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Mathematical and Statistical Aspects of Estimating Small Oscillations Parameters in a Conservative Mechanical System Using Inaccurate Observations

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Abstract: This paper selects a set of reference points in the form of an arithmetic progression for planning an experiment to evaluate the parameters of systems of differential equations. This choice makes it possible to construct estimates of the parameters of a system of first-order differential equations based on the reversibility of the observation matrix, as well as estimates of the parameters of a system of second-order differential equations describing vibrations in a mechanical system by switching to a system of first-order differential equations. In turn, the reversibility of the observation matrix used in parameter estimation is established using the Vandermonde formula. A volumetric computational experiment has been carried out showing how, with an increase in the number of observations in the vicinity of reference points and with a decrease in the step of arithmetic progression, the accuracy of estimates of the parameters of the analyzed system increases. Among the estimated parameters, the most important are the oscillation frequencies of a conservative mechanical system, which establish its proximity to resonance, and therefore, determine the stability and reliability of the system.

Keywords: reference points; experiment planning; Vandermonde determinant; matrix exponents; diagonal matrices

MSC: 60J28

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1. Introduction

In [1], an algorithm was constructed for estimating the parameters of a system of first-order ordinary differential equations by a large number of inaccurate observations in the vicinity of one selected point. We will further call such points reference points. Estimates of the solution of the system and its derivative at the reference point are constructed by the method of linear regression analysis. According to them, estimates of the system parameters are determined by the method of moments, and their consistency is proved. However, the development of this topic requires the selection of several reference points for estimating the parameters of a linear system of ordinary differential equations. Such a problem arises, for example, if the number of unknown parameters is greater than the number of equations in the system and if the order of the system is higher than the first.

The estimation of the parameters of a system of linear differential equations with constant coefficients based on inaccurate observations of its solution is of particular interest. The choice of reference points forming a finite arithmetic progression allows us, in this case, to construct ratios for evaluating the elements of the coefficient matrix, including the inverse matrix, the square root of the matrix, etc. [2–4]. At the same time, it turns out that the Vandermonde determinant plays the main role in the circulation of the matrices considered in the paper. The reversibility of the Vandermonde matrix requires the realness and difference of the eigenvalues of the coefficients matrix in the system of equations.

However, consideration of a system of first-order differential equations is insufficient in the study of oscillatory dynamical systems, described by systems of the second order. Meanwhile, the extension of the problem of estimating the parameters of systems by inaccurate data to the class of oscillatory systems is of interest. For example, it is an inverse problem of oscillations theory [5,6] when it is necessary to restore the parameters of a distributed system by one or another full-scale data. This formulation of the question is closely related to the models of mechanical systems in the problems of mechatronics and robotics, which have received great development and dissemination in recent years [7–11]. This task is also closely related to the problems of technical systems reliability, containing similar mechanical components. If we talk about reliability, then the issues related to the possibility of resonance [12–16] are of particular interest. Therefore, it is desirable to obtain more accurate estimates of the parameters of such differential equations systems, especially frequencies, which play an important role in the analysis of resonant phenomena.

Direct reduction in general oscillatory systems to first-order systems (in the case of linear systems with constant coefficients) leads to matrices whose spectrum can contain multiples of eigenvalues. The reversal of the matrices arising, in this case, requires additional research. In the case of conservative mechanical systems, this leads to a matrix inversion algorithm based on the Vandermonde determinant.

In this paper, we consider two problems of estimating the parameters of systems of first and second-order differential equations using inaccurate observations from analytical and computational points of view. The solution to both problems is reduced to matrix calculations, in which the reversibility of the corresponding matrices plays an important role. Moreover, an important element of solving these problems is the choice of reference points in the vicinity of which numerous measurements are made. These points form an arithmetic progression, which makes it possible to use the Vandermonde determinant to estimate the parameters of the models under consideration. For models of mechanical systems, this problem is solved by switching from a system of second-order differential equations to a system of first-order equations and estimating the parameters of this system by inaccurate observations. Analytical calculations are supplemented by computational experiments confirming the possibility of using the proposed methods.

The main result of the work is the selection of reference points for estimating parameters, an analytical study of the constructed estimates, and a computational experiment to determine the errors of the estimates obtained. These results can be used to analyze vibrations in mechatronics and robotics systems and to determine the reliability of these systems in terms of their protection from resonance. Estimates of matrices containing solutions of systems at reference points and matrices containing derivatives of solutions at reference points were constructed. Estimates of the coefficient matrices of the systems under consideration were based on them. At the same time, the conditions for the reversibility of the matrices were established using the Vandermonde formula. The constructed estimates of the coefficient matrices are consistent. This is confirmed in the course of computational experiments. The method of parameter estimation proposed in the article allows us not only to build sufficiently accurate estimates of parameters but also to control this accuracy by choosing the necessary number of observations and reference points in the vicinity of which observations are carried out, i.e., ensuring that these estimates are consistent (converging in probability with an increase in the number of observations).

It is worth mentioning that this paper is devoted to the estimation of the frequencies and amplitudes of linear oscillations in a conservative system. This task is complex and requires both the study of the properties of the differential equations themselves, describing a conservative system, and statistical estimates of the oscillation parameters. A large number of papers have been devoted to the study of the properties of such differential equations. Among them, one should point to the classical monograph [2], in which linear Lagrange equations, describing a conservative system, are solved by methods of matrix theory. As for the estimates of the parameters of differential equations of a fairly general form, we should point to the monographs [17–19] and articles developing this direction (see, for example, [20–23]). They give estimates of the parameters of differential equations using least squares error minimization between the response of the model and the actual response of the system. In some cases, the asymptotic normality of the obtained estimates is established.

In this paper, the question of adapting such estimates to differential equations, describing conservative systems, is raised. To do this, we had to move from differential equations in Lagrange variables, studied in [2], to differential equations in Hamilton variables. This made it possible to obtain solutions to the equations in the form of exponential matrices and to establish the conditions for the reversibility of the matrices used to construct statistical estimates. As for the properties of statistical estimates, by constructing consistent estimates of the values of the observed functions and their derivatives, using linear regression analysis for a large number of observations in the vicinity of selected (reference) points, it is possible to construct estimates of matrices, included in linear differential equations, describing conservative systems, and with their help to estimate the frequencies and amplitudes of oscillations.

2. Systems of Linear Differential Equations of the First Order

Consider a system of linear differential equations of the form

$$\dot{X}(t) = A \cdot X(t), \tag{1}$$

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{pmatrix}, \ \dot{X}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_m(t) \end{pmatrix}, \ A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix}$$

Suppose that the time points $0 = t_1 < \ldots < t_m$ are given (let us call these moments reference points) and then the equalities are fulfilled

$$\dot{X}(t_1) = AX(t_1), \ldots, \dot{X}(t_m) = AX(t_m). \tag{2}$$

Let $2n + 1$ inaccurate observations be made in the vicinity of each reference point. It is required to construct consistent estimates (converging in probability at $n \to \infty$ to the estimated parameter) of all elements of the matrix A, according to the constructed consistent estimates of matrix elements $X(t_1), \ldots, X(t_m), \ \dot{X}(t_1), \ldots \dot{X}(t_m)$.

Let us rewrite the system of Equalities (2) in matrix form

$$\dot{Y} = A \cdot Y, \text{ where } Y = (X(t_1), \ldots, X(t_m)), \ \dot{Y} = \left(\dot{X}(t_1), \ldots, \dot{X}(t_m)\right). \tag{3}$$

Then, the matrix A can be reconstructed using the matrix relation

$$A = \dot{Y} \cdot Y^{-1} \tag{4}$$

by matrices $Y, \ \dot{Y}$, if the matrix Y is revisable. Let us find out the conditions under which the matrix Y is revisable.

Suppose that the matrix A has m of various real eigenvalues $\lambda_1, \ldots, \lambda_m$ and their corresponding eigenvectors-columns $u_1, \ldots, u_m$, forming a basis in m-dimensional space.

Let us rewrite the system of differential Equation (1) in the basis $\mathbf{u_1}, \ldots, \mathbf{u_m}$, assuming $\tilde{X} = U^{-1} \cdot X$, where $U = (\mathbf{u_1}, \ldots, \mathbf{u_m}):$

$$\dot{\tilde{X}} = \Lambda \cdot \tilde{X},\ \Lambda = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_m \end{pmatrix},\ \tilde{X}(t) = \begin{pmatrix} c_1 \exp(\lambda_1 t) \\ c_2 \exp(\lambda_2 t) \\ \vdots \\ c_m \exp(\lambda_m t) \end{pmatrix}. \tag{5}$$

Here, the multipliers $c_k,\ k = 1, \ldots, m$, are components of the column vector $\tilde{X}(0)$, defining the initial conditions for a system of differential Equation (5).

Let us define a set of reference points $0 = t_1 < \ldots < t_m$ for some $\Delta > 0$ by the relations $t_j = (j-1)\Delta,\ j = 1, \ldots, m$. Let us define the square matrix $\tilde{Y}$ by the equality $\tilde{Y} = (\tilde{X}(t_1), \ldots, \tilde{X}(t_m)) = U^{-1} \cdot Y$. Then, the matrix Y for $\alpha_k = \exp(\lambda_k \Delta),\ k = 1, \ldots, m$, satisfies the following relations:

$$Y = U \cdot \tilde{Y},\ \tilde{Y} = \begin{pmatrix} c_1 & c_1\alpha_1 & c_1\alpha_1^2 & \ldots & c_1\alpha_1^{m-1} \\ c_2 & c_2\alpha_2 & c_2\alpha_2^2 & \ldots & c_2\alpha_2^{m-1} \\ \vdots & \vdots & \vdots & & \vdots \\ c_m & c_m\alpha_m & c_m\alpha_m^2 & \ldots & c_m\alpha_m^{m-1} \end{pmatrix}. \tag{6}$$

From the Formula (6) and the formula for calculating the Vandermond determinant (see [24], for example), we obtain

$$\det Y = \det U \prod_{k=1}^{m} c_k \prod_{1 \le j < k \le m} (\alpha_k - \alpha_j). \tag{7}$$

Since the eigenvalues $\lambda_1, \ldots, \lambda_m$ are pairwise different and $\det U \neq 0$, when performing the relations $c_k \neq 0,\ k = 1, \ldots, m,$, the matrix Y is reversible.

Suppose that at time points $t_i,\quad i = 1, \ldots, m,\ k = 0, 1, \ldots, n,\ h > 0$, inaccurate observations are obtained. We construct estimates $\widehat{x}_j(t_i),\ \widehat{\dot{x}}_j(t_i),\ i,\ j = 1, \ldots, m$, for matrix elements $X(t_i),\ \dot{X}(t_i):$

$$\widehat{x}_j(t_i) = \frac{\sum_{k=-n}^{n} (x_j(t_i + kh) + \varepsilon_j(t_i + kh))}{2n+1},$$

$$\widehat{\dot{x}}_j(t_i) = \frac{\sum_{k=-n}^{n} (x_j(t_i + kh) + \varepsilon_j(t_i + kh))kh}{\sum_{-n}^{n} (kh)^2}. \tag{8}$$

Here, $\varepsilon_j(t_i + kh),\ i,\ j = 1, \ldots, m,\ k = 0, 1, \ldots, n$, are independent identically distributed random variables with zero mean and finite variance. For $h = n^{-\alpha},\ 1 < \alpha < 3/2$, in [1], convergence in probability (and hence the consistency of estimates) is proved

$$\widehat{x}_j(t_i) \to x_j(t_i),\ \widehat{\dot{x}}_j(t_i) \to \dot{x}_j(t_i),\ n \to \infty,\ i, j = 1, \ldots, m.$$

Denote the matrices $\widehat{Y} = (\widehat{X}(t_1), ..., \widehat{X}(t_m)),\ \widehat{\dot{Y}} = (\widehat{\dot{X}}(t_1), ..., \widehat{\dot{X}}(t_m))$. Each element of the matrix $\widehat{\dot{Y}}$ (matrix element $\widehat{Y}$) is a consistent estimate of the corresponding matrix element $\dot{Y}$ (matrix element Y). The inverse matrix Y^{-1} coincides with the transposed matrix of algebraic complements corresponding to the elements of the matrix Y divided by the determinant of the matrix Y (see, for example, [24]). Therefore, each element of the matrix $\widehat{Y}^{-1}$ is a consistent estimate of the corresponding element of the matrix Y^{-1}. It follows that each element of the matrix

$$\widehat{A} = \widehat{\dot{Y}} \cdot \widehat{Y}^{-1}. \tag{9}$$

is a consistent estimate of the corresponding element of the matrix A.

Computational Experiment

A computational experiment was conducted for the Cauchy problem

$$\dot{X} = A \cdot X,\ X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

The solution of this system of equations has the form

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{3t} + e^t \\ e^{3t} - e^t \end{pmatrix}.$$

We assumed that by observing the process described by this system of equations, inaccurate observations were obtained at time points $t_i \pm kh,\ i = 1,2,\ t_1 = 0,\ t_2 = 0,5,$ $k = 0,1,\ldots,n,\ n = 100{,}000,\ h = n^{-5/4}:$

$$\begin{pmatrix} x_1(t_i \pm kh) + \varepsilon_1(t_i \pm kh) \\ x_2(t_i \pm kh) + \varepsilon_2(t_i \pm kh) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{3(t_i \pm kh)} + e^{t_i \pm kh} + \varepsilon_1(t_i \pm kh) \\ e^{3(t_i \pm kh)} - e^{t_i \pm kh} + \varepsilon_2(t_i \pm kh) \end{pmatrix}.$$

Here, $\varepsilon_j(t_i),\ i = 1,2,\ j = 1,2,\ k = 0,1,..,n,$ are independent random variables distributed uniformly on the segment $[-1/8, 1/8]$. Using Formula (9), the matrix A is evaluated by the following matrices

$$\widehat{Y} = \begin{pmatrix} \widehat{x}_1(t_0) & \widehat{x}_1(t_1) \\ \widehat{x}_2(t_0) & \widehat{x}_2(t_1) \end{pmatrix},\ \widehat{\dot{Y}} = \begin{pmatrix} \widehat{\dot{x}}_1(t_0) & \widehat{\dot{x}}_1(t_1) \\ \widehat{\dot{x}}_2(t_0) & \widehat{\dot{x}}_2(t_1) \end{pmatrix}.$$

As a result

$$\widehat{A} = \begin{pmatrix} 1.99517 & 1.00781 \\ 1.00025 & 1.99168 \end{pmatrix}.$$

Next, a computational experiment was conducted for the Cauchy problem

$$\dot{X} = A \cdot X,\ X(0) = \begin{pmatrix} 0 \\ -4 \\ 2 \end{pmatrix},\ A = \begin{pmatrix} 1 & -3 & 1 \\ 3 & -3 & -1 \\ 3 & -5 & 1 \end{pmatrix}.$$

The solution of this system of equations has the form

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} -2e^{-t} + 4e^{2t} - 2e^{-2t} \\ -2e^{-t} + e^{2t} - 3e^{-2t} \\ -2e^{-t} + 7e^{2t} - 3e^{-2t} \end{pmatrix}.$$

We assumed that by observing the process described by this system of equations, inaccurate observations were obtained at time points $t_i \pm kh,\ i = 1,2,3,\ t_1 = 0,\ t_2 = 0,5,$ $t_3 = 1,\ k = 0,1,\ldots,n,\ n = 100{,}000,\ h = n^{-5/4}:$

$$x_j(t_i \pm kh) + \varepsilon_j(t_i \pm kh),\ j = 1,2,3,$$

where $\varepsilon(t_i \pm kh),\ i = 1,2,3,\ k = 0,1,..,n,$ are independent random variables distributed uniformly on the segment $[-1/8, 1/8]$. As a result

$$\widehat{A} \approx \begin{pmatrix} 1.0638 & -3.0157 & 0.9650 \\ 2.9178 & -2.9764 & -0.9569 \\ 2.8818 & -4.9635 & 1.0605 \end{pmatrix}.$$

It should be noted that for the proposed version of the parameter estimation, it is necessary to establish the conditions for the reversibility of the matrices involved in this assessment and correctly select the step of the arithmetic progression between the reference points and the step between neighboring observation points in the vicinity of the reference points.

3. Equations of Oscillations of a Conservative System

Consider a mechanical conservative system described by a vector of generalized coordinates $q = (q_1, \ldots, q_m)$, with kinetic energy $\sum_{i,j=1}^{m} c_{ij}\dot{q}_i\dot{q}_j$ and potential energy $\sum_{i,j=1}^{m} d_{ij}q_iq_j$. We assume that the matrices $C = \|c_{ij}\|_{i,j=1}^{m}$, $D = \|d_{ij}\|_{i,j=1}^{m}$ are matrices of positive definite quadratic forms and, therefore, reversible. Then, the Lagrange equation for the system under consideration has the form $C\ddot{q} + Dq = 0$ or

$$\ddot{q} + Kq = 0,\ K = C^{-1}D. \tag{10}$$

Suppose that the eigenvalues $\lambda_1, \ldots, \lambda_m$ of the matrix K are positive and distinct. It should be noted that the requirement of the uniqueness of the eigenvalues of the matrix K can be relaxed.

Let us put $p = \dot{q}$, using the transition from Lagrangian variables q to Hamiltonian variables $q,\ p$, we obtain

$$X(t) = \begin{pmatrix} q(t) \\ p(t) \end{pmatrix}, \quad A = \begin{pmatrix} 0 & I \\ -K & 0 \end{pmatrix}.$$

Then, the system (10) will take the form (1). Its solution is described by a matrix exponent:

$$X(t) = \exp(tA)X(0). \tag{11}$$

Let us write down this solution with respect to the basis of the eigenvectors of the matrix A. To do this, we denote by $\theta_1, \ldots, \theta_{2m}$ the eigenvalues of the matrix A. The numbers $\theta_1, \ldots, \theta_{2m}$ are the roots of the equation $\det(A - \theta I) = \det(K + \theta^2 I) = 0$ and satisfy the equalities $\theta_1 = i\sqrt{\lambda_1} = i\omega_1, \ldots, \theta_m = i\sqrt{\lambda_m} = i\omega_m,\ \theta_{m+1} = -\theta_1, \ldots, \theta_{2m} = -\theta_m$. Let us put $\Omega = \sqrt{\Lambda} = \mathrm{d}iag[\omega_1, \ldots, \omega_m],\ \Lambda = \mathrm{d}iag[\lambda_1, \ldots, \lambda_m]$ and $\Theta = \mathrm{d}iag[i\Omega, -i\Omega]$.

Denote R and T transition matrices as the basis of the eigenvectors of the matrices K and A,, respectively. We can consider R to be real since such are the eigenvalues of the matrix K. By definition of the transition matrix, $KR = R\Lambda$ and $AT = T\Theta$. A non-trivial solution to these equations is $T = \begin{pmatrix} R & R \\ iR\Omega & -iR\Omega \end{pmatrix}$. Indeed, let us check the equality $AT = T\Omega :$

$$AT = \begin{pmatrix} 0 & I \\ -K & 0 \end{pmatrix}\begin{pmatrix} R & R \\ iR\Omega & -iR\Omega \end{pmatrix} = \begin{pmatrix} iR\Omega & -iR\Omega \\ -KR & -KR \end{pmatrix} = \begin{pmatrix} iR\Omega & -iR\Omega \\ -R\Lambda & -R\Lambda \end{pmatrix} =$$

$$= \begin{pmatrix} iR\Omega & -iR\Omega \\ -R\Omega^2 & -R\Omega^2 \end{pmatrix} = \begin{pmatrix} R & R \\ iR\Omega & -iR\Omega \end{pmatrix}\begin{pmatrix} i\Omega & 0 \\ 0 & -i\Omega \end{pmatrix} = T\Theta,$$

and, therefore, $A = T\Theta T^{-1}$. Elementary calculations check the equality

$$\frac{1}{2}\begin{pmatrix} R^{-1} & -i\Omega^{-1}R^{-1} \\ R^{-1} & i\Omega^{-1}R^{-1} \end{pmatrix}\begin{pmatrix} R & R \\ iR\Omega & -iR\Omega \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{12}$$

Let us denote now

$$\exp(i\Omega t) = \mathrm{d}iag[e^{\omega_1 t}, \ldots, e^{\omega_m t}],\ \exp(\Theta t) = \mathrm{d}iag[\exp(i\Omega t), \exp(-i\Omega t)],$$

$$\cos(\Omega t) = \frac{1}{2}[\exp(i\Omega t) + \exp(-i\Omega t)] = \mathrm{d}iag[\cos(\omega_1 t), \ldots, \cos(\omega_m t)],$$

$$\sin(\Omega t) = \frac{1}{2i}[\exp(i\Omega t) - \exp(-i\Omega t)] = \mathrm{d}iag[\sin(\omega_1 t), \ldots, \sin(\omega_m t)].$$

Obviously, all the introduced functions from matrices are diagonal. It is not difficult to verify that using the equality $A = T\Theta T^{-1}$ and Formula (12), it is possible to rewrite Formula (11) as $X(t) = \exp(tT\Theta T^{-1})X(0) = T\exp(t\Theta)T^{-1}X(0)$ and, then,

$$X(t) = \begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} R & R \\ iR\Omega & -iR\Omega \end{pmatrix} \begin{pmatrix} \exp(i\Omega t) & 0 \\ 0 & \exp(-i\Omega t) \end{pmatrix} \begin{pmatrix} R & R \\ iR\Omega & -iR\Omega \end{pmatrix}^{-1} \begin{pmatrix} q(0) \\ p(0) \end{pmatrix} =$$

$$= \frac{1}{2} \begin{pmatrix} R & R \\ iR\Omega & -iR\Omega \end{pmatrix} \begin{pmatrix} \exp(i\Omega t) & 0 \\ 0 & \exp(-i\Omega t) \end{pmatrix} \begin{pmatrix} R^{-1} & -i\Omega^{-1}R^{-1} \\ R^{-1} & i\Omega^{-1}R^{-1} \end{pmatrix} \begin{pmatrix} q(0) \\ p(0) \end{pmatrix} =$$

$$= \begin{pmatrix} R\cos(\Omega t)R^{-1} & R\Omega^{-1}\sin(\Omega t)R^{-1} \\ -R\Omega\sin(\Omega t)R^{-1} & R\cos(\Omega t)R^{-1} \end{pmatrix} \begin{pmatrix} q(0) \\ p(0) \end{pmatrix}. \tag{13}$$

Let us now investigate the reversibility of the matrix $Y = [X(t_1), \ldots, X(t_{2m})]$, $t_j = (j-1)\Delta,\ j = 1, \ldots, 2m$. Using (13), it is easy to check that

$$Y = T\mathrm{d}iag[T^{-1}X(0)][e^{\Theta t_1} \cdot \mathbf{1}, \ldots, e^{\Theta t_{2m}} \cdot \mathbf{1}],$$

where **1** is a column of units. Thus, the reversibility of Y is determined by the reversibility of the matrices $\mathrm{d}iag[T^{-1}X(0)]$ and $[e^{\Theta t_1} \cdot \mathbf{1}, \ldots, e^{\Theta t_{2m}} \cdot \mathbf{1}]$. The matrix $\mathrm{d}iag[T^{-1}X(0)]$ is reversible if and only if all its diagonal elements are not zero, i.e., when all components of the vector $T^{-1}X(0)$ are non-zero (vector $X(0)$ with the components $q(0)$ and $p(0)$, subjected to the specified condition, is called the general position vector). In turn, the matrix

$$V = [e^{t_1\Theta}\mathbf{1}, \ldots, e^{t_{2m}\Theta}\mathbf{1}] = \begin{pmatrix} e^{\omega_1 t_1} & \ldots & e^{\omega_1 t_{2m}} \\ \vdots & \ddots & \vdots \\ e^{\omega_{2m} t_1} & \ldots & e^{\omega_{2m} t_{2m}} \end{pmatrix}$$

is a Vandermonde matrix, the determinant of which is $\det V = \prod\limits_{1 \le k < l \le 2m} (e^{\Delta\theta_k} - e^{\Delta\theta_l})$ is not zero if and only if all exponents $e^{\Delta\theta_1}, \ldots, e^{\Delta\theta_{2m}}$ are different. If $\Delta < \pi/\omega_{\max}$, where $\omega_{\max} = \max(\omega_1, \ldots, \omega_m)$, then all the exponents will be different.

By analogy with Formulas (8) and (9), it is now possible to construct estimates of the matrix A and estimates of the matrix K based on them. In turn, the estimates of the matrix K allow us to construct estimates of the frequencies $\omega_1, \ldots, \omega_m$ and coefficients for $\cos\omega_1 t, \ldots, \cos\omega_m t,\ \sin\omega_1 t, \ldots, \sin\omega_m t$.

Computational Experiment for a System of Two Coupled Pendulums

As an example, consider a system of two identical connected linear pendulums (Figure 1).

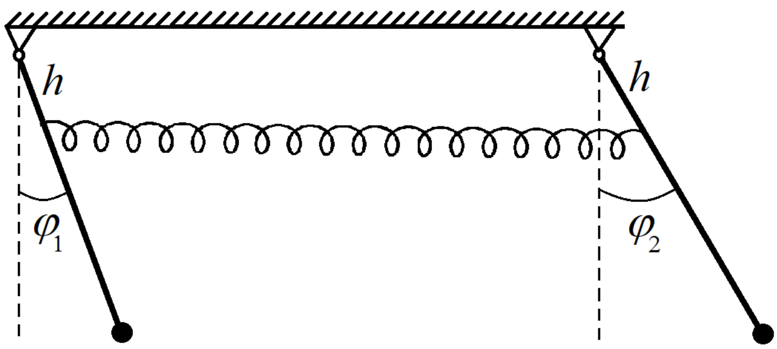

Figure 1. Diagram of two connected pendulums.

In this case, the kinetic and potential energies are given by the expressions

$$\frac{ml^2}{2}(\dot{\varphi}_1^2+\dot{\varphi}_2^2), \quad \frac{mgl}{2}(\varphi_1^2+\varphi_2^2)+\frac{\gamma h^2}{2}(\varphi_2-\varphi_1)^2.$$

Here, m—mass of each pendulum, l—length, φ_1 and φ_2—angles of deviation from the vertical axis of the first and second pendulums, respectively, γ—stiffness of the spring connecting the pendulums, h—the distance from the pendulum suspension point to the spring attachment point. Then

$$K=\begin{pmatrix} a & b \\ b & a \end{pmatrix},\ a=\frac{mgl+\gamma h^2}{ml^2},\ b=-\frac{\gamma h^2}{ml^2}.$$

Solving the problem of eigenvalues and eigenvectors for the matrix K, we find

$$R=\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \Omega=\begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}=\begin{pmatrix} \sqrt{a-b} & 0 \\ 0 & \sqrt{a+b} \end{pmatrix}.$$

Substituting these expressions into (13), we obtain

$$\begin{aligned} X(t)=\frac{1}{2}&\begin{pmatrix} \cos\omega_1 t & -\cos\omega_1 t & \omega_1^{-1}\sin\omega_1 t & -\omega_1^{-1}\sin\omega_1 t \\ -\cos\omega_1 t & \cos\omega_1 t & -\omega_1^{-1}\sin\omega_1 t & \omega_1^{-1}\sin\omega_1 t \\ -\omega_1\sin\omega_1 t & \omega_1\sin\omega_1 t & \cos\omega_1 t & -\cos\omega_1 t \\ \omega_1\sin\omega_1 t & -\omega_1\sin\omega_1 t & -\cos\omega_1 t & \cos\omega_1 t \end{pmatrix}\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{pmatrix} \\ +\frac{1}{2}&\begin{pmatrix} \cos\omega_2 t & \cos\omega_2 t & \omega_2^{-1}\sin\omega_2 t & \omega_2^{-1}\sin\omega_2 t \\ \cos\omega_2 t & \cos\omega_2 t & \omega_2^{-1}\sin\omega_2 t & \omega_2^{-1}\sin\omega_2 t \\ -\omega_2\sin\omega_2 t & -\omega_2\sin\omega_2 t & \cos\omega_2 t & \cos\omega_2 t \\ -\omega_2\sin\omega_2 t & -\omega_2\sin\omega_2 t & \cos\omega_2 t & \cos\omega_2 t \end{pmatrix}\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{pmatrix}. \end{aligned} \tag{14}$$

The conditions for the reversibility of the observation matrix are formulated as follows. The vector of the initial data of the general position is an arbitrary vector of the form $X(0)=[c_1+c_2,-c_1+c_2,d_1+d_2,-d_1+d_2]^T$ with non-zero pairs (c_1,d_1) and (c_2,d_2). For example, such is the vector $X(0)=[1,0,0,0]$. Meanwhile, the vector $X(0)=[1,-1,0,0]$ is not a vector of the general position. When choosing a general position vector as the initial data vector, reversibility obviously takes place under the condition $\Delta<\pi/\omega_1$. To conduct a computational experiment, we choose $q_1(0)=1,\ q_2(0)=p_1(0)=p_2(0)=0$, then

$$X(t)=\frac{1}{2}\begin{pmatrix} \cos\ \omega_1 t+\cos\ \omega_2 t \\ -\cos\ \omega_1 t+\cos\ \omega_2 t \\ -\omega_1\ \sin\ \omega_1 t-\omega_2\ \sin\ \omega_2 t \\ \omega_1\ \sin\ \omega_1 t-\omega_2\ \sin\ \omega_2 t \end{pmatrix}.$$

Estimates of $\widehat{x}_j(t_i),\ \widehat{\dot{x}}_j(t_i),\ i,\ j=1,\ldots,m,$ (see formula (8)) can be used to construct estimates of matrix elements A, and so estimates of matrix elements K. Therefore, it is possible to estimate the frequencies $\omega_1=\sqrt{\lambda_1},\ \omega_2=\sqrt{\lambda_2}$. To perform the calculations, we additionally put $a=2,\ b=-1,\ \Delta=\pi/3$, then $\omega_1=\sqrt{3}\approx 1.732,\ \omega_2=1$. We assume that by observing the process, described by this system of equations, inaccurate observations are obtained at time points $t_i=(i-1)\Delta\pm kh,\ i=1,2,3,4,\ k=0,1,\ldots,n,\ n=$100,000, $h=n^{-5/4}: x_j(t_i\pm kh)+\varepsilon_j(t_i\pm kh)$, where $\varepsilon_i(t_i\pm kh),\ i=1,2,3,4,\ k=0,1,\ldots,n$, are independent random variables distributed uniformly on the segment $[-1/8,1/8]$. The segment $[-1/8,1/8]$ characterizes the spread of random observation errors and their variation. As a result, we have

$$\widehat{A} \approx \begin{pmatrix} -0.0052 & 0.0021 & 1.0058 & 0.0089 \\ -0.0242 & -0.0022 & 0.00092 & 1.0213 \\ -2.0049 & 0.9846 & -0.0077 & 0.0013 \\ 0.9937 & -2.0211 & -0.0024 & 0.0127 \end{pmatrix}, \widehat{K} \approx \begin{pmatrix} 2.0049 & -0.9846 \\ -0.9937 & 2.0211 \end{pmatrix},$$

$$\widehat{\omega_1} \approx 1.7255, \ \widehat{\omega_2} \approx 0.9956.$$

Thus, we obtained estimates $\widehat{\omega_1}, \ \widehat{\omega_2}$ of frequencies $\omega_1 = 1.73205, \ \omega_2 = 1$. It is interesting to investigate the oscillation amplitudes (multipliers at $\cos\omega_k t, \ \sin\omega_k t, \ k = 1,2$) of the components of the vector $X(t)$. To do this, using Formula (14), we obtain the equality

$$X(t) = \begin{pmatrix} q_1(t) \\ q_2(t) \\ p_1(t) \\ p_2(t) \end{pmatrix} = F \begin{pmatrix} \cos\omega_1 t \\ \sin\omega_1 t \\ \cos\omega_2 t \\ \sin\omega_2 t \end{pmatrix},$$

in which the matrix F is representable as

$$F = \frac{1}{2} \begin{pmatrix} q_1(0) - q_2(0) & (p_1(0) - p_2(0))\omega_1^{-1} & q_1(0) + q_2(0) & (p_1(0) + p_2(0))\omega_2^{-1} \\ -q_1(0) + q_2(0) & (-p_1(0) + p_2(0))\omega_1^{-1} & q_1(0) + q_2(0) & (p_1(0) + p_2(0))\omega_2^{-1} \\ p_1(0) - p_2(0) & (-q_1(0) + q_2(0))\omega_1 & p_1(0) + p_2(0) & (-q_1(0) - q_2(0))\omega_2 \\ -p_1(0) + p_2(0) & (q_1(0) - q_2(0))\omega_1 & p_1(0) + p_2(0) & (-q_1(0) - q_2(0))\omega_2 \end{pmatrix}.$$

Let us now construct an estimate of $\widehat{F}$ of the matrix F, replacing the values $\omega_1, \ \omega_2, \ q_1(0)$, $q_2(0), \ p_1(0), \ p_2(0)$ in it by their estimates $\widehat{\omega_1}, \ \widehat{\omega_2}, \ \widehat{q_1(0)}, \ \widehat{q_2(0)}, \ \widehat{p_1(0)}, \ \widehat{p_2(0)}$. Then the elements of the matrix $\widehat{F}$ determine estimates of oscillation amplitudes with frequencies $\widehat{\omega_1}, \ \widehat{\omega_2}$. For $n \to \infty$, these estimates converge in probability to the corresponding elements of the matrix A. In our case

$$F \approx \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1.7321 & 0 & -1 \\ 0 & 1.7321 & 0 & -1 \end{pmatrix}, \ \widehat{F} \approx \frac{1}{2} \begin{pmatrix} 0.9955 & 0.0016 & 0.9984 & -0.0081 \\ -0.9955 & -0.0016 & 0.9984 & -0.0081 \\ 0.0028 & -1.7254 & -0.0081 & -1.0008 \\ -0.0028 & 1.7254 & -0.0081 & -1.0008 \end{pmatrix}.$$

4. Discussion

When analyzing a system of differential equations, describing oscillations in a conservative mechanical system, a transition is used from (Lagrangian) variables included in a system of second-order differential equations to (Hamiltonian) variables included in a system of first-order differential equations. As a result of such a replacement, it becomes possible to use exponents from diagonal matrices. This circumstance makes it possible to establish conditions for the reversibility of the matrix used in estimating the parameters of a conservative system. During the computational experiment, it was found that the step of the arithmetic progression, which determines the set of reference points (in the vicinity of which observations are carried out), significantly affects the accuracy of the estimates obtained. In turn, the estimation of the parameters of a conservative mechanical system is mainly needed to estimate the oscillation frequencies. These frequencies allow us to determine how close this mechanical system is to resonance, and, therefore, what is its stability and reliability.

5. Conclusions

The paper selects a set of reference points in the form of an arithmetic sequence for estimating the parameters of a system of ordinary differential equations, based on inaccurate observations. This makes it possible to obtain conditions for the reversibility of the matrix involved in estimating the parameters of a system of first-order differential equations. The estimation of the parameters of a mechanical conservative system, based on inaccurate observations, is carried out by switching to a system of first-order differential equations. This transition is made by replacing Lagrangian variables describing the analyzed conservative system with Hamiltonian variables. As a result, the reversibility of the matrix used in parameter estimation is established using the Vandermonde formula. A rather voluminous computational experiment has been carried out showing how, with an increase in the number of observations in the vicinity of reference points, the accuracy of parameter estimates of the analyzed systems increases. The influence of the arithmetic progression step, determining the set of reference points on the accuracy of parameter estimates, is investigated. The problem of estimating the characteristics of oscillations in a conservative system, considered in this paper, is multi-parametric. Therefore, to solve it, it is necessary to combine the methods of the theory of ordinary differential equations with constant coefficients and methods for constructing consistent estimates of coefficients based on the definition of reference points for planning an experiment. This circumstance makes it possible to construct modern measuring systems based on strict mathematical methods and to increase the accuracy of determining the parameters of the conservative system based on inaccurate observations. It should be noted that the proposed method of parameter estimation can be applied to systems consisting of a large number of elements since it is based on classical calculations of such systems, using Lagrange equations and matrix methods. However, in order to construct and study the quality of the estimates obtained, it is necessary to switch from Lagrange variables q (generalized coordinates) to generalized Hamilton variables, $q,\ p = \dot{q}$.

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