{\Omega})^3 \cap H^1(\Omega)^3.$$

Let $H_{DC\Sigma_\tau}(\Omega)$ be the closure of $C_{\Sigma_\tau 0}(\overline{\Omega})^3 \cap H^1(\Omega)^3$ with respect to norm $\| \cdot \|_{DC}$ in (4). Set

$$\mathcal{H}_{\Sigma_\tau}(\Omega) = \{\mathbf{h} \in L^2(\Omega)^3 : \operatorname{div} \mathbf{h} = 0, \ \operatorname{curl} \mathbf{h} = \mathbf{0} \text{ in } \Omega, \ \mathbf{h} \cdot \mathbf{n}|_{\Sigma_\tau} = 0, \ \mathbf{h} \times \mathbf{n}|_{\Sigma_\nu} = \mathbf{0}\},$$

$$\mathcal{H}_{\Sigma_\nu}(\Omega) = \{\mathbf{h} \in L^2(\Omega)^3 : \operatorname{div} \mathbf{h} = 0, \ \operatorname{curl} \mathbf{h} = \mathbf{0} \text{ in } \Omega, \ \mathbf{h} \cdot \mathbf{n}|_{\Sigma_\nu} = 0, \ \mathbf{h} \times \mathbf{n}|_{\Sigma_\tau} = \mathbf{0}\},$$

$$V_{\Sigma_\tau}(\Omega) = \{\mathbf{v} \in H_{DC\Sigma_\tau}(\Omega) : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega\} \cap \mathcal{H}_{\Sigma_\tau}(\Omega)^\perp. \tag{7}$$

Equalities $\mathbf{h} \cdot \mathbf{n}|_{\Sigma_\tau} = 0$, $\mathbf{h} \times \mathbf{n}|_{\Sigma_\nu} = 0$ or $\mathbf{h} \cdot \mathbf{n}|_{\Sigma_\nu} = 0$, $\mathbf{h} \times \mathbf{n}|_{\Sigma_\tau} = 0$ in (7) are understood in the weak sense defined above.

We remind that the spaces $\mathcal{H}_{\Sigma_\tau}(\Omega)$ and $\mathcal{H}_{\Sigma_\nu}(\Omega)$ are finite dimensional [37]. In particular, the dimension of $\mathcal{H}_{\Sigma_\tau}(\Omega)$ under conditions (i), (ii) is exactly $\hat{m} + n_2$, and the basis of the space $\mathcal{H}_{\Sigma_\tau}(\Omega)$ consists of gradients $\nabla \hat{z}_j$ of harmonic functions $\hat{z}_j \in H^1(\Omega)$, $j = 1, \ldots, \hat{m}$ satisfying boundary conditions

$$\partial \hat{z}_j / \partial n|_{\Sigma_\tau} = 0, \ \hat{z}_j|_{\Sigma_\nu \setminus \Sigma_j} = 0, \ \hat{z}_j|_{\Sigma_j} = 1, \ \ j = 1, \ldots, \hat{m},$$

and harmonic vector fields $\mathbf{y}_l \in \mathcal{H}_{\Sigma_\tau}(\Omega)$, $l = 1, \ldots, n_2$ satisfying the condition $\int_{\varsigma_k} \mathbf{y}_l \cdot \mathbf{n} d\sigma = \delta_{lk}$ for any cycle ς_k, $k = 1, \ldots, n_2$, contained in Σ_τ and not homotopic to zero in Ω. Here and below, δ_{lk} is the Christoffel symbol equal to 1 at $l = k$ and 0 at $l \neq k$.

Similarly, the basis of $\mathcal{H}_{\Sigma_\nu}(\Omega)$ consists of gradients $\nabla \hat{z}_j$ of harmonic functions $\hat{z}_j \in H^1(\Omega)$, $j = \hat{m} + 1, \ldots, m$ satisfying the boundary conditions

$$\partial \hat{z}_j / \partial n|_{\Sigma_\nu} = 0, \ \ \hat{z}_j|_{\Sigma_\tau \setminus \Sigma_j} = 0, \ \ \hat{z}_j|_{\Sigma_j} = 1, \ \ j = \hat{m} + 1, \ldots, m$$

and harmonic vector fields $\mathbf{y}_l \in \mathcal{H}_{\Sigma_\nu}(\Omega)$ satisfying the condition $\int_{\zeta_k} \mathbf{y}_l \cdot \mathbf{n} d\sigma = \delta_{lk} \ \ \forall \zeta_k$, $k, l = n_2 + 1, \ldots, n$. Here ζ_k denotes a cicle contained in Σ_ν that is not homotopic to zero in Ω.

A number of important properties inherent in the function spaces defined above and proved in [25] are presented in the following lemma.

Lemma 1. *We assume that conditions (i), (ii) hold. Then:*

(1) $H_{DC\Sigma_\tau}(\Omega) \subset H^1(\Omega)^3$ and the norm $\|\cdot\|_{DC}$ is equivalent to the norm $\|\cdot\|_{1,\Omega}$;

(2) there is a positive constant δ_1 depending on domain Ω and Σ_τ such that the following holds:

$$\|\mathrm{curl}\, \mathbf{h}\|^2 \geq \delta_1 \|\mathbf{h}\|^2_{1,\Omega} \ \ \forall \mathbf{h} \in V_{\Sigma_\tau}(\Omega); \tag{8}$$

(3) the orthogonal decomposition of the space $L^2(\Omega)^3$ holds:

$$L^2(\Omega)^3 = \nabla H^1(\Omega, \Sigma_\tau) \oplus \mathrm{curl}\, H_{DC\Sigma_\tau}(\Omega) \oplus \mathcal{H}_{\Sigma_\nu}(\Omega). \tag{9}$$

Along with the spaces $H_{DC}(\Omega)$ and $H^0(\mathrm{curl}, \Omega)$, their subspaces

$$\mathcal{H}^{s+1/2}_{\mathrm{div}}(\Omega) := H^{s+1/2}(\Omega)^3 \cap \{\mathbf{h} \in H_{DC}(\Omega) : \mathrm{div}\mathbf{h} = 0\} \cap \mathcal{H}_{\Sigma_\tau}(\Omega)^\perp,$$

$$H^0_{\Sigma_\tau}(\mathrm{curl}, \Omega) := \{\mathbf{e} \in H^0(\mathrm{curl}, \Omega) : \mathbf{e} \times \mathbf{n}|_{\Sigma_\tau} \in L^2_T(\Sigma_\tau)\}, \ 0 \leq s \leq 1/2,$$

will be used equipped, respectively, with norms

$$\|\mathbf{h}\|_{\mathcal{H}^{s+1/2}_{\mathrm{div}}(\Omega)} = \|\mathbf{h}\|_{s+1/2,\Omega} + \|\mathrm{curl}\, \mathbf{h}\|_\Omega, \ \ \|\mathbf{e}\|_{H^0_{\Sigma_\tau}(\mathrm{curl},\Omega)} := \|\mathbf{e}\|_\Omega + \|\mathbf{e} \times \mathbf{n}\|_{\Sigma_\tau}.$$

The spaces $\mathcal{H}^{s+1/2}_{\mathrm{div}}(\Omega)$ and $H^0_{\Sigma_\tau}(\mathrm{curl}, \Omega)$ will be used below as the solution spaces for magnetic and electric components, respectively. In turn, the spaces

$$H^1_T(\Omega) = \{\mathbf{v} \in H^1(\Omega) : \mathbf{v} \cdot \mathbf{n}|_\Sigma = 0\}, \ H^1_{\mathrm{div}}(\Omega) := \{\mathbf{v} \in H^1_T(\Omega) : \ \mathrm{div}\, \mathbf{v} = 0\},$$

will play the role of the solution spaces for velocity $\mathbf{u}$. Besides Lemma 1, we also will use the following lemma (for details, see [25,26,38]).

Lemma 2. *Under condition (i) there exist constants $C_1 = C_1(\Omega)$, $\delta_i = \delta_i(\Omega) > 0$, $\gamma_i'(\Omega)$, $\gamma_i = \gamma_i(\Omega) > 0$, $i = 0, 1$, $\beta = \beta(\Omega)$, depending on Ω such that*

$$|(\nabla \mathbf{u}, \nabla \mathbf{w})| \leq \|\mathbf{u}\|_{1,\Omega} \|\mathbf{w}\|_{1,\Omega} \ \forall \mathbf{u}, \mathbf{w} \in H^1(\Omega)^3, \ (\nabla \mathbf{v}, \nabla \mathbf{v}) \geq \delta_0 \|\mathbf{v}\|^2_{1,\Omega} \ \ \forall \mathbf{v} \in H^1_0(\Omega)^3, \tag{10}$$

$$(\mathrm{curl}\, \boldsymbol{\Psi}, \mathrm{rot}\, \boldsymbol{\Psi}) \geq \delta_1 \|\boldsymbol{\Psi}\|^2_{1,\Omega} \ \forall \boldsymbol{\Psi} \in V_{\Sigma_\tau}(\Omega), \tag{11}$$

$$\|\mathrm{curl}\mathbf{H}\|_\Omega \leq \|\mathbf{H}\|_{\mathcal{H}^{s+1/2}_{\mathrm{div}}(\Omega)} \leq C_1 \|\mathbf{H}\|_{1,\Omega} \quad \forall\, \mathbf{H} \in H^1(\Omega)^3. \tag{12}$$

$$\begin{gathered} |((\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{w})| \leq \gamma_0' \|\mathbf{u}\|_{L^3(\Omega)^3} \|\mathbf{v}\|_{1,\Omega} \|\mathbf{w}\|_{1,\Omega} \leq \\ \leq \gamma_0 \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \|\mathbf{w}\|_{1,\Omega} \ \ \forall \mathbf{u}, \ \mathbf{v}, \ \mathbf{w} \in H^1(\Omega)^3, \end{gathered} \tag{13}$$

$$\begin{gathered} |(\mathrm{curl}\mathbf{H}_1 \times \mathbf{H}_2, \mathbf{v})| \leq \|\mathrm{curl}\mathbf{H}_1\|_\Omega \|\mathbf{H}_2\|_{L^3(\Omega)} \|\mathbf{v}\|_{L^6(\Omega)} \leq \\ \leq \gamma_1 \|\mathbf{H}_1\|_{1,\Omega} \|\mathbf{H}_2\|_{s+1/2,\Omega} \|\mathbf{v}\|_{1,\Omega} \ \forall \mathbf{H}_1 \in H(\mathrm{curl}, \Omega), \ \mathbf{H}_2 \in H^{s+1/2}(\Omega)^3, \end{gathered} \tag{14}$$

$$\begin{gathered} |(\mathrm{curl}\boldsymbol{\Psi} \times \mathbf{h}, \mathbf{v})| \leq \|\mathrm{rot}\boldsymbol{\Psi}\|_\Omega \|\mathbf{h}\|_{L^3(\Omega)^3} \|\hat{\mathbf{u}}\|_{L^6(\Omega)^3} \leq \\ \leq \gamma_1 \|\boldsymbol{\Psi}\|_{1,\Omega} \|\mathbf{H}\|_{s+1/2,\Omega} \|\mathbf{v}\|_{1,\Omega} \ \forall \mathbf{H} \in \mathcal{H}^{s+1/2}_{\mathrm{div}}(\Omega), \mathbf{v}, \boldsymbol{\Psi} \in H^1(\Omega)^3. \end{gathered} \tag{15}$$

Moreover, the following equality holds:

$$((\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{v}) = 0 \ \forall \mathbf{u} \in H^1_{\mathrm{div}}(\Omega), \ \ \mathbf{v} \in H^1_0(\Omega)^3. \tag{16}$$

The bilinear form defined by $-(\mathrm{div}\cdot,\cdot)$ *satisfies*

$$\sup_{\mathbf{v}\in H_0^1(\Omega)^3,\mathbf{v}\neq 0} -(\mathrm{div}\mathbf{v},p)/\|\mathbf{v}\|_{1,\Omega}\geq \beta\|p\|_{\Omega}\ \ \forall p\in L_0^2(\Omega). \tag{17}$$

Below, when formulating a result on the existence of magnetic lifting, we will need the space

$$H_T^{-1/2}(\mathrm{div}_{\Sigma_\nu};\Sigma_\nu)=\{\mathbf{h}\in H_T^{-1/2}(\Sigma_\nu):\mathrm{div}_{\Sigma_\nu}\mathbf{h}\in H^{-1/2}(\Sigma_\nu)\},$$

endowed with a norm

$$\|\mathbf{h}\|^2_{-1/2,\mathrm{div},\Sigma_\nu}=\|\mathbf{h}\|^2_{-1/2,\Sigma_\nu}+\|\mathrm{div}_{\Sigma_\nu}\mathbf{h}\|^2_{-1/2,\Sigma_\nu},$$

and its subspace

$$\tilde{H}_T^s(\Sigma_\nu)=\{\mathbf{h}\in H_T^{-1/2}(\mathrm{div}_{\Sigma_\nu};\Sigma_\nu):\mathrm{div}_{\Sigma_\nu}\mathbf{h}=0\text{ in }H^{-1/2}(\Sigma_\nu),$$

$$(\mathbf{h},\mathbf{m})_{\Sigma_\nu}=0\ \forall\mathbf{m}\in\mathcal{H}_{\Sigma_\nu}(\Omega),\ (\mathbf{h},\mathbf{y}_l)_{\Sigma_\nu}=0,\ l=n_2+1,\dots,n\}\cap H_T^s(\Sigma_\nu), \tag{18}$$

$s\geq 0$, with the norm $\|\mathbf{h}\|_{\tilde{H}_T^s(\Sigma_\nu)}=\|\mathbf{h}\|_{s,\Sigma_\nu}$. Here $\mathrm{div}_{\Sigma_\nu}$ is the linear surface divergence operator on the part Σ_ν of the boundary Σ (see [15,37]). In the case $s=0$, instead of $\tilde{H}_T(\Sigma_\nu)$, we will write $\tilde{L}_T^2(\Sigma_\nu)$.

Let the following conditions for the data take place in addition to conditions (i), (ii):

(iii) $\mathbf{f}\in H^{-1}(\Omega)^3$, $\mathbf{g}\in H_T^{1/2}(\Sigma)$, $\mathbf{k}\in(\gamma_\tau|_{\Sigma_\tau})H^0_{\Sigma_\tau}(\mathrm{curl},\Omega)$,

(iv) $q\in H^s(\Sigma_\tau)$, $\mathbf{q}\in\tilde{H}_T^s(\Sigma_\nu)$, $\mathbf{j}\in L^2(\Omega)^3$, $s\in[0,1/2]$.

As usual, while studying control problems for the MHD system, we will deal with a weak form of Problem 1. This consists of finding a triple of functions $(\mathbf{u},\mathbf{H},p)\in H_T^1(\Omega)\times\mathcal{H}_{\mathrm{div}}^{s+1/2}(\Omega)\times L_0^2(\Omega)$ satisfying

$$\begin{gathered}\nu(\nabla\mathbf{u},\nabla\mathbf{v})+\nu_1(\mathrm{rot}\mathbf{H},\mathrm{rot}\mathbf{\Psi})+((\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{v})+\kappa[(\mathrm{rot}\mathbf{\Psi}\times\mathbf{H},\mathbf{u})-(\mathrm{rot}\mathbf{H}\times\mathbf{H},\mathbf{v})]-\\-(\mathrm{div}\mathbf{v},p)=F(\mathbf{v},\mathbf{\Psi})\equiv\langle\mathbf{f},\mathbf{v}\rangle+\nu_1(\mathbf{j},\mathrm{rot}\,\mathbf{\Psi})+\rho_0^{-1}(\mathbf{k},\mathbf{\Psi})_{\Sigma_\tau}\ \forall(\mathbf{v},\mathbf{\Psi})\in H_0^1(\Omega)^3\times V_{\Sigma_\tau}(\Omega),\end{gathered} \tag{19}$$

$$\mathrm{div}\mathbf{u}=0\text{ in }\Omega,\ \ \mathbf{u}=\mathbf{g}\text{ on }\Sigma,\ \ \mathbf{H}\cdot\mathbf{n}=q\text{ on }\Sigma_\tau,\ \ \mathbf{H}\times\mathbf{n}=\mathbf{q}\text{ on }\Sigma_\nu. \tag{20}$$

In order to obtain (19), one should multiply the first relation in (1) by $\mathbf{v}\in H_0^1(\Omega)^3$, the first relation in (2) by rot $\mathbf{\Psi}$ where $\mathbf{\Psi}\in V_{\Sigma_\tau}(\Omega)$, to integrate over Ω, to apply Green's formulas, to add the obtained results and to make use of the identity (see details in [27])

$$(\mathbf{E},\mathrm{curl}\,\mathbf{\Psi})=\int_{\Sigma_\tau}(\mathbf{E}\times\mathbf{n}|_{\Sigma_\tau})\cdot\mathbf{\Psi}_T d\sigma=(\mathbf{k},\mathbf{\Psi}_T)_{\Sigma_\tau}\ \ \forall\mathbf{\Psi}\in V_{\Sigma_\tau(\Omega)}. \tag{21}$$

The identity (19) does not contain electric field $\mathbf{E}\in H^0_{\Sigma_\tau}(\mathrm{curl},\Omega)$, which was eliminated with the help of (21). However, using a condition on a boundary vector $\mathbf{k}$ in (iii), vector $\mathbf{E}$ can be uniquely recovered from triple $(\mathbf{u},\mathbf{H},p)\in H^1_{\mathrm{div}}(\Omega)\times\mathcal{H}^{s+1/2}_{\mathrm{div}}(\Omega)\times L_0^2(\Omega)$ satisfying (19) such that the first equation in relations (2) holds a.e. in Ω (see [27]). This allows us to refer, below, to the mentioned triple $(\mathbf{u},\mathbf{H},p)$ satisfying relations (19), (20) as a weak solution to Problem 1.

3. Lifting of the Velocity and Magnetic Field, Solvability of Problem 1 and Its Linear Analogue

The proof of the existence of a solution of the inhomogeneous boundary value problem (1)–(3) essentially uses the results for the existence of lifting of the velocity and magnetic field corresponding to the boundary conditions in (3). By velocity lifting, we mean the function $\mathbf{u}_0\in H^1_{\mathrm{div}}(\Omega)$ satisfying the boundary condition $\mathbf{u}_0|_\Sigma=\mathbf{g}$. By magnetic field lifting, we mean the function $\mathbf{H}_0\in\mathcal{H}^{s+1/2}_{\mathrm{div}}(\Omega)$ satisfying the mixed boundary conditions $\mathbf{H}_0 \cdot \mathbf{n}|_{\Sigma_\tau} = q$ and $\mathbf{H}_0 \times \mathbf{n}|_{\Sigma_\nu} = \mathbf{q}$. The existence of velocity lifting is ensured by the following lemma, proved in [3].

Lemma 3. *Under condition (i) for each function $\mathbf{g} \in H_T^{1/2}(\Sigma)$ and arbitrary number $\varepsilon > 0$, there is a function $\mathbf{u}_\varepsilon \in H_T^1(\Omega)$ such that*

$$\operatorname{div} \mathbf{u}_\varepsilon = 0 \text{ in } \Omega,\ \mathbf{u}_\varepsilon = \mathbf{g} \text{ on } \Gamma,\ \|\mathbf{u}_\varepsilon\|_{1,\Omega} \le C_\varepsilon \|\mathbf{g}\|_{1/2,\Sigma},\ \|\mathbf{u}_\varepsilon\|_{L^4(\Omega)^3} \le \varepsilon \|\mathbf{g}\|_{1/2,\Sigma}. \quad (22)$$

Here C_ε is a constant depending on ε and Ω.

The existence of the lifting of the magnetic part of the solution is ensured by the following result, which is a generalization of Theorem 4.2 in [37], where it was proved in the case $s = 0$.

Lemma 4. *Let conditions (i), (ii) be satisfied. Then for any pair $q \in H^s(\Sigma_\tau)$ and $\mathbf{q} \in \tilde{H}_T^s(\Sigma_\nu)$, where $s \in [0, 1/2]$, there exists a unique function $\mathbf{H}_0 \in \mathcal{H}_{\text{div}}^{s+1/2}(\Omega)$ such that*

$$\operatorname{rot} \mathbf{H}_0 = \mathbf{0},\ \operatorname{div} \mathbf{H}_0 = 0 \text{ in } \Omega,\ \ \mathbf{H}_0 \cdot \mathbf{n} = q \text{ on } \Sigma_\tau,\ \mathbf{H}_0 \times \mathbf{n} = \mathbf{q} \text{ on } \Sigma_\nu,$$

$$\|\mathbf{H}_0\|_{\mathcal{H}_{\text{div}}^{s+1/2}(\Omega)} = \|\mathbf{H}_0\|_{s+1/2,\Omega} \le C_\Sigma (\|q\|_{s,\Sigma_\tau} + \|\mathbf{q}\|_{s,\Sigma_\nu}). \quad (23)$$

Here C_Σ is a constant independent of q and $\mathbf{q}$.

In what follows, the vector $\mathbf{H}_0$ alone, defined in Lemma 4, will play the role of magnetic lifting for Problem 1.

Let us define two linear subspaces of $\mathcal{H}_{\text{div}}^{s+1/2}(\Omega)$:

$$\mathcal{H}_{\text{div}}^{s+1/2}(\Omega, \Sigma_\nu) = \{\mathbf{h} \in \mathcal{H}_{\text{div}}^{s+1/2}(\Omega) : \mathbf{h} \times \mathbf{n} = 0 \text{ on } \Sigma_\nu\},$$

$$\mathcal{H}_{\text{div}}^{s+1/2}(\Omega, \Sigma_\tau) = \{\mathbf{h} \in \mathcal{H}_{\text{div}}^{s+1/2}(\Omega) : \mathbf{h} \cdot \mathbf{n} = 0 \text{ on } \Sigma_\tau\}. \quad (24)$$

From Lemma 4, applied for the case $\mathbf{q} = 0$ on Σ_ν, it follows that for any function $q \in H^s(\Sigma_\tau)$, there exists a unique solution $\mathbf{h} \in \mathcal{H}_{\text{div}}^{s+1/2}(\Omega, \Sigma_\nu)$ of problem (23) for $\mathbf{q} = \mathbf{0}$, while for any function $\mathbf{q} \in \tilde{H}_T^s(\Sigma_\nu)$ there exists a unique solution $\mathbf{h} \in \mathcal{H}_{\text{div}}^{s+1/2}(\Omega, \Sigma_\tau)$ of problem (23) for the case $q = 0$. Moreover, when $\mathbf{h}$ runs through $\mathcal{H}_{\text{div}}^{s+1/2}(\Omega, \Sigma_\nu)$, its normal component $\mathbf{h} \cdot \mathbf{n}|_{\Sigma_\tau}$ runs through the space $H^s(\Sigma_\tau)$. Similarly, when $\mathbf{h}$ runs through the space $\mathcal{H}_{\text{div}}^{s+1/2}(\Omega, \Sigma_\tau)$, its tangential component $\mathbf{h} \times \mathbf{n}|_{\Sigma_\nu}$ runs through the space $\tilde{H}_T^s(\Sigma_\nu)$.

Define the next products of spaces:

$$X = \mathbf{H}_T^1(\Omega) \times \mathcal{H}_{\text{div}}^{s+1/2}(\Omega) \times L_0^2(\Omega),\ \ H = H_T^1(\Omega) \times \mathcal{H}_{\text{div}}^{s+1/2}(\Omega),$$

$$W = H_0^1(\Omega)^3 \times V_{\Sigma_\tau}(\Omega),\ Y = W^* \times L_0^2(\Omega) \times H_T^{1/2}(\Sigma) \times H^s(\Sigma_\tau) \times \tilde{H}_T^s(\Sigma_\nu). \quad (25)$$

Along with Problem 1, when studying the control problems below, an important role will be played by a linear analogue of Problem 1. This consists of finding a triple $(\mathbf{u}, \mathbf{H}, p) \in X \equiv H_T^1(\Omega) \times \mathcal{H}_{\text{div}}^{s+1/2}(\Omega) \times L_0^2(\Omega)$ satisfying the functions

$$a_{\hat{\mathbf{u}}\hat{\mathbf{H}}}((\mathbf{u}, \mathbf{H}), (\mathbf{v}, \mathbf{\Psi})) \equiv \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + \nu_1(\operatorname{rot}\mathbf{H}, \operatorname{rot}\mathbf{\Psi}) + ((\hat{\mathbf{u}} \cdot \nabla)\mathbf{u}, \mathbf{v}) + \kappa(\operatorname{rot}\mathbf{\Psi} \times \hat{\mathbf{H}}, \mathbf{w}) -$$

$$- (\operatorname{rot}\mathbf{H} \times \hat{\mathbf{H}}, \mathbf{v}) = \langle \mathbf{F}, (\mathbf{v}, \mathbf{\Psi}) \rangle + (\operatorname{div}\mathbf{v}, p)\ \forall (\mathbf{v}, \mathbf{\Psi}) \in W, \quad (26)$$

$$\operatorname{div}\mathbf{u} = \chi \text{ in } \Omega,\ \mathbf{u}|_\Sigma = \mathbf{g},\ \ \mathbf{H} \cdot \mathbf{n}|_{\Sigma_\tau} = q \text{ and } \mathbf{H} \times \mathbf{n}|_{\Sigma_\nu} = \mathbf{q}. \quad (27)$$

Here $\mathbf{F} \in W^*$ is an arbitrary functional, "velocity" $\hat{\mathbf{u}} \in H_{\text{div}}^1(\Omega)$, "magnetic field" $\hat{\mathbf{H}} \in \mathcal{H}_{\text{div}}^{s+1/2}(\Omega)$ and $\chi \in L_0^2(\Omega)$ are given functions. In fact, we have somewhat generalized the linear analogue of Problem 1 by replacing the solenoidality condition $\operatorname{div}\mathbf{u} = 0$ by the more general condition $\operatorname{div}\mathbf{u} = \chi$. Another generalization is that by the functional $\mathbf{F}$ in (26), we mean an arbitrary functional from W^*, which does not necessarily coincide with the functional $\mathbf{F}$ defined in (19).

The following lemma about the unique solvability of problems (26), (27) holds.

Lemma 5. *Let conditions (i), (ii) be satisfied. Then for any quintuple $(\mathbf{F}, \chi, \mathbf{g}, q, \mathbf{q}) \in Y$ for arbitrary $s \in [0, 1/2]$, there exists a unique solution $(\mathbf{u}, \mathbf{H}, p) \in X$ of problems (26), (27), and for the solution the following estimates hold:*

$$\|\mathbf{u}\|_{1,\Omega} \leq \hat{M}^0_{\mathbf{u}}, \ \|\mathbf{H}\|_{\mathcal{H}^{s+1/2}_{\mathrm{div}}(\Omega)} \leq \hat{M}^0_{\mathbf{H}}, \ \|p\|_\Omega \leq \hat{M}^0_p. \tag{28}$$

Here $\hat{M}^0_{\mathbf{u}}$, $\hat{M}^0_{\mathbf{H}}$, $\hat{M}^0_p$ are nondecreasing continuous functions of $\|\chi\|_\Omega$, $\|\mathbf{F}\|_{W^*}$, $\|\mathbf{g}\|_{1/2,\Sigma}$, $\|q\|_{s,\Sigma_\tau}$, $\|\mathbf{q}\|_{s,\Sigma_\nu}$.

Proof of Lemma 5. The existence of the solution of (26), (27) and estimate (28) are proved using the scheme proposed in [3] (see also [15], Chapter 6). For proving the uniqueness, let us assume that there exist two solutions $(\mathbf{u}_i, \mathbf{H}_i, p_i) \in X$, $i = 1, 2$, of problem (26), (27). Then the quantities $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, $\mathbf{H} = \mathbf{H}_1 - \mathbf{H}_2$ and $p = p_1 - p_2$ belong to $V \times V_{\Sigma_\tau}(\Omega) \times L^2_0(\Omega)$ and satisfy

$$\begin{gathered}\nu(\nabla\mathbf{u}, \nabla\mathbf{v}) + \nu_1(\mathrm{curl}\,\mathbf{H}, \mathrm{curl}\boldsymbol{\Psi}) + ((\hat{\mathbf{u}}\cdot\nabla)\mathbf{u}, \mathbf{v}) + \\ + \kappa[(\mathrm{curl}\boldsymbol{\Psi} \times \hat{\mathbf{H}}, \mathbf{w}) - (\mathrm{curl}\,\mathbf{H} \times \hat{\mathbf{H}}, \mathbf{v})] - (\mathrm{div}\,\mathbf{v}, p) = 0 \ \ \forall(\mathbf{v}, \boldsymbol{\Psi}) \in W,\end{gathered} \tag{29}$$

$$\mathrm{div}\,\mathbf{u} = 0 \text{ in } \Omega, \ \mathbf{u}|_\Sigma = 0, \ \mathbf{H}\cdot\mathbf{n}|_{\Sigma_\tau} = 0, \ \mathbf{H}\times\mathbf{n}|_{\Sigma_\nu} = 0. \tag{30}$$

Setting $\mathbf{v} = \mathbf{u}$, $\boldsymbol{\Psi} = \mathbf{H}$ in (29) and using (16), we arrive at the relation

$$\nu(\nabla\mathbf{u}, \nabla\mathbf{u}) + \nu_1(\mathrm{curl}\,\mathbf{H}, \mathrm{curl}\,\mathbf{H}) = 0. \tag{31}$$

where (31) takes place by (10), (11) if $\mathbf{u} = \mathbf{0}$ and $\mathbf{H} = \mathbf{0}$ or $\mathbf{u}_1 = \mathbf{u}_2$ and $\mathbf{H}_1 = \mathbf{H}_2$ in Ω. Then from (29), it follows that $(\mathrm{div}\mathbf{v}, p) = 0$ for all $\mathbf{v} \in H^1_0(\Omega)^3$. This means by inf-sup condition (17), that $p = 0$ or $p_1 = p_2$ in Ω. □

Let us rewrite the problem (26), (27) in an equivalent operator form. To this end, we put in correspondence with the bilinear continuous forms

$$a_{\hat{\mathbf{u}},\hat{\mathbf{H}}} : H \times W \equiv (H^1_T(\Omega) \times \mathcal{H}^{s+1/2}_{\mathrm{div}}(\Omega)) \times (H^1_0(\Omega)^3 \times V_{\Sigma_\tau}(\Omega)) \to \mathbb{R}, \ b : W \times L^2_0(\Omega) \to \mathbb{R},$$

linear continuous operators $A : H \to W^*$, $B : W \to L^2_0(\Omega)^* \equiv L^2_0(\Omega)$, $B^* : L^2_0(\Omega) \to W^*$, acting by

$$\begin{gathered}\langle A(\mathbf{u}, \mathbf{H}), (\mathbf{v}, \boldsymbol{\Psi})\rangle = a_{\hat{\mathbf{u}},\hat{\mathbf{H}}}((\mathbf{u}, \mathbf{H}), (\mathbf{v}, \boldsymbol{\Psi})) \ \forall(\mathbf{u}, \mathbf{H}) \in H \text{ and } (\mathbf{v}, \boldsymbol{\Psi}) \in W, \\ \langle B(\mathbf{v}, \boldsymbol{\Psi}), r\rangle = \tilde{b}((\mathbf{v}, \boldsymbol{\Psi}), r) \equiv b(\mathbf{v}, r) = \langle B^* r, (\mathbf{v}, \boldsymbol{\Psi})\rangle \ \ \forall r \in L^2_0(\Omega), \ (\mathbf{v}, \boldsymbol{\Psi}) \in W.\end{gathered} \tag{32}$$

Using (32), one can rewrite the identity (26) in the equivalent operator form

$$A(\mathbf{u}, \mathbf{H}) + B^* p = \mathbf{F}.$$

Setting $\mathbf{x} = (\mathbf{u}, \mathbf{H}, p)$, we define the operator $\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5) : X \to Y$ by

$$\begin{gathered}\Phi_1(\mathbf{x}) = A(\mathbf{u}, \mathbf{H}) + B^* p, \ \Phi_2(\mathbf{x}) = \mathrm{div}\mathbf{u}, \ \Phi_3(\mathbf{x}) = \mathbf{u}|_\Sigma, \\ \Phi_4(\mathbf{x}) = \mathbf{H}\cdot\mathbf{n}|_{\Sigma_\tau}, \ \Phi_5(\mathbf{x}) = \mathbf{H}\times\mathbf{n}|_{\Sigma_\nu}.\end{gathered} \tag{33}$$

By construction, Φ belongs to the space of continuous linear operators $\mathcal{L}(X, Y)$, and the original linear problems (26), (27) is equivalent to the operator equation

$$\Phi(\mathbf{x}) = \mathbf{y} \equiv (\mathbf{F}, \chi, \mathbf{g}, q, \mathbf{q}) \in Y. \tag{34}$$

From Lemma 5, it follows that (34) has a unique solution for any element $\mathbf{y} \in Y$. This means that the operator $\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5)$ is surjective and invertible. Then it follows from Banach's inverse operator theorem that the operator $\Phi : X \to Y$ is an isomorphism. Therefore, the following theorem holds.

Theorem 1. *Let assumptions (i), (ii) be satisfied. Then the operator $\Phi : X \to Y$, which is defined by (32), (33), is the isomorphism of the space X into Y.*

We now formulate the following result concerning the sufficient conditions of the existence of the solution to Problem 1, which was proved in [27].

Theorem 2. *Under assumptions (i)–(iv) there exists a weak solution $(\mathbf{u}, \mathbf{H}, p)$ to Problem 1, and for this solution the following estimates hold:*

$$\|\mathbf{u}\|_{1,\Omega} \le M_{\mathbf{u}}, \ \|\mathbf{H}\|_{\mathcal{H}^{s+1/2}_{\rm div}(\Omega)} \le M_{\mathbf{H}}, \ \|p\|_{\Omega} \le M_p. \tag{35}$$

Here $M_{\mathbf{u}}$, M_p and $M_{\mathbf{H}}$ are nondecreasing continuous functions of norms $\|\mathbf{f}\|_{-1,\Omega}$, $\|\mathbf{j}\|_{\Omega}$, $\|\mathbf{k}\|_{\Sigma_\tau}$, $\|\mathbf{g}\|_{1/2,\Sigma}$, $\|q\|_{s,\Sigma_\tau}$, $\|\mathbf{q}\|_{s,\Sigma_\nu}$. If, besides, elements $\mathbf{f}, \mathbf{j}, \mathbf{k}, \mathbf{g}, q$ and $\mathbf{q}$ are small (or "viscosity coefficients" ν, ν_m are, vice versa, great) in the sense

$$\gamma_0 M_{\mathbf{u}} + \gamma_1(\sqrt{\kappa}/2)M_{\mathbf{H}} < \delta_0 \nu, \ \ \gamma_1 M_{\mathbf{u}} + \gamma_1(\sqrt{\kappa}/2)M_{\mathbf{H}} < \delta_1 \nu_m, \tag{36}$$

where constants δ_0, δ_1, γ_0, γ_1 were defined in Lemmas 1 and 2, then the weak solution $(\mathbf{u}, \mathbf{H}, p)$ is unique.

Remark 1. *Theorem 2 was proved in [27] under an additional condition on function $\mathbf{g}$, namely that $\mathbf{g}$ is a tangential vector on Σ. If this condition does not hold, i.e., $\mathbf{g} \cdot \mathbf{n}|_\Sigma \neq 0$, one can prove only the local solvability of Problem 1. Thus, the problem of global solvability of inhomogeneous problem (1)–(3) in the general case when $\mathbf{g} \cdot \mathbf{n}|_\Sigma \neq 0$ is still an open problem (see related discussion in [15,22]).*

4. Statement of Control Problems, Optimality System and Additional Properties of Optimal Solutions

We note that problems (1)–(3) contain constant parameters $\nu, \nu_1, \rho_0, \kappa$ and functional parameters—boundary functions $\mathbf{g}, \mathbf{k}, q, \mathbf{q}$ and "volume" source densities $\mathbf{f}$ and $\mathbf{j}$. To solve problems (1)–(3), one must specify values of respective parameters, boundary functions and sources. In practice, however, some of their specific elements may be unknown, and one should determine them and the solution $(\mathbf{u}, \mathbf{H}, p)$ using certain information about the solution.

In this section, control problems for the MHD system (1)–(3) will be considered, for which we prove their global solvability. These problems are to minimize the so-called cost functionals, which depend on the variables $(\mathbf{u}, \mathbf{H}, p)$ of the main state and other unknown functions (controls) satisfying the state Equations (1)–(3). We will choose one specific cost functional:

$$I_1(\mathbf{u}) = \|\mathbf{u} - \mathbf{u}_d\|^2_Q, \ I_2(\mathbf{H}) = \|\mathbf{H} - \mathbf{H}_d\|^2_Q, \ I_3(p) = \|p - p_d\|^2_Q. \tag{37}$$

Here the function $\mathbf{u}_d \in L^2(Q)^3$ describes, in a subdomain $Q \subset \Omega$ velocity field, given functions $\mathbf{H}_d \in L^2(Q)^3$ and $p_d \in L^2(Q)$ have a similar sense for magnetic field or pressure. We note that functionals I_1, I_2, I_3 are used to solve inverse problems for the MHD system (1)–(3) using the optimization method. This method was developed by A.N. Tikhonov in the process of creating the famous Tikhonov regularization method [39]. Currently, the optimization method is one of the fundamental methods for solving inverse problems arising in electromagnetism, acoustics, fluid mechanics, heat and mass transfer, design of complicated technical devices and in other fields of physics, natural science and engineering (for more detail, see [40–51]).

As controls in this paper, we choose three functions: q, $\mathbf{q}$ and $\mathbf{j}$. The function $\mathbf{j}$ will play the role of distributed control, while q and $\mathbf{q}$ will play the role of boundary controls in the control problems stated below. We assume that the functions q, $\mathbf{q}$ and $\mathbf{j}$ change over sets K_1, K_2 and K_3 satisfying the following conditions:

(j) $K_1 \subset H^s(\Sigma_\tau)$, $K_2 \subset \tilde{H}^s_T(\Sigma_\nu)$, $K_3 \subset L^2(\Omega)^3$ are nonempty convex closed sets.

Setting $u = (q, \mathbf{q}, \mathbf{j}) \in K = K_1 \times K_2 \times K_3$, $\mathbf{x} = (\mathbf{u}, \mathbf{H}, p) \in X \equiv H^1_T(\Omega) \times \mathcal{H}^{s+1/2}_{\rm div}(\Omega) \times L^2_0(\Omega)$, we define an operator $F \equiv (F_1, F_2, F_3, F_4, F_5) : X \times K \to Y \equiv (H^{-1}(\Omega)^3 \times V_{\Sigma_\tau}(\Omega)^*) \times L^2_0(\Omega) \times H^{1/2}_T(\Sigma) \times H^s(\Sigma_\tau) \times \tilde{H}^s_T(\Sigma_\nu)$ by

$$\langle F_1(\mathbf{x}, u), (\mathbf{v}, \mathbf{\Psi})\rangle = \nu(\nabla\mathbf{u}, \nabla\mathbf{v}) + \nu_1(\text{rot}\mathbf{H}, \text{rot}\mathbf{\Psi}) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{v}) - (\text{div}\mathbf{v}, p) +$$

$$+ \kappa[(\text{rot}\mathbf{\Psi} \times \mathbf{H}, \mathbf{u}) - (\text{rot}\mathbf{H} \times \mathbf{H}, \mathbf{v})] - \langle \mathbf{f}, \mathbf{v}\rangle - \nu_1(\mathbf{j}, \text{rot}\,\mathbf{\Psi}) -$$

$$- \rho_0^{-1}(\mathbf{k}, \mathbf{\Psi}_T)_{\Sigma_\tau}\ \forall(\mathbf{v}, \mathbf{\Psi}) \in H^1_0(\Omega)^3 \times V_{\Sigma_\tau}(\Omega),$$

$$\langle F_2(\mathbf{x}), r\rangle = -(\text{div}\mathbf{u}, r)\ \forall r \in L^2_0(\Omega),\ \ F_3(\mathbf{x}) = \mathbf{u}|_\Sigma - \mathbf{g} \in H^{1/2}_T(\Sigma),$$

$$F_4(\mathbf{x}, u) = \mathbf{H}\cdot\mathbf{n}|_{\Sigma_\tau} - q \in H^s(\Sigma_\tau),\ \ F_5(\mathbf{x}, u) = \mathbf{H}\times\mathbf{n}|_{\Sigma_\nu} - \mathbf{q} \in \tilde{H}^s_T(\Sigma_\nu) \tag{38}$$

and rewrite the weak form (19), (20) of Problem 1 as operator equation

$$F(\mathbf{x}, u) \equiv F(\mathbf{u}, \mathbf{H}, p, q, \mathbf{q}, \mathbf{j}) = 0. \tag{39}$$

Let us introduce a functional $I : X \to \mathbb{R}$ and nonnegative parameters μ_0, μ_1, μ_2, μ_3, and let us presuppose that the following conditions take place, in addition to (j):

(jj) $I : X \to \mathbb{R}$ is a weakly lower semicontinuous cost functional;

(jjj) $\mu_0 > 0$, $\mu_1 \ge 0$, $\mu_2 \ge 0$, $\mu_3 \ge 0$ and K_1, K_2, K_3 are limited sets, or $\mu_1 > 0$, $\mu_2 > 0$, $\mu_3 > 0$ and the functional $I : X \to \mathbb{R}$ is bounded below.

The following problem will be considered below:

$$J(\mathbf{x}, u) = \frac{\mu_0}{2} I(\mathbf{x}) + \frac{\mu_1}{2}\|q\|^2_{s,\Sigma_\tau} + \frac{\mu_2}{2}\|\mathbf{q}\|^2_{s,\Sigma_\nu} + \frac{\mu_3}{2}\|\mathbf{j}\|^2_\Omega \to \inf,\ F(\mathbf{x}, u) = 0,\ (\mathbf{x}, u) \in X \times K. \tag{40}$$

The parameters μ_0, μ_1, μ_2, μ_3 serve in regulating the relative contribution of each of the terms in (40) and, moreover, to adjust their dimensions. Another purpose of using μ_l is to ensure the uniqueness and stability for the control problems under study (see below). The last three terms in the structure of J are penalization terms. In what follows, we will refer to these terms as regularizers (strong in the case $s > 0$ and weak for $s = 0$). As will be shown in Section 5, just the presence of these regularizers in the structure of the functionals under minimization will allow us to prove the theorems concerning both the uniqueness and stability for optimal solutions.

Let $Z_{ad} = \{(\mathbf{x}, u) \in X \times K, F(\mathbf{x}, u) = 0, J(\mathbf{x}, u) < \infty\}$ be a set of possible pairs for problem (40).

Theorem 3. *Let the assumptions (i)–(iii), (j)–(jjj) hold, and let the set Z_{ad} be nonempty. Then for arbitrary $s \in [0, 1/2]$, the problem (40) has at least one solution $(\mathbf{x}, u) \in X \times K$.*

Proof of Theorem 3. Let us denote by $(\mathbf{x}_m, u_m) \equiv (\mathbf{u}_m, \mathbf{H}_m, p_m, q_m, \mathbf{q}_m, \mathbf{j}_m) \in Z_{ad}$, $m \in \mathbb{N} = \{1, 2, \ldots\}$ a minimizing sequence, for which

$$\lim_{m\to\infty} J(\mathbf{x}_m, q_m, \mathbf{q}_m, \mathbf{j}_m) = \inf_{(\mathbf{x},q,\mathbf{q},\mathbf{j})\in Z_{ad}} J(\mathbf{x}, q, \mathbf{q}, \mathbf{j}) \equiv J^*.$$

By the conditions of Theorem 3 and in virtue of Theorem 2, we have the following estimates for q_m, $\mathbf{q}_m$, $\mathbf{j}_m$, $\mathbf{u}_m$, $\mathbf{H}_m$, p_m:

$$\|q_m\|_{s,\Sigma_\tau} \le c_1, \; \|\mathbf{q}_m\|_{s,\Sigma_\nu} \le c_2, \; \|\mathbf{j}_m\|_\Omega \le c_3, \; \|\mathbf{u}_m\|_{1,\Omega} \le c_4,$$

$$\|\mathbf{H}_m\|_{\mathcal{H}^{s+1/2}_{\mathrm{div}}(\Omega)} \le c_5, \; \|p_m\|_\Omega \le c_6.$$

Here $c_1, c_2, \dots$ are some constants that do not depend on m. From the given estimates, it follows that there are weak limits $q^* \in H^s(\Sigma_\tau)$, $\mathbf{q}^* \in H^s_T(\Sigma_\nu)$, $\mathbf{j}^* \in L^2(\Omega)^3$, $\mathbf{u}^* \in H^1_T(\Omega)$, $\mathbf{H}^* \in \mathcal{H}^{s+1/2}_{\mathrm{div}}(\Omega)$, $p^* \in L^2_0(\Omega)$ of some subsequences of sequences $\{q_m\}$, $\{\mathbf{q}_m\}$, $\{\mathbf{j}_m\}$, $\{\mathbf{u}_m\}$, $\{\mathbf{H}_m\}$, $\{p_m\}$. As usual, one should consider that as $m \to \infty$

$$q_m \to q^* \text{ weakly in } H^s(\Sigma_\tau), \; \mathbf{q}_m \to \mathbf{q}^* \text{ weakly in } H^s_T(\Sigma_\nu), \; \mathbf{j}_m \to \mathbf{j}^* \text{ weakly in } L^2(\Omega)^3,$$

$$p_m \to p^* \text{ weakly in } L^2(\Omega), \; \mathbf{u}_m \to \mathbf{u}^* \text{ weakly in } H^1(\Omega)^3 \text{ and strongly in } L^4(\Omega)^3,$$

$$\mathbf{H}_m \to \mathbf{H}^* \text{ weakly in } \mathcal{H}^{s+1/2}_{\mathrm{div}}(\Omega) \cap L^3(\Omega)^3 \text{ and strongly in } L^2(\Omega)^3. \tag{41}$$

It is clear that $u^* \equiv (q^*, \mathbf{q}^*, \mathbf{j}^*) \in K$, $F_2(\mathbf{x}^*) = 0$, $F_3(\mathbf{x}^*) = \mathbf{0}$, $F_4(\mathbf{x}^*, u^*) = 0$ and $F_5(\mathbf{x}^*, u^*) = \mathbf{0}$ where $\mathbf{x}^* \equiv (\mathbf{u}^*, \mathbf{H}^*, p^*)$. Let us show that $F_1(\mathbf{x}^*, u^*) = 0$, i.e., that

$$\begin{aligned} &\nu(\nabla\mathbf{u}^*, \nabla\mathbf{v}) + \nu_1(\mathrm{rot}\,\mathbf{H}^*, \mathrm{rot}\,\mathbf{\Psi}) + ((\mathbf{u}^* \cdot \nabla)\mathbf{u}^*, \mathbf{v}) - (\mathrm{div}\,\mathbf{v}, p^*) + \\ &+ \kappa[(\mathrm{rot}\,\mathbf{\Psi} \times \mathbf{H}^*, \mathbf{u}^*) - (\mathrm{rot}\,\mathbf{H}^* \times \mathbf{H}^*, \mathbf{v})] \equiv \\ &= \langle \mathbf{f}, \mathbf{v}\rangle + \nu(\mathbf{j}^*, \mathrm{rot}\,\mathbf{\Psi}) + \rho_0^{-1}(\mathbf{k}, \mathbf{\Psi}_T)_{\Sigma_\tau} \;\; \forall(\mathbf{v}, \mathbf{\Psi}) \in H^1_0(\Omega)^3 \times V_{\Sigma_\tau}(\Omega). \end{aligned} \tag{42}$$

To this end, we note that $\mathbf{u}_m, \mathbf{H}_m$, p_m and $\mathbf{j}_m$ satisfy the identity

$$\begin{aligned} &\nu(\nabla\mathbf{u}_m, \nabla\mathbf{v}) + \nu_1(\mathrm{rot}\,\mathbf{H}_m, \mathrm{rot}\,\mathbf{\Psi}) + ((\mathbf{u}_m \cdot \nabla)\mathbf{u}_m, \mathbf{v}) - (\mathrm{div}\,\mathbf{v}, p_m) + \\ &+ \kappa[(\mathrm{rot}\,\mathbf{\Psi} \times \mathbf{H}_m, \mathbf{u}_m) - (\mathrm{rot}\,\mathbf{H}_m \times \mathbf{H}_m, \mathbf{v})] = \\ &= \langle \mathbf{f}, \mathbf{v}\rangle + \nu_1(\mathbf{j}_m, \mathrm{rot}\,\mathbf{\Psi}) + \rho_0^{-1}(\mathbf{k}, \mathbf{\Psi}_T)_{\Sigma_\tau} \;\; \forall(\mathbf{v}, \mathbf{\Psi}) \in H^1_0(\Omega)^3 \times V_{\Sigma_\tau}(\Omega). \end{aligned} \tag{43}$$

Let us pass to a limit in (43) as $m \to \infty$. From (41), it follows that all linear terms in (43) pass to corresponding linear terms in (42) as $m \to \infty$. Let us treat nonlinear terms beginning with $((\mathbf{u}_m \cdot \nabla)\mathbf{u}_m, \mathbf{v})$. Since $\mathbf{u}_m \in H^1_T(\Omega)$, then arguing as in [3], we easily derive that

$$((\mathbf{u}_m \cdot \nabla)\mathbf{u}_m, \mathbf{v}) \to ((\mathbf{u}^* \cdot \nabla)\mathbf{u}^*, \mathbf{v}) \; \forall \mathbf{v} \in H^1_0(\Omega)^3 \; \text{ as } m \to \infty. \tag{44}$$

We now consider the second nonlinear term $(\mathrm{rot}\,\mathbf{H}_m \times \mathbf{H}_m, \mathbf{v})$ in (43). It is clear that

$$\begin{aligned} &(\mathrm{rot}\,\mathbf{H}_m \times \mathbf{H}_m, \mathbf{v}) - (\mathrm{rot}\,\mathbf{H}^* \times \mathbf{H}^*, \mathbf{v}) = \\ &= (\mathrm{rot}\,\mathbf{H}_m \times (\mathbf{H}_m - \mathbf{H}^*), \mathbf{v}) + (\mathrm{rot}\,(\mathbf{H}_m - \mathbf{H}^*) \times \mathbf{H}^*, \mathbf{v}). \end{aligned} \tag{45}$$

Let us prove firstly that for the first term in (45), we have

$$(\mathrm{rot}\,\mathbf{H}_m \times (\mathbf{H}_m - \mathbf{H}^*), \mathbf{v}) \to 0 \; \forall \mathbf{v} \in H^1_0(\Omega)^3 \text{ as } m \to \infty. \tag{46}$$

For this purpose, it is sufficient to prove that for an arbitrary pair, a number $\varepsilon > 0$ and a test function $\mathbf{v} \in H^1_0(\Omega)^3$, there exists a number $M = M(\varepsilon, \mathbf{v})$ such that

$$|(\mathrm{rot}\,\mathbf{H}_m \times (\mathbf{H}_m - \mathbf{H}^*), \mathbf{v})| \le \varepsilon \;\; \forall \mathbf{v} \in H^1_0(\Omega)^3, \; \forall m \ge M. \tag{47}$$

As $H_0^1(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ by norm $\|\cdot\|_{1,\Omega}$, then for mentioned function $\mathbf{v} \in H_0^1(\Omega)^3$ there exists a sequence $\mathbf{v}_n \in \mathcal{D}(\Omega)^3$, converging to $\mathbf{v}$ by norm $\|\cdot\|_{1,\Omega}$ as $n \to \infty$. It is clear that for all $m, n \in \mathbb{N}$, we have

$$|(\text{rot}\,\mathbf{H}_m \times (\mathbf{H}_m - \mathbf{H}^*), \mathbf{v})| \leq |(\text{rot}\,\mathbf{H}_m \times (\mathbf{H}_m - \mathbf{H}^*), \mathbf{v}_n)| +$$
$$+ |(\text{rot}\,\mathbf{H}_m \times (\mathbf{H}_m - \mathbf{H}^*), \mathbf{v}_n - \mathbf{v})| \; \forall m, n \in \mathbb{N}. \tag{48}$$

From (41) follows that the norms $\|\text{rot}\,\mathbf{H}_m\|_\Omega$ and $\|\mathbf{H}_m - \mathbf{H}^*\|_{L^3(\Omega)^3}$ are uniformly bounded. Therefore for all $m \in \mathbb{N}$ there exists a number $N = N(\varepsilon, \mathbf{v})$ such that for the second term in the right-hand side of (48) we have

$$|(\text{rot}\,\mathbf{H}_m \times (\mathbf{H}_m - \mathbf{H}^*), \mathbf{v}_n - \mathbf{v})| \leq \varepsilon/2 \;\; \forall n \geq N, \; \forall m \in \mathbb{N}. \tag{49}$$

Let us consider the first term on the right-hand side of (48). Based on Hölder inequality for three functions, we have

$$|(\text{rot}\,\mathbf{H}_m \times (\mathbf{H}_m - \mathbf{H}^*), \mathbf{v}_n)| \leq \|\text{rot}\,\mathbf{H}_m\|_\Omega \, \|\mathbf{H}_m - \mathbf{H}^*\|_\Omega \, \|\mathbf{v}_n\|_{L^\infty(\Omega)^3}.$$

Since $\|\text{rot}\,\mathbf{H}_m\|_\Omega \leq \|\mathbf{H}_m\|_{\mathcal{H}^{s+1/2}_{\text{div}}(\Omega)} \leq c_5$ and $\mathbf{H}_m \to \mathbf{H}^*$ in $L^2(\Omega)^3$ as $m \to \infty$ by (41), then there exists a number $M = M(\varepsilon, \mathbf{v}, N)$ such that the estimate $|(\text{rot}\,\mathbf{H}_m \times (\mathbf{H}_m - \mathbf{H}^*), \mathbf{v}_n)| \leq \varepsilon/2$ holds for all $m \geq M$, $n = N$. From the obtained inequality and from (48), (49), the estimate (47) follows. Since ε is an arbitrary positive number, the relation (46) is proved.

Now we turn to the second term on the right-hand side of (45) and prove that

$$|((\text{rot}(\mathbf{H}_m - \mathbf{H}^*) \times \mathbf{H}^*), \mathbf{v})| \to 0 \; \forall \mathbf{v} \in H_0^1(\Omega)^3 \;\; \text{as } m \to \infty. \tag{50}$$

From the weak convergence of the sequence $\mathbf{H}_m$ in the space $\mathcal{H}^{s+1/2}_{\text{div}}(\Omega)$, the weak convergence of $\text{rot}\,\mathbf{H}_m$ to $\text{rot}\,\mathbf{H}^*$ in $L^2(\Omega)^3$ follows. As $\mathbf{H}^* \times \mathbf{v} \in L^2(\Omega)^3$, we have

$$(\text{rot}\,(\mathbf{H}_m - \mathbf{H}^*) \times \mathbf{H}^*, \mathbf{v}) = (\text{rot}\,\mathbf{H}_m - \text{rot}\mathbf{H}^*), \mathbf{H}^* \times \mathbf{v}) \to 0 \; \forall \mathbf{v} \in H_0^1(\Omega)^3 \text{ as } m \to 0.$$

Therefore, (50) is proved. From (46) and (50), it follows from (45) that

$$(\text{rot}\,\mathbf{H}_m \times \mathbf{H}_m, \mathbf{v}) \to (\text{rot}\,\mathbf{H}^* \times \mathbf{H}^*, \mathbf{v}) \; \forall \mathbf{v} \in H_0^1(\Omega)^3 \text{ as } m \to \infty. \tag{51}$$

Using the analogous scheme, one can show that for the last nonlinear term in (43), we have

$$(\text{rot}\,\boldsymbol{\Psi} \times \mathbf{H}_m, \mathbf{u}_m) \to (\text{rot}\,\boldsymbol{\Psi} \times \mathbf{H}^*, \mathbf{u}^*) \; \forall \boldsymbol{\Psi} \in V_{\Sigma_\tau}(\Omega) \text{ as } m \to \infty. \tag{52}$$

As a result, passing to the limit in (43) as $m \to \infty$, we arrive by (44), (51) and (52) at (42). Finally, since J below is weakly semicontinuous on $X \times K$ functional, we have that $J(\mathbf{x}^*, u^*) = J^*$. □

According to Theorem 3, the solution of the control problem (40) exists for any value $s \in [0, 1/2]$. The case $s = 0$, corresponding to weak regularizers when $q \in L^2(\Sigma_\tau)$, $\mathbf{q} \in L^2_T(\Sigma_\nu)$, is physically the most interesting. Another physically interesting case is the control problem with weak regularizers having the form

$$J(\mathbf{x}, q) = (\mu_0/2) I(\mathbf{x}) + (\mu_1/2)\|q\|^2_{\Sigma_\tau} + (\mu_2/2)\|\mathbf{q}\|^2_{\Sigma_\nu} +$$
$$+ (\mu_3/2)\|\mathbf{j}\|^2_\Omega \to \inf, \; F(\mathbf{x}, u) = 0, \; (\mathbf{x}, u) \in X \times K \tag{53}$$

provided that $(\mathbf{x}, u) \in X \times K$ for $s > 0$ as in the situation of problem (40) with strong regularizers. Let the following additional condition apply:

(jv) $\mu_0 > 0$; $\mu_1 \geq 0$, $\mu_2 \geq 0$, $\mu_3 \geq 0$ and K is a bounded set in $H^s(\Sigma_\tau) \times \tilde{H}^s_T(\Sigma_\nu) \times L^2(\Omega)^3$ for fixed $s \in (0, 1/2]$.

The proof of the following theorem is carried out analogously to the proof of Theorem 3.

Theorem 4. *Let the assumptions (i)–(iii) and (j), (jj), (jv) hold, and let the set Z_{ad} be nonempty. Then the problem (53) has a solutuion $(\mathbf{x}, u) \in X \times K$ for any $s \in (0, 1/2]$.*

Our next goal is to analyze the uniqueness and stability of the solutions of our control problems. For this purpose, we apply the technique developed in [5], which is based on using the additional properties of optimal solutions obtained by analyzing the optimality system. Therefore, the next stage of our study is to obtain the required optimality system. For concreteness, we will consider the case of control problem (40). Preliminarily, we define the spaces

$$X^* = H^1_T(\Omega)^* \times \mathcal{H}^{s+1/2}_{\text{div}}(\Omega)^* \times L^2_0(\Omega),$$

$$Y^* = (H^1_0(\Omega)^3 \times V_{\Sigma_\tau}(\Omega)) \times L^2_0(\Omega) \times H^{1/2}_T(\Sigma)^* \times H^s(\Sigma_\tau)^* \times \tilde{H}^s_T(\Sigma_\nu)^*,$$

which are dual of spaces X and Y defined in (25).

By [13], the derivation of the optimality system uses finding of the Fréchet partial derivative with respect to $\mathbf{x}$ of $F :\equiv (F_1, F_2, F_3, F_4, F_5) : X \times K \to Y$ defined in (38). The simple analysis shows that the mentioned partial derivative at any point $(\hat{\mathbf{x}}, \hat{u}) \equiv (\hat{\mathbf{u}}, \hat{\mathbf{H}}, \hat{p}, \hat{q}, \hat{\mathbf{q}}, \hat{\mathbf{j}}) \in X \times K$ is a linear continuous operator $F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{u}) : X \to Y$, which associates every element $(\mathbf{w}, \mathbf{h}, r) \in X$ with the element $F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{u})(\mathbf{w}, \mathbf{h}, r) = (\hat{y}_1, \hat{y}_2, \hat{y}_3, \hat{y}_4, \hat{y}_5) \in Y$. Here the elements $\hat{y}_1 \in W^* = H^{-1}(\Omega)^3 \times V_{\Sigma_\tau}(\Omega)^*$, $\hat{y}_2 \in L^2_0(\Omega)$, $\hat{y}_3 \in H^{1/2}_T(\Sigma)$, $\hat{y}_4 \in H^s(\Sigma_\tau)$, $\hat{y}_5 \in \tilde{H}^s_T(\Sigma_\nu)$ are defined by the triples $(\hat{\mathbf{u}}, \hat{\mathbf{H}}, \hat{p})$ and $(\mathbf{w}, \mathbf{h}, r)$ from relations

$$\langle \hat{y}_1, (\mathbf{v}, \mathbf{\Psi}) \rangle = \nu(\nabla \mathbf{w}, \nabla \mathbf{v}) + \nu_1(\text{rot}\, \mathbf{h}, \text{rot} \mathbf{\Psi}) + ((\mathbf{w} \cdot \nabla)\hat{\mathbf{u}}, \mathbf{v}) +$$

$$+ ((\hat{\mathbf{u}} \cdot \nabla)\mathbf{w}, \mathbf{v}) + \kappa[(\text{rot} \mathbf{\Psi} \times \mathbf{h}, \hat{\mathbf{u}}) + (\text{rot} \mathbf{\Psi} \times \hat{\mathbf{H}}, \mathbf{w}) - (\text{rot}\, \mathbf{h} \times \hat{\mathbf{H}}, \mathbf{v})$$

$$-(\text{rot}\, \hat{\mathbf{H}} \times \mathbf{h}, \mathbf{v})] + b(\mathbf{v}, r) \;\; \forall (\mathbf{v}, \mathbf{\Psi}) \in H^1_0(\Omega)^3 \times V_{\Sigma_\tau}(\Omega),$$

$$\langle \hat{y}_2, r \rangle = b(\mathbf{w}, r) \;\; \forall r \in L^2_0(\Omega),\; \hat{y}_3 = \mathbf{w}|_\Sigma,\; \hat{y}_4 = \mathbf{h} \cdot \mathbf{n}|_{\Sigma_\tau},\; \hat{y}_5 = \mathbf{h} \times \mathbf{n}|_{\Sigma_\nu}. \tag{54}$$

By $F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{u})^* : Y^* \to X^*$, we denote the operator adjoint to $F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{u})$, which is defined by

$$\langle F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{u})^* \mathbf{y}^*, \mathbf{x} \rangle_{X^* \times X} = \langle \mathbf{y}^*, F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{u}) \mathbf{x} \rangle_{Y^* \times Y} \;\; \forall (\mathbf{x}, \mathbf{y}^*) \in X \times Y^*.$$

Following to [13] (Chapter 1), let us introduce an element $\mathbf{y}^* = ((\xi, \eta), \sigma, \zeta_1, \zeta_2, \zeta_3) \in Y^*$ having the sense of an adjoint state and define the Lagrangian $\mathcal{L} : X \times K \times \mathbb{R}^+ \times Y^* \to \mathbb{R}$ where $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$ by

$$\mathcal{L}(\mathbf{x}, u, \lambda_0, \mathbf{y}^*) = \lambda_0 J(\mathbf{x}, u) + \langle \mathbf{y}^*, F(\mathbf{x}, u) \rangle_{Y^* \times Y} \equiv \lambda_0 J(\mathbf{x}, u) + \langle F_1(\mathbf{x}, u), (\xi, \eta) \rangle +$$

$$+ (F_2(\mathbf{x}), \sigma) + \langle \zeta_1, F_3(\mathbf{x}) \rangle_\Sigma + \langle \zeta_2, F_4(\mathbf{x}, u) \rangle_{\Sigma_\tau} + \langle \zeta_3, F_5(\mathbf{x}, u) \rangle_{\Sigma_\nu}. \tag{55}$$

Here and below, $\langle \cdot, \cdot \rangle_\Sigma$, $\langle \cdot, \cdot \rangle_{\Sigma_\tau}$ or $\langle \cdot, \cdot \rangle_{\Sigma_\nu}$ denote the duality pairings between $H^{1/2}_T(\Sigma)$ and $H^{-1/2}_T(\Sigma)$, between $H^s(\Sigma_\tau)$ and $H^{-s}(\Sigma_\tau)$ or between $H^s_T(\Sigma_\nu)$ and $H^{-s}_T(\Sigma_\nu)$, respectively. Based on the results of [13] (Chapter 1), one can prove the following theorem about the justification of the Lagrange principle for problem (40) and the regularity for the Lagrange multiplier $(\lambda_0, \mathbf{y}^*)$.

Theorem 5. *Let, under assumptions (i)–(iii) and (j), (jjj), the element $(\hat{\mathbf{x}}, \hat{u}) \equiv (\hat{\mathbf{u}}, \hat{\mathbf{H}}, \hat{p}, \hat{q}, \hat{\mathbf{q}}, \hat{\mathbf{j}}) \in X \times K$ be a local minimizer for problem (40), and let the cost functional I have the continuous Fréchet derivative with respect to $\mathbf{x}$ at $\hat{\mathbf{x}}$. Then:*

(1) there is a nonvanishing Lagrange multiplier $(\lambda_0, \mathbf{y}^*) = (\lambda_0, (\xi, \eta), \sigma, \zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^+ \times Y^*$ for which the Euler–Lagrange equation

$$F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{u})^* \mathbf{y}^* = -\lambda_0 (\mu_0/2) I'(\hat{\mathbf{x}}) \text{ in } X^* \tag{56}$$

for adjoint state $\mathbf{y}^*$ is satisfied and the minimum principle holds, having the form

$$\mathcal{L}(\hat{\mathbf{x}}, \hat{u}, \lambda_0, \mathbf{y}^*) \leq \mathcal{L}(\hat{\mathbf{x}}, u, \lambda_0, \mathbf{y}^*) \quad \forall u \in K. \tag{57}$$

(2) If, besides, condition (36) holds for each triple $u \equiv (q, \mathbf{q}, \mathbf{j}) \in K$, then any nonvanishing Lagrange multiplier $(\lambda_0, \mathbf{y}^*)$ satisfying (56) is regular, i.e., $\lambda_0 = 1$. Moreover, it is determined uniquely if the value is given.

Proof of Theorem 5. To prove statement 1 of Theorem 5 by [13] Chapter 1 (see also [15], Chapter 6), it is sufficient to prove that the operator $F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{u}) : X \to Y$ is a Fredholm operator. By virtue of (54), the operator $\hat{F} \equiv F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{u}) : X \to Y$ can be written in the form

$$\hat{F} = \Phi + \hat{\Phi} \equiv (\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5) + (\hat{\Phi}_1, 0, 0, 0, 0). \tag{58}$$

Here the operators $\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5 : X \to Y$ are defined by relations (32) and (33), while the operator $\hat{\Phi}_1 : X \to W^* = H^{-1}(\Omega)^3 \times V_{\Sigma_\tau}(\Omega)^*$ is defined by

$$\langle \hat{\Phi}_1(\mathbf{w}, \mathbf{h}, r), (\mathbf{v}, \mathbf{\Psi}) \rangle = ((\mathbf{w} \cdot \nabla)\hat{\mathbf{u}}, \mathbf{v}) + \kappa[(\mathrm{rot}\, \mathbf{\Psi} \times \mathbf{h}, \hat{\mathbf{u}}) - (\mathrm{rot}\, \hat{\mathbf{H}} \times \mathbf{h}, \mathbf{v})]. \tag{59}$$

Using estimates (13)–(15), we deduce that

$$|((\mathbf{w} \cdot \nabla)\hat{\mathbf{u}}, \mathbf{v})| \leq \gamma'_0 \|\mathbf{w}\|_{L^3(\Omega)^3} \|\hat{\mathbf{u}}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \; \forall \mathbf{v}, \mathbf{w} \in H^1_T(\Omega), \tag{60}$$

$$|(\mathrm{rot}\mathbf{\Psi} \times \mathbf{h}, \hat{\mathbf{u}})| \leq \|\mathrm{rot}\mathbf{\Psi}\|_\Omega \|\mathbf{h}\|_{L^3(\Omega)^3} \|\hat{\mathbf{u}}\|_{L^6(\Omega)^3} \; \forall \mathbf{\Psi} \in \mathcal{H}^{s+1/2}_{\mathrm{div}}(\Omega), \; \forall \mathbf{h} \in \mathcal{H}^{s+1/2}_{\mathrm{div}}(\Omega), \tag{61}$$

$$|(\mathrm{rot}\hat{\mathbf{H}} \times \mathbf{h}, \mathbf{v})| \leq \|\mathrm{rot}\hat{\mathbf{H}}\|_\Omega \|\mathbf{h}\|_{L^3(\Omega)^3} \|\mathbf{v}\|_{L^6(\Omega)^3} \; \forall \mathbf{v} \in H^1_T(\Omega), \; \mathbf{h} \in \mathcal{H}^{s+1/2}_{\mathrm{div}}(\Omega). \tag{62}$$

Since the space $H^1(\Omega)^3$ and the space $\mathcal{H}^{s+1/2}_{\mathrm{div}}(\Omega)$ for any $s > 0$ are continuously and compactly embedded into the space $L^3(\Omega)^3$, then estimates (60)–(62) imply that the operator $\hat{\Phi} = (\hat{\Phi}_1, 0, 0, 0, 0) : X \to Y$ where $\hat{\Phi}_1$ is defined by (59) is continuous and compact. In addition, it follows from Theorem 1 that the operator $\Phi : X \to Y$, defined by relations (32) and (33) is an isomorphism. This means that the operator $F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{u})$ is Fredholm as the sum of the isomorphism Φ and the continuous compact operator $\hat{\Phi}$. Therefore, statement 1 about existence of nonzero Lagrange multiplier $(\lambda_0, \mathbf{y}^*)$ is proved.

It remains to prove the regularity of the multiplier $(\lambda_0, \mathbf{y}^*)$, i.e., that $\lambda_0 \neq 0$ under conditions (36). Arguing as in paper [3], one can verify that (56) is equivalent to three identities with respect to the adjoint state $(\xi, \eta, \sigma, \zeta_1, \zeta_2, \zeta_3)$ having the form:

$$\begin{gathered} \nu(\nabla \mathbf{w}, \nabla \xi) + \nu_1(\mathrm{rot}\mathbf{h}, \mathrm{rot}\eta) + ((\hat{\mathbf{u}} \cdot \nabla)\mathbf{w}, \xi) + ((\mathbf{w} \cdot \nabla)\hat{\mathbf{u}}, \xi) + \\ + \kappa[(\mathrm{rot}\eta \times \hat{\mathbf{H}}, \mathbf{w}) + (\mathrm{rot}\eta \times \mathbf{h}, \hat{\mathbf{u}})] - \\ - \kappa[(\mathrm{rot}\hat{\mathbf{H}} \times \mathbf{h}, \xi) + (\mathrm{rot}\mathbf{h} \times \hat{\mathbf{H}}, \xi)] - (\mathrm{div}\mathbf{w}, \sigma) + \langle \zeta_1, \mathbf{w} \rangle_\Sigma + \langle \zeta_2, \mathbf{h} \cdot \mathbf{n} \rangle_{\Sigma_\tau} + \\ + \langle \zeta_3, \mathbf{h} \times \mathbf{n} \rangle_{\Sigma_\nu} = \\ = -\lambda_0(\mu_0/2)\langle I'_{\mathbf{u}}(\hat{\mathbf{x}}), \mathbf{w} \rangle - \lambda_0(\mu_0/2)\langle I'_{\mathbf{H}}(\hat{\mathbf{x}}), \mathbf{h} \rangle \;\; \forall (\mathbf{w}, \mathbf{h}) \in H^1_T(\Omega) \times \mathcal{H}^{s+1/2}_{\mathrm{div}}(\Omega), \end{gathered} \tag{63}$$

$$(\mathrm{div}\xi, r) = \lambda_0(\mu_0/2)(I'_p(\hat{\mathbf{x}}), r) \;\; \forall r \in L^2_0(\Omega). \tag{64}$$

The mentioned statement about the regularity of the multiplier $(\lambda_0, \mathbf{y}^*)$ is equivalent to the statement about the nonexistence of nonvanishing solutions of the systems (63), (64) at $\lambda_0 = 0$, in which the elements $\hat{\mathbf{x}} = (\hat{\mathbf{u}}, \hat{\mathbf{H}}, \hat{p})$ and $\hat{u} = (\hat{q}, \hat{\mathbf{q}}, \hat{\mathbf{j}})$ are connected by the relation $F(\hat{\mathbf{x}}, \hat{u}) = 0$. To prove this, one should denote by $\mathbf{y}^* = (\xi, \eta, \sigma, \zeta_1, \zeta_2, \zeta_3) \in Y^*$ an arbitrary solution of problems (63), (64) at $\lambda_0 = 0$. Setting $\mathbf{w} = \xi$, $\mathbf{h} = \eta$ and $\lambda_0 = 0$ in (63), (64) and using (16), we have

$$\nu(\nabla\xi, \nabla\xi) + \nu_1(\text{rot}\eta, \text{rot}\eta) + ((\xi\cdot\nabla)\hat{\mathbf{u}}, \xi) + \kappa[(\text{rot}\eta\times\eta, \hat{\mathbf{u}}) - (\text{rot}\hat{\mathbf{H}}\times\eta, \xi)] = 0,$$

$$\text{div}\,\xi = 0 \text{ in } \Omega. \tag{65}$$

Arguing as in [8], i.e., applying inequalities (8), (10), (13)–(15) for estimating all terms in (65) and using estimates (35) at $\mathbf{u} = \hat{\mathbf{u}}$, $\mathbf{H} = \hat{\mathbf{H}}$, we derive the following inequality from (65):

$$(\delta_0\nu - \gamma_0 M_{\hat{\mathbf{u}}} - \gamma_1(\sqrt{\kappa}/2)M_{\hat{\mathbf{H}}})\|\xi\|^2_{1,\Omega} + (\delta_1\nu_m - \gamma_1 M_{\hat{\mathbf{u}}} - \gamma_1(\sqrt{\kappa}/2)M_{\hat{\mathbf{H}}})\kappa\|\eta\|^2_{1,\Omega} \le 0. \tag{66}$$

From (66), it follows under smallness conditions (36) at $\mathbf{u} = \hat{\mathbf{u}}$, $\mathbf{H} = \hat{\mathbf{H}}$ that $\xi = \mathbf{0}$ and $\eta = \mathbf{0}$ in Ω.

Setting $\xi = \mathbf{0}$, $\eta = \mathbf{0}$ in (63) at $\lambda_0 = 0$, we have

$$-(\text{div}\mathbf{w}, \sigma) + \langle\zeta_1, \mathbf{w}\rangle_\Sigma + \langle\zeta_2, \mathbf{h}\cdot\mathbf{n}\rangle_{\Sigma_\tau} + \langle\zeta_3, \mathbf{h}\times\mathbf{n}\rangle_{\Sigma_\nu} = 0\ \forall(\mathbf{w}, \mathbf{h}) \in H^1_T(\Omega)\times\mathcal{H}^{s+1/2}_{\text{div}}(\Omega) \tag{67}$$

which in the case $\mathbf{h} = 0$, transforms to

$$-(\text{div}\mathbf{w}, \sigma) + \langle\zeta_1, \mathbf{w}\rangle_\Sigma = 0\ \forall\mathbf{w} \in H^1_T(\Omega). \tag{68}$$

From (68), it follows that

$$(\text{div}\mathbf{w}, \sigma) = 0\ \forall\mathbf{w} \in H^1_0(\Omega)^3. \tag{69}$$

By inf-sup condition (17), this identity holds if $\sigma = 0$. Setting $\sigma = 0$ in (68), we obtain $\langle\zeta_1, \mathbf{w}\rangle_\Sigma = 0$ for all $\mathbf{w} \in H^1_T(\Omega)$. This means that $\zeta_1 = 0$ in $H^{-1/2}_T(\Sigma)$.

Setting $\sigma = 0$, $\zeta_1 = 0$ in (67), we arrive at

$$\langle\zeta_2, \mathbf{h}\cdot\mathbf{n}\rangle_{\Sigma_\tau} + \langle\zeta_3, \mathbf{h}\times\mathbf{n}\rangle_{\Sigma_\nu} = 0\ \forall\mathbf{h} \in \mathcal{H}^{s+1/2}_{\text{div}}(\Omega). \tag{70}$$

Choosing $\mathbf{h} \in \mathcal{H}^{s+1/2}_{\text{div}}(\Omega, \Sigma_\nu)$ in (70), where $\mathcal{H}^{s+1/2}_{\text{div}}(\Omega, \Sigma_\nu)$ is defined in (24), we obtain $\langle\zeta_2, \mathbf{h}\cdot\mathbf{n}\rangle_{\Sigma_\tau} = 0$ for all $\mathbf{h} \in \mathcal{H}^{s+1/2}_{\text{div}}(\Omega, \Sigma_\nu)$. This means that $\zeta_2 = 0$ in $H^{-s}(\Sigma_\tau)$. Analogously, we deduce that $\zeta_3 = 0$ in $H^{-s}_T(\Sigma_\nu)$. Thus, we obtained $\mathbf{y}^* = \mathbf{0}$, and the regularity of the Lagrage multiplier is proved. Concerning the uniqueness property of the Lagrange multiplier $(1, \mathbf{y}^*_i)$, it is a consequence of Fredholm property of the linear operator $F'_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{u}) : X \to Y$. □

We note that the Lagrangian $\mathcal{L}$ defined in (55) is a continuously differentiable function of controls $q \in K_1$, $\mathbf{q} \in K_2$ and $\mathbf{j} \in K_3$, and its partial derivatives $\mathcal{L}'_q(\hat{\mathbf{x}}, \hat{u}, \lambda_0, \mathbf{y}^*)$ with respect to q, $\mathcal{L}'_{\mathbf{q}}(\hat{\mathbf{x}}, \hat{u}, \lambda_0, \mathbf{y}^*)$ with respect to $\mathbf{q}$ and $\mathcal{L}'_{\mathbf{j}}(\hat{\mathbf{x}}, \hat{u}, \lambda_0, \mathbf{y}^*)$ with respect to $\mathbf{j}$ in any point $(\hat{\mathbf{x}}, \hat{u}, \lambda_0, \mathbf{y}^*)$ are determined by

$$\langle\mathcal{L}'_q(\hat{\mathbf{x}}, \hat{u}, \lambda_0, \mathbf{y}^*), q\rangle = \lambda_0\mu_1(\hat{q}, q)_{s,\Sigma_\tau} - \langle\zeta_2, q\rangle_{\Sigma_\tau}\quad \forall q \in K_1,$$

$$\langle\mathcal{L}'_{\mathbf{q}}(\hat{\mathbf{x}}, \hat{u}, \lambda_0, \mathbf{y}^*), \mathbf{q}\rangle = \lambda_0\mu_2(\hat{\mathbf{q}}, \mathbf{q})_{s,\Sigma_\nu} - \langle\zeta_3, \mathbf{q}\rangle_{\Sigma_\nu}\quad \forall\mathbf{q} \in K_2,$$

$$\langle\mathcal{L}'_{\mathbf{j}}(\hat{\mathbf{x}}, \hat{u}, \lambda_0, \mathbf{y}^*), \mathbf{j}\rangle = \lambda_0\mu_3(\hat{\mathbf{j}}, \mathbf{j}) - \nu_1(\mathbf{j}, \text{rot}\,\eta)\quad \forall\mathbf{j} \in K_3.$$

As the triple $\hat{u} \equiv (\hat{q}, \hat{\mathbf{q}}, \hat{\mathbf{j}})$ is a minimizer of the function $\mathcal{L}(\hat{\mathbf{x}}, \cdot, \lambda_0, \mathbf{y}^*)$ on a closed convex set $K = K_1 \times K_2 \times K_3$ by (57), the following inequalities hold (see, e.g., [52]):

$$\langle\mathcal{L}'_q(\hat{\mathbf{x}}, \hat{u}, \lambda_0, \mathbf{y}^*), q - \hat{q}\rangle \equiv \lambda_0\mu_1(\hat{q}, q - \hat{q})_{s,\Sigma_\tau} - \langle\zeta_2, q - \hat{q}\rangle_{\Sigma_\tau} \ge 0\quad \forall q \in K_1, \tag{71}$$

$$\langle\mathcal{L}'_{\mathbf{q}}(\hat{\mathbf{x}}, \hat{u}, \lambda_0, \mathbf{y}^*), \mathbf{q} - \hat{\mathbf{q}}\rangle \equiv \lambda_0\mu_2(\hat{\mathbf{q}}, \mathbf{q} - \hat{\mathbf{q}})_{s,\Sigma_\nu} - \langle\zeta_3, \mathbf{q} - \hat{\mathbf{q}}\rangle_{\Sigma_\nu} \ge 0\quad \forall\mathbf{q} \in K_2, \tag{72}$$

$$\langle \mathcal{L}'_{\mathbf{j}}(\hat{\mathbf{x}}, \hat{u}, \lambda_0, \mathbf{y}^*), \mathbf{j} - \hat{\mathbf{j}} \rangle \equiv \lambda_0 \mu_3 (\hat{\mathbf{j}}, \mathbf{j} - \hat{\mathbf{j}}) - \nu_1 (\mathbf{j} - \hat{\mathbf{j}}, \mathrm{rot}\, \eta) \geq 0 \quad \forall \mathbf{j} \in K_3. \tag{73}$$

Identities (63) and (64), the variational inequalities (71)–(73) and the operator constraint (state equation) (39), which is equivalent to the weak forms (19) and (20) of Problem 1, constitute the optimality system for our control problem (40). It describes the first-order necessary conditions of optimality.

Below we will assume that multiplier λ_0 is dimensionless. Dimensions of adjoint state variables $\xi, \eta, \sigma, \zeta_1, \zeta_2, \zeta_3$ clearly depend on dimension $[\mu_0]$ of parameter μ_0. We assume that the dimension $[\mu_0]$ is chosen so that the dimensions of the adjoint state variables ξ, η and σ coincide with those of $\mathbf{u}, \mathbf{H}$ and p in the main state $\mathbf{x} = (\mathbf{u}, \mathbf{H}, p)$, i.e., so that

$$[\xi] = [\mathbf{u}] = L_0/T_0, \ [\eta] = [\mathbf{H}] = I_0/L_0, \ [\sigma] = [p] = L_0^2/T_0^2. \tag{74}$$

Here and below, L_0, T_0, I_0 and M_0 denote the SI dimensions of the length, time, electric current and mass units expressed in meters, seconds, amperes and kilograms, respectively. This allows us to refer to ξ, η and σ as "adjoint velocity", "adjoint magnetic field" and "adjoint pressure".

Remark 2. *It follows from conditions $\xi \in H_0^1(\Omega)^3$, $\eta \in V_{\Sigma_\tau}(\Omega)$ and (64) that the adjoint velocity ξ and adjoint magnetic field η possess properties*

$$\xi|_\Sigma = 0, \ \eta \cdot \mathbf{n}|_{\Sigma_\tau} = 0, \ \eta \times \mathbf{n}|_{\Sigma_\nu} = \mathbf{0}, \ \mathrm{div}\, \eta = 0 \textit{ and } \mathrm{div}\, \xi = \lambda_0(\mu_0/2)\chi_Q I'_p(\hat{\mathbf{x}}) \textit{ in } \Omega. \tag{75}$$

Here χ_Q is a characteristic function of the set Q. We emphasize that the adjoint velocity ξ, unlike η, is in a general case a nonsolenoidal vector function except for the case when the cost functional I is independent of pressure p. Only in this case $\mathrm{div}\, \xi = 0$ and, moreover, $\xi \in V$.

To prove the uniqueness and also stability of the solution of (40), we need to introduce additional conditions for the data depending on the cost functional. Below, we preliminarily derive one important inequality with respect to the difference of a solution $(\mathbf{x}_1, u_1)$ of problem (40) and a solution $(\mathbf{x}_2, u_2)$ of the perturbed problem (40). In addition, we derive the estimates for the difference $\mathbf{x}_1 - \mathbf{x}_2$ via the difference $u_1 - u_2 = (q_1 - q_2, \mathbf{q}_1 - \mathbf{q}_2, \mathbf{j}_1 - \mathbf{j}_2)$.

Let us denote by $(\mathbf{x}_1, u_1) \equiv (\mathbf{u}_1, \mathbf{H}_1, p_1, q_1, \mathbf{q}_1, \mathbf{j}_1) \in X \times K$ an arbitrary solution to problem (40). By $(\mathbf{x}_2, u_2) \equiv (\mathbf{u}_2, \mathbf{H}_2, p_2, q_2, \mathbf{q}_2, \mathbf{j}_2) \in X \times K$, we denote the solution to problem

$$\tilde{J}(\mathbf{x}, u) = \frac{\mu_0}{2}\tilde{I}(\mathbf{x}) + \frac{\mu_1}{2}\|q\|^2_{s,\Sigma_\tau} + \frac{\mu_2}{2}\|\mathbf{q}\|^2_{s,\Sigma_\nu} + \frac{\mu_3}{2}\|\mathbf{j}\|^2_\Omega \to \inf, \ F(\mathbf{x}, u) = 0, \ (\mathbf{x}, u) \in X \times K. \tag{76}$$

Here $\tilde{I}$ is a functional which is close to functional I. By virtue of Theorem 2, we have for triples $(\mathbf{u}_i, \mathbf{H}_i, p_i)$, $i = 1, 2$:

$$\|\mathbf{u}_i\|_{1,\Omega} \leq M^0_{\mathbf{u}} = \sup_{u \in K} M_{\mathbf{u}}(u), \ \|\mathbf{H}_i\|_{\mathcal{H}^{s+1/2}_{\mathrm{div}}(\Omega)} \leq M^0_{\mathbf{H}} = \sup_{u \in K} M_{\mathbf{H}}(u),$$
$$\|p_i\|_\Omega \leq M^0_p = \sup_{u \in K} M_p(u). \tag{77}$$

Let the values $M^0_{\mathbf{u}}$ and $M^0_{\mathbf{H}}$ be such that

$$(\gamma_0/\delta_0\nu)M^0_{\mathbf{u}} + (\gamma_1/\delta_0\nu)(\sqrt{\kappa}/2)M^0_{\mathbf{H}} < 1/2,$$
$$(\gamma_1/\delta_1\nu_m)M^0_{\mathbf{u}} + (\gamma_1/\delta_1\nu_m)(\sqrt{\kappa}/2)M^0_{\mathbf{H}} < 1/2. \tag{78}$$

To make conditions (78) more illustrative and to simplify the subsequent presentation, we define the parameters

$$\mathcal{R}e = \gamma_0 M^0_{\mathbf{u}}/(\delta_0\nu), \ \mathcal{R}m = \gamma_1 M^0_{\mathbf{u}}/(\delta_1\nu_m), \ \mathcal{H}a = \gamma_1\sqrt{\kappa}M^0_{\mathbf{H}}/(\delta_0\nu), \ \mathcal{P}m = \delta_0\nu/(\delta_1\nu_m). \tag{79}$$

They are analogues of the hydrodynamic dimensionless parameters [53], namely the Reynolds number Re, the magnetic Reynolds number Rm, the Hartman number Ha and the magnetic Prandtl number Pm. We emphasize that parameters $\mathcal{R}e$, $\mathcal{R}m$, $\mathcal{H}a$ and $\mathcal{P}m$ are dimensionless. To demonstrate this fact, one should know the dimensions of all parameters $\delta_0, \delta_1, \gamma_0, \gamma_1$ and β defined in Lemma 2 and also of $M_{\mathbf{u}}^0$ and $M_{\mathbf{H}}^0$ entering into (78). In order to determine the dimensions of $\mathcal{R}e$, $\mathcal{R}m$, $\mathcal{H}a$ and $\mathcal{P}m$, it will be assumed, below, the norms $\|\cdot\|_\Omega$ and $\|\cdot\|_{1,\Omega}$ of a function u in $L^2(\Omega)$ and in $H^1(\Omega)$ and seminorm $|\cdot|_{1,\Omega}$ in $H^1(\Omega)$ are defined as follows:

$$\|u\|_\Omega^2 = \int_\Omega u^2 d\Omega, \ \ |u|_{1,\Omega}^2 = \int_\Omega |\nabla u|^2 d\Omega, \ \ \|u\|_{1,\Omega}^2 = l^{-2}\|u\|_\Omega^2 + |u|_{1,\Omega}^2, \ \ [l] = L_0. \tag{80}$$

Here l is the dimensional factor having the dimension $[l] = L_0$ with value equal to 1. Using (80), one can verify that the dimensions of $\|u\|_\Omega, |u|_{1,\Omega}$ and $\|u\|_{1,\Omega}$ are connected with the dimension $[u]$ of u by the formulas

$$[\|u\|_\Omega] = [u]L_0^{3/2} \text{ and } [|u|_{1,\Omega}] = [\|u\|_{1,\Omega}] = [u]L_0^{1/2}.$$

We recall also (see, e.g., [15] (p. 272)) that

$$[\mathbf{g}] = [\mathbf{u}] = L_0/T_0, \ [q] = [\mathbf{q}] = [\mathbf{H}] = I_0/L_0, \ [\mathbf{j}] = I_0/L_0^2,$$

$$[\nu_m] = [\nu] = L_0^2/T_0, \ [\kappa] = L_0^4/T_0^2 I_0^2.$$

Combining this with (10)–(15), (17) and (74) yields

$$[\delta_i] = 1, \ [\gamma_i] = L_0^{1/2}, \ [\beta] = 1,$$

$$[M_{\mathbf{u}}^0] = [\|\mathbf{u}\|_{1,\Omega}] = [\|\xi\|_{1,\Omega}] = [\|\mathbf{g}\|_{1/2,\Sigma}] = L_0^{3/2}/T_0, \ [M_p^0] = L_0^{7/2}/T_0^2,$$

$$[M_{\mathbf{H}}^0] = [\|\mathbf{H}\|_{\mathcal{H}_{\mathrm{div}}^{s+1/2}(\Omega)}] = [\|\eta\|_{1,\Omega}] = [\|q\|_{\Sigma_\tau}] = [\|\mathbf{q}\|]_{\Sigma_\nu} = [\|\mathbf{j}\|_\Omega] = I_0/L_0^{1/2},$$

$$[\sqrt{\kappa}\|\mathbf{H}\|_{\mathcal{H}_{\mathrm{div}}^{s+1/2}(\Omega)}] = L_0^{3/2}/T_0 = [M_{\mathbf{u}}^0]. \tag{81}$$

Using (81), we have that

$$[\mathcal{R}e] = [\mathcal{R}m] = [\mathcal{H}a] = [\mathcal{P}m] = 1,$$

i.e., all the parameters $\mathcal{R}e$, $\mathcal{R}m$, $\mathcal{H}a$ and $\mathcal{P}m$ defined in (79) are dimensionless. Since parameters $\mathcal{R}e$ and $\mathcal{R}m$ are connected by relation $\mathcal{R}m = (\gamma_1/\gamma_0)\mathcal{P}m\mathcal{R}e$, we can rewrite conditions (78) in the following form containing only three nondimensional parameters $\mathcal{R}e$, $\mathcal{H}a$ and $\mathcal{P}m$:

$$\mathcal{R}e + (1/2)\mathcal{H}a < 1/2, \ \ (\gamma_1/\gamma_0)\mathcal{P}m\mathcal{R}e + (1/2)\mathcal{P}m\,\mathcal{H}a < 1/2. \tag{82}$$

Denote, by $(1, \mathbf{y}_i^*) \equiv (1, \xi_i, \eta_i, \sigma_i, \zeta_1^{(i)}, \zeta_2^{(i)}, \zeta_3^{(i)})$, $i = 1,2$, the Lagrange multipliers that correspond to solutions $(\mathbf{x}_1, u_1)$ and $(\mathbf{x}_2, u_2)$ of problems (40) and (76), respectively (these multipliers are determined uniquely under conditions (82)). By definition, they satisfy identities

$$\begin{gathered}\nu(\nabla\mathbf{w}, \nabla\xi_i) + \nu_1(\mathrm{rot}\mathbf{h}, \mathrm{rot}\eta_i) + ((\mathbf{u}_i\cdot\nabla)\mathbf{w}, \xi_i) + ((\mathbf{w}\cdot\nabla)\mathbf{u}_i, \xi_i) + \kappa[(\mathrm{rot}\eta_i \times \mathbf{H}_i, \mathbf{w}) \\ +(\mathrm{rot}\eta_i \times \mathbf{h}, \mathbf{u}_i)] - \kappa[(\mathrm{rot}\mathbf{H}_i \times \mathbf{h}, \xi_i) + (\mathrm{rot}\mathbf{h} \times \mathbf{H}_i, \xi_i)] - (\mathrm{div}\mathbf{w}, \sigma_i) + \langle\zeta_1^{(i)}, \mathbf{w}\rangle_\Sigma + \\ + \langle\zeta_2^{(i)}, \mathbf{h}\cdot\mathbf{n}\rangle_{\Sigma_\tau} + \langle\zeta_3^{(i)}, \mathbf{h}\times\mathbf{n}\rangle_{\Sigma_\nu} = \\ = -(\mu_0/2)\langle(I^i)'_{\mathbf{u}}(\mathbf{x}_i), \mathbf{w}\rangle - (\mu_0/2)\langle(I^i)'_{\mathbf{H}}(\mathbf{x}_i), \mathbf{h}\rangle \ \forall(\mathbf{w}, \mathbf{h}) \in H_T^1(\Omega)\times\mathcal{H}_{\mathrm{div}}^{s+1/2}(\Omega), \ i = 1,2,\end{gathered} \tag{83}$$

$$(\mathrm{div}\xi_i, r) = (\mu_0/2)((I^i)'_p(\mathbf{x}_i), r) \ \ \forall r \in L_0^2(\Omega), \ \ i = 1,2. \tag{84}$$

Here we renamed $I = I^1$, $\tilde{I} = I^2$. We define the following differences:

$$\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2,\ \ \mathbf{H} = \mathbf{H}_1 - \mathbf{H}_2,\ \ p = p_1 - p_2,\ q = q_1 - q_2,\ \ \mathbf{q} = \mathbf{q}_1 - \mathbf{q}_2,\ \ \mathbf{j} = \mathbf{j}_1 - \mathbf{j}_2,$$

$$\xi = \xi_1 - \xi_2,\ \ \sigma = \sigma_1 - \sigma_2,\ \ \zeta_1 = \zeta_1^{(1)} - \zeta_1^{(2)},\ \ \zeta_2 = \zeta_2^{(1)} - \zeta_2^{(2)},\ \ \zeta_3 = \zeta_3^{(1)} - \zeta_3^{(2)}. \tag{85}$$

We now deduce an important inequality for differences (85). While obtaining the inequality, we will use some ideas and results from [5], which we will sketch here for the reader's convenience.

Subtract relations (19), (20), written for $\mathbf{u}_2, \mathbf{H}_2, p_2, u_2$, from (19), (20) for $\mathbf{u}_1, \mathbf{H}_1, p_1, u_1$. Using (85), we obtain

$$\nu(\nabla\mathbf{u}, \nabla\mathbf{v}) + \nu_1(\mathrm{rot}\mathbf{H}, \mathrm{rot}\boldsymbol{\Psi}) + [((\mathbf{u}\cdot\nabla)\mathbf{u}_1, \mathbf{v}) + ((\mathbf{u}_2\cdot\nabla)\mathbf{u}, \mathbf{v})] + \kappa[(\mathrm{rot}\boldsymbol{\Psi}\times\mathbf{H}, \mathbf{u}_1) +$$

$$+ (\mathrm{rot}\boldsymbol{\Psi}\times\mathbf{H}_2, \mathbf{u})] - \kappa[(\mathrm{rot}\mathbf{H}_1\times\mathbf{H}, \mathbf{v}) + (\mathrm{rot}\mathbf{H}\times\mathbf{H}_2, \mathbf{v})] - (\mathrm{div}\mathbf{v}, p) =$$

$$= \nu_1(\mathbf{j}, \mathrm{rot}\,\boldsymbol{\Psi})\ \forall(\mathbf{v}, \boldsymbol{\Psi}) \in H_0^1(\Omega)^3 \times V_{\Sigma_\tau}(\Omega), \tag{86}$$

$$\mathrm{div}\,\mathbf{u} = 0 \text{ in } \Omega,\ \ \mathbf{u}|_\Sigma = \mathbf{0},\ \ \mathbf{H}\cdot\mathbf{n}|_{\Sigma_\tau} = q,\ \ \mathbf{H}\times\mathbf{n}|_{\Sigma_\nu} = \mathbf{q}. \tag{87}$$

Set $\mathbf{j} = \mathbf{j}_2$ in the inequality (73) under $\lambda_0 = 1$, written at $\hat{\mathbf{j}} = \mathbf{j}_1$, $\eta = \eta_1$, and then we set $\mathbf{j} = \mathbf{j}_1$ in (73) under $\lambda_0 = 1$, written at $\hat{\mathbf{j}} = \mathbf{j}_2$, $\eta = \eta_2$. We obtain $-\mu_3(\mathbf{j}_1, \mathbf{j}) + \nu_1(\mathbf{j}, \mathrm{rot}\,\eta_1) \geq 0$ and $\mu_3(\mathbf{j}_2, \mathbf{j}) - \nu_1(\mathbf{j}, \mathrm{rot}\,\eta_2) \geq 0$. Adding these inequalities yields the relation

$$-\nu_1(\mathbf{j}, \mathrm{rot}\,\eta) \leq -\mu_3\|\mathbf{j}\|_\Omega^2. \tag{88}$$

In the same manner, we derive from (71) and (72) the inequalities

$$-\langle\zeta_2, q\rangle_{\Sigma_\tau} \leq -\mu_1\|q\|_{s,\Sigma_\tau}^2,\ \ -\langle\zeta_3, \mathbf{q}\rangle_{\Sigma_\nu} \leq -\mu_2\|\mathbf{q}\|_{s,\Sigma_\nu}^2. \tag{89}$$

Let us subtract identities (83), (84) for $i = 2$ from (83), (84) for $i = 1$ and set $\mathbf{w} = \mathbf{u}, \mathbf{h} = \mathbf{H}, r = p$. Adding the results and using conditions $\mathbf{u}|_\Sigma = 0$, $\mathbf{H}\cdot\mathbf{n}|_{\Sigma_\tau} = q$, $\mathbf{H}\times\mathbf{n}|_{\Sigma_\nu} = \mathbf{q}$, we obtain

$$\nu(\nabla\mathbf{u}, \nabla\xi) + \nu_1(\mathrm{rot}\mathbf{H}, \mathrm{rot}\eta) + ((\mathbf{u}_1\cdot\nabla)\mathbf{u}, \xi) + 2((\mathbf{u}\cdot\nabla)\mathbf{u}, \xi_2) + ((\mathbf{u}\cdot\nabla)\mathbf{u}_1, \xi) +$$

$$+\kappa[(\mathrm{rot}\eta\times\mathbf{H}_1, \mathbf{u}) + 2(\mathrm{rot}\eta_2\times\mathbf{H}, \mathbf{u}) + (\mathrm{rot}\eta\times\mathbf{H}, \mathbf{u}_1)] -$$

$$-\kappa[(\mathrm{rot}\mathbf{H}_1\times\mathbf{H}, \xi) + 2(\mathrm{rot}\mathbf{H}\times\mathbf{H}, \xi_2) + (\mathrm{rot}\mathbf{H}\times\mathbf{H}_1, \xi)] - (\mathrm{div}\xi, p) =$$

$$= -\langle\zeta_2, q\rangle_{\Sigma_\tau} - \langle\zeta_3, \mathbf{q}\rangle_{\Sigma_\nu} - (\mu_0/2)\langle I'_\mathbf{u}(\mathbf{x}_1) - \tilde{I}'_\mathbf{u}(\mathbf{x}_2), \mathbf{u}\rangle -$$

$$-(\mu_0/2)\langle I'_\mathbf{H}(\mathbf{x}_1) - \tilde{I}'_\mathbf{H}(\mathbf{x}_2), \mathbf{H}\rangle - (\mu_0/2)\langle I'_p(\mathbf{x}_1) - \tilde{I}'_p(\mathbf{x}_2), p\rangle. \tag{90}$$

We now set $\mathbf{v} = \xi$, $\boldsymbol{\Psi} = \eta$ in (86) and subtract from (90). Using (88), (89) and relations

$$2((\mathbf{u}\cdot\nabla)\mathbf{u}, \xi_2) + ((\mathbf{u}_1\cdot\nabla)\mathbf{u}, \xi) - ((\mathbf{u}_2\cdot\nabla)\mathbf{u}, \xi) = 2((\mathbf{u}\cdot\nabla)\mathbf{u}, \xi_2) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \xi) =$$

$$= ((\mathbf{u}\cdot\nabla)\mathbf{u}, \xi_1 + \xi_2),$$

$$-2(\mathrm{rot}\mathbf{H}\times\mathbf{H}, \xi_2) + (\mathrm{rot}\mathbf{H}\times\mathbf{H}_2, \xi) - (\mathrm{rot}\mathbf{H}\times\mathbf{H}_1, \xi) = -(\mathrm{rot}\mathbf{H}\times\mathbf{H}, \xi_1 + \xi_2),$$

$$2(\mathrm{rot}\eta_2\times\mathbf{H}, \mathbf{u}) + (\mathrm{rot}\eta\times\mathbf{H}_1, \mathbf{u}) - (\mathrm{rot}\eta\times\mathbf{H}_2, \mathbf{u}) = (\mathrm{rot}(\eta_1 + \eta_2)\times\mathbf{H}, \mathbf{u}),$$

we arrive at

$$((\mathbf{u}\cdot\nabla)\mathbf{u}, \xi_1 + \xi_2) - \kappa(\mathrm{rot}\mathbf{H}\times\mathbf{H}, \xi_1 + \xi_2) + \kappa(\mathrm{rot}(\eta_1 + \eta_2)\times\mathbf{H}, \mathbf{u}) +$$

$$+(\mu_0/2)\langle I'_\mathbf{u}(\mathbf{x}_1) - \tilde{I}'_\mathbf{u}(\mathbf{x}_2), \mathbf{u}\rangle + (\mu_0/2)\langle I'_\mathbf{H}(\mathbf{x}_1) - \tilde{I}'_\mathbf{H}(\mathbf{x}_2), \mathbf{H}\rangle + (\mu_0/2)(I'_p(\mathbf{x}_1) - \tilde{I}'_p(\mathbf{x}_2), p) \leq$$

$$\leq -\mu_1\|q\|^2_{s,\Sigma_\tau} - \mu_2\|\mathbf{q}\|^2_{s,\Sigma_\nu} - \mu_3\|\mathbf{j}\|^2_{\Omega}. \tag{91}$$

This inequality alone will take the important role in Section 5 when studying the local uniqueness and stability for optimal solutions. It is appropriate to formulate this result as the next theorem.

Theorem 6. *Let, under assumptions of Theorem 3 for cost functionals I and $\tilde{I}$ and under (82), pairs $(\mathbf{x}_1, u_1) = (\mathbf{u}_1, \mathbf{H}_1, p_1, q_1, \mathbf{q}_1, \mathbf{j}_1) \in X \times K$ and $(\mathbf{x}_2, u_2) = (\mathbf{u}_2, \mathbf{H}_2, p_2, q_2, \mathbf{q}_2, \mathbf{j}_2) \in X \times K$ be solutions to (40) and (76), respectively. Let $\mathbf{y}_i^* = (\xi_i, \eta_i, \sigma_i, \zeta_1^{(i)}, \zeta_2^{(i)}, \zeta_3^{(i)}) \in Y^*$, $i = 1, 2$, be adjoint states which correspond to these solutions. Then the relation (91) for differences $\mathbf{u}, \mathbf{H}, p, q, \mathbf{q}, \mathbf{j}$ defined in (85) holds.*

We now consider problems (86), (87) with respect to differences $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, $\mathbf{H} = \mathbf{H}_1 - \mathbf{H}_2$ and $p = p_1 - p_2$. In this problem, differences $\mathbf{j} = \mathbf{j}_1 - \mathbf{j}_2$, $q = q_1 - q_2$ and $\mathbf{q} = \mathbf{q}_1 - \mathbf{q}_2$ play the role of the data together with functions $\mathbf{u}_1, \mathbf{u}_2, \mathbf{H}_1$ and $\mathbf{H}_2$. Below, we will need estimates for norms of the differences $\mathbf{u}$, $\mathbf{H}$ and p via norms of the differences q, $\mathbf{q}$ and $\mathbf{j}$. In order to derive these estimates, we present the difference $\mathbf{H} \equiv \mathbf{H}_1 - \mathbf{H}_2$ in the form $\mathbf{H} = \mathbf{H}_0 + \tilde{\mathbf{H}}$ where $\mathbf{H}_0$ is a unique solution of (23) corresponding to differences $q = q_1 - q_2$ and $\mathbf{q} = \mathbf{q}_1 - \mathbf{q}_2$, while $\tilde{\mathbf{H}} \in V_{\Sigma_\tau}(\Omega)$ is a certain function. We set $\mathbf{v} = \mathbf{u}$ and $\mathbf{\Psi} = \tilde{\mathbf{H}}$ in (86). Taking into account (16) and condition $\text{rot}\mathbf{H}_0 = 0$, we obtain

$$\nu(\nabla\mathbf{u}, \nabla\mathbf{u}) + \nu_1(\text{rot}\tilde{\mathbf{H}}, \text{rot}\tilde{\mathbf{H}}) = -((\mathbf{u}\cdot\nabla)\mathbf{u}_1, \mathbf{u}) - \kappa(\text{rot}\,\tilde{\mathbf{H}} \times \tilde{\mathbf{H}}, \mathbf{u}_1) - \kappa(\text{rot}\,\tilde{\mathbf{H}} \times \mathbf{H}_0, \mathbf{u}_1) -$$

$$-\kappa(\text{rot}\,\mathbf{H}_1 \times \tilde{\mathbf{H}}, \mathbf{u}) - \kappa(\text{rot}\,\mathbf{H}_1 \times \mathbf{H}_0, \mathbf{u}) + \nu_1(\mathbf{j}, \text{rot}\,\tilde{\mathbf{H}}). \tag{92}$$

Using estimates (10)–(15), (77) and setting $\nu_1 = \nu_m\kappa$, we obtain from (92) that

$$\delta_0\nu\|\mathbf{u}\|^2_{1,\Omega} + \delta_1\nu_m\kappa\|\tilde{\mathbf{H}}\|^2_{1,\Omega} \leq \gamma_0 M^0_{\mathbf{u}}\|\mathbf{u}\|^2_{1,\Omega} + \kappa\gamma_1 M^0_{\mathbf{u}}\|\tilde{\mathbf{H}}\|^2_{1,\Omega}$$

$$+\kappa\gamma_1 M^0_{\mathbf{H}}\|\mathbf{u}\|_{1,\Omega}\|\tilde{\mathbf{H}}\|_{1,\Omega} + \kappa\gamma_1 M^0_{\mathbf{H}}\|\mathbf{u}\|_{1,\Omega}\|\mathbf{H}_0\|_{s+1/2,\Omega}$$

$$+\kappa\gamma_1 M^0_{\mathbf{u}}\|\tilde{\mathbf{H}}\|_{1,\Omega}\|\mathbf{H}_0\|_{s+1/2,\Omega} + C_1\nu_m\kappa\|\mathbf{j}\|_{\Omega}\|\tilde{\mathbf{H}}\|_{1,\Omega}. \tag{93}$$

Applying (79) and Young's inequality $|ab| \leq \varepsilon a^2/2 + b^2/(2\varepsilon)$ or $2ab \leq \varepsilon a^2/2 + 2b^2/\varepsilon$, $\varepsilon = \text{const} > 0$, at $\varepsilon = \delta_0\nu/2$, $\varepsilon = \delta_1\nu_m\kappa/4$ or $\varepsilon = 1$, we consequently derive that

$$\kappa\gamma_1 M^0_{\mathbf{H}}\|\mathbf{u}\|_{1,\Omega}\|\tilde{\mathbf{H}}\|_{1,\Omega} \leq (\sqrt{\kappa}/2)\gamma_1(\|\mathbf{u}\|^2_{1,\Omega} + \kappa\|\tilde{\mathbf{H}}\|^2_{1,\Omega}), \tag{94}$$

$$\kappa\gamma_1 M^0_{\mathbf{H}}\|\mathbf{H}_0\|_{s+1/2,\Omega}\|\mathbf{u}\|_{1,\Omega} \leq \frac{\varepsilon}{2}\|\mathbf{u}\|^2_{1,\Omega} + \frac{(\gamma_1\kappa M^0_{\mathbf{H}}\|\mathbf{H}_0\|_{s+1/2,\Omega})^2}{2\varepsilon} =$$

$$= \frac{\delta_0\nu}{4}\|\mathbf{u}\|^2_{1,\Omega} + \kappa\delta_0\nu\mathcal{H}a^2\|\mathbf{H}_0\|^2_{s+1/2,\Omega}, \tag{95}$$

$$\kappa\gamma_1 M^0_{\mathbf{H}}\|\mathbf{H}_0\|_{s+1/2,\Omega}\|\tilde{\mathbf{H}}\|_{1,\Omega} \leq \frac{\varepsilon}{2}\|\tilde{\mathbf{H}}\|^2_{1,\Omega} + \frac{(\gamma_1\kappa M^0_{\mathbf{u}})^2}{2\varepsilon}\|\mathbf{H}_0\|^2_{s+1/2,\Omega}) =$$

$$= (\kappa\delta_1\nu_m/8)\|\tilde{\mathbf{H}}\|^2_{1,\Omega} + 2\kappa\delta_0\nu\mathcal{P}m(\gamma_1/\gamma_0)^2\mathcal{R}e\|\mathbf{H}_0\|^2_{s+1/2,\Omega}, \tag{96}$$

$$C_1\kappa\nu_m\|\mathbf{j}\|_{\Omega}\|\tilde{\mathbf{H}}\|_{1,\Omega} \leq 2\kappa\nu_m C_1^2\delta_1^{-1}\|\mathbf{j}\|^2_{\Omega} + \frac{\delta_1\kappa\nu_m}{8}\|\tilde{\mathbf{H}}\|_{1,\Omega}. \tag{97}$$

Taking into account (94)–(97), we obtain from (93) that

$$(\delta_0\nu - \gamma_0 M^0_{\mathbf{u}} + \gamma_1(\sqrt{\kappa}/2)M^0_{\mathbf{H}})\|\mathbf{u}\|^2_{1,\Omega} + (\delta_1\nu_m - \gamma_1 M^0_{\mathbf{H}} + \gamma_1(\sqrt{\kappa}/2)M^0_{\mathbf{H}})\kappa\|\tilde{\mathbf{H}}\|^2_{1,\Omega} \leq$$

$$\leq (\delta_0\nu/4)\|\mathbf{u}\|^2_{1,\Omega} + (\delta_1\nu_m\kappa/4)\|\tilde{\mathbf{H}}\|^2_{1,\Omega} + \kappa\delta_0\nu(\mathcal{H}a^2 +$$

$$+2\mathcal{P}m(\gamma_1/\gamma_0)^2\mathcal{R}^2)\|\mathbf{H}_0\|^2_{s+1/2,\Omega} + 2\kappa\nu_m C_1^2\delta_1^{-1}\|\mathbf{j}\|^2_\Omega. \tag{98}$$

It follows from (78) that

$$(\delta_0\nu/2) < \delta_0\nu - \gamma_0 M^0_{\mathbf{u}} - \gamma_1(\sqrt{\kappa}/2)M^0_{\mathbf{H}},\ (\delta_1\nu_m/2) < \delta_1\nu_m - \gamma_1 M^0_{\mathbf{u}} - \gamma_1(\sqrt{\kappa}/2)M^0_{\mathbf{H}}. \tag{99}$$

Using (99), we derive from (98) that

$$\delta_0\nu\|\mathbf{u}\|^2_{1,\Omega} + \delta_1\nu_m\kappa\|\tilde{\mathbf{H}}\|^2_{1,\Omega} \le 2\kappa\delta_0\nu\mathcal{R}^2\|\mathbf{H}_0\|^2_{s+1/2,\Omega} + 8\kappa\nu_m C_1^2\delta_1^{-1}\|\mathbf{j}\|^2_\Omega) \tag{100}$$

where

$$\mathcal{R} = 2\sqrt{\mathcal{H}a^2 + 2\mathcal{P}m(\gamma_1/\gamma_0)^2\mathcal{R}e^2}. \tag{101}$$

From (100), we conclude that

$$\|\mathbf{u}\|_{1,\Omega} \le (2\kappa\mathcal{R}^2\|\mathbf{H}_0\|^2_{s+1/2,\Omega} + \frac{8\kappa\nu_m C_1^2\delta_1^{-1}}{\delta_0\nu}\|\mathbf{j}\|^2_\Omega)^{1/2} \le$$

$$\le \sqrt{\kappa}\mathcal{R}\|\mathbf{H}_0\|_{s+1/2,\Omega} + 2\frac{C_1}{\delta_1}\sqrt{\frac{\kappa}{\mathcal{P}m}}\|\mathbf{j}\|_\Omega, \tag{102}$$

$$\|\tilde{\mathbf{H}}\|_{1,\Omega} \le (2\frac{\delta_0\nu}{\delta_1\nu_m}\mathcal{R}^2\|\mathbf{H}_0\|^2_{s+1/2,\Omega} + \frac{8C_1^2}{\delta_1^2}\|\mathbf{j}\|^2_\Omega)^{1/2} \le \sqrt{\mathcal{P}m}\mathcal{R}\|\mathbf{H}_0\|_{s+1/2,\Omega} + 2\frac{C_1}{\delta_1}\|\mathbf{j}\|_\Omega. \tag{103}$$

Since $\mathbf{H} = \mathbf{H}_0 + \tilde{\mathbf{H}}$, from (102), (103), using (11), we obtain the needed estimates for differences $\mathbf{u}$ and $\mathbf{H}$:

$$\|\mathbf{u}\|_{1,\Omega} \le C_{\mathbf{u}}\sqrt{\kappa}(\|q\|_{\Sigma_\tau} + \|\mathbf{q}\|_{\Sigma_\nu} + \|\mathbf{j}\|_\Omega), \tag{104}$$

$$\|\mathbf{H}\|_{\mathcal{H}^{s+1/2}_{\mathrm{div}}(\Omega)} \le C_{\mathbf{H}}(\|q\|_{\Sigma_\tau} + \|\mathbf{q}\|_{\Sigma_\nu} + \|\mathbf{j}\|_\Omega). \tag{105}$$

Here $C_{\mathbf{u}}$ and $C_{\mathbf{H}}$ are dimensionless constants defined by

$$C_{\mathbf{u}} = \max\{C_\Sigma\mathcal{R}, \frac{2C_1}{\delta_1\sqrt{\mathcal{P}m}}\},\ C_{\mathbf{H}} = \max\{C_\Sigma(1 + C_1\sqrt{\mathcal{P}m}\mathcal{R}), 2\frac{C_1^2}{\delta_1}\}. \tag{106}$$

Based on (17), we now derive a similar estimate for the difference $p = p_1 - p_2$. In view of (17) for function p and for any (small enough) number $\delta > 0$, there exists a function $\mathbf{v}_0 \in H^1_0(\Omega)^3$, $\mathbf{v}_0 \ne \mathbf{0}$ such that

$$-(\mathrm{div}\mathbf{v}_0, p) \ge \beta_0\|\mathbf{v}_0\|_{1,\Omega}\|p\|_\Omega,\ \ \beta_0 = (\beta - \delta) > 0. \tag{107}$$

Setting $\mathbf{v} = \mathbf{v}_0$, $\mathbf{\Psi} = \mathbf{0}$ in (86), we obtain

$$\nu(\nabla\mathbf{u}, \nabla\mathbf{v}_0) + [((\mathbf{u}\cdot\nabla)\mathbf{u}_1, \mathbf{v}_0) + ((\mathbf{u}_2\cdot\nabla)\mathbf{u}, \mathbf{v}_0)] - \kappa[(\mathrm{rot}\mathbf{H}_1 \times \mathbf{H}, \mathbf{v}_0) +$$

$$+(\mathrm{rot}\mathbf{H} \times \mathbf{H}_2, \mathbf{v}_0)] - (\mathrm{div}\mathbf{v}_0, p) = 0. \tag{108}$$

Using the previous estimate (107) for $-(\mathrm{div}\mathbf{v}_0, p)$ and (10), (13)–(15) from (108), we deduce that

$$\beta_0\|\mathbf{v}_0\|_{1,\Omega}\|p\|_\Omega \le -(\mathrm{div}\mathbf{v}_0, p) \le \nu\|\mathbf{v}_0\|_{1,\Omega}\|\mathbf{u}\|_{1,\Omega} +$$

$$+2\gamma_0 M^0_{\mathbf{u}}\|\mathbf{v}_0\|_{1,\Omega}\|\mathbf{u}\|_{1,\Omega} + 2\kappa\gamma_1 M^0_{\mathbf{H}}\|\mathbf{v}_0\|_{1,\Omega}\|\mathbf{H}\|_{\mathcal{H}^{s+1/2}_{\mathrm{div}}(\Omega)}. \tag{109}$$

Dividing (109) by $\|\mathbf{v}_0\|_{1,\Omega} \ne 0$ and using (104), (105) yields

$$\|p\|_\Omega \le \beta_0^{-1}[(\nu + 2\gamma_0 M^0_{\mathbf{u}})\|\mathbf{u}\|_{1,\Omega} + 2\kappa\gamma_1 M^0_{\mathbf{H}}\|\mathbf{H}\|_{\mathcal{H}^{s+1/2}_{\mathrm{div}}(\Omega)}] =$$

$$= \beta_0^{-1}\delta_0\nu[(\delta_0^{-1} + 2\mathcal{R}e)\|\mathbf{u}\|_{1,\Omega} + 2\sqrt{\kappa}\mathcal{H}a\|\mathbf{H}\|_{\mathcal{H}^{s+1/2}_{\rm div}(\Omega)}]. \quad (110)$$

Taking into account (104) and (105), we obtain the following estimate:

$$\|p\|_\Omega \le C_p\nu\sqrt{\kappa}(\|q\|_{\Sigma_\tau} + \|\mathbf{q}\|_{\Sigma_\nu} + \|\mathbf{j}\|_\Omega). \quad (111)$$

Here C_p is a dimensionless constant defined by

$$C_p \equiv \delta_0\beta_0^{-1}[(\delta_0^{-1} + 2\mathcal{R}e)C_{\mathbf{u}} + 2\mathcal{H}aC_{\mathbf{H}}]. \quad (112)$$

Remark 3. *We note two peculiarities of the mathematical apparatus used in this paper. On the one hand, we use the dimensional MHD system (1)–(3) so that all formulae used or obtained in the paper are dimensional. On the other hand, we essentially use three dimensionless parameters $\mathcal{R}e$, $\mathcal{H}a$, $\mathcal{P}m$ defined in (79) and three dimensionless constants $C_{\mathbf{u}}$, $C_{\mathbf{H}}$, C_p defined in (106), (112). These constants depend on $\mathcal{R}e$, $\mathcal{H}a$ and $\mathcal{P}m$ snf contain important information on the MHD system (1), (2). In particular, using $C_{\mathbf{u}}, C_{\mathbf{H}}$ and C_p, we could write estimates of the norms of differences $\mathbf{u}, \mathbf{H}$ and p via norms of differences $q, \mathbf{q}$ and $\mathbf{j}$ in the simple form (104), (105) and (111). We recall that these estimates hold under the condition (82). Based on these estimates and Theorem 6, in Section 5, we establish similar stability estimates for a number of specific control problems for the MHD system (1)–(3) under study.*

5. Analysis of Uniqueness and Stability for Solutions to Control Problems

We begin with consideration of the case when $I = I_1$ in (40), i.e., we consider control problem

$$J(\mathbf{x}, u) \equiv \frac{\mu_0}{2}\|\mathbf{u} - \mathbf{u}_d\|^2_Q + \frac{\mu_1}{2}\|q\|^2_{s,\Sigma_\tau} + \frac{\mu_2}{2}\|\mathbf{q}\|^2_{s,\Sigma_\nu} + \frac{\mu_3}{2}\|\mathbf{j}\|^2_\Omega \to \inf,$$
$$F(\mathbf{x}, u) = 0,\ (\mathbf{x}, u) \in X \times K. \quad (113)$$

Here $\mathbf{x} = (\mathbf{u}, \mathbf{H}, p)$, $u = (q, \mathbf{q}, \mathbf{j})$. Denote by $(\mathbf{x}_1, u_1) \equiv (\mathbf{u}_1, \mathbf{H}_1, p_1, q_1, \mathbf{q}_1, \mathbf{j}_1)$ a solution to problem (113) that corresponds to the function $\mathbf{u}_d \equiv \mathbf{u}_d^{(1)} \in L^2(Q)^3$. By $(\mathbf{x}_2, u_2) \equiv (\mathbf{u}_2, \mathbf{H}_2, p_2, q_2, \mathbf{q}_2, \mathbf{j}_2)$, we denote a solution to problem (113) that corresponds to another function $\tilde{\mathbf{u}}_d \equiv \mathbf{u}_d^{(2)} \in L^2(Q)^3$.

We define dimensionless parameter $\mathcal{R}e^0$ (Reynolds number for the data $\mathbf{u}_d^{(i)}$) by

$$\mathcal{R}e^0 = (\gamma_0/\delta_0\nu l)\max(\|\mathbf{u}_d^{(1)}\|_Q, \|\mathbf{u}_d^{(2)}\|_Q) \quad (114)$$

where l is a dimensional constant defined in Section 2. We assume that the data for problem (113) or parameters μ_0, μ_1, μ_2 and μ_3 are such that the following condition with some sufficiently small $\varepsilon > 0$ takes place:

$$\min\{(1-\varepsilon)\mu_1, (1-\varepsilon)\mu_2, (1-\varepsilon)\mu_3\} \ge 12\mu_0\kappa\gamma(\mathcal{R}e + \mathcal{R}e^0)(\gamma_0C^2_{\mathbf{u}} + \gamma_1C^2_{\mathbf{H}} + \gamma_1\sqrt{\mathcal{P}m}C_{\mathbf{u}}C_{\mathbf{H}}). \quad (115)$$

Here $C_{\mathbf{u}}$, $C_{\mathbf{H}}$ are dimensionless constants defined in (106).

Lemma 6. *Let, under assumptions (i)–(iii), (j) and (82), a pair $(\mathbf{x}_i, u_i) = (\mathbf{u}_i, \mathbf{H}_i, p_i, q_i, \mathbf{q}_i, \mathbf{j}_i)$ be a solution to problem (113) corresponding to function $\mathbf{u}_d^{(i)} \in L^2(Q)^3$, $i = 1, 2$, where $Q \subset \Omega$ is an arbitrary nonempty open subset. Assume that condition (115) is satisfied. Then the following estimate for $\mathbf{u} \equiv \mathbf{u}_1 - \mathbf{u}_2$ holds:*

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_Q \le \Delta \equiv \|\mathbf{u}_d^{(1)} - \mathbf{u}_d^{(2)}\|_Q. \quad (116)$$

Proof of Lemma 6. Setting $\mathbf{u}_d = \mathbf{u}_d^{(1)} - \mathbf{u}_d^{(2)}$, in addition to (85), we have that $(I_1)'_p = 0$, $(I_1)'_{\mathbf{H}} = 0$,

$$\langle (I_1)'_{\mathbf{u}}(\mathbf{x}_i), \mathbf{w}\rangle = 2(\mathbf{u}_i - \mathbf{u}_d^{(i)}, \mathbf{w})_Q,\ \langle (I_1)'_{\mathbf{u}}(\mathbf{x}_1) - (\tilde{I}_1)'_{\mathbf{u}}(\mathbf{x}_2), \mathbf{u}\rangle =$$

$$= 2(\mathbf{u} - \mathbf{u}_d, \mathbf{u})_Q = 2(\|\mathbf{u}\|_Q^2 - (\mathbf{u}, \mathbf{u}_d)_Q). \tag{117}$$

In view of (117), identities (83), (84) for adjoint states $\mathbf{y}_i^* = (\xi_i, \eta_i, \sigma_i, \zeta_1^{(i)}, \zeta_2^{(i)}, \zeta_3^{(i)})$, $i = 1, 2$, corresponding to solutions $(\mathbf{u}_i, \mathbf{H}_i, p_i, u_i)$ and the main inequality (91) for differences $\mathbf{u}$, $\mathbf{H}$, p, q, $\mathbf{q}$, $\mathbf{j}$ defined in (85), take by Remark 2 the form

$$\nu(\nabla\mathbf{w}, \nabla\xi_i) + \nu_1(\mathrm{rot}\mathbf{h}, \mathrm{rot}\eta_i) + ((\mathbf{u}_i\cdot\nabla)\mathbf{w}, \xi_i) + ((\mathbf{w}\cdot\nabla)\mathbf{u}_i, \xi_i) + \kappa[(\mathrm{rot}\eta_i \times \mathbf{H}_i, \mathbf{w}) +$$
$$+ (\mathrm{rot}\eta_i \times \mathbf{h}, \mathbf{u}_i)] - \kappa[(\mathrm{rot}\mathbf{H}_i \times \mathbf{h}, \xi_i) + (\mathrm{rot}\mathbf{h} \times \mathbf{H}_i, \xi_i)] - (\mathrm{div}\mathbf{w}, \sigma_i) + \langle \zeta_1^{(i)}, \mathbf{w}\rangle_\Sigma +$$
$$+ \langle \zeta_2^{(i)}, \mathbf{h}\cdot\mathbf{n}\rangle_{\Sigma_\tau} + \langle \zeta_3^{(i)}, \mathbf{h}\times\mathbf{n}\rangle_{\Sigma_\nu} =$$
$$= -\mu_0(\mathbf{u}_i - \mathbf{u}_d^{(i)}, \mathbf{w})_Q\ \forall(\mathbf{w}, \mathbf{h}) \in H_T^1(\Omega)\times\mathcal{H}_{\mathrm{div}}^{s+1/2}(\Omega);\ \xi_i \in V,\ i = 1, 2, \tag{118}$$

$$((\mathbf{u}\cdot\nabla)\mathbf{u}, \xi_1 + \xi_2) - \kappa(\mathrm{rot}\mathbf{H}\times\mathbf{H}, \xi_1 + \xi_2) + \kappa(\mathrm{rot}(\eta_1 + \eta_2)\times\mathbf{H}, \mathbf{u}) +$$
$$+ \mu_0(\|\mathbf{u}\|_Q^2 - (\mathbf{u}, \mathbf{u}_d)_Q) \le -\mu_1\|q\|_{s,\Sigma_\tau}^2 - \mu_2\|\mathbf{q}\|_{s,\Sigma_\nu}^2 - \mu_3\|\mathbf{j}\|_\Omega^2. \tag{119}$$

Our nearest purpose is to estimate adjoint state variables ξ_i and η_i via $(\mathcal{R}e + \mathcal{R}e^0)$. To this end, we set $\mathbf{w} = \xi_i$, $\mathbf{h} = \eta_i$ in (118). Using (16) and conditions $\mathrm{div}\xi_i = 0$ in Ω, $\xi_i|_\Sigma = \mathbf{0}$, $\eta_i\cdot\mathbf{n}|_{\Sigma_\tau} = 0$ and $\eta_i\times\mathbf{n}|_{\Sigma_\nu} = \mathbf{0}$, we obtain

$$\nu(\nabla\xi_i, \nabla\xi_i) + \nu_1(\mathrm{rot}\eta_i, \mathrm{rot}\eta_i) + ((\xi_i\cdot\nabla)\mathbf{u}_i, \xi_i) + \kappa(\mathrm{rot}\eta_i\times\eta_i, \mathbf{u}_i) -$$
$$-\kappa(\mathrm{rot}\mathbf{H}_i\times\eta_i, \xi_i) = -\mu_0(\mathbf{u}_i - \mathbf{u}_d^{(i)}, \xi_i)_Q. \tag{120}$$

Taking into account estimates (10)–(15), (77) and (79), (80), (114), we have

$$(\nabla\xi_i, \nabla\xi_i) \ge \delta_0\|\xi_i\|_{1,\Omega}^2,\ |((\xi_i\cdot\nabla)\mathbf{u}_i, \xi_i)| \le \gamma_0\|\mathbf{u}_i\|_{1,\Omega}\|\xi_i\|_{1,\Omega}^2 \le \gamma_0 M_{\mathbf{u}}^0\|\xi_i\|_{1,\Omega}^2, \tag{121}$$

$$\kappa|(\mathrm{rot}\mathbf{H}_i\times\eta_i, \xi_i)| \le \gamma_1\kappa M_{\mathbf{H}}^0\|\eta_i\|_{1,\Omega}\|\xi_i\|_{1,\Omega} \le \gamma_1(\sqrt{\kappa}/2)M_{\mathbf{H}}^0(\|\xi_i\|_{1,\Omega}^2 + \kappa\|\eta_i\|_{1,\Omega}^2), \tag{122}$$

$$(\mathrm{rot}\eta_i, \mathrm{rot}\eta_i) \ge \delta_1\|\eta_i\|_{1,\Omega}^2,\ |(\mathrm{rot}\eta_i\times\eta_i, \mathbf{u}_i)| \le \gamma_1 M_{\mathbf{u}}^0\|\eta_i\|_{1,\Omega}^2, \tag{123}$$

$$\|\mathbf{u}_i - \mathbf{u}_d^{(i)}\|_Q \le \|\mathbf{u}_i\|_Q + \|\mathbf{u}_d^{(i)}\|_Q \le lM_{\mathbf{u}}^0 + \|\mathbf{u}_d^{(i)}\|_Q \le \delta_0\nu l\gamma_0^{-1}(\mathcal{R}e + \mathcal{R}e^0). \tag{124}$$

In virtue of (121)–(124) and (99), we infer from (120) that

$$(\delta_0\nu/2)\|\xi_i\|_{1,\Omega}^2 + (\delta_1\nu_m/2)\kappa\|\eta_i\|_{1,\Omega}^2 \le [\delta_0\nu - \gamma_0 M_{\mathbf{u}}^0 - \gamma_1(\sqrt{\kappa}/2)M_{\mathbf{H}}^0]\|\xi_i\|_{1,\Omega}^2 +$$
$$+ [\delta_1\nu_m - \gamma_1 M_{\mathbf{u}}^0 - \gamma_1(\sqrt{\kappa}/2)M_{\mathbf{H}}^0]\kappa\|\eta_i\|_{1,\Omega}^2 \le$$
$$\le \mu_0 l(lM_{\mathbf{u}}^0 + \|\mathbf{u}_d^{(i)}\|_Q)\|\xi_i\|_{1,\Omega} \le \mu_0\delta_0\nu\gamma(\mathcal{R}e + \mathcal{R}e^0)\|\xi_i\|_{1,\Omega}$$

where $\gamma = l^2\gamma_0^{-1}$. Using (79) and (124), we conclude from the last inequality that

$$\|\xi_i\|_{1,\Omega} \le 2\mu_0\gamma(\mathcal{R}e + \mathcal{R}e^0),\ \ \sqrt{\kappa}\|\eta_i\|_{1,\Omega} \le 2\mu_0\gamma\sqrt{\mathcal{P}m}(\mathcal{R}e + \mathcal{R}e^0),\ \ i = 1, 2. \tag{125}$$

Using (13)–(15), (104), (105), (115) and (125), we have

$$|((\mathbf{u}\cdot\nabla)\mathbf{u}, \xi_1 + \xi_2) - \kappa(\mathrm{rot}\mathbf{H}\times\mathbf{H}, \xi_1 + \xi_2) + \kappa(\mathrm{rot}(\eta_1 + \eta_2)\times\mathbf{H}, \mathbf{u})| \le$$
$$\le \gamma_0\|\mathbf{u}\|_{1,\Omega}^2(\|\xi_1\|_{1,\Omega} + \|\xi_2\|_{1,\Omega}) + \gamma_1\kappa\|\mathbf{H}\|_{\mathcal{H}_{\mathrm{div}}^{s+1/2}(\Omega)}^2(\|\xi_1\|_{1,\Omega} + \|\xi_2\|_{1,\Omega}) +$$

$$+\gamma_1\kappa\|\mathbf{H}\|_{\mathcal{H}^{s+1/2}_{\text{div}}(\Omega)}\|\mathbf{u}\|_{1,\Omega}(\|\eta_1\|_{1,\Omega}+\|\eta_2\|_{1,\Omega})\le$$

$$\le 12\mu_0\kappa\gamma(\mathcal{R}e+\mathcal{R}e^0)(\gamma_0 C_{\mathbf{u}}^2+\gamma_1 C_{\mathbf{H}}^2+\gamma_1\sqrt{\mathcal{P}m}C_{\mathbf{u}}C_{\mathbf{H}})(\|q\|^2_{s,\Sigma_\tau}+\|\mathbf{q}\|^2_{s,\Sigma_\nu}+\|\mathbf{j}\|^2_\Omega\le$$

$$\le(1-\varepsilon)\mu_1\|q\|^2_{s,\Sigma_\tau}+(1-\varepsilon)\mu_2\|\mathbf{q}\|^2_{s,\Sigma_\nu}+(1-\varepsilon)\mu_3\|\mathbf{j}\|^2_\Omega. \tag{126}$$

Taking into account (126) from (119), we arrive at

$$\mu_0(\|\mathbf{u}\|^2_Q-(\mathbf{u},\mathbf{u}_d)_Q)\le-((\mathbf{u}\cdot\nabla)\mathbf{u},\xi_1+\xi_2)+\kappa(\text{rot}\mathbf{H}\times\mathbf{H},\xi_1+\xi_2)-$$

$$-\kappa(\text{rot}(\eta_1+\eta_2)\times\mathbf{H},\mathbf{u})-\mu_1\|q\|^2_{\Sigma_\tau}-\mu_2\|\mathbf{q}\|^2_{\Sigma_\nu}-\mu_3\|\mathbf{j}\|^2_\Omega\le$$

$$\le-\varepsilon\mu_1\|q\|^2_{s,\Sigma_\tau}-\varepsilon\mu_2\|\mathbf{q}\|^2_{s,\Sigma_\nu}-\varepsilon\mu_3\|\mathbf{j}\|^2_\Omega. \tag{127}$$

Omitting the nonpositive term $-\varepsilon\mu_1\|q\|^2_{s,\Sigma_\tau}-\varepsilon\mu_2\|\mathbf{q}\|^2_{s,\Sigma_\nu}-\varepsilon\mu_3\|\mathbf{j}\|^2_\Omega$, we derive from (127) that $\|\mathbf{u}\|^2_Q-\|\mathbf{u}\|_Q\|\mathbf{u}_d\|_Q\le 0$ or

$$\|\mathbf{u}\|_Q\le\mathbf{u}_d\|_Q. \tag{128}$$

Since $\mathbf{u}=\mathbf{u}_1-\mathbf{u}_2$, $\mathbf{u}_d=\mathbf{u}_d^{(1)}-\mathbf{u}_d^{(2)}$, Lemma 6 is proved. □

If $Q=\Omega$, the estimate (116) is the stability estimate in $L^2(\Omega)^3$ norm for the component $\hat{\mathbf{u}}$ of the solution $(\hat{\mathbf{u}},\hat{\mathbf{H}},\hat{p},\hat{q},\hat{\mathbf{q}},\hat{\mathbf{j}})$ of problem (113) with respect to small disturbances of function $\mathbf{u}_d\in L^2(\Omega)^3$ in the norm of $L^2(\Omega)^3$. The same estimate was obtained in viscous hydrodynamics [16]. Additionally, if $\mathbf{u}_d^{(1)}=\mathbf{u}_d^{(2)}$ we conclude from (116) that $\mathbf{u}_1=\mathbf{u}_2$ in Q. This yields together with (127) and (86) that $q_1=q_2$, $\mathbf{q}_1=\mathbf{q}_2$, $\mathbf{j}_1=\mathbf{j}_2$. In turn, it follows from this fact and (104), (105), (111) that $\mathbf{u}_1=\mathbf{u}_2$, $\mathbf{H}_1=\mathbf{H}_2$, $p_1=p_2$ in Ω. The latter is equivalent to the uniqueness for the solution of (113).

We cannot prove the estimates for differences $\mathbf{H}=\mathbf{H}_1-\mathbf{H}_2$ and $p=p_1-p_2$, which are analogous to (116). Based on (128), however, one can obtain coarser stability estimates for all differences $\mathbf{u},\mathbf{H}$ and p even in situations in which $Q\Subset\Omega$, i.e., Q is only a part of Ω. Indeed, let us consider inequality (127). Using (128), we deduce from (127) that

$$\varepsilon\mu_1\|q\|^2_{s,\Sigma_\tau}+\varepsilon\mu_2\|\mathbf{q}\|^2_{s,\Sigma_\nu}+\varepsilon\mu_3\|\mathbf{j}\|^2_\Omega\le\mu_0(-\|\mathbf{u}\|^2_Q+\|\mathbf{u}\|_Q\|\mathbf{u}_d\|_Q)\le\mu_0\|\mathbf{u}_d\|^2_Q=\mu_0\Delta^2 \tag{129}$$

where Δ is defined in (116). From (104), (105), (111) and (129), we arrive at

$$\|q_1-q_2\|_{s,\Sigma_\tau}\le\sqrt{\mu_0/\varepsilon\mu_1}\Delta,\ \|q_1-q_2\|_{s,\Sigma_\nu}\le\sqrt{\mu_0/\varepsilon\mu_2}\Delta,\ \|\mathbf{j}_1-\mathbf{j}_2\|_\Omega\le\sqrt{\mu_0/\varepsilon\mu_3}\Delta,$$

$$\|\mathbf{u}_1-\mathbf{u}_2\|_{1,\Omega}\le C_{\mathbf{u}}\sqrt{\kappa}(\sqrt{\mu_0/\varepsilon\mu_1}+\sqrt{\mu_0/\varepsilon\mu_2}+\sqrt{\mu_0/\varepsilon\mu_3})\Delta,$$

$$\|\mathbf{H}_1-\mathbf{H}_2\|_{\mathcal{H}^{s+1/2}_{\text{div}}(\Omega)}\le C_{\mathbf{H}}(\sqrt{\mu_0/\varepsilon\mu_1}+\sqrt{\mu_0/\varepsilon\mu_2}+\sqrt{\mu_0/\varepsilon\mu_3})\Delta,$$

$$\|p_1-p_2\|_\Omega\le C_p\nu\sqrt{\kappa}(\sqrt{\mu_0/\varepsilon\mu_1}+\sqrt{\mu_0/\varepsilon\mu_2}+\sqrt{\mu_0/\varepsilon\mu_3})\Delta. \tag{130}$$

Let us describe the obtained result as the theorem:

Theorem 7. *Let parameter $\mathcal{R}e^0$ be defined by relation (114) and let assumptions of Lemma 6 be fulfilled. Then the stability estimates (130) for problem (113) hold where $\Delta=\|\mathbf{u}_d^{(1)}-\mathbf{u}_d^{(2)}\|_Q$.*

We now study the uniqueness and stability for solutions to the control problem (40) in the case when $I=I_2$, i.e., we consider control problem

$$J(\mathbf{x},u)\equiv\frac{\mu_0}{2}\|\mathbf{H}-\mathbf{H}_d\|^2_Q+\frac{\mu_1}{2}\|q\|^2_{s,\Sigma_\tau}+\frac{\mu_2}{2}\|\mathbf{q}\|^2_{s,\Sigma_\nu}+\frac{\mu_2}{2}\|\mathbf{j}\|^2_\Omega\to\inf,$$

$$F(\mathbf{x},u)=0,\ (\mathbf{x},u)\in X\times K. \tag{131}$$

Here, as usual, $\mathbf{x}=(\mathbf{u},\mathbf{H},p)$, $u=(q,\mathbf{q},\mathbf{j})$. Denote by $(\mathbf{x}_1,u_1)\equiv(\mathbf{u}_1,\mathbf{H}_1,p_1,q_1,\mathbf{q}_1,\mathbf{j}_1)$ a solution to the problem (131) that corresponds to the function $\mathbf{H}_d\equiv\mathbf{H}_d^{(1)}\in L^2(Q)^3$. By

$(\mathbf{x}_2, u_2) \equiv (\mathbf{u}_2, \mathbf{H}_2, p_2, q_2, \mathbf{q}_2, \mathbf{j}_2)$, we denote the solution to problem (131) that corresponds to another function $\tilde{\mathbf{H}}_d \equiv \mathbf{H}_d^{(2)} \in L^2(Q)^3$.

Let us define a parameter γ and dimensionless Hartman number $\mathcal{H}a^0$ for the data $\mathbf{H}_d^{(i)}$ by

$$\gamma = l^2\gamma_1^{-1}, \;\; \mathcal{H}a^0 = (\gamma_1\sqrt{\kappa}/\delta_0\nu l)\max(\|\mathbf{H}_d^{(1)}\|_Q, \|\mathbf{H}_d^{(2)}\|_Q). \tag{132}$$

We assume that the data for problem (131) or parameters μ_0, μ_1, μ_2 and μ_3 are such that the following condition with some sufficiently small $\varepsilon > 0$ takes place:

$$\min\{(1-\varepsilon)\mu_1, (1-\varepsilon)\mu_2, (1-\varepsilon)\mu_3\}$$
$$\geq 12\mu_0\gamma\sqrt{\mathcal{P}m}(\mathcal{H}a + \mathcal{H}a^0)(\gamma_0 C_{\mathbf{u}}^2 + \gamma_1 C_{\mathbf{H}}^2 + \gamma_1\sqrt{\mathcal{P}m}C_{\mathbf{u}}C_{\mathbf{H}}). \tag{133}$$

Lemma 7. *Let, under assumptions (i)–(iii), (j) and (82), a pair* $(\mathbf{x}_i, u_i) = (\mathbf{u}_i, \mathbf{H}_i, p_i, q_i, \mathbf{q}_i, \mathbf{j}_i)$ *be a solution to problem (131) that corresponds to the function* $\mathbf{H}_d^{(i)} \in L^2(Q)^3$, $i = 1,2$, *where* $Q \subset \Omega$ *is an arbitrary nonempty open subset. Assume that the condition (133) is satisfied. Then the following estimate for difference* $\mathbf{H} \equiv \mathbf{H}_1 - \mathbf{H}_2$ *holds:*

$$\|\mathbf{H}_1 - \mathbf{H}_2\|_Q \leq \Delta, \;\; \Delta \equiv \|\mathbf{H}_d^{(1)} - \mathbf{H}_d^{(2)}\|_Q. \tag{134}$$

Proof of Lemma 7. Setting $\mathbf{H}_d = \mathbf{H}_d^{(1)} - \mathbf{H}_d^{(2)}$, in addition to (85), we have that $(I_2)'_{\mathbf{u}} = 0$, $(I_2)'_p = 0$,

$$\langle (I_2)'_{\mathbf{H}}(\mathbf{x}_i), \mathbf{h}\rangle = 2(\mathbf{H}_i - \mathbf{H}_d^{(i)}, \mathbf{h})_Q, \; \langle (I_2)'_{\mathbf{H}}(\mathbf{x}_1) - (\tilde{I}_2)'_{\mathbf{H}}(\mathbf{x}_2), \mathbf{H}\rangle = 2(\|\mathbf{H}\|_Q^2 - (\mathbf{H}, \mathbf{H}_d)_Q). \tag{135}$$

In view of (135), identities (83), (84) and the main inequality (91) become

$$\nu(\nabla\mathbf{w}, \nabla\xi_i) + \nu_1(\text{rot}\mathbf{h}, \text{rot}\eta_i) + ((\mathbf{u}_i\cdot\nabla)\mathbf{w}, \xi_i) + ((\mathbf{w}\cdot\nabla)\mathbf{u}_i, \xi_i) + \kappa[(\text{rot}\eta_i \times \mathbf{H}_i, \mathbf{w}) +$$
$$+ (\text{rot}\eta_i \times \mathbf{h}, \mathbf{u}_i)] - \kappa[(\text{rot}\mathbf{H}_i \times \mathbf{h}, \xi_i) + (\text{rot}\mathbf{h} \times \mathbf{H}_i, \xi_i)] - (\text{div}\mathbf{w}, \sigma_i) + \langle \zeta_1^{(i)}, \mathbf{w}\rangle_\Sigma +$$
$$+ \langle \zeta_2^{(i)}, \mathbf{h}\cdot\mathbf{n}\rangle_{\Sigma_\tau} + \langle \zeta_3^{(i)}, \mathbf{h}\times\mathbf{n}\rangle_{\Sigma_\nu} =$$
$$= -\mu_0(\mathbf{H}_i - \mathbf{H}_d^{(i)}, \mathbf{h})_Q \;\forall(\mathbf{w}, \mathbf{h}) \in H_T^1(\Omega) \times \mathcal{H}_{\text{div}}^{s+1/2}(\Omega);\; \xi_i \in V,\; i = 1,2, \tag{136}$$

$$((\mathbf{u}\cdot\nabla)\mathbf{u}, \xi_1 + \xi_2) - \kappa(\text{rot}\mathbf{H} \times \mathbf{H}, \xi_1 + \xi_2) + \kappa(\text{rot}(\eta_1 + \eta_2) \times \mathbf{H}, \mathbf{u}) +$$
$$+ \mu_0(\|\mathbf{H}\|_Q^2 - (\mathbf{H}, \mathbf{H}_d)_Q) \leq -\mu_1\|q\|_{s,\Sigma_\tau}^2 - \mu_2\|\mathbf{q}\|_{s,\Sigma_\nu}^2 - \mu_3\|\mathbf{j}\|_\Omega^2. \tag{137}$$

We firstly estimate adjoint state variables ξ_i and η_i via $(\mathcal{H}a + \mathcal{H}a^0)$. To this end, we set $\mathbf{w} = \xi_i$, $\mathbf{h} = \eta_i$ in (136). In virtue of (16) and conditions $\xi_i|_\Sigma = \mathbf{0}$, $\eta_i\cdot\mathbf{n}|_{\Sigma_\tau} = 0$ and $\eta_i \times \mathbf{n}|_{\Sigma_\nu} = \mathbf{0}$, we obtain

$$\nu(\nabla\xi_i, \nabla\xi_i) + \nu_1(\text{rot}\eta_i, \text{rot}\eta_i) + ((\xi_i\cdot\nabla)\mathbf{u}_i, \xi_i) + \kappa(\text{rot}\eta_i \times \eta_i, \mathbf{u}_i)$$
$$-\kappa(\text{rot}\mathbf{H}_i \times \eta_i, \xi_i) = -\mu_0(\mathbf{H}_i - \mathbf{H}_d^{(i)}, \eta_i)_Q. \tag{138}$$

Using (79), (80), (132), we derive, in addition to (121)–(123), that

$$\|\mathbf{H}_i - \mathbf{H}_d^{(i)}\|_Q \leq l\|\mathbf{H}_i\|_{\mathcal{H}_{\text{div}}^{s+1/2}(\Omega)} + \|\mathbf{H}_d^{(i)}\|_Q \leq l(M_{\mathbf{H}}^0 + l^{-1}\|\mathbf{H}_d^{(i)}\|_Q) \leq$$
$$\leq \delta_0\nu l(\mathcal{H}a + \mathcal{H}a^0)/(\gamma_1\sqrt{\kappa}). \tag{139}$$

Taking into account (99), (121)–(123) and (139), we infer from (138) that

$$(\delta_0\nu/2)\|\xi_i\|_{1,\Omega}^2 + (\delta_1\nu_m/2)\kappa\|\eta_i\|_{1,\Omega}^2 \leq \mu_0(\|\mathbf{H}_i\|_Q) + \|\mathbf{H}_d^{(i)}\|_Q)\|\eta_i\|_Q \leq$$

$$\leq \mu_0 l^2 (M^0_{\mathbf{H}} + l^{-1} \|\mathbf{H}_d^{(i)}\|_Q) \|\eta_i\|_{1,\Omega} \leq \mu_0 \delta_0 \nu \gamma \kappa^{-1/2} (\mathcal{H}a + \mathcal{H}a^0) \|\eta_i\|_{1,\Omega}$$

where $\gamma = l^2 \gamma_1^{-1}$. Using (79) and (139), we conclude from the last inequality that

$$\|\xi_i\|_{1,\Omega} \leq 2\mu_0 \gamma \kappa^{-1} \sqrt{\mathcal{P}m} (\mathcal{H}a + \mathcal{H}a^0), \quad \sqrt{\kappa} \|\eta_i\|_{1,\Omega} \leq 2\mu_0 \gamma \kappa^{-1} \mathcal{P}m (\mathcal{H}a + \mathcal{H}a^0). \tag{140}$$

Combining (15), (104), (105), (115) and (140) gives the estimate

$$|((\mathbf{u} \cdot \nabla)\mathbf{u}, \xi_1 + \xi_2) - \kappa(\mathrm{rot}\mathbf{H} \times \mathbf{H}, \xi_1 + \xi_2) + \kappa(\mathrm{rot}(\eta_1 + \eta_2) \times \mathbf{H}, \mathbf{u})| \leq$$

$$\leq 12\mu_0 \gamma \sqrt{\mathcal{P}m} (\mathcal{H}a + \mathcal{H}a^0)(\gamma_0 C^2_{\mathbf{u}} + \gamma_1 C^2_{\mathbf{H}} + \gamma_1 \sqrt{\mathcal{P}m} C_{\mathbf{u}} C_{\mathbf{H}})(\|q\|^2_{s,\Sigma_\tau} + \|\mathbf{q}\|^2_{s,\Sigma_\nu} + \|\mathbf{j}\|^2_\Omega) \leq$$

$$\leq (1-\varepsilon)\mu_1 \|q\|^2_{s,\Sigma_\tau} + (1-\varepsilon)\mu_2 \|\mathbf{q}\|^2_{s,\Sigma_\nu} + (1-\varepsilon)\mu_3 \|\mathbf{j}\|^2_\Omega. \tag{141}$$

Using (141) from (137), we arrive at

$$\mu_0(\|\mathbf{H}\|^2_Q - (\mathbf{H}, \mathbf{H}_d)_Q) \leq -((\mathbf{u} \cdot \nabla)\mathbf{u}, \xi_1 + \xi_2) +$$

$$+\kappa(\mathrm{rot}\mathbf{H} \times \mathbf{H}, \xi_1 + \xi_2) - \kappa(\mathrm{rot}(\eta_1 + \eta_2) \times \mathbf{H}, \mathbf{u}) - \mu_1 \|q\|^2_{s,\Sigma_\tau} - \mu_2 \|\mathbf{q}\|^2_{s,\Sigma_\nu} - \mu_3 \|\mathbf{j}\|^2_\Omega \leq$$

$$\leq -\varepsilon\mu_1 \|q\|^2_{s,\Sigma_\tau} - \varepsilon\mu_2 \|\mathbf{q}\|^2_{s,\Sigma_\nu} - \varepsilon\mu_3 \|\mathbf{j}\|^2_\Omega. \tag{142}$$

Omitting the nonpositive term $-\varepsilon\mu_1 \|q\|^2_{s,\Sigma_\tau} - \varepsilon\mu_2 \|\mathbf{q}\|^2_{s,\Sigma_\nu} - \varepsilon\mu_3 \|\mathbf{j}\|^2_\Omega$, we infer from (142) that $\|\mathbf{H}\|^2_Q - \|\mathbf{H}\|_Q \|\mathbf{H}_d\|_Q \leq 0$ or $\|\mathbf{H}\|_Q \leq \|\mathbf{H}_d\|_Q$. □

If $Q = \Omega$, the estimate (134) is the stability estimate in the $L^2(\Omega)^3$ norm for the magnetic component $\hat{\mathbf{H}}$ of the solution $(\hat{\mathbf{u}}, \hat{\mathbf{H}}, \hat{p}, \hat{q}, \hat{\mathbf{q}}, \hat{\mathbf{j}})$ of problem (131) with respect to small disturbances of function $\mathbf{H}_d \in L^2(\Omega)^3$ in (131). If, besides, $\mathbf{H}_d^{(1)} = \mathbf{H}_d^{(2)}$ in Q it follows from (134) that $\mathbf{H}_1 = \mathbf{H}_2$ in Q. This, together with (104), (105), (111) and (142), yields that $q_1 = q_2$, $\mathbf{q}_1 = \mathbf{q}_2$, $\mathbf{j}_1 = \mathbf{j}_2$, $\mathbf{u}_1 = \mathbf{u}_2$, $\mathbf{H}_1 = \mathbf{H}_2$, $p_1 = p_2$. The latter is equivalent to the uniqueness for the solution of (131).

Again, we note that we cannot prove the estimates for differences $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$ and $p = p_1 - p_2$, which are analogous to (134), but we can obtain coarser estimates for $\mathbf{u}, \mathbf{H}$ and p even if Q is only a part of Ω. Indeed, using (134), we infer from (142) that

$$\varepsilon\mu_1 \|q\|^2_{s,\Sigma_\tau} + \varepsilon\mu_2 \|\mathbf{q}\|_{s,\Sigma_\nu} + \varepsilon\mu_3 \|\mathbf{j}\|^2_\Omega \leq \mu_0(-\|\mathbf{H}\|^2_Q + \|\mathbf{H}\|_Q \|\mathbf{H}_d\|_Q)$$

$$\leq \mu_0(-\|\mathbf{H}\|^2_Q + \|\mathbf{H}\|_Q \|\mathbf{H}_d\|_Q) \leq \mu_0 \|\mathbf{H}_d\|^2_Q = \mu_0 \Delta^2 \tag{143}$$

where Δ is defined in (134). From (104), (105), (111) and (143), we come to the required stability estimates having the form (130). Thus, the following result was proved.

Theorem 8. *Let parameters γ and $\mathcal{H}a^0$ be defined by relations (132) and let conditions of Lemma 7 be fulfilled. Then the stability estimates (130) for control problem (131) hold where $\Delta = \|\mathbf{H}_d^{(1)} - \mathbf{H}_d^{(2)}\|_Q$.*

Using an analogous scheme, similar theorems can be proved for control problems corresponding to weak regularizers $(\mu_1/2)\|q\|^2_{\Sigma_\tau}$ and $(\mu_2/2)\|\mathbf{q}\|^2_{\Sigma_\nu}$ as in (53), or to the third cost functional $I_s(p)$ defined in (37). We leave the formulations and proofs of the corresponding theorems to the reader.

6. Conclusions

In this paper, the optimal control problems for stationary magnetohydrodynamic equations considered under the inhomogeneous mixed boundary conditions for a magnetic field and the Dirichlet condition for velocity were analyzed. The normal and tangential magnetic components at different parts of the boundary and the exterior current density within the flow region are the controls. A general theory of investigation of the control problems with special tracking-type cost functionals has been developed and applied for studying the uniqueness and stability for optimal solutions.

The use of this theory made it possible to prove the existence of solutions of control problems in the wide class of cost functionals, to deduce an optimality system and, based on its analysis, to prove theorems on local uniqueness and stability of optimal solutions for a number of tracking-type functionals. In particular, for problems (113) and (131), it is shown that the uniqueness as well as stability of their solutions occurs, provided the conditions (115) or (133) for the data are accordingly fulfilled. This means that the presence of summands with squared norms of controls, with positive coefficients μ_1, μ_2, μ_3 in the expressions of functionals to be minimized in (113) and in (131), contributes a regularizing effect to the control problems under study.

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