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Solitonic View of Generic Contact CR-Submanifolds of Sasakian Manifolds with Concurrent Vector Fields

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Abstract: This paper mainly devotes to the study of some solitons such as Ricci and Yamabe solitons and also their combination called Ricci-Yamabe solitons. In the geometry of solitons, a fundamental question is to identify the conditions under which these solitons can be trivial. Firstly, in this paper we study some extensive results on generic contact CR-submanifolds of Sasakian manifolds endowed with concurrent vector fields. Then some applications of solitons such as Ricci and Ricci-Yamabe solitons on such submanifolds with concurrent vector fields in the same ambient manifold have been discussed.

Keywords: Sasakian manifold; generic submanifolds; contact CR-submanifolds; concurrent vector field; Ricci soliton; Ricci–Yamabe soliton

MSC: 53C44; 53B30; 53C50; 53C80

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1. Introduction

The origin of the theory of submanifolds is in the study of geometry of plane curves. Since then it has a great impact on differential geometry of space and also emerging in different directions of this subject. This field of differential geometry has a vast research area which plays a significant role in developing of modern differential geometry. In this research paper, we will discuss a special type of isometrically immersed submanifold, namely contact CR(Cauchy-Riemannian)-submanifolds. In Kähler geometry, CR-submanifolds have been introduced to act as a link between invariant and anti-invariant submanifolds.

A. Bejancu introduced CR-submanifolds of Kähler manifolds with Riemannian metric [1,2]. After that, Shahid investigated CR-submanifolds of almost contact manifold and trans-Sasakian manifolds in [3,4]. In 1981, CR-submanifolds of Sasakian manifolds were studied by Kobayashi [5]. Later, in the context of indefinite Sasakian manifolds, contact CR-submanifolds were studied by Tarafdar et al. in [6]. Thus, in this present work we extend the study of the generic contact CR-submanifold as an exceptional category of generic submanifolds.

The notion of Ricci flow was presented by Hamilton [7] in 1982, to find a canonical metric on smooth manifolds which become a powerful tool for the analysis of Riemannian manifolds. The self-similar solution of Ricci flow is known as a Ricci soliton.

A smooth vector field I on a Riemannian manifold $(\overline{M}, g)$ is said to define a Ricci soliton if it satisfies

$$\pounds_{I} g + 2S + 2\lambda g = 0, \tag{1}$$

and it is denoted by $(\overline{\mathrm{M}}, g, \mathrm{I}, \lambda)$. Here, the Lie-derivative operator along with vector field I on $\overline{\mathrm{M}}$ is denoted by $\pounds_{\mathrm{I}}$, S defines Ricci tensor, and λ is some constant. Several geometers have studied Ricci solitons in different classes of contact geometry.

In addition, a Yamabe soliton is a self-similar solution of Yamabe flow, which was again defined by Hamilton [8] in 1988 as

$$\pounds_{\mathrm{I}} g + 2(\lambda - r)g = 0, \tag{2}$$

where r denotes the scalar curvature of the metric g.

Remark 1. *For dim $\overline{\mathrm{M}} = 2$, Ricci soliton and Yamabe soliton are equivalent, but when dim $\overline{\mathrm{M}} > 2$, both are different.*

Guler and Crasmareanu [9] gave a new class of geometric flows of the type (α, β) in 2019, namely, Ricci–Yamabe flow (in short, RY flow), and then Dey [10] proposed the self-similar solution of RY flow, called Ricci–Yamabe soliton (in short, RYS) of type (α, β), given below

$$\pounds_{\mathrm{I}} g + 2\alpha \mathrm{S} + (2\lambda - \beta r)g = 0, \tag{3}$$

where $\alpha, \beta \in \mathbb{R}$.

Remark 2. *Note that α-Ricci soliton and β-Yamabe soliton are illustrated from RYS if it is a type of $(\alpha, 0)$ and $(0, \beta)$ respectively. In addition, RYS is shrinking and steadily expanding according to $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$, respectively.*

In modern geometry, many papers have been published on Riemannian manifolds with concurrent vector field [11,12]. In the paper [13], generic submanifolds of Sasakian manifolds with concurrent vector fields were discussed.

From the above discussion, the present work is organized as follows: Section 1 gives the introduction. We define some basic definitions, formulas, and notions of almost contact metric manifolds in Section 2. In Section 3, we establish some interesting results on generic contact CR-submanifolds of Sasakian manifolds with concurrent vector fields which are helpful in finding the results of the last section. We find the normal and tangent components of vector field tangent to generic contact CR-submanifolds under different conditions. In Section 4, we deal with generic contact CR-submanifolds of Sasakian manifolds that admit Ricci solitons and Ricci–Yamabe solitons endowed with concurrent vector fields.

2. Preliminaries

In this section, we introduce some general formulas and notion of a Sasakian manifold and its submanifolds [14].

Definition 1. *An odd dimensional $m(= 2a + 1)$ differentiable manifold $\overline{\mathrm{M}}$ is said to be an almost contact metric manifold with the structure (ψ, ζ, η, g) such that ψ is a $(1, 1)$ type tensor field, ζ is a structural vector field, η is 1-form, and g denotes Riemannian metric, holding the following conditions:*

$$\psi^2 \mathrm{I} = -\mathrm{I} + \eta(\mathrm{I})\zeta, \quad \eta(\zeta) = 1, \quad \psi\zeta = 0, \quad \eta \circ \psi = 0, \quad \eta(\mathrm{I}) = g(\mathrm{I}, \zeta), \tag{4}$$

and

$$g(\psi \mathrm{I}, \psi \mathrm{J}) = g(\mathrm{I}, \mathrm{J}) - \eta(\mathrm{I})\eta(\mathrm{J}), g(\psi \mathrm{I}, \mathrm{J}) = -g(\mathrm{I}, \psi \mathrm{J}), \tag{5}$$

for all $\mathrm{I}, \mathrm{J} \in T\overline{\mathrm{M}}$.

Definition 2. *A contact metric manifold $(\overline{\mathrm{M}}, \psi, \zeta, \eta, g)$ is said to be a Sasakian manifold if the contact metric structure (ψ, ζ, η, g) is normal.*

An almost contact metric manifold $(\overline{\mathrm{M}}, \psi, \zeta, \eta, g)$ becomes a Sasakian manifold if $\overline{\mathrm{M}}$ if and only if it satisfies the following

$$(\overline{\nabla}_{\mathrm{I}}\psi)\mathrm{J} = g(\mathrm{I},\mathrm{J})\zeta - \eta(\mathrm{J})\mathrm{I}, \tag{6}$$

for any $\mathrm{I},\mathrm{J} \in T\overline{\mathrm{M}}$ and Levi–Civita connection is denoted by $\overline{\nabla}$.

In addition, Sasakian manifolds have the following relation [14]

$$\overline{\nabla}_{\mathrm{I}}\zeta = -\psi\mathrm{I}. \tag{7}$$

Now, let M be a submanifold of a Sasakian manifold $\overline{\mathrm{M}}$ with the induced metric g (using the same symbol as that of ambient manifold). In addition, assuming induced metric connection on the tangent bundle $T\mathrm{M}$ and the normal bundle $T^{\perp}\mathrm{M}$ of M, which are shown by ∇ and $\nabla^{\perp}$, respectively. The Gauss and Weingarten formulas are respectively represented by [14]

$$\overline{\nabla}_{\mathrm{I}}\mathrm{J} = \nabla_{\mathrm{I}}\mathrm{J} + h(\mathrm{I},\mathrm{J}), \tag{8}$$

and

$$\overline{\nabla}_{\mathrm{I}}\mathrm{V} = -A_{\mathrm{V}}\mathrm{I} + \nabla^{\perp}_{I}\mathrm{V}, \tag{9}$$

for all $\mathrm{I},\mathrm{J} \in T\mathrm{M}$ and $\mathrm{V} \in T^{\perp}\mathrm{M}$, where h and A_{V} are second fundamental form and the shape operator (corresponding to the normal vector field V) for the immersion of M into $\overline{\mathrm{M}}$.

The relation between the second fundamental form h and the shape operator A_{V} is given by

$$g(h(\mathrm{I},\mathrm{J}),\mathrm{V}) = g(A_{\mathrm{V}}\mathrm{I},\mathrm{J}), \tag{10}$$

for any $\mathrm{I},\mathrm{J} \in T\mathrm{M}$ and $\mathrm{V} \in T^{\perp}\mathrm{M}$. If $h = 0$, then M is known by totally geodesic submanifold [14,15].

A submanifold M of $\overline{\mathrm{M}}$ is known by totally umbilical [14,15], if

$$h(\mathrm{I},\mathrm{J}) = g(\mathrm{I},\mathrm{J})\mathrm{H}, \tag{11}$$

where H indicates the mean curvature on M for any $\mathrm{I},\mathrm{J} \in T\mathrm{M}$. Also, if $\mathrm{H} = 0$, then M is minimal in $\overline{\mathrm{M}}$ [14,15].

Now, the decomposition of tangent bundle $T\overline{\mathrm{M}}$ is given by

$$T\overline{\mathrm{M}} = T\mathrm{M} \oplus T^{\perp}\mathrm{M}. \tag{12}$$

Definition 3. *A submanifold* M *of a Sasakian manifold* $\overline{\mathrm{M}}$ *is said to be a contact CR-submanifold [14] if*

1. *ζ is tangent to* M*;*
2. *Tangent bundle divided into two differentiable distribution* D *and* $\mathrm{D}^{\perp}$ *such that* $T\mathrm{M} = \mathrm{D} \oplus \mathrm{D}^{\perp}$*;*
3. *The distribution* $\mathrm{D} : u \to \mathrm{D}_u \subset T_u\mathrm{M}$ *is invariant to* ψ*, that is,* $\psi\mathrm{D}_{\mathrm{u}} \subset \mathrm{D}_{\mathrm{u}}$ *for each* $u \in \mathrm{M}$*,=;*
4. *The distribution* $\mathrm{D}^{\perp} : u \to \mathrm{D}^{\perp}{}_u \subset T_u\mathrm{M}$ *is anti-invariant to* ψ*, that is,* $\psi\mathrm{D}^{\perp}_{\mathrm{u}} \subset T^{\perp}_u\mathrm{M}$ *for each* $u \in \mathrm{M}$*.*

We call D and $\mathrm{D}^{\perp}$ a horizontal and vertical distribution, respectively, and also P and Q are the projection operators on D and $\mathrm{D}^{\perp}$, respectively, such that

$$\mathrm{P}^2 = \mathrm{P}, \quad \mathrm{Q}^2 = \mathrm{Q}, \quad \mathrm{PQ} = \mathrm{QR} = 0, \quad \mathrm{P} + \mathrm{Q} = \mathbb{I}, \tag{13}$$

where $\mathbb{I}$ stands for the identity transformation M.

Next, we have [14]

$$\psi\mathrm{I} = \mathrm{PI} + \mathrm{QI}, \tag{14}$$

and

$$\psi\mathrm{N} = \mathrm{BN} + \mathrm{CN}, \tag{15}$$

for any vector field I tangent to M and vector field N normal to M. Here PI and QI are tangent and normal components of ψI; BN and CN are tangent and normal components of ψN.

Consider that dim $\overline{\mathrm{M}} = m$, dim $\mathrm{M} = n$, dim $\mathrm{D} = p$, dim $\mathrm{D}^{\perp} = q$, and codim $\mathrm{M} = s$. A contact CR-submanifold M is said to be generic submanifold [14] of $\overline{\mathrm{M}}$ if $q = s$. In this case, $\psi T_u^{\perp}\mathrm{M} \subset T_u\mathrm{M}$ for every point $u \in \mathrm{M}$.

According to the definition of a contact CR-submanifold M of a Sasakian manifold $\overline{\mathrm{M}}$, the orthogonal decompositions of tangent and normal bundles of M can be written as

$$T\mathrm{M} = \mathrm{D} \oplus \mathrm{D}^{\perp} \oplus \{\zeta\}, \quad T^{\perp}\mathrm{M} = \psi(\mathrm{D}^{\perp}) \oplus \mu, \tag{16}$$

where μ is the complementary subbundle orthogonal to $\psi(\mathrm{D}^{\perp})$ in $T^{\perp}\mathrm{M}$.

Furthermore, $\mu = 0$ in (16) if the contact CR-submanifold of Sasakian manifolds is the generic contact CR-submanifold with the condition $q = s$. Then we have the following decomposition

$$T\mathrm{M} = \mathrm{D} \oplus \mathrm{D}^{\perp} \oplus \{\zeta\}, \quad T^{\perp}\mathrm{M} = \psi(\mathrm{D}^{\perp}). \tag{17}$$

For a generic contact CR-submanifold, from (17) we can write the following

$$\mathrm{V} = \mathrm{V}^{\mathrm{T}} + \mathrm{V}^{\perp} + \psi(\mathrm{V}^{\perp}) + f\zeta, \tag{18}$$

where f is constant function, $\mathrm{V} \in T\overline{\mathrm{M}}$, $\mathrm{V}^{\mathrm{T}} \in \mathrm{D}$, and $\mathrm{V}^{\perp} \in \mathrm{D}^{\perp}$.

Note that the contact CR-submanifold M is called D-geodesic ($\mathrm{D}^{\perp}$-geodesic) if $h(\mathrm{I}, \mathrm{J}) = 0$ for any $\mathrm{I}, \mathrm{J} \in \mathrm{D}$ ($\mathrm{I}, \mathrm{J} \in \mathrm{D}^{\perp}$). Again, for $\mathrm{I} \in \mathrm{D}$, $\mathrm{J} \in \mathrm{D}^{\perp}$, the contact CR-submanifold M is said to be $(\mathrm{D}, \mathrm{D}^{\perp})$-geodesic or mixed geodesic if $h(\mathrm{I}, \mathrm{J}) = 0$.

In addition, the covariant derivative of the tensor fields ψ, P, and Q are defined as

$$(\overline{\nabla}_{\mathrm{I}}\psi)\mathrm{J} = \overline{\nabla}_{\mathrm{I}}\psi\mathrm{J} - \psi\overline{\nabla}_{\mathrm{I}}\mathrm{J}, \tag{19}$$

$$(\overline{\nabla}_{\mathrm{I}}\mathrm{P})\mathrm{J} = \nabla_{\mathrm{I}}\mathrm{PJ} - \mathrm{P}\nabla_{\mathrm{I}}\mathrm{J}, \tag{20}$$

and

$$(\overline{\nabla}_{\mathrm{I}}\mathrm{Q})\mathrm{J} = \nabla_{\mathrm{I}}^{\perp}\mathrm{QJ} - \mathrm{Q}\nabla_{\mathrm{I}}\mathrm{J}, \tag{21}$$

for all $\mathrm{I}, \mathrm{J} \in T\mathrm{M}$.

3. Generic Contact CR-Submanifolds of Sasakian Manifolds

This section contains some important results related to generic contact CR-submanifolds, which helps to deduct our main results.

Definition 4. *A vector field* V *on a (semi-)Riemannian manifold* $(\overline{\mathrm{M}}, g)$ *is said to be a concircular vector field [16] if it satisfies*

$$\overline{\nabla}_{\mathrm{I}}\mathrm{V} = \gamma\mathrm{I}, \tag{22}$$

for any $\mathrm{I} \in T\overline{\mathrm{M}}$ *and* γ *is a nontrivial smooth function on* $\overline{\mathrm{M}}$*. The concircular vector field* V *is said to be a concurrent vector field if we particularly take* $\gamma = 1$ *in (22).*

First, we have the following

Proposition 1. *Let* M *be a generic contact CR-submanifold of Sasakian manifold* $\overline{\mathrm{M}}$*. Then we have the following*

1. ζ *is a killing vector field in* $\overline{\mathrm{M}}$*.*
2. M *is minimal in* $\overline{\mathrm{M}}$*.*

Proof. Since we have $\overline{\nabla}_I\zeta = -\psi I$, for any $I \in TM$. From which we get

$$\nabla_I\zeta = -PI, \tag{23}$$

and

$$h(I,\zeta) = -QI. \tag{24}$$

Now, we use Lie-derivative operator and find that

$$\begin{aligned}(\pounds_\zeta g)(I,J) &= g(\nabla_I\zeta, J) + g(I, \nabla_J\zeta)\\ &= g(-PI, J) + g(J, -PJ)\\ &= 0.\end{aligned}$$

As we know that a vector field is killing if Lie-derivative operator along that vector field is zero. Thus, we conclude that ζ is killing. Also in particular, we put $I = \zeta$ in (24) and by (11) we get $h(\zeta,\zeta) = 0$ which says that $H = 0$, that is, M is minimal. □

Now, we prove the following lemma which is quite useful in deriving the new results of this section and also that of section 4.

Lemma 1. *Let* M *be a generic contact CR-submanifold of Sasakian manifolds* $\overline{M}$ *with concurrent vector field* V*, then tangent and normal component are, respectively, given by*

$$\begin{aligned}PI &= \nabla_I PV^T - A_{QV^\perp}I - \nabla_I V^\perp\\ &\quad -g(I, V^T + V^\perp)\zeta - f\psi^2 I.\end{aligned} \tag{25}$$

and

$$QI = h(I, PV^T) + \nabla_I^\perp QV^\perp - h(I, V^\perp). \tag{26}$$

Proof. Since V is a concurrent vector field, then by definition of concurrent vector field, we have

$$\psi I = \psi\overline{\nabla}_I V = \overline{\nabla}_I\psi V - (\overline{\nabla}_I\psi)V, \tag{27}$$

for any $I \in TM$.

Using (6) and (18) in (27), we have

$$\begin{aligned}\psi I &= \overline{\nabla}_I\psi V^T + \overline{\nabla}_I\psi V^\perp + \overline{\nabla}_I\psi^2 V^\perp + \overline{\nabla}_I\psi f\zeta - g(I,V^T)\zeta - g(I,V^\perp)\zeta\\ &\quad -g(I,\psi(V^\perp))\zeta - g(I,f\zeta)\zeta + \eta(V^T)I + \eta(V^\perp)I + \eta(\psi(V^\perp))I\\ &\quad +\eta(f\zeta)I.\end{aligned} \tag{28}$$

Since ζ is tangent to M and by using (4), (7), (8), (9), and (14) in (28), we yield

$$\begin{aligned}PI + QI &= \nabla_I PV^T + h(I, PV^T) + \nabla_I^\perp QV^\perp - A_{QV^\perp}I - \nabla_I V^\perp\\ &\quad -h(I, V^\perp) - g(I, V^T)\zeta - g(I, V^\perp)\zeta\\ &\quad -f\eta(I)\zeta + fI.\end{aligned} \tag{29}$$

Upon comparing the tangential and normal components, we obtain relations (25) and (26). □

Proposition 2. *Let* M *be a generic contact CR-submanifold of Sasakian manifold* $\overline{M}$ *with concurrent vector field* V*. If M is totally geodesic, then*

$$\nabla_I PV^T - \nabla_I V^\perp - P(\nabla_I V^T) - P(\nabla_I V^\perp) = g(I, V^T + V^\perp)\zeta \tag{30}$$

and

$$\nabla_{\mathrm{I}}^{\perp}\mathrm{QV}^{\perp} - \mathrm{Q}(\nabla_{\mathrm{I}}\mathrm{V}^{\mathrm{T}}) - \mathrm{Q}(\nabla_{\mathrm{I}}\mathrm{V}^{\perp}) - \psi(\nabla_{\mathrm{I}}^{\perp}\mathrm{QV}^{\perp}) = 0. \tag{31}$$

Proof. We input $\mathrm{J} = \mathrm{V}$ in (6), then

$$\overline{\nabla}_{\mathrm{I}}\psi\mathrm{V} - \psi\overline{\nabla}_{\mathrm{I}}\mathrm{V} = g(\mathrm{I},\mathrm{V})\zeta - \eta(\mathrm{V})\mathrm{I},$$

for any $\mathrm{I} \in T\mathrm{M}$.

$$\begin{aligned} & \overline{\nabla}_{\mathrm{I}}\psi\mathrm{V}^{\mathrm{T}} + \overline{\nabla}_{\mathrm{I}}\psi\mathrm{V}^{\perp} + \overline{\nabla}_{\mathrm{I}}\psi^{2}\mathrm{V}^{\perp} + \overline{\nabla}_{\mathrm{I}}\psi f\zeta \\ & -\psi(\overline{\nabla}_{\mathrm{I}}\mathrm{V}^{\mathrm{T}} + \overline{\nabla}_{\mathrm{I}}\mathrm{V}^{\perp} + \overline{\nabla}_{\mathrm{I}}\psi\mathrm{V}^{\perp} + \overline{\nabla}_{\mathrm{I}}f\zeta) \\ = \; & g(\mathrm{I},\mathrm{V})\zeta - \eta(\mathrm{V})\mathrm{I}, \end{aligned} \tag{32}$$

Using (8), (14), (20), and (21) in (32), we have

$$\begin{aligned} & \nabla_{\mathrm{I}}\mathrm{PV}^{\mathrm{T}} + h(\mathrm{I},\mathrm{PV}^{\mathrm{T}}) + \nabla_{\mathrm{I}}^{\perp}\mathrm{QV}^{\perp} - A_{\mathrm{QV}^{\perp}}\mathrm{I} - \nabla_{\mathrm{I}}\mathrm{V}^{\perp} - h(\mathrm{I},\mathrm{V}^{\perp}) \\ & -\psi(\overline{\nabla}_{\mathrm{I}}\mathrm{V}^{\mathrm{T}} + \overline{\nabla}_{\mathrm{I}}\mathrm{V}^{\perp} + \overline{\nabla}_{\mathrm{I}}\psi\mathrm{V}^{\perp} - f\psi\mathrm{I}) \\ = \; & g(\mathrm{I},\mathrm{V}^{\mathrm{T}} + \mathrm{V}^{\perp})\zeta + f\eta(\mathrm{I})\zeta - f\mathrm{I}. \end{aligned}$$

Further, we arrive at

$$\begin{aligned} & \nabla_{\mathrm{I}}\mathrm{PV}^{\mathrm{T}} + h(\mathrm{I},\mathrm{PV}^{\mathrm{T}}) + \nabla_{\mathrm{I}}^{\perp}\mathrm{QV}^{\perp} - A_{\mathrm{QV}^{\perp}}\mathrm{I} - \nabla_{\mathrm{I}}\mathrm{V}^{\perp} - h(\mathrm{I},\mathrm{V}^{\perp}) \\ & -\psi(\nabla_{\mathrm{I}}\mathrm{V}^{\mathrm{T}}) - \psi h(\mathrm{I},\mathrm{V}^{\mathrm{T}}) - \psi(\nabla_{\mathrm{I}}\mathrm{V}^{\perp}) - \psi h(\mathrm{I},\mathrm{V}^{\perp}) - \psi(\nabla_{\mathrm{I}}^{\perp}\mathrm{QV}^{\perp}) \\ & +\psi A_{\mathrm{QV}^{\perp}}\mathrm{I} + \psi f\psi\mathrm{I}) \\ = \; & g(\mathrm{I},\mathrm{V}^{\mathrm{T}} + \mathrm{V}^{\perp})\zeta + f\eta(\mathrm{I})\zeta - f\mathrm{I}. \end{aligned}$$

After comparing tangent and normal components from the above equation, we yield the relations (30) and (31). □

Note that the distributions D and $\mathrm{D}^{\perp}$ are said to be parallel with respect to ∇ if $\nabla_{\mathrm{I}}\mathrm{J} \in \mathrm{D}$ for any $\mathrm{I} \in T\mathrm{M}, \mathrm{J} \in \mathrm{D}$ and $\nabla_{\mathrm{I}}\mathrm{J} \in \mathrm{D}^{\perp}$ for any $\mathrm{I} \in T\mathrm{M}, \mathrm{J} \in \mathrm{D}^{\perp}$, respectively.

Proposition 3. *Let* M *be a totally geodesic generic contact CR-submanifold of Sasakian manifold* $\overline{\mathrm{M}}$ *with concurrent vector field* V*. If* D *and* $\mathrm{D}^{\perp}$ *are parallel with respect to* ∇*, then*

$$(\pounds_{\mathrm{V}^{\mathrm{T}}}g)(\mathrm{I},\mathrm{J}) + (\pounds_{\mathrm{V}^{\perp}}g)(\mathrm{I},\mathrm{J}) = 2g(\mathrm{I},\mathrm{J}) - 2\eta(\mathrm{I})\eta(\mathrm{J}). \tag{33}$$

Proof. We derive the desired result by using Lemma 1 and Proposition 2. Upon combining (25), (26), (30), and (31), we obtain

$$\mathrm{P}(\nabla_{\mathrm{I}}\mathrm{V}^{\mathrm{T}}) + \mathrm{P}(\nabla_{\mathrm{I}}\mathrm{V}^{\perp}) + \mathrm{Q}(\nabla_{\mathrm{I}}\mathrm{V}^{\mathrm{T}}) + \mathrm{Q}(\nabla_{\mathrm{I}}\mathrm{V}^{\perp}) = f\psi^{2}\mathrm{I} + \mathrm{PI} + \mathrm{QI} - \psi\mathrm{QI},$$

which further gives

$$\psi(\nabla_{\mathrm{I}}\mathrm{V}^{\mathrm{T}} + \nabla_{\mathrm{I}}\mathrm{V}^{\perp}) = f\psi^{2}\mathrm{I} + \psi\mathrm{I}.$$

Applying ψ on both the sides, we have the following:

$$-(\nabla_{\mathrm{I}}\mathrm{V}^{\mathrm{T}} + \nabla_{\mathrm{I}}\mathrm{V}^{\perp}) + \eta(\nabla_{\mathrm{I}}\mathrm{V}^{\mathrm{T}} + \nabla_{\mathrm{I}}\mathrm{V}^{\perp})\zeta = f\psi^{3}\mathrm{I} - \mathrm{I} + \eta(\mathrm{I})\zeta,$$

which can be rewritten as

$$-(\nabla_{\mathrm{I}}\mathrm{V}^{\mathrm{T}} + \nabla_{\mathrm{I}}\mathrm{V}^{\perp} - \mathrm{I}) + \eta(\nabla_{\mathrm{I}}\mathrm{V}^{\mathrm{T}} + \nabla_{\mathrm{I}}\mathrm{V}^{\perp} - \mathrm{I})\zeta = f\psi^{3}\mathrm{I}.$$

In addition,

$$-g(\nabla_I V^T + \nabla_I V^\perp - I, J) + g(\nabla_I V^T + \nabla_I V^\perp - I, \zeta)g(\zeta, J) = -fg(\psi I, J), \quad (34)$$

for any $J \in TM$.

Interchanging I and J in (34), we obtain

$$-g(\nabla_J V^T + \nabla_J V^\perp - J, I) + g(\nabla_J V^T + \nabla_J V^\perp - J, \zeta)g(\zeta, I) = -fg(\psi J, I). \quad (35)$$

Upon adding (34) and (35), we arrive at (33). □

4. Solitonic View

This section contains some results related to generic CR-submanifold admitting Ricci soliton and Ricci–Yamabe soliton of Sasakian manifolds with concurrent vector fields.

Theorem 1. *Let M be a generic contact CR-submanifold admitting RS of Sasakian manifold $\overline{M}$ with concurrent vector field V. Then, the Ricci tensor $S_{D^\perp}$ for $D^\perp$ is given by*

$$\begin{aligned} S_{D^\perp}(I, J) &= -(\lambda + f)g(I, J) + g(h(J, I), QV^\perp) \\ &\quad -\frac{1}{2}[g(\nabla_I PV^T, J) + g(\nabla_J PV^T, I)]. \end{aligned} \quad (36)$$

Proof. By definition of the Lie-derivative operator, we have

$$£_{V^\perp}(I, J) = g(\nabla_I V^\perp, J) + g(\nabla_J V^\perp, I),$$

for any $I, J \in D^\perp$.

Here, we use Lemma 1

$$\begin{aligned} (£_{V^\perp} g)(I, J) &= -g(PI, J) + g(\nabla_I PV^T, J) - g(A_{QV^\perp} I, J) - fg(\psi^2 I, J) \\ &\quad -g(PJ, I) + g(\nabla_J PV^T, I) - g(A_{QV^\perp} J, I) - fg(\psi^2 J, I). \end{aligned} \quad (37)$$

With the help of $g(PI, J) + g(I, PJ) = 0$ and (10), the equation (37) becomes

$$\begin{aligned} (£_{V^\perp} g)(I, J) &= g(\nabla_I PV^T, J) + g(\nabla_J PV^T, I) - 2g(h(J, I), QV^\perp) - 2fg(\psi^2 I, J) \\ &= g(\nabla_I PV^T, J) + g(\nabla_J PV^T, I) - 2g(h(J, I), QV^\perp) + 2fg(I, J). \end{aligned} \quad (38)$$

By definition of Ricci soliton and (38), we have (36). □

Theorem 2. *Let M be a totally geodesic generic contact CR-submanifold admitting RS of Sasakian manifold $\overline{M}$ with concurrent vector field V. If D and $D^\perp$ are parallel with respect to ∇, then Ricci tensor S_D for D is given by*

$$S_D(I, J) = -(\lambda + 1 - f)g(I, J) + \frac{1}{2}[g(\nabla_I PV^T, J) + g(\nabla_J PV^T, I)]. \quad (39)$$

Proof. By using Proposition 3 and Theorem 1, we can easily compute $S_D(I, J)$ for any $I, J \in D$ □

Theorem 3. *Let $(M, g, \zeta, \lambda, \alpha, \beta)$ be a generic contact CR-submanifold admitting RYS of Sasakian manifold $\overline{M}$. Then we have the following*

1. *S is given by $S = -\frac{1}{\alpha}(\lambda - \frac{\beta r}{2})g$.*
2. *M is Einstein.*

Proof. By using Proposition 1, we derive the desired Ricci tensor and hence M is Einstein. □

Now, by following the same steps as adopted in proving Theorems 1 and 2 and also RYS (3), we conclude the following results

Theorem 4. *Let* M *be a generic contact CR-submanifold admitting RYS of Sasakian manifold* $\overline{\mathrm{M}}$ *with concurrent vector field* V*. Then, Ricci tensor* $S_{\mathrm{D}^\perp}$ *for* $\mathrm{D}^\perp$ *is given by*

$$\begin{aligned} S_{\mathrm{D}^\perp}(\mathrm{I},\mathrm{J}) &= -\frac{1}{\alpha}(\lambda - \frac{\beta r}{2} + f)g(\mathrm{I},\mathrm{J}) + g(h(\mathrm{J},\mathrm{I}),\mathrm{QV}^\perp) \\ &\quad -\frac{1}{2\alpha}[g(\nabla_{\mathrm{I}}\mathrm{PV}^{\mathrm{T}},\mathrm{J}) + g(\nabla_{\mathrm{J}}\mathrm{PV}^{\mathrm{T}},\mathrm{I})]. \end{aligned} \tag{40}$$

Theorem 5. *Let* M *be a totally geodesic generic contact CR-submanifold admitting RYS of Sasakian manifold* $\overline{\mathrm{M}}$ *with concurrent vector field* V*. If* D *and* $\mathrm{D}^\perp$ *are parallel with respect to* ∇*, then Ricci tensor* S_{D} *for* D *is given by*

$$S_{\mathrm{D}}(\mathrm{I},\mathrm{J}) = -\frac{1}{\alpha}(\lambda - \frac{\beta r}{2} + 1 - f)g(\mathrm{I},\mathrm{J}) + \frac{1}{2\alpha}[g(\nabla_{\mathrm{I}}\mathrm{PV}^{\mathrm{T}},\mathrm{J}) + g(\nabla_{\mathrm{J}}\mathrm{PV}^{\mathrm{T}},\mathrm{I})]. \tag{41}$$

Remark 3. *As we can see, Theorems 1 and 2 can be easily obtained from Theorems 4 and 5 by using* $\alpha = 1$ *and* $\beta = 0$*. It is also noted that Theorems 4 and 5 can be stated for* α*-RS, where* $\beta = 0$*.*

5. Conclusions

Ricci solitons are helpful in biology, physics, economics and chemistry. In fact, Ricci flows and Ricci solitons are able to show their existence in medical imaging for brain surfaces. In the field of differential geometry, we can execute major geometric findings by selecting a suitable soliton and a vector field. In this regard, by considering immersions into a sufficiently enormous family of manifolds (including spaces with constant sectional curvature) is a convenient way to extend the foregoing findings to a wider class of ambient spaces. Our results are important steps in this direction.

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