mathematics

MDPI

Article

Applications of q-Calculus Multiplier Operators and Subordination for the Study of Particular Analytic Function Subclasses

Ekram E. Ali [1,2,*], Georgia Irina Oros [3,*], Shujaat Ali Shah [4] and Abeer M. Albalahi [1]

[1] Department of Mathematics, College of Science, University of Ha'il, Ha'il 81451, Saudi Arabia; a.albalahi@uoh.edu.sa
[2] Department of Mathematics and Computer Science, Faculty of Science, Port Said University, Port Said 42521, Egypt
[3] Department of Mathematics, University of Oradea, Universitatii 1, 410087 Oradea, Romania
[4] Department of Mathematics and Statistics, Quaid-e-Awam University of Engineering, Science and Technology (QUEST), Nawabshah 67450, Pakistan; shahglike@yahoo.com
* Correspondence: e.ahmad@uoh.edu.sa or ekram_008eg@yahoo.com (E.E.A.); georgia_oros_ro@yahoo.co.uk (G.I.O.)

Abstract: In this article, a new linear extended multiplier operator is defined utilizing the q-Choi–Saigo–Srivastava operator and the q-derivative. Two generalized subclasses of q—uniformly convex and starlike functions of order δ—are defined and studied using this new operator. Necessary conditions are derived for functions to belong in each of the two subclasses, and subordination theorems involving the Hadamard product of such particular functions are stated and proven. As applications of those findings using specific values for the parameters of the new subclasses, associated corollaries are provided. Additionally, examples are created to demonstrate the conclusions' applicability in relation to the functions from the newly introduced subclasses.

Keywords: subordination; uniformly starlike function; uniformly convex function; convolution (Hadamard) product; subordinating factor sequence; q-derivative operator; q-Choi–Saigo–Srivastava operator

MSC: 30C45; 30A10

check for updates

Citation: Ali, E.E.; Oros, G.I.; Ali Shah, S.; Albalahi, A.M. Applications of q-Calculus Multiplier Operators and Subordination for the Study of Particular Analytic Function Subclasses. *Mathematics* **2023**, *11*, 2705. https://doi.org/10.3390/math11122705

Academic Editor: Jay Jahangiri

Received: 15 May 2023
Revised: 11 June 2023
Accepted: 12 June 2023
Published: 14 June 2023

1. Introduction

The outcome of this work is connected to geometric function theory, and techniques based on subordination are utilized to obtain those results, combined with aspects regarding q-calculus operators.

Let the class denoted by A contain all functions of the form

$$f(z) = z + \sum_{\nu=2}^{\infty} a_\nu z^\nu,\ z \in U, \tag{1}$$

where $U = \{z \in \mathbb{C} : |z| < 1\}$.

As given in [1–3], if f and $\hbar$ are analytic in U, f is *subordinate* to $\hbar$, denoted as $f(z) \prec \hbar(z)$, if there exists an analytic function ϖ, with $\varpi(0) = 0$ and $|\varpi(z)| < 1$ for all $z \in U$, such that $f(z) = \hbar(\varpi(z))$, $z \in U$. In the case when the function $\hbar$ is univalent in U, $f(z) \prec \hbar(z)$ is interpreted as:

$$f(0) = \hbar(0) \text{ and } f(U) \subset \hbar(U).$$

For a function $f \in$ A written as (1) and $\hbar$ described as

$$\hbar(z) = z + \sum_{\nu=2}^{\infty} b_\nu z^\nu,\ z \in U,$$

the well-known *convolution product* is

$$(f * \hbar)(z) := z + \sum_{\nu=2}^{\infty} a_\nu b_\nu z^\nu,\ z \in U.$$

If a function $f \in$ A satisfies

$$\Re\left[\frac{zf'(z)}{f(z)}\right] > \delta,\ (0 \leq \delta < 1).$$

then f is said to be *starlike of order* δ, written as $f \in S^*(\delta)$, where $S^*(\delta)$ denotes the class of all such functions.

If the function $f \in$ A has the property

$$\Re\left[1 + \frac{zf''(z)}{f'(z)}\right] > \delta,\ z \in U.$$

then f is said to be *convex of order* δ, written as $f \in K(\delta)$, where $K(\delta)$ denotes the class of all such functions.

For $\delta = 0$, $S^*(\delta) = S^*$ and $K(\delta) = K$ refer to the regular classes of *starlike and convex functions* in U, respectively.

In [4], $UCV(\rho, \delta)$ was designated to represent the class of *uniformly convex functions of order* δ *and type* ρ containing all functions $f \in$ A satisfying:

$$\Re\left[1 + \frac{zf''(z)}{f'(z)} - \delta\right] > \rho\left|\frac{zf''(z)}{f'(z)}\right|,\ z \in U,$$

where $\rho \geq 0$, $\delta \in [-1, 1)$ and $\rho + \delta \geq 0$.

Similarly, $UST(\rho, \delta)$ represents the class of all functions $f \in$ A satisfying:

$$\Re\left[\frac{zf'(z)}{f(z)} - \delta\right] > \rho\left|\frac{zf'(z)}{f(z)} - 1\right|,\ z \in U,$$

where $\rho \geq 0$, $\delta \in [-1, 1)$ and $\rho + \delta \geq 0$.

If follows that $f \in UCV(\rho, \delta)$ iff $zf'(z) \in UST(\rho, \delta)$. We emphasize that these classes generalize other various subclasses defined by several authors, and for $\rho = 0$, we obtain the classes $K(\delta)$ and $S^*(\delta)$, respectively.

(i) Thus, the class of *uniformly convex functions*, $UCV(1, 0) = UCV$, was investigated by Goodman and has an interesting geometric property (see [5]).
(ii) The class $UST(1, 0) = UST$ was defined by Rønning in [6], while the classes $UCV(1, \delta) = UCV(\delta)$ and $UST(1, \delta) = UST(\delta)$ were introduced and investigated by Rønning in [7].
(iii) For $\delta = 0$, the classes $UCV(\rho, 0) =: \rho - UCV$ and $UST(\rho, 0) =: \rho - UST$ were defined by Kanas and Wiśniowska in [8,9], respectively.

The investigation on the q-derivative, which has applications in various branches of mathematics and other related fields, has inspired scholars to use it in geometric function theory, too. Jackson [10,11] described the q-derivative and the q-integral, and certain incipient applications of those functions can be seen in [12]. By applying the idea of convolution, Ķanas and Răducanu [13] presented the q-analogue of the Ruscheweyh differential operator, obtaining the first characteristics of this new operator. Several types of analytical functions defined by the q-analogue of the Ŕuscheweyh differential operator were investigated by Aldweby and Darus [14], Mahmood and Sokol [15], and others. Furthermore, q-difference operators were investigated in [16–18]; fractional calculus aspects were added to the studies regarding q-calculus in [19–21]; and a q-integral operator was used for studies in [22]. The q-Srivastava–Attiya operator is used for investigation on the class of close-to-convex functions in [23], and a q-analogue integral operator is applied for a family of non-Bazilevič functions in [24]. A q-analogue of a multiplier transformation is used for obtaining new differential subordination and superordination results in [25].

We will now introduce the fundamental idea of the q-calculus established by Ĵackson [10] and useful for our research. Additionally, this technique can be used to higher-dimensional domains.

Definition 1 ([10,11])**.** *The q-derivative, or the Jackson derivative, of a function f is defined by*

$$D_q f(z) := \partial_q f(z) = \frac{f(qz) - f(z)}{(q-1)z},\ q \in (0,1),\ z \neq 0.$$

As a remark, for a function $f \in \mathrm{A}$, it follows that

$$D_q f(z) = D_q\left(z + \sum_{\nu=2}^{\infty} a_\nu z^\nu\right) = 1 + \sum_{\nu=2}^{\infty} [\nu]_q a_\nu z^{\nu-1}, \tag{2}$$

where $[\nu]_q$ is the *q-bracket of ν*; that is,

$$[\nu]_q := \frac{1-q^\nu}{1-q} = 1 + \sum_{\ell=1}^{\nu-1} q^\ell,\ [0]_q := 0, \tag{3}$$

and

$$\lim_{q\to 1^-} [\nu]_q = \nu.$$

Definition 2 ([10,11])**.** *For $v \in \mathbb{C}$ and $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, the q-shifted factorial is defined by*

$$(v;q)_0 = 1, \quad (v;q)_k := \prod_{\ell=0}^{k-1}(1 - vq^\ell),$$

and in terms of basic or q-gamma function

$$(q^v;q)_k = \frac{(1-q^k)\Gamma_q(v+k)}{\Gamma_q(v)},\ k \in \mathbb{N}_0,$$

where the q-gamma function is defined by

$$\Gamma_q(z) := \frac{(1-q)^{1-z}(q;q)_\infty}{(q^z;q)_\infty}, \quad |q| < 1,$$

and

$$(v;q)_\infty = \prod_{\ell=0}^{\infty}(1 - vq^\ell),\ |q| < 1.$$

For the q-gamma function, Γ_q, it is known that

$$\Gamma_q(z+1) = [z]_q \Gamma_q(z),$$

where $[z]_q$ is defined by (3), and in terms of the classical gamma function Γ, we have $\lim\limits_{q\to 1^-} \Gamma_q(z) = \Gamma(z)$.

Wang et al. developed in [26], based on the the concept of the convolution and the notion of q-derivative, the *q-analogue Choi–Saigo–Srivastava operator* $I^{q}_{\alpha,\beta} : \mathrm{A} \to \mathrm{A}$,

$$I^{q}_{\alpha,\beta} f(z) := f(z) * \mathcal{F}_{q,\alpha+1,\beta}(z),\ z \in U \quad (\alpha > -1,\ \beta > 0), \tag{4}$$

where

$$\mathcal{F}_{q,\alpha+1,\beta}(z) = z + \sum_{\nu=2}^{\infty} \frac{\Gamma_q(\beta+\nu-1)\Gamma_q(\alpha+1)}{\Gamma_q(\beta)\Gamma_q(\alpha+\nu)} z^{\nu} = z + \sum_{\nu=2}^{\infty} \frac{[\beta,q]_{\nu-1}}{[\alpha+1,q]_{\nu-1}} z^{\nu},\ z \in U,$$

where $[\beta,q]_{\nu}$ stands for the *q-generalized Pochhammer symbol* for $\beta > 0$ defined by

$$[\beta,q]_{\nu} := \begin{cases} 1, & \text{if} \quad \nu = 0, \\ [\beta]_q\,[\beta+1]_q \ldots [\beta+\nu-1]_q, & \text{if} \quad \nu \in \mathbb{N}. \end{cases}$$

Thus,

$$I^{q}_{\alpha,\beta} f(z) = z + \sum_{\nu=2}^{\infty} \frac{[\beta,q]_{\nu-1}}{[\alpha+1,q]_{\nu-1}} a_{\nu} z^{\nu},\ z \in U, \tag{5}$$

while

$$I^{q}_{0,2} f(z) = zD_q f(z) \quad \text{and} \quad I^{q}_{1,2} f(z) = f(z).$$

Definition 3. *For $\mu \geq 0$ and $\tau > -1$, with the aid of the operator $I^{q}_{\alpha,\beta}$, we will define the new linear extended multiplier q-Choi–Saigo–Srivastava operator $D^{m,q}_{\alpha,\beta}(\mu,\tau) : \mathrm{A} \to \mathrm{A}$ as follows:*

$$\begin{aligned} D^{0,q}_{\alpha,\beta}(\mu,\tau) f(z) =&: D^{q}_{\alpha,\beta}(\mu,\tau) f(z) = f(z), \\ D^{1,q}_{\alpha,\beta}(\mu,\tau) f(z) =& \left(1 - \frac{\mu}{\tau+1}\right) I^{q}_{\alpha,\beta} f(z) + \frac{\mu}{\tau+1} zD_q\left(I^{q}_{\alpha,\beta} f(z)\right) \\ =& z + \sum_{\nu=2}^{\infty} \left(\frac{[\beta,q]_{\nu-1}}{[\alpha+1,q]_{\nu-1}} \cdot \frac{\tau+1+\mu([\nu]_q - 1)}{\tau+1} \right) a_{\nu} z^{\nu}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ D^{m,q}_{\alpha,\beta}(\mu,\tau) f(z) =& D^{q}_{\alpha,\beta}(\mu,\tau)\left(D^{m-1,q}_{\alpha,\beta}(\mu,\tau) f(z)\right),\ m \geq 1, \end{aligned}$$

where $\mu \geq 0$, $\tau > -1$, $m \in \mathbb{N}_0$, $\alpha > -1$, $\beta > 0$ and $0 < q < 1$.

If $f \in \mathrm{A}$ has the form (1), from (5) and the above definition, it follows that

$$D^{m,q}_{\alpha,\beta}(\mu,\tau) f(z) = z + \sum_{\nu=2}^{\infty} \aleph^{m,q}_{\alpha,\beta}(\nu,\mu,\tau) a_{\nu} z^{\nu},\ z \in U, \tag{6}$$

where

$$\aleph^{m,q}_{\alpha,\beta}(\nu,\mu,\tau) := \left(\frac{[\beta,q]_{\nu-1}}{[\alpha+1,q]_{\nu-1}} \cdot \frac{\tau+1+\mu([\nu]_q - 1)}{\tau+1} \right)^{m}. \tag{7}$$

From (4) and (7), the operator $D^{m,q}_{\alpha,\beta}(\mu,\tau)$ can be expressed using convolution product as

$$D^{m,q}_{\alpha,\beta}(\mu,\tau) f(z) = \\ \underbrace{\left[\left(I^{q}_{\alpha,\beta} f(z) * \wp^{q}_{\mu,\tau}(z)\right) * \ldots * \left(I^{q}_{\alpha,\beta} f(z) * \wp^{q}_{\mu,\tau}(z)\right)\right]}_{n-\text{times}} * f(z),$$

where

$$\wp^{q}_{\mu,\tau}(z) := \frac{z - \left(1 - \frac{\mu}{\tau+1}\right)qz^2}{(1-z)(1-qz)}.$$

Remark 1. *The following operators, which have been investigated by various authors, are obtained by specifying the parameters q, m, α, β, τ, and μ:*

(i) For $q \to 1^-$, $\alpha = 1$, $\beta = 2$, and $\tau = 0$, the operator D^m_μ was defined and studied by Al-Oboudi [27];
(ii) If $q \to 1^-$, $\alpha = 1$, $\beta = 2$, $\mu = 1$, and $\tau = 0$, the operator D^m was introduced by Sălăgean [28];
(iii) Taking $q \to 1^-$, $\alpha = 1$, and $\beta = 2$, the operator $I^m(\lambda, \ell)$ was studied Cătaş [29];
(iv) Considering $\alpha = 1$, $\beta = 2$, and $\tau = 0$, the operator $D^m_{\mu,q}$ was introduced and studied by Aouf et al. [30];
(v) For $\alpha = 1$, $\beta = 2$, $\mu = 1$, and $\tau = 0$, the operator S^m_q was studied by Govindaraj and Sivasubramanian [18];
(vi) If $q \to 1^-$, the operator $D^{m,\alpha}_{\mu,\tau,\beta}$ was defined and studied by El-Ashwah et al. [31] for $q = 2$, $s = 1$, $\alpha_1 = \beta$, $\alpha_2 = 1$, $\beta_1 = \alpha + 1$;
(vii) Taking $q \to 1^-$, $\alpha = 1$, $\beta = 2$, and $\mu = 1$, the operator I^m_τ, $\tau \geq 0$, was studied by Cho and Srivastava [32];
(viii) Considering $q \to 1^-$, $\mu = \tau = 0$ and $m = 1$, the operator $I^q_{\alpha,\beta}$ was defined and investigated by Wang et al. [26];
(ix) For $q \to 1^-$, $\alpha := 1 - \alpha$, $\beta = 2$, and $\tau = 0$, the operator $D^{m,\alpha}_\mu$ was introduced and studied by Al-Oboudi and Al-Amoudi [33];
(x) If we take $\alpha := 1 - \varrho$ and $\beta = 2$, we obtain the operator $D^{m,\lambda,\ell}_{q,\varrho}$ studied by Kota and El-Ashwah [19];
(xi) Taking $\beta = 2$, $\mu = 0$, and $\tau = 0$, the q-analogue integral operator of Noor $I^q_{\alpha,2}$ was defined and studied in [26];
(xii) Considering $q \to 1^-$, $\beta = 2$, $\mu = 0$, and $\tau = 0$, the differential operator I^ν was studied in [34,35];
(xiii) For $q \to 1^-$, $\beta = 2$, $\alpha := 1 - \alpha$, $\mu = 0$, and $\tau = 0$, the Owa–Srivastava operator $I_{1-\alpha,2}$ was introduced and investigated in [36].

Implementing the linear multiplier q-derivative operator provided by (6), for $\mu \geq 0$, $\tau > -1$, $m \in \mathbb{N}_0$, $\alpha > -1$, $\beta > 0$, $\rho \geq 0$, and $0 < q < 1$, new subclasses $\Pi^{m,q}_{\alpha,\beta}(\mu,\tau,\rho,\delta)$ of *q—uniformly convex functions of order δ* in U, and $\Omega^{m,q}_{\alpha,\beta}(\mu,\tau,\rho,\delta)$ of *q—uniformly starlike functions of order δ* in U are introduced as follows:

Definition 4. *A function $f \in \mathrm{A}$ belongs to $\Pi^{m,q}_{\alpha,\beta}(\mu,\tau,\rho,\delta)$ if:*

$$\Re\left[1 + \frac{D_q\left(zD_q(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(z))\right)}{D_q\left(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(z)\right)} - \delta\right] > \rho\left|\frac{D_q\left(zD_q(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(z))\right)}{D_q\left(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(z)\right)}\right|, \ z \in U, \tag{8}$$

and $f \in \mathrm{A}$ belongs to $\Omega^{m,q}_{\alpha,\beta}(\mu,\tau,\rho,\delta)$ if:

$$\Re\left[\frac{zD_q\left(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(z)\right)}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(z)} - \delta\right] > \rho\left|\frac{zD_q\left(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(z)\right)}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(z)} - 1\right|, \ z \in U. \tag{9}$$

From (8) and (9), we have the next equivalence

$$f \in \Pi^{m,q}_{\alpha,\beta}(\mu,\tau,\rho,\delta) \Leftrightarrow zD_q\left(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(z)\right) \in \Omega^{m,q}_{\alpha,\beta}(\mu,\tau,\rho,\delta). \tag{10}$$

Remark 2. *(i)* $\Omega^{1,q}_{\alpha,\beta}(0,0,\rho,\delta) = \Omega^{q}_{\alpha,\beta}(\rho,\delta)$ *and* $\Pi^{1,q}_{\alpha,\beta}(0,0,\rho,\delta) = \Pi^{q}_{\alpha,\beta}(\rho,\delta)$

$$\left\{ f \in \mathrm{A} : \Re\left\{ \frac{zD_q(D^q_{\alpha,\beta}f(z))}{D^q_{\alpha,\beta}f(z)} - \delta \right\} > \rho \left| \frac{zD_q(D^q_{\alpha,\beta}f(z))}{D^q_{\alpha,\beta}f(z)} - 1 \right|, \right.$$

$$\left. -1 \leq \delta < 1,\ \rho \geq 0,\ m > -1, z \in U \right\},$$

and

$$\left\{ f \in \mathrm{A} : \Re\left\{ \frac{D_q\Big(zD_q(D^q_{\alpha,\beta}f(z))\Big)}{D_q\Big(D^q_{\alpha,\beta}f(z)\Big)} - \delta \right\} > \rho \left| \frac{D_q\Big(zD_q(D^q_{\alpha,\beta}f(z))\Big)}{D_q\Big(D^q_{\alpha,\beta}f(z)\Big)} - 1 \right|, \right.$$

$$\left. -1 \leq \delta < 1,\ \rho \geq 0,\ m > -1, z \in U \right\}.$$

(ii) $\Omega^{m,q}_{1,2}(\mu,\tau,\rho,\delta) = \Omega^{m,q}(\mu,\tau,\rho,\delta)$ *and* $\Pi^{m,q}_{1,2}(\mu,\tau,\rho,\delta) = \Pi^{m,q}(\mu,\tau,\rho,\delta)$

$$\left\{ f \in \mathrm{A} : \Re\left\{ \frac{zD_q(D^m_q(\mu,\tau)f(z))}{D^m_q(\mu,\tau)f(z)} - \delta \right\} > \rho \left| \frac{zD_q(D^m_q(\mu,\tau)f(z))}{D^m_q(\mu,\tau)f(z)} - 1 \right|, \right.$$

$$\left. -1 \leq \delta < 1,\ \rho \geq 0,\ m > -1, z \in U \right\},$$

and

$$\left\{ f \in \mathrm{A} : \Re\left\{ \frac{D_q\Big(zD_q(D^m_q(\mu,\tau)f(z))\Big)}{D_q\Big(D^m_q(\mu,\tau)f(z)\Big)} - \delta \right\} > \rho \left| \frac{D_q\Big(zD_q(D^m_q(\mu,\tau)f(z))\Big)}{D_q\Big(D^m_q(\mu,\tau)f(z)\Big)} - 1 \right|, \right.$$

$$\left. -1 \leq \delta < 1,\ \rho \geq 0,\ m > -1, z \in U \right\}.$$

(iii) $\lim\limits_{q\to 1^-} \Omega^{1}_{\alpha,\beta}(0,0,\rho,\delta) = \Omega_{\alpha,\beta}(\rho,\delta)$ *and* $\lim\limits_{q\to 1^-} \Pi^{1}_{\alpha,\beta}(0,0,\rho,\delta) = \Pi_{\alpha,\beta}(\rho,\delta)$

$$\left\{ f \in \mathrm{A} : \Re\left\{ \frac{z(D_{\alpha,\beta}f(z))'}{D_{\alpha,\beta}f(z)} - \delta \right\} > \rho \left| \frac{z(D_{\alpha,\beta}f(z))'}{D_{\alpha,\beta}f(z)} - 1 \right|, \right.$$

$$\left. -1 \leq \delta < 1,\ \rho \geq 0,\ m > -1, z \in U \right\},$$

and

$$\left\{ f \in \mathrm{A} : \Re\left\{ \frac{z(D_{\alpha,\beta}f(z))''}{(D_{\alpha,\beta}f(z))'} - \delta \right\} > \rho \left| \frac{z(D_{\alpha,\beta}f(z))''}{(D_{\alpha,\beta}f(z))'} - 1 \right|, \right.$$

$$\left. -1 \leq \delta < 1,\ \rho \geq 0,\ m > -1, z \in U \right\}.$$

(iv) $\lim\limits_{q\to 1^-} \Omega^{m,q}_{1,2}(\mu,\tau,\rho,\delta) = \Omega^{m}(\mu,\tau,\rho,\delta)$ *and* $\lim\limits_{q\to 1^-} \Pi^{m,q}_{1,2}(\mu,\tau,\rho,\delta) = \Pi^{m}(\mu,\tau,\rho,\delta)$

$$\left\{ f \in \mathrm{A} : \Re\left\{ \frac{z(D^m(\mu,\tau)f(z))'}{D^m(\mu,\tau)f(z)} - \delta \right\} > \rho \left| \frac{z(D^m(\mu,\tau)f(z))'}{D^m(\mu,\tau)f(z)} - 1 \right|, \right.$$

$$\left. -1 \leq \delta < 1,\ \rho \geq 0,\ m > -1, z \in U \right\},$$

and

$$\left\{ f \in \mathrm{A} : \Re\left\{ \frac{z(D^m(\mu,\tau)f(z))''}{(D^m(\mu,\tau)f(z))'} - \delta \right\} > \rho \left| \frac{z(D^m(\mu,\tau)f(z))''}{(D^m(\mu,\tau)f(z))'} - 1 \right|, \right.$$

$$-1 \leq \delta < 1,\ \rho \geq 0,\ m > -1, z \in U \Big\}.$$

(v) $\Omega^{m,q}_{\alpha,2}(\mu,\tau,\rho,\delta) = \Omega^{m,q}_{\alpha,2}(\mu,\tau,\rho,\delta)$ *and* $\Pi^{m,q}_{\alpha,2}(\mu,\tau,\rho,\delta) = \Pi^{m,q}_{\alpha,2}(\mu,\tau,\rho,\delta)$

$$\left\{ f \in \mathrm{A} : \Re\left\{ \frac{zD_q(D^{m,q}_{\alpha}(\mu,\tau)f(z))}{D^{m,q}_{\alpha}(\mu,\tau)f(z)} - \delta \right\} > \rho \left| \frac{zD_q(D^{m,q}_{\alpha}(\mu,\tau)f(z))}{D^{m,q}_{\alpha}(\mu,\tau)f(z)} - 1 \right|, \right.$$

$$\left. -1 \leq \delta < 1,\ \rho \geq 0,\ m > -1, z \in U \right\},$$

and

$$\left\{ f \in \mathrm{A} : \Re\left\{ \frac{D_q\Big(zD_q(D^{m,q}_{\alpha}(\mu,\tau)f(z)\Big))}{D_q\Big(D^{m,q}_{\alpha}(\mu,\tau)f(z)\Big)} - \delta \right\} > \rho \left| \frac{D_q\Big(zD_q(D^{m,q}_{\alpha}(\mu,\tau)f(z)\Big))}{D_q\Big(D^{m,q}_{\alpha}(\mu,\tau)f(z)\Big)} - 1 \right|, \right.$$

$$\left. -1 \leq \delta < 1,\ \rho \geq 0,\ m > -1, z \in U \right\},$$

where

$$D^{m,q}_{\alpha}(\mu,\tau)f(z) = z + \sum_{\nu=2}^{\infty} \left(\frac{[\nu,q]!}{[\alpha+1,q]_{\nu-1}} \frac{\tau+1+\mu([\nu]_q - 1)}{\tau+1} \right)^m a_\nu z^\nu.$$

(vi) $\lim\limits_{q\to 1^-} \Omega^{m,1}_{\alpha,2}(\mu,\tau,\rho,\delta) = \Omega^{m}_{\alpha,2}(\mu,\tau,\rho,\delta)$ *and* $\Pi^{m,1}_{\alpha,2}(\mu,\tau,\rho,\delta) = \Pi^{m}_{\alpha,2}(\mu,\tau,\rho,\delta)$

$$\left\{ f \in \mathrm{A} : \Re\left\{ \frac{zD_q(D^{m}_{\alpha}(\mu,\tau)f(z))}{D^{m}_{\alpha}(\mu,\tau)f(z)} - \delta \right\} > \rho \left| \frac{zD_q(D^{m}_{\alpha}(\mu,\tau)f(z))}{D^{m}_{\alpha}(\mu,\tau)f(z)} - 1 \right|, \right.$$

$$\left. -1 \leq \delta < 1,\ \rho \geq 0,\ m > -1, z \in U \right\},$$

and

$$\left\{ f \in \mathrm{A} : \Re\left\{ \frac{D_q(zD_q(D^{m}_{\alpha}(\mu,\tau)f(z))))}{D_q(D^{m}_{\alpha}(\mu,\tau)f(z))} - \delta \right\} > \rho \left| \frac{D_q(zD_q(D^{m}_{\alpha}(\mu,\tau)f(z)))}{D_q(D^{m}_{\alpha}(\mu,\tau)f(z))} - 1 \right|, \right.$$

$$\left. -1 \leq \delta < 1,\ \rho \geq 0,\ m > -1, z \in U \right\},$$

where

$$D^{m}_{\alpha}(\mu,\tau)f(z) = z + \sum_{\nu=2}^{\infty} \left(\frac{\nu!}{(\alpha+1)_{\nu-1}} \frac{\tau+1+\mu(\nu - 1)}{\tau+1} \right)^m a_\nu z^\nu.$$

(vii) $\lim\limits_{q\to 1^-} \Omega^{m,1}_{1-\alpha,2}(\mu,\tau,\rho,\delta) = \Omega^{m}_{1-\alpha,2}(\mu,\tau,\rho,\delta)$ *and* $\Pi^{m,1}_{1-\alpha,2}(\mu,\tau,\rho,\delta) = \Pi^{m}_{1-\alpha,2}(\mu,\tau,\rho,\delta)$

$$\left\{ f \in \mathrm{A} : \Re\left\{ \frac{zD_q(D^{m}_{1-\alpha}(\mu,\tau)f(z))}{D^{m}_{1-\alpha}(\mu,\tau)f(z)} - \delta \right\} > \rho \left| \frac{zD_q(D^{m}_{1-\alpha}(\mu,\tau)f(z))}{D^{m}_{1-\alpha}(\mu,\tau)f(z)} - 1 \right|, \right.$$

$$\left. -1 \leq \delta < 1,\ \rho \geq 0,\ m > -1, z \in U \right\},$$

and

$$\left\{ f \in \mathrm{A} : \Re\left\{ \frac{D_q(zD_q(D^{m}_{1-\alpha}(\mu,\tau)f(z))))}{D_q(D^{m}_{1-\alpha}(\mu,\tau)f(z))} - \delta \right\} > \rho \left| \frac{D_q(zD_q(D^{m}_{1-\alpha}(\mu,\tau)f(z)))}{D_q(D^{m}_{1-\alpha}(\mu,\tau)f(z))} - 1 \right|, \right.$$

$$\left. -1 \leq \delta < 1,\ \rho \geq 0,\ m > -1, z \in U \right\},$$

where

$$D^m_{1-\alpha}(\mu,\tau)f(z) = z + \sum_{\nu=2}^{\infty}\left(\frac{\Gamma(\nu+1)\Gamma(2-\alpha)}{\Gamma(1-\alpha+\nu)}\frac{\tau+1+\mu(\nu-1)}{\tau+1}\right)^m a_\nu z^\nu.$$

The following definition and lemma are required to demonstrate our original results.

Definition 5 ([37], p. 690, (Subordinating factor sequence))**.** *A sequence $\{b_\nu\}_{\nu=1}^{\infty}$ of complex numbers is called a subordinating factor sequence if, whenever f of the Form* (1) *is convex (univalent) in U, the following subordination holds:*

$$\sum_{\nu=1}^{\infty} a_\nu b_\nu z^\nu \prec f(z), \quad (a_1 := 1).$$

Lemma 1 ([37], Theorem 2, p. 690)**.** *The sequence $\{b_\nu\}_{\nu=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\Re\left(1 + 2\sum_{\nu=1}^{\infty} b_\nu z^\nu\right) > 0,\ z \in U.$$

The first new outcome, obtained using the operator given by (6) and the related results, presents conditions for a function $f \in \mathrm{A}$ to belong to the newly introduced class $\Omega^{m,q}_{\alpha,\beta}(\mu,\tau,\rho,\delta)$. This first proven theorem is followed by a corollary stating the conditions for a function $f \in \mathrm{A}$ to be in the class $\Pi^{m,q}_{\alpha,\beta}(\mu,\tau,\rho,\delta)$. An example shows that the classes are not empty. A subordination result involving the convolution product of functions from class $\Omega^{m,q}_{\alpha,\beta}(\mu,\tau,\rho,\delta)$ is described in Theorem 2. It is highlighted that this result generalizes known results, and the following corollary proves similar subordination results regarding the class $\Pi^{m,q}_{\alpha,\beta}(\mu,\tau,\rho,\delta)$. An example accompanies the proved results employing the technique used earlier by Attiya [38], Srivastava and Attiya [39], and Singh [40]. Some special cases of this operator are also obtained by Aouf and Mostafa [41] and Frasin [42].

2. Main Results

Unless explicitly stated, it will be presumed throughout this article that $\mu \geq 0$, $\tau > -1$, $m \in \mathbb{N}_0$, $\alpha > -1$, $\beta > 0$, and $0 < q < 1$.

Our initial finding provides a sufficient condition such that the function $f \in \mathrm{A}$ to be considered a member of the class $\Omega^{m,q}_{\alpha,\beta}(\mu,\tau,\rho,\delta)$.

Theorem 1. *If a function $f \in \mathrm{A}$ satisfies the following inequalities:*

$$\sum_{\nu=2}^{\infty}\left|\aleph^{m,q}_{\alpha,\beta}(\nu,\mu,\tau)\right| |a_\nu| < 1, \tag{11}$$

$$\sum_{\nu=2}^{\infty}\left[\rho([\nu]_q - 1) + [\nu]_q - \delta\right]\left|\aleph^{m,q}_{\alpha,\beta}(\nu,\mu,\tau)\right| |a_\nu| \leq 1-\delta, \tag{12}$$

then $f \in \Omega^{m,q}_{\alpha,\beta}(\mu,\tau,\rho,\delta)$.

Proof. For the proof of the assertions of the theorem, it is necessary to show that the following inequality, equivalent to (9), holds:

$$\rho\left|\frac{zD_q\left(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(z)\right)}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(z)} - 1\right| - \Re\left[\frac{zD_q\left(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(z)\right)}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(z)} - 1\right] < 1-\delta,\ z \in U.$$

From the assumption (11), using the *principle of the maximum of the module* of an analytic function and triangle's inequality, it follows that

$$
\begin{aligned}
&\rho\left|\frac{zD_q\left(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(z)\right)}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(z)}-1\right|-\Re\left[\frac{zD_q\left(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(z)\right)}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(z)}-1\right]\\
\leq\ &(1+\rho)\left|\frac{zD_q\left(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(z)\right)}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(z)}-1\right|=(1+\rho)\left|\frac{\sum\limits_{\nu=2}^{\infty}([\nu]_q-1)\aleph^{m,q}_{\alpha,\beta}(\nu,\mu,\tau)a_\nu z^{\nu-1}}{1+\sum\limits_{\nu=2}^{\infty}\aleph^{m,q}_{\alpha,\beta}(\nu,\mu,\tau)a_\nu z^{\nu-1}}\right|\\
<\ &(1+\rho)\left|\frac{\sum\limits_{\nu=2}^{\infty}([\nu]_q-1)\aleph^{m,q}_{\alpha,\beta}(\nu,\mu,\tau)a_\nu e^{i\theta(\nu-1)}}{1+\sum\limits_{\nu=2}^{\infty}\aleph^{m,q}_{\alpha,\beta}(\nu,\mu,\tau)a_\nu e^{i\theta(\nu-1)}}\right|\\
\leq\ &(1+\rho)\frac{\sum\limits_{\nu=2}^{\infty}([\nu]_q-1)\left|\aleph^{m,q}_{\alpha,\beta}(\nu,\mu,\tau)\right||a_\nu|}{1-\sum\limits_{\nu=2}^{\infty}\left|\aleph^{m,q}_{\alpha,\beta}(\nu,\mu,\tau)\right||a_\nu|},\ z\in U,
\end{aligned}
$$

for some $\theta \in \mathbb{R}$. It is easy to check that the last expression is bounded above by $1-\delta$ if the assumption inequalities (11) and (12) are satisfied; hence, $f \in \Omega^{m,q}_{\alpha,\beta}(\mu,\tau,\rho,\delta)$. □

By virtue of (10) and Theorem 1, the subsequent sufficient condition for the function $f \in \mathrm{A}$ to be included in the class $\Pi^{m,q}_{\alpha,\beta}(\mu,\tau,\rho,\delta)$ is shown.

Corollary 1. *Since the function $f \in \mathrm{A}$ given by (1) satisfies the following inequalities:*

$$
\begin{aligned}
&\sum_{\nu=2}^{\infty}[\nu]_q\left|\aleph^{m,q}_{\alpha,\beta}(\nu,\mu,\tau)\right||a_\nu|<1,\\
&\sum_{\nu=2}^{\infty}[\nu]_q\left[\rho([\nu]_q-1)+[\nu]_q-\delta\right]\left|\aleph^{m,q}_{\alpha,\beta}(\nu,\mu,\tau)\right||a_\nu|\leq 1-\delta,
\end{aligned}
\tag{13}
$$

then $f \in \Pi^{m,q}_{\alpha,\beta}(\mu,\tau,\rho,\delta)$.

Proof. If $f \in \mathrm{A}$, using (2) and (6), the following can be stated:

$$g(z):=zD_q\left(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(z)\right)=z+\sum_{\nu=2}^{\infty}[\nu]_q\aleph^{m,q}_{\alpha,\beta}(\nu,\mu,\tau)a_\nu z^\nu,\ z\in U.$$

Therefore, if the assumptions of this theorem hold, according to Theorem 1 it follows that $g \in \Omega^{m,q}_{\alpha,\beta}(\mu,\tau,\rho,\delta)$. According to the equivalence (10), we conclude that $f \in \Pi^{m,q}_{\alpha,\beta}(\mu,\tau,\rho,\delta)$. □

For the particular case $f(z) = z + \lambda z^2$, $\lambda \in \mathbb{C}$, the above two results reduce to the next examples:

Example 1. *1. If*

$$
\begin{aligned}
&\left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right||\lambda|<1,\\
&[(\rho+1)q+1-\delta]\left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right||\lambda|\leq 1-\delta,
\end{aligned}
$$

then $z+\lambda z^2 \in \Omega^{m,q}_{\alpha,\beta}(\mu,\tau,\rho,\delta)$, $\lambda \in \mathbb{C}$.

2. *If*

$$(1+q)\left|\aleph_{\alpha,\beta}^{m,q}(2,\mu,\tau)\right||\lambda|<1,$$
$$(1+q)[(\rho+1)q+1-\delta]\left|\aleph_{\alpha,\beta}^{m,q}(2,\mu,\tau)\right||\lambda|\le 1-\delta,$$

then $z+\lambda z^2\in\Pi_{\alpha,\beta}^{m,q}(\mu,\tau,\rho,\delta)$, $\lambda\in\mathbb{C}$.

Remark 3. *Replacing in the assumptions of the Theorem 1 and of the Corollary 1 the values*

$$a_\nu=\lambda^{\nu-1},\ a_\nu=\frac{\lambda^{\nu-1}}{(\nu-1)!},\ a_\nu=\frac{\lambda(\lambda-1)\dots(\lambda-\nu+2)}{(\nu-1)!},\ a_\nu=\frac{(-1)^\nu}{\nu-1},$$

we obtain sufficient conditions for the functions

$$f(z)=\frac{z}{1-\lambda z},\ f(z)=z\exp(\lambda z),\ f(z)=z(1+z)^\lambda,\ f(z)=z\log(1+z),\ \lambda\in\mathbb{C},$$

to be members of the classes $\Omega_{\alpha,\beta}^{m,q}(\mu,\tau,\rho,\delta)$, *and* $\Pi_{\alpha,\beta}^{m,q}(\mu,\tau,\rho,\delta)$, *respectively.*

Based on the implications of Theorem 1 and Corollary 1, we define the subclasses $\Omega_{\alpha,\beta}^{*m,q}(\mu,\tau,\rho,\delta)\subset\Omega_{\alpha,\beta}^{m,q}(\mu,\tau,\rho,\delta)$ and $\Pi_{\alpha,\beta}^{*m,q}(\mu,\tau,\rho,\delta)\subset\Pi_{\alpha,\beta}^{m,q}(\mu,\tau,\rho,\delta)$, which consist of functions $f\in A$ whose coefficients meet the requirements (12) and (13), respectively.

Certain subordination results for the functions in classes $\Omega_{\alpha,\beta}^{*m,q}(\mu,\tau,\rho,\delta)$ and $\Pi_{\alpha,\beta}^{*m,q}$ (μ,τ,ρ,δ) are provided in the next theorem by applying the techniques previously used by Attiya [38], Srivastava and Attiya [39], and Singh [40].

Theorem 2. *If the function* $f\in A$ *is a member of the class* $\Omega_{\alpha,\beta}^{*m,q}(\mu,\tau;\rho,\delta)$, *then for all* $\phi\in K$, *we have*

$$\frac{\left[\rho\left([2]_q-1\right)+[2]_q-\delta\right]\left|\aleph_{\alpha,\beta}^{m,q}(2,\mu,\tau)\right|}{2\left\{\left[\rho\left([2]_q-1\right)+[2]_q-\delta\right]\left|\aleph_{\alpha,\beta}^{m,q}(2,\mu,\tau)\right|+(1-\delta)\right\}}(f*\phi)(z)\prec\phi(z), \tag{14}$$

and

$$\Re(f(z))>-\frac{\left[\rho\left([2]_q-1\right)+[2]_q-\delta\right]\left|\aleph_{\alpha,\beta}^{m,q}(2,\mu,\tau)\right|+(1-\delta)}{\left[\rho\left([2]_q-1\right)+[2]_q-\delta\right]\left|\aleph_{\alpha,\beta}^{m,q}(2,\mu,\tau)\right|},\ z\in U. \tag{15}$$

The above constant $\dfrac{\left[\rho\left([2]_q-1\right)+[2]_q-\delta\right]\left|\aleph_{\alpha,\beta}^{m,q}(2,\mu,\tau)\right|}{2\left\{\left[\rho\left([2]_q-1\right)+[2]_q-\delta\right]\left|\aleph_{\alpha,\beta}^{m,q}(2,\mu,\tau)\right|+(1-\delta)\right\}}$ *is the best estimate.*

Proof. If $f\in\Omega_{\alpha,\beta}^{*m,q}(\mu,\tau;\rho,\delta)$, and $\phi(z)=z+\sum\limits_{\nu=2}^{\infty}c_\nu z^\nu$ is an arbitrary function of the class K, then

$$\begin{aligned}&\frac{\left[\rho\left([2]_q-1\right)+[2]_q-\delta\right]\left|\aleph_{\alpha,\beta}^{m,q}(2,\mu,\tau)\right|}{2\left\{\left[\rho([2]_q-1)+[2]_q-\delta\right]\left|\aleph_{\alpha,\beta}^{m,q}(2,\mu,\tau)\right|+(1-\delta)\right\}}(f*\phi)(z)\\ =\ &\frac{\left[\rho\left([2]_q-1\right)+[2]_q-\delta\right]\left|\aleph_{\alpha,\beta}^{m,q}(2,\mu,\tau)\right|}{2\left\{\left[\rho([2]_q-1)+[2]_q-\delta\right]\left|\aleph_{\alpha,\beta}^{m,q}(2,\mu,\tau)\right|+(1-\delta)\right\}}\left(z+\sum_{\nu=2}^{\infty}a_\nu c_\nu z^\nu\right).\end{aligned} \tag{16}$$

Thus, by Definition 3, the claim of the theorem is true if the sequence

$$\left\{ \frac{\left[\rho\left([2]_q - 1\right) + [2]_q - \delta\right] \left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right|}{2\left\{ \left[\rho([2]_q - 1) + [2]_q - \delta\right] \left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right| + (1-\delta) \right\}} a_\nu \right\}_{\nu=1}^{\infty} \tag{17}$$

is a subordination factor sequence, with $a_1 = 1$. According to Lemma 1, the following equivalent relation must be proven:

$$\Re\left\{ 1 + \sum_{\nu=1}^{\infty} \frac{\left[\rho([2]_q - 1) + [2]_q - \delta\right] \left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right|}{\left[\rho\left([2]_q - 1\right) + [2]_q - \delta\right] \left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right| + (1-\delta)} a_\nu z^\nu \right\} > 0,\ z \in U. \tag{18}$$

Now,

$$\begin{aligned}
&\Re\left\{ 1 + \sum_{\nu=1}^{\infty} \frac{\left[\rho\left([2]_q - 1\right) + [2]_q - \delta\right] \left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right|}{\left[\rho([2]_q - 1) + [2]_q - \delta\right] \left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right| + (1-\delta)} a_\nu z^\nu \right\} \\
= &\Re\left\{ 1 + \frac{\left[\rho\left([2]_q - 1\right) + [2]_q - \delta\right] \left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right|}{\left[\rho([2]_q - 1) + [2]_q - \delta\right] \left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right| + (1-\delta)} z + \right. \\
&\left. \frac{\sum_{\nu=2}^{\infty} \left[\rho\left([2]_q - 1\right) + [2]_q - \delta\right] \left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right| a_\nu z^\nu}{\left[\rho\left([2]_q - 1\right) + [2]_q - \delta\right] \left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right| + (1-\delta)}. \right\} \\
\geq\ & 1 - \frac{\left[\rho\left([2]_q - 1\right) + [2]_q - \delta\right] \left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right|}{\left[\rho([2]_q - 1) + [2]_q - \delta\right] \left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right| + (1-\delta)} r - \\
&\frac{\sum_{\nu=2}^{\infty} \left[\rho\left([2]_q - 1\right) + [2]_q - \delta\right] \left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right| |a_\nu| r^\nu}{\left[\rho\left([2]_q - 1\right) + [2]_q - \delta\right] \left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right| + (1-\delta)} \\
>\ & 1 - \frac{\left[\rho\left([2]_q - 1\right) + [2]_q - \delta\right] \left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right|}{\left[\rho([2]_q - 1) + [2]_q - \delta\right] \left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right| + (1-\delta)} r - \\
&\frac{1-\delta}{\left[\rho\left([2]_q - 1\right) + [2]_q - \delta\right] \left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right| + (1-\delta)} r. \\
=\ & 1 - r > 0,
\end{aligned}$$

Thus, (17) holds true in U. The proof of (14) follows by considering $\phi(z) = \dfrac{z}{1-z}$ in (13). Next, choosing the function $f_0(z) \in \Omega^{*m,q}_{\alpha,\beta}(\mu,\tau;\rho,\delta)$ given by

$$f_0(z) = z - \frac{(1-\delta)}{\left[\rho\left([2]_q - 1\right) + [2]_q - \delta\right] \left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right|} z^2 \quad (-1 \leq \delta < 1; \rho \geq 0) \tag{19}$$

and by using (13), we have

$$\frac{\left[\rho\left([2]_q - 1\right) + [2]_q - \delta\right] \left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right|}{2\left\{ \left[\rho([2]_q - 1) + [2]_q - \delta\right] \left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right| + (1-\delta) \right\}} f_0(z) \prec \frac{z}{1-z}.\ (z \in U) \tag{20}$$

It can be easily verified that

$$\min_{|z|\leq 1} \Re\left[\frac{\left[\rho([2]_q-1)+[2]_q-\delta\right]\left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right|}{2\left\{\left[\rho\left([2]_q-1\right)+[2]_q-\delta\right]\left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right|+(1-\delta)\right\}}f_0(z)\right]=-\frac{1}{2},\ (z\in U) \quad (21)$$

then the constant $\frac{\left[\rho\left([2]_q-1\right)+[2]_q-\delta\right]\left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right|}{2\left\{\left[\rho([2]_q-1)+[2]_q-\delta\right]\left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right|+(1-\delta)\right\}}$ is the best possible. The theorem's proof is now complete. □

Remark 4. *Employing $q\to 1^-, \alpha=1, \beta=2, \mu=1$ and $\tau=0$ in Theorem 2, the results previously obtained by Aouf and Mostafa ([41], Theorem 2.4); are found.*

Similarly, we can demonstrate the following corollary by using (10) and Theorem 2.

Corollary 2. *Consider the function $f(z)\in$ A from the class $\Pi^{*m,q}_{\alpha,\beta}(\mu,\tau;\rho,\delta)$. In this case, the following relation is true:*

$$\frac{\left[\rho\left([2]_q-1\right)+[2]_q-\delta\right][2]_q\left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right|}{\left\{2\left[\rho([2]_q-1)+[2]_q-\delta\right][2]_q\left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right|+(1-\delta)\right\}}(f*\phi)(z)\prec\phi(z)\ \ (z\in U;\phi\in CV), \quad (22)$$

and

$$\Re(f(z))>-\frac{\left[\rho([2]_q-1)+[2]_q-\delta\right][2]_q\left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right|+(1-\delta)}{2\left[\rho([2]_q-1)+[2]_q-\delta\right][2]_q\left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right|}\ \ (z\in U). \quad (23)$$

The constant $\frac{[2]_q\left[\rho([2]_q-1)+[2]_q-\delta\right]\left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right|}{\left\{2\left[\rho\left([2]_q-1\right)+[2]_q-\delta\right][2]_q\left|\aleph^{m,q}_{\alpha,\beta}(2,\mu,\tau)\right|+(1-\delta)\right\}}$ is the best estimate.

Putting $\mu=\tau=0$ and $m=1$ in Theorem 2, the subsequent corollary emerges.

Corollary 3. *Consider the function $f(z)\in$ A a member of the class $\Omega^{*m,q}_{\alpha,\beta}(\mu,\tau,\rho,\delta)$. The following subordination is satisfied:*

$$\frac{\left[\rho\left([2]_q-1\right)+[2]_q-\delta\right]\left|D^q_{\alpha,\beta}f(z)\right|}{2\left\{\left[\rho\left([2]_q-1\right)+[2]_q-\delta\right]\left|D^q_{\alpha,\beta}f(z)\right|+(1-\delta)\right\}}(f*\phi)(z)\prec\phi(z)\ \ (z\in U;\phi\in CV),$$

and

$$\Re\{f(z)\}>-\frac{\left[\rho([2]_q-1)+[2]_q-\delta\right]\left|D^q_{\alpha,\beta}f(z)\right|+(1-\delta)}{\left[\rho\left([2]_q-1\right)+[2]_q-\delta\right]\left|D^q_{\alpha,\beta}f(z)\right|}\ \ (z\in U).$$

The constant $\frac{\left[\rho\left([2]_q-1\right)+[2]_q-\delta\right]\left|D^q_{\alpha,\beta}f(z)\right|}{2\left\{\left[\rho([2]_q-1)+[2]_q-\delta\right]\left|D^q_{\alpha,\beta}f(z)\right|+(1-\delta)\right\}}$ is the best estimate.
Putting $\alpha=1$ and $\beta=2$ in Theorem 2, the next corollary can be stated.

Corollary 4. *Let the function $f(z)\in$ A be in the class $\Omega^{*m,q}_{\alpha,\beta}(\mu,\tau,\rho,\delta)$. Then*

$$\frac{\left[\rho\left([2]_q-1\right)+[2]_q-\delta\right]\left|\frac{\tau+1+\mu([2]_q-1)}{\tau+1}\right|^m}{2\left\{\left[\rho([2]_q-1)+[2]_q-\delta\right]\left|\frac{\tau+1+\mu([2]_q-1)}{\tau+1}\right|^m+(1-\delta)\right\}}(f*\phi)(z)\prec\phi(z)\ \ (z\in U;\phi\in CV),$$

and

$$\Re(f(z))>-\frac{\left[\rho([2]_q-1)+[2]_q-\delta\right]\left|\frac{\tau+1+\mu([2]_q-1)}{\tau+1}\right|^m f(z)+(1-\delta)}{\left[\rho\left([2]_q-1\right)+[2]_q-\delta\right]\left|\frac{\tau+1+\mu([2]_q-1)}{\tau+1}\right|^m f(z)}\ \ (z\in U).$$

The constant $\frac{\left[\rho\left([2]_q-1\right)+[2]_q-\delta\right]\left|\frac{\tau+1+\mu([2]_q-1)}{\tau+1}\right|^m}{2\left\{\left[\rho([2]_q-1)+[2]_q-\delta\right]\left|\frac{\tau+1+\mu([2]_q-1)}{\tau+1}\right|^m+(1-\delta)\right\}}$ is the best estimate.

Employing $\mu=1,\tau=0$ and $m=1$ in Corollary 4, we obtain the following particular case as an example.

Example 2. *(i) Let the function $f(z)\in A$ defined by (1) be in the class $\Omega^{*m,q}_{\alpha,\beta}(\mu,\tau,\rho,\delta)$. Then,*

$$\frac{\left[\rho\left([2]_q-1\right)+[2]_q-\delta\right][2]_q}{2\left\{\left[\rho\left([2]_q-1\right)+[2]_q-\delta\right][2]_q+(1-\delta)\right\}}(f*\phi)(z)\prec\phi(z)\ \ (z\in U;\phi\in CV),$$

and

$$\Re\{f(z)\}>-\frac{\left[\rho([2]_q-1)+[2]_q-\delta\right][2]_q+(1-\delta)}{\left[\rho([2]_q-1)+[2]_q-\delta\right][2]_q}\ \ (z\in U).$$

The constant $\frac{\left[\rho\left([2]_q-1\right)+[2]_q-\delta\right][2]_q}{2\left\{\left[\rho\left([2]_q-1\right)+[2]_q-\delta\right][2]_q+(1-\delta)\right\}}$ *is the best estimate. (ii) For $\rho=0$ in (i) then*

$$\frac{\left([2]_q-\delta\right)[2]_q}{2\left\{([2]_q-\delta)[2]_q+(1-\delta)\right\}}(f*\phi)(z)\prec\phi(z)\ \ (z\in U;\phi\in CV),$$

and

$$\Re\{f(z)\}>-\frac{([2]_q-\delta)[2]_q+(1-\delta)}{([2]_q-\delta)[2]_q}\ \ (z\in U).$$

The constant $\frac{\left([2]_q-\delta\right)[2]_q}{2\left\{([2]_q-\delta)[2]_q+(1-\delta)\right\}}$ *is the best estimate.*

Remark 5. *Letting $q\to1^-$ and $m=0$ in Corollary 2, we have the results proved by Frasin ([42], Corollaries, 2.5).*

3. Conclusions

This study employs means of q-operators combined with differential subordination techniques and the notion of convolution. A new linear extended multiplier q-Choi–Saigo–Srivastava operator in the open unit disk U is introduced in Definition 3. This operator is used for introducing and investigating the subclasses of normalized analytic functions presented in Definition 4, $\Omega^{*m,q}_{\alpha,\beta}(\mu,\tau,\rho,\delta)$ and $\Pi^{*m,q}_{\alpha,\beta}(\mu,\tau,\rho,\delta)$. Subordination results involving the Hadamard product of the associated functions are established in two theorems. Interesting corollaries and particular cases are shown for each of those theorems for particular choices of parameters found in the definition of the classes. Examples are also associated with the theorems to highlight the relevance of the new results.

In future investigations, the new linear extended multiplier q-Choi–Saigo–Srivastava given in Definition 3 can be applied for further developments in the theories of differential subordination and its dual, differential superordination introduced by Miller and Mocanu in 2003 [43] as performed in [20,21]. The newer theories of strong differential subordination and superordination can be considered for investigations involving the new operator, as presented in [44]. In addition, the theories of fuzzy differential subordination and superordination can be applied as was done recently in [45,46]. The q-operator employed in this study can be used for defining other subclasses of analytic functions as it has been done for α-convex functions in [47] or for multivalent functions in [48].

Author Contributions: Conceptualization, E.E.A. and G.I.O.; methodology, E.E.A. and A.M.A.; validation, G.I.O., S.A.S. and E.E.A.; formal analysis, S.A.S., A.M.A. and G.I.O.; investigation, E.E.A., A.M.A. and S.A.S.; writing–original draft preparation, E.E.A., A.M.A. and S.A.S.; writing–review and editing, G.I.O., S.A.S. and E.E.A.; supervision, G.I.O. and E.E.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Miller, S.S.; Mocanu, P.T. Differential subordinations and univalent functions. *Mich. Math. J.* **1981**, *28*, 157–171. [CrossRef]
2. Miller, S.S.; Mocanu, P.T. *Differential Subordinations: Theory and Applications*; Series on Monographs and Texbooks in Pure and Applied Mathematics; Marcel Dekker: New York, NY, USA; Basel, Switzerland, 2000; Volume 225.
3. Bulboaca, T. *Differential Subordinations and Superordinations, Recent Results*; House of Scientific Book Publication: Cluj-Napoca, Romania, 2005.
4. Bharati, R.; Parvatham, R.; Swaminathan, A. On subclasses of uniformly cunvex functions and corresponding class of starlike functions. *Tamkang J. Math.* **1927**, *28*, 17–32. [CrossRef]
5. Goodmen, A.W. On uniformly convex functions. *Ann. Polon. Math.* **1991**, *56*, 87–92. [CrossRef]
6. Ronning, F. Uniformly convex functions and a corresponding class of starlike functions. *Proc. Am. Math. Soc.* **1993**, *118*, 189–196. [CrossRef]
7. Ronning, F. On starlike functions associated with parabolic regions. *Ann. Univ. Mariae Curiesklodowska Sect. A* **1991**, *45*, 117–122.
8. Kanas, S.; Wisniowska, A. Conic regions and k-uniform convexity. *Comput. Appl. Math.* **1999**, *105*, 327–336. [CrossRef]
9. Kanas, S.; Wisniowska, A. Conic domains and starlike functions. *Rev. Roum. Math. Pures Appl.* **2000**, *45*, 647–657.
10. Jackson, F.H. On q-functions and a certain difference operator. *Earth Environ. Sci. Trans. R. Soc. Edinb.* **1909**, *46*, 253–281. [CrossRef]
11. Jackson, F.H. On q-definite integrals. *Quart. J. Pure Appl. Math.* **1910**, *41*, 193–203.
12. Mason, T.E. On properties of the solution of linear q-difference equations with entire function coefficients. *Am. J. Math.* **1915**, *37*, 439–444. [CrossRef]
13. Kanas, S.; Raducanu, D. Some class of analytic functions related to conic domains. *Math. Slovaca* **2014**, *64*, 1183–1196. [CrossRef]
14. Aldweby, H.; Darus, M. Some subordination results on q-analogue of Ruscheweyh differential operator. *Abstr. Appl. Anal.* **2014**, *2014*, 958563. [CrossRef]
15. Mahmood, S.; Sokol, J. New subclass of analytic functions in conical domain associated with Ruscheweyh q-differential operator. *Results Math.* **2017**, *71*, 1345–1357. [CrossRef]
16. Ali, E.E.; Bulboaca, T. Subclasses of Multivalent Analytic Functions Associated with a q-Difference Operator. *Mathematics* **2020**, *8*, 2184. [CrossRef]
17. Ali, E.E.; Lashin, A.M.; Albalahi, A.M. Coefficient Estimates for Some Classes of Bi-univalent Function Associated with Jackson q-Difference Operator. *J. Funct. Spac.* **2022**, *2022*, 2365918.
18. Govindaraj, M.; Sivasubramanian, S. On a class of analytic functions related to conic domains involving q-calculus. *Anal. Math.* **2017**, *43*, 475–487. [CrossRef]
19. Kota, W.Y.; El-Ashwah, R.M. Some application of subordination theorems associated with fractional q-calculus operator. *Math. Bohem.* **2022**, *148*, 18.
20. Alb Lupaş, A.; Oros, G.I. Differential sandwich theorems involving Riemann-Liouville fractional integral of q-hypergeometric function. *AIMS Math.* **2023**, *8*, 4930–4943. [CrossRef]
21. Alb Lupaş, A.; Oros, G.I. Sandwich-type results regarding Riemann-Liouville fractional integral of q-hypergeometric function. *Demonstr. Math.* **2023**, *56*, 20220186. [CrossRef]

22. Srivastava, H.M.; Khan, S.; Ahmad, Q.Z.; Khan, N.; Hussain, S. The Faber polynomial expansion method and its application to the general coefficient problem for some subclasses of bi-univalent functions associated with a certain q-integral operator. *Stud. Univ. Babecs-Bolyai Math.* **2018**, *63*, 419–436. [CrossRef]
23. Breaz, D.; Alahmari, A.A.; Cotîrlă, L.-I.; Ali Shah, S. On Generalizations of the Close-to-Convex Functions Associated with q-Srivastava-Attiya Operator. *Mathematics* **2023**, *11*, 22. [CrossRef]
24. Hadi, S.H.; Darus, M.; Ghanim, F.; Alb Lupaş, A. Sandwich-Type Theorems for a Family of Non-Bazilevič Functions Involving a q-Analog Integral Operator. *Mathematics* **2023**, *11*, 2479. [CrossRef]
25. Alb Lupaş, A.; Cătaş, A. Differential Subordination and Superordination Results for q-Analogue of Multiplier Transformation. *Fractal Fract.* **2023**, *7*, 199. [CrossRef]
26. Wang, Z.G.; Hussain, S.; Naeem, M.; Mahmood, T.; Khan, S. A subclass of univalent functions associated with q-analogue of Choi-Saigo-Srivastava operator. *Hacet. J. Math. Stat.* **2019**, *49*, 1471–1479. [CrossRef]
27. Al-Oboudi, F.M. On univalent functions defined by a generalized Sălăgean operator. *Int. J. Math. Math Sci.* **2004**, *27*, 1429–1436. [CrossRef]
28. Sălăgean, G.S. Subclasses of univalent functions. In *Complex Analysis-Fifth Romanian-Finish Seminar, Part-I, Bucharest, 1981*; Lecture notes in Math; Springer: Berlin/Heidelberg, Germany, 1983; Volume 1013, pp. 362–372.
29. Cătaş, A. On certain classes of p-valent functions defined by multiplier transformations. In *Proceedings of the International Symposium on Geometric Function Theory and Applications: GFTA 2007 Proceedings, Istanbul, Turkey, 20–24 August 2007*; Owa, S., Polatoġ lu, Y., Eds.; TC Istanbul Kűltűr University Publications, TC Istanbul Kűltűr University: Istanbul, Turkey, 2008; Volume 91, pp. 241–250.
30. Aouf, M.K.; Mostafa, A.O.; Elmorsy, R.E. Certain subclasses of analytic functions with varying arguments associated with q -difference operator. *Afr. Mat.* **2021**, *32*, 621–630. [CrossRef]
31. El-Ashwah, R.M.; Aouf, M.K.; Shamandy, A.; Ali, E.E. Subordination results for some subclasses of analytic functions. *Math. Bohem.* **2011**, *136*, 311–331. [CrossRef]
32. Cho, N.E.; Srivastava, H.M. Argument estimates of certain analytic functions defined by a class of multiplier transformations. *Math. Comput. Model.* **2003**, *37*, 39–49. [CrossRef]
33. Al-Oboudi, F.M.; Al-Amoudi, K.A. On classes of analytic functions related to conic domain. *J. Math. Anal. Appl.* **2008**, *339*, 655–667. [CrossRef]
34. Noor, K.I. On new classes of integral operator. *J. Nat. Geom.* **1999**, *16*, 71–80.
35. Noor, K.I.; Noor, M.A. On integral operators. *J. Math. Anal. Appl.* **1999**, *238*, 341–352. [CrossRef]
36. Shareef, Z.; Hussain, S.; Darus, M. Convolution operator in geometric functions theory. *J. Inequal. Appl.* **2012**, *2012*, 213. [CrossRef]
37. Wilf, H.S. Subordinating factor sequence for convex maps of the unit circle. *Proc. Am. Math. Soc.* **1961**, *12*, 689–693. [CrossRef]
38. Attiya, A.A. On some applications of a subordination theorem. *J. Math. Anal. Appl.* **2005**, *311*, 489–494. [CrossRef]
39. Srivastava, H.M.; Attiya, A.A. Some subordination result associated with certain subclasses of analytic function. *J. Inequal. Pure Appl. Math.* **2004**, *5*, 82.
40. Singh, S. A subordination theorems for spirallike functions. *Int. J. Math. Math. Sci.* **2004**, *24*, 433–435. [CrossRef]
41. Aouf, M.K.; Mostafa, A.O. Some Subordinating Results for Classes of Functions Defined by Sălăgean Type q-Derivative Operator. *Filomat* **2020**, *7*, 2283–2292. [CrossRef]
42. Frasin, B.A. Subordination results for a class of analytic functions defined by linear operator. *J. Inequal. Pure. Appl. Math.* **2006**, *7*, 134.
43. Miller, S.S.; Mocanu, P.T. Subordinations of differential superordinations. *Complex Var.* **2003**, *48*, 815–826.
44. Alb Lupaş, A.; Ghanim, F. Strong Differential Subordination and Superordination Results for Extended q-Analogue of Multiplier Transformation. *Symmetry* **2023**, *15*, 713. [CrossRef]
45. Alb Lupaş, A.; Oros, G.I. Fuzzy Differential Subordination and Superordination Results Involving the q-Hypergeometric Function and Fractional Calculus Aspects. *Mathematics* **2022**, *10*, 4121. [CrossRef]
46. Alb Lupaş, A.; Shah, S.A.; Iambor, L.F. Fuzzy differential subordination and superordination results for q-analogue of multiplier transformation. *AIMS Math.* **2023**, *8*, 15569–15584. [CrossRef]
47. Azzam, A.F.; Ali Shah, S.; Alburaikan, A.; El-Deeb, S.M. Certain Inclusion Properties for the Class of q-Analogue of Fuzzy α-Convex Functions. *Symmetry* **2023**, *15*, 509. [CrossRef]
48. Noor, S.; Al-Sa'di, S.; Hussain, S. Some Subordination Results Defined by Using the Symmetric q-Differential Operator for Multivalent Functions. *Axioms* **2023**, *12*, 313. [CrossRef]