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The Study of Bicomplex-Valued Controlled Metric Spaces with Applications to Fractional Differential Equations

Gunaseelan Mani [1], Salma Haque [2], Arul Joseph Gnanaprakasam [3], Ozgur Ege [4] and Nabil Mlaiki [2,*]

[1] Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 602105, Tamil Nadu, India; mathsguna@yahoo.com
[2] Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia; shaque@psu.edu.sa
[3] Department of Mathematics, Faculty of Engineering and Technology, SRM Institute of Science and Technology, SRM Nagar, Kattankulathur 603203, Tamil Nadu, India; aruljoseph.alex@gmail.com
[4] Department of Mathematics, Ege University, Bornova, 35100 Izmir, Turkey; ozgur.ege@ege.edu.tr
* Correspondence: nmlaiki2012@gmail.com

Abstract: In this paper, we introduce the concept of bicomplex-valued controlled metric spaces and prove fixed point theorems. Our results mainly focus on generalizing and expanding some recently established results. Finally, we explain an application of our main result to a certain type of fractional differential equation.

Keywords: controlled-type metric spaces; bicomplex-valued controlled metric spaces; integral equation; fixed point

MSC: 47H09; 47H10; 30G35; 46N99; 54H25

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1. Introduction

Fixed point theory is an important branch of non-linear analysis. After the celebrated Banach contraction principle [1], a number of authors have been working in this area of research. Fixed point theorems (FPTs) are important instruments for proving the existence and uniqueness of solutions to variational inequalities. Metric FPTs expanded after the well-known Banach contraction theorem was established. From this point forward, there have been numerous results related to maps fulfilling various contractive conditions and many types of metric spaces (see, for example, [2–9]).

The authors of [10,11] presented a novel extension of the b-metric space known as controlled metric spaces (CMSs) and demonstrated the FPTs on the CMSs, providing an example by employing a control function $\aleph(x, y)$ in the triangle inequality.

Serge [12] made a pioneering attempt at developing special algebra. He conceptualized commutative generalizations of complex numbers as briefly bicomplex numbers (BCN), briefly tricomplex numbers (tcn), etc., as elements of an infinite set of algebra. Subsequently, many researchers contributed in this area, (see, for example, [13–19]).

In 2021, the authors of [20] proved a common fixed point for a pair of contractive-type maps in bicomplex-valued metric spaces. Later, several authors discussed their results using this concept, see [21–24]. Guechi [25] introduced the concept of optimal control of ϕ-Hilfer fractional equations and proved the fixed point results. For details, see [26–28] and the references therein.

In this paper, we introduce the notion of bicomplex-valued CMSs (BVCMSs) and prove FPT under Banach, Kannan and Fisher contractions on BVCMSs. Then, we give an application to solve a fractional differential equation (FDE) and show that this extension is different from bicomplex-valued metric spaces in terms of Beg, Kumar Datta and Pal [20].

2. Preliminaries

We use standard notations throughout this paper: The real, complex, and bicomplex number sets are represented by $\mathbb{C}_0$, $\mathbb{C}_1$ and $\mathbb{C}_2$, respectively. The following complex numbers were described by Segre [12].

$$z = \vartheta_1 + \vartheta_2 i_1,$$

where $\vartheta_1, \vartheta_2 \in \mathbb{C}_0$, $i_1^2 = -1$. We represent $\mathbb{C}_1$ as:

$$\mathbb{C}_1 = \{z : z = \vartheta_1 + \vartheta_2 i_1, \vartheta_1, \vartheta_2 \in \mathbb{C}_0\}.$$

Let $z \in \mathbb{C}_1$, then $|z| = (\vartheta_1^2 + \vartheta_2^2)^{\frac{1}{2}}$. Every element in $\mathbb{C}_1$ with a positive real-valued norm function $\|.\| : \mathbb{C}_1 \to \mathbb{C}_0^+$ is defined by

$$\|z\| = (\vartheta_1^2 + \vartheta_2^2)^{\frac{1}{2}}.$$

Segre [12] described the bicomplex number (BCN) as:

$$f = \vartheta_1 + \vartheta_2 i_1 + \vartheta_3 i_2 + \vartheta_4 i_1 i_2,$$

where $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \in \mathbb{C}_0$, and the independent units i_1, i_2 satisfy $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$. We represent the BCN set $\mathbb{C}_2$ as:

$$\mathbb{C}_2 = \{f : f = \vartheta_1 + \vartheta_2 i_1 + \vartheta_3 i_2 + \vartheta_4 i_1 i_2, \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \in \mathbb{C}_0\},$$

that is,

$$\mathbb{C}_2 = \{f : f = z_1 + i_2 z_2, z_1, z_2 \in \mathbb{C}_1\},$$

where $z_1 = \vartheta_1 + \vartheta_2 i_1 \in \mathbb{C}_1$ and $z_2 = \vartheta_3 + \vartheta_4 i_1 \in \mathbb{C}_1$. If $f = z_1 + i_2 z_2$ and $\nu = \omega_1 + i_2 \omega_2$ are any two BCNs, then their sum is

$$\begin{aligned} f \pm \nu &= (z_1 + i_2 z_2) \pm (\omega_1 + i_2 \omega_2) \\ &= z_1 \pm \omega_1 + i_2(z_2 \pm \omega_2), \quad \text{and the product is} \\ f.\nu &= (z_1 + i_2 z_2)(\omega_1 + i_2 \omega_2) \\ &= (z_1 \omega_1 - z_2 \omega_2) + i_2(z_1 \omega_2 + z_2 \omega_1). \end{aligned}$$

There are four idempotent elements in $\mathbb{C}_2$. They are $0, 1, \mathfrak{e}_1 = \frac{1+i_1 i_2}{2}, \mathfrak{e}_2 = \frac{1-i_1 i_2}{2}$ of which $\mathfrak{e}_1$ and $\mathfrak{e}_2$ are non-trivial, such that $\mathfrak{e}_1 + \mathfrak{e}_2 = 1$ and $\mathfrak{e}_1 \mathfrak{e}_2 = 0$. Every BCN $z_1 + i_2 \mathfrak{z}_2$ can be uniquely expressed as a combination of $\mathfrak{e}_1$ and $\mathfrak{e}_2$, namely,

$$f = z_1 + i_2 z_2 = (z_1 - i_1 z_2)\mathfrak{e}_1 + (z_1 + i_1 z_2)\mathfrak{e}_2.$$

This representation of f is known as the idempotent representation of a BCN, and the complex coefficients $f_1 = (z_1 - i_1 z_2)$ and $f_2 = (z_1 + i_1 z_2)$ are known as the idempotent components of the BCN f.

Each element in $\mathbb{C}_2$ with a positive real-valued norm function $\|.\| : \mathbb{C}_2 \to \mathbb{C}_0^+$ is defined by

$$\begin{aligned} \|f\| = \|z_1 + i_2 z_2\| &= \{\|z_1\|^2 + \|z_2\|^2\}^{\frac{1}{2}} \\ &= \left[\frac{|z_1 - i_1 z_2|^2 + |z_1 + i_1 z_2|^2}{2}\right]^{\frac{1}{2}} \\ &= (\vartheta_1^2 + \vartheta_2^2 + \vartheta_3^2 + \vartheta_4^2)^{\frac{1}{2}}, \end{aligned}$$

where $f = \vartheta_1 + \vartheta_2 i_1 + \vartheta_3 i_2 + \vartheta_4 i_1 i_2 = z_1 + i_2 z_2 \in \mathbb{C}_2$.

The linear space $\mathbb{C}_2$ with respect to a defined norm is a normed linear space, and $\mathbb{C}_2$ is complete. Therefore, $\mathbb{C}_2$ is a Banach space. If $f, \nu \in \mathbb{C}_2$, then $\|f\nu\| \leq \sqrt{2}\|f\|\|\nu\|$ holds instead of $\|f\nu\| \leq \|f\|\|\nu\|$, and therefore $\mathbb{C}_2$ is not a Banach algebra. For any two BCN $f, \nu \in \mathbb{C}_2$, then

1. $f \preceq_{i_2} \nu \iff \|f\| \leq \|\nu\|$;
2. $\|f + \nu\| \leq \|f\| + \|\nu\|$;
3. $\|\vartheta f\| = |\vartheta|\|f\|$, where ϑ is in $\mathbb{C}_0$;
4. $\|f\nu\| \leq \sqrt{2}\|f\|\|\nu\|$, and $\|f\nu\| = \sqrt{2}\|f\|\|\nu\|$ holds when only one of f or ν is degenerated;
5. $\|f^{-1}\| = \|f\|^{-1}$, if f is degenerated with $f \succ 0$;
6. $\|\frac{f}{\nu}\| = \frac{\|f\|}{\|\nu\|}$, if ν is a degenerated BCN.

The relation $\preceq_{i_2}$ (partial order) is defined on $\mathbb{C}_2$ as given below. Let $\mathbb{C}_2$ be a set of BCNs and $f = z_1 + i_2 z_2$ and $\nu = \omega_1 + i_2\omega_2 \in \mathbb{C}_2$. Then, $f \preceq_{i_2} \nu$ if and only if $z_1 \preceq_{i_2} \omega_1$ and $z_2 \preceq_{i_2} \omega_2$, i.e., $f \preceq_{i_2} \nu$, if one of the following conditions is fulfilled:

1. $z_1 = \omega_1, z_2 = \omega_2$;
2. $z_1 \prec_{i_2} \omega_1, z_2 = \omega_2$;
3. $z_1 = \omega_1, z_2 \prec_{i_2} \omega_2$;
4. $z_1 \prec_{i_2} \omega_1, z_2 \prec_{i_2} \omega_2$.

Clearly, we can write $f \lnsim_{i_2} \nu$ if $f \preceq_{i_2} \nu$ and $f \neq \nu$, i.e., if 2, 3 or 4 are satisfied, and we will write $f \prec_{i_2} \nu$ if only 4 is satisfied.

Definition 1 ([10]). *Let $\zeta \neq \emptyset$ and $\varphi\colon \zeta \times \zeta \to [1, \infty)$. The functional $\pounds_{cm}\colon \zeta \times \zeta \to [0, \infty)$ is called the briefly controlled-type metric CMT if*

(CMT_1) $\pounds_{cm}(\aleph, \mathfrak{i}) = 0 \iff \aleph = \mathfrak{i}$,
(CMT_2) $\pounds_{cm}(\aleph, \mathfrak{i}) = \pounds_{cm}(\mathfrak{i}, \aleph)$,
(CMT_3) $\pounds_{cm}(\aleph, \flat) \leq \varphi(\aleph, \mathfrak{i})\pounds_{cm}(\aleph, \mathfrak{i}) + \varphi(\mathfrak{i}, \flat)\pounds_{cm}(\mathfrak{i}, \flat)$,

for all $\aleph, \mathfrak{i}, \flat \in \zeta$. Then, the doublet $(\zeta, \pounds_{cm})$ is called a CMT space.

Several researchers have proven FPTs using this notion (see [3,4,6,11]).

Definition 2. *Let $\zeta \neq \emptyset$ and consider $\varphi\colon \zeta \times \zeta \to [1, \infty)$. The functional $\pounds_{bvcms}\colon \zeta \times \zeta \to \mathbb{C}_2$ is said to be a BVCMS if*

$(BCCMS_1)$ $0 \precsim_{i_2} \pounds_{bvcms}(\aleph, \mathfrak{i})$ *also* $\pounds_{bvcms}(\aleph, \mathfrak{i}) = 0 \iff \aleph = \mathfrak{i}$,
$(BCCMS_2)$ $\pounds_{bvcms}(\aleph, \mathfrak{i}) = \pounds_{bvcms}(\mathfrak{i}, \aleph)$,
$(BCCMS_3)$ $\pounds_{bvcms}(\aleph, \flat) \precsim_{i_2} \varphi(\aleph, \mathfrak{i})\pounds_{bvcms}(\aleph, \mathfrak{i}) + \varphi(\mathfrak{i}, \flat)\pounds_{bvcms}(\mathfrak{i}, \flat)$,

for all $\aleph, \mathfrak{i}, \flat \in \zeta$, Then, the pair $(\zeta, \pounds_{bvcms})$ is known as a BVCMS.

Example 1. *Let $\zeta = [0, \infty)$ and $\varphi\colon \zeta \times \zeta \to [1, \infty)$ be defined as*

$$\varphi(\aleph, \mathfrak{i}) = \begin{cases} 1, & \text{if } \aleph, \mathfrak{i} \in [0, 1], \\ 1 + \aleph + \mathfrak{i}, & \text{otherwise.} \end{cases}$$

and $\pounds_{bvcms}\colon \zeta \times \zeta \to [0, \infty)$ be defined as follows

$$\pounds_{bvcms}(\aleph, \mathfrak{i}) := \begin{cases} 0, & \aleph = \mathfrak{i} \\ i_2, & \aleph \neq \mathfrak{i} \end{cases}$$

Then $(\zeta, \pounds_{bvcms})$ is a bvcms.

Remark 1. *If we take $\varphi(\aleph, \mathfrak{i}) = \mathfrak{t} \geq 1$, for all $\aleph, \mathfrak{i} \in \zeta$, then $(\zeta, \pounds_{bvcms})$ is a bicomplex-valued b-metric space, that is, every bicomplex-valued b-metric space is a BVCMS.*

Example 2. *Let $\zeta = \mathcal{V} \cup \mathcal{W}$ with $\mathcal{V} = \{(\frac{1}{v}) | v \in \mathbb{N}\}, \mathcal{W}$ is the set of all positive integers and $\varphi\colon \zeta \times \zeta \to [1, \infty)$ is defined for all $\aleph, \mathfrak{i} \in \zeta$ as*

$$\varphi(\aleph, \mathfrak{i}) = 5\mathfrak{p}$$

where $\mathfrak{p} > 0$ and $\pounds_{bvcms}\colon \zeta \times \zeta \to \mathbb{C}_2$ defined as follows

$$\pounds_{bvcms}(\aleph, \mathfrak{i})\colon = \begin{cases} 0, & iff\ \aleph = \mathfrak{i} \\ 2\mathfrak{p}i_2, & if\ \aleph, \mathfrak{i} \in \mathcal{V} \\ \frac{\mathfrak{p}i_2}{2}, & otherwise \end{cases}$$

where $\mathfrak{p} > 0$.

Now, the conditions $(BCCMS_1)$ and $(BCCMS_2)$ hold. Furthermore, $(BCCMS_3)$ holds under the following cases.

Case 1. *If $\aleph = \mathfrak{i}$ and $\mathfrak{i} = \flat$;*

Case 2. *If $\aleph = \mathfrak{i} \neq \flat$ or if $\aleph \neq \mathfrak{i} = \flat$ or if $\aleph = \flat \neq \mathfrak{i}$ or if $\aleph \neq \mathfrak{i} \neq \flat$;*

- ***SubCase 1.*** *If $\aleph \in \mathcal{V}$ and $\mathfrak{i}, \flat \in \mathcal{W}$;*
- ***SubCase 2.*** *If $\mathfrak{i} \in \mathcal{V}$ and $\aleph, \flat \in \mathcal{W}$;*
- ***SubCase 3.*** *If $\flat \in \mathcal{V}$ and $\aleph, \mathfrak{i} \in \mathcal{W}$;*
- ***SubCase 4.*** *If $\aleph, \mathfrak{i} \in \mathcal{V}$ and $\flat \in \mathcal{W}$;*
- ***SubCase 5.*** *If $\aleph, \flat \in \mathcal{V}$ and $\mathfrak{i} \in \mathcal{W}$;*
- ***SubCase 6.*** *If $\mathfrak{i}, \flat \in \mathcal{V}$ and $\aleph \in \mathcal{W}$;*
- ***SubCase 7.*** *If $\aleph, \mathfrak{i}, \flat \in \mathcal{V}$;*
- ***SubCase 8.*** *If $\aleph, \mathfrak{i}, \flat \in \mathcal{W}$.*

Then $(\zeta, \pounds_{bvcms})$ is a BVCMS.

Remark 2. *If $\varphi(\aleph, \mathfrak{i}) = \varphi(\mathfrak{i}, \flat)$ (as in the above example) for all $\aleph, \mathfrak{i}, \flat \in \zeta$, then $(\zeta, \pounds_{bvcms})$ is a bicomplex-valued extended b-metric space. We can conclude that every bicomplex-valued extended b-metric space is a BVCMS. However, the converse may not true in general.*

Example 3. *Let $\zeta = \{1, 2, 3\}$ and $\pounds_{bvcms}\colon \zeta \times \zeta \to \mathbb{C}_2$ be defined as*

$$\pounds_{bvcms}(1,1) = \pounds_{bvcms}(2,2) = \pounds_{bvcms}(3,3) = 0,$$

$$\begin{aligned} \pounds_{bvcms}(2,1) = \pounds_{bvcms}(1,2) &= 4 + 4i_2, \\ \pounds_{bvcms}(3,2) = \pounds_{bvcms}(2,3) &= 1 + 2i_2, \\ \pounds_{bvcms}(3,1) = \pounds_{bvcms}(1,3) &= 1 - i_2, \end{aligned}$$

and $\varphi\colon \zeta \times \zeta \to [1, \infty)$ be defined as

$$\varphi(1,1) = \varphi(2,2) = \varphi(3,3) = 3,$$

$$\begin{aligned} \varphi(1,2) = \varphi(2,1) &= 2, \\ \varphi(2,3) = \varphi(3,2) &= 4, \\ \varphi(1,3) = \varphi(3,1) &= 1. \end{aligned}$$

Clearly, the conditions $(BCCMS_1)$ and $(BCCMS_2)$ hold. Now,

Case 1. *If $\aleph = \flat$ the condition $(BCCMS_3)$ holds.*

Case 2. *If* $\aleph = 1$ *and* $\flat = 3$ *(same as* $\flat = 1$ *and* $\aleph = 3$*) and* $\mathfrak{i} = 2$

$$\begin{aligned}
\pounds_{bvcms}(\aleph,\flat) = |\pounds_{bvcms}(1,3)| = |1 - i_2| &\precsim_{i_2} |12 + 16i_2| \\
&= |2(4 + 4i_2) + 4(1 + 2i_2)| \\
&\precsim_{i_2} 2|(4 + 4i_2)| + 4|(1 + 2i_2)| \\
&= \varphi(1,2)\pounds_{bvcms}(1,2) + \varphi(2,3)\pounds_{bvcms}(2,3) \\
&= \varphi(\aleph,\mathfrak{i})\pounds_{bvcms}(\aleph,\mathfrak{i}) + \varphi(\mathfrak{i},\flat)\pounds_{bvcms}(\mathfrak{i},\flat).
\end{aligned}$$

Case 3. *If* $\aleph = 1$ *and* $\flat = 2$ *(same as* $\flat = 1$ *and* $\aleph = 2$*) and* $\mathfrak{i} = 3$

$$\begin{aligned}
\pounds_{bvcms}(\aleph,\flat) = |\pounds_{bvcms}(1,2)| = |4 + 4i_2| &\precsim_{i_2} |5 + 7i_2| \\
&= |1(1 - i_2) + 4(1 + 2i_2)| \\
&\precsim_{i_2} 1|(1 - i_2)| + 4|(1 + 2i_2)| \\
&= \varphi(1,3)\pounds_{bvcms}(1,3) + \varphi(3,2)\pounds_{bvcms}(3,2) \\
&= \varphi(\aleph,\mathfrak{i})\pounds_{bvcms}(\aleph,\mathfrak{i}) + \varphi(\mathfrak{i},\flat)\pounds_{bvcms}(\mathfrak{i},\flat).
\end{aligned}$$

Case 4. *If* $\aleph = 2$ *and* $\flat = 3$ *(same as* $\flat = 3$ *and* $\aleph = 2$*) and* $\mathfrak{i} = 1$

$$\begin{aligned}
\pounds_{bvcms}(\aleph,\flat) = |\pounds_{bvcms}(2,3)| = |1 + 2i_2| &\precsim_{i_2} |9 + 7i_2| \\
&= |2(4 + 4i_2) + 1(1 - i_2)| \\
&\precsim_{i_2} 2|(4 + 4i_2)| + 1|(1 - i_2)| \\
&= \varphi(2,1)\pounds_{bvcms}(2,1) + \varphi(1,3)\pounds_{bvcms}(1,3) \\
&= \varphi(\aleph,\mathfrak{i})\pounds_{bvcms}(\aleph,\mathfrak{i}) + \varphi(\mathfrak{i},\flat)\pounds_{bvcms}(\mathfrak{i},\flat).
\end{aligned}$$

Then, $(\zeta, \pounds_{bvcms})$ *is a BVCMS.*

Definition 3. *Let* $(\zeta, \pounds_{bvcms})$ *be a BVCMS with a sequence* $\{\aleph_v\}$ *in* ζ *and* $\aleph \in \zeta$*. Then,*

(i) *A sequence* $\{\aleph_v\}$ *in* ζ *is convergent to* $\aleph \in \zeta$ *if* $\forall\ 0 \prec_{i_2} \alpha \in \mathbb{C}_2$, $\exists$ *a natural number* $\mathtt{N}$ *so that* $\pounds_{bvcms}(\aleph_v, \aleph) \prec_{i_2} \alpha$ *for each* $v \geq \mathtt{N}$*. Then,* $\lim_{v\to\infty} \aleph_v = \aleph$ *or* $\aleph_v \to \aleph$ *as* $v \to \infty$*.*

(ii) *If, for each* $0 \prec_{i_2} \alpha$ *where* $\alpha \in \mathbb{C}_2$, $\exists$ *a natural number* $\mathtt{N}$ *so that* $\pounds_{bvcms}(\aleph_v, \aleph_{v+\varsigma}) \prec_{i_2} \alpha$ *for each* $\varsigma \in \mathbb{N}$ *and* $v > \mathtt{N}$*. Then,* $\{\aleph_v\}$ *is called a Cauchy sequence in* $(\zeta, \pounds_{\alpha\varepsilon\alpha})$*.*

(iii) *BVCMS* $(\zeta, \pounds_{bvcms})$ *is termed complete if every Cauchy sequence is convergent.*

Lemma 1. *Let* $(\zeta, \pounds_{bvcms})$ *be a BVCMS. Then a sequence* $\{\aleph_v\}$ *in* ζ *is a Cauchy sequence, such that* $\aleph_\varsigma \neq \aleph_v$*, with* $\varsigma \neq v$*. Then,* $\{\aleph_v\}$ *converges to one point at most.*

Proof. Let $\aleph^*$ and $\mathfrak{i}^*$ be two limits of the sequence $\{\aleph_v\} \in \zeta$ and $\lim_{v\to\infty} \pounds_{bvcms}(\aleph_v, \aleph^*) = 0 = \pounds_{bvcms}(\aleph_v, \mathfrak{i}^*)$. Since $\{\aleph_v\}$ is a Cauchy sequence, from ($BCCMS_3$), for $\aleph_\varsigma \neq \aleph_v$, whenever $\varsigma \neq v$, we can write

$$\begin{aligned}
\|\pounds_{bvcms}(\aleph^*, \mathfrak{i}^*)\| \precsim_{i_2} [&\varphi(\aleph^*, \aleph_v)\|\pounds_{bvcms}(\aleph^*, \aleph_v)\| \\
&+ \varphi(\aleph_v, \mathfrak{i}^*)\|\pounds_{bvcms}(\aleph_v, \mathfrak{i}^*)\|] \to 0 \quad \text{as} \quad v \to \infty.
\end{aligned}$$

We obtain $\|\pounds_{bvcms}(\aleph^*, \mathfrak{i}^*)\| = 0$, i.e., $\aleph^* = \mathfrak{i}^*$. Thus, $\{\aleph_v\}$ converges to one point at most. □

Lemma 2. *For a given BVCMS* $(\zeta, \pounds_{bvcms})$*, the tricomplex-valued controlled metric map* $\pounds_{bvcms}\colon \zeta \times \zeta \to \mathbb{C}_2$ *is continuous with respect to "*$\precsim_{i_2}$*".*

Proof. Let $\mathfrak{s}, \mathfrak{q} \in \mathbb{C}_2$, such that $\mathfrak{s} \succ \mathfrak{q}$, then we show that the set $\pounds^{-1}_{bvcms}(\mathfrak{q}, \mathfrak{s})$ given by

$$\pounds^{-1}_{bvcms}(\mathfrak{q}, \mathfrak{s}) := \{(\aleph, \mathfrak{i}) \in \zeta \times \zeta | \mathfrak{q} \prec_{i_2} \pounds_{bvcms}(\aleph, \mathfrak{i}) \prec_{i_2} \mathfrak{s}\},$$

is open in the product topology on $\zeta \times \zeta$. Then, let $(\aleph, \mathfrak{i}) \in £^{-1}_{bvcms}(\mathfrak{q}, \mathfrak{s})$. We choose $\epsilon = \frac{1}{200}\min(£_{bvcms}(\aleph, \mathfrak{i}) - \mathfrak{q}, \mathfrak{s} - £_{bvcms}(\aleph, \mathfrak{i}))$. Then, for $(\varphi, \lambda) \in \beta(\aleph, \epsilon) \times \beta(\mathfrak{i}, \epsilon)$ we obtain

$$\begin{aligned} £_{bvcms}(\varphi, \lambda) &\precsim_{i_2} £_{bvcms}(\lambda, \aleph) + £_{bvcms}(\aleph, \mathfrak{i}) + £_{bvcms}(\mathfrak{i}, \lambda) \\ &\prec_{i_2} 2\epsilon + £_{bvcms}(\aleph, \mathfrak{i}) \prec_{i_2} \mathfrak{s} \end{aligned}$$

and

$$\begin{aligned} \mathfrak{q} \precsim_{i_2} £_{bvcms}(\aleph, \mathfrak{i}) - 2\epsilon &\prec_{i_2} £_{bvcms}(\varphi, \lambda) + £_{bvcms}(\aleph, \lambda) - \epsilon + £_{bvcms}(\mathfrak{i}, \lambda) - \epsilon \\ &\prec_{i_2} £_{bvcms}(\varphi, \lambda). \end{aligned}$$

Then, $(\aleph, \mathfrak{i}) \in \beta(\varkappa, \epsilon) \times \beta(\mathfrak{i}, \epsilon) \subseteq £^{-1}_{bvcms}(\mathfrak{q}, \mathfrak{s})$. □

Defining Fix $\eta :\ = \{\aleph^* \in \zeta | \aleph^* = \eta(\aleph^*)\}$ will be the set of fixed points.

In this paper, we introduce the notion of BVCMS and FPT in the context of BVCMSs.

3. Main Results

Now, we prove the Banach-type contraction principle.

Theorem 1. *Let $(\zeta, £_{bvcms})$ be a complete BVCMS and $\eta : \zeta \to \zeta$ a continuous map, such that*

$$£_{bvcms}(\eta\aleph, \eta\mathfrak{i}) \precsim_{i_2} \mathfrak{a}£_{bvcms}(\aleph, \mathfrak{i}), \tag{1}$$

for all $\aleph, \mathfrak{i} \in \zeta$, where $0 < \mathfrak{a} < 1$. For $\aleph_0 \in \zeta$, we denote $\aleph_v = \eta^v \aleph_0$. Suppose that

$$\max_{\varsigma \geq 1} \lim_{\mathfrak{q} \to \infty} \frac{\varphi(\aleph_{\mathfrak{q}+1}, \aleph_{\mathfrak{q}+2})}{\varphi(\aleph_{\mathfrak{q}}, \aleph_{\mathfrak{q}+1})} \varphi(\aleph_{\mathfrak{q}+1}, \aleph_{\varsigma}) < \frac{1}{\mathfrak{a}}, \tag{2}$$

Moreover, for every $\aleph \in \zeta$ the limits

$$\lim_{v \to \infty} \varphi(\aleph_v, \aleph) \quad \text{and} \quad \lim_{v \to \infty} \varphi(\aleph, \aleph_v) \quad \text{exists and are finite.} \tag{3}$$

Then η has a unique fixed point (UFP).

Proof. Let $\{\aleph_v = \eta^v \aleph_0\}$. By (1), we obtain

$$\begin{aligned} £_{bvcms}(\aleph_v, \aleph_{v+1}) &\precsim_{i_2} \mathfrak{a}£_{bvcms}(\aleph_{v-1}, \aleph_v) \\ &\precsim_{i_2} \\ &\quad \cdots \\ &\precsim_{i_2} \mathfrak{a}^v £_{bvcms}(\aleph_0, \aleph_1), \quad \forall\, v \geq 0. \end{aligned}$$

For all $v < \varsigma$, where $v, \varsigma \in \mathbb{N}$, we have

$$\begin{aligned} £_{bvcms}(\aleph_v, \aleph_\varsigma) &\precsim_{i_2} \varphi(\aleph_v, \aleph_{v+1})£_{bvcms}(\aleph_v, \aleph_{v+1}) + \varphi(\aleph_{v+1}, \aleph_\varsigma £_{bvcms}(\aleph_{v+1}, \aleph_\varsigma \\ &\precsim_{i_2} \varphi(\aleph_v, \aleph_{v+1})£_{bvcms}(\aleph_v, \aleph_{v+1}) \\ &\quad + \varphi(\aleph_{v+1}, \aleph_\varsigma \varphi(\aleph_{v+1}, \aleph_{v+2})£_{bvcms}(\aleph_{v+1}, \aleph_{v+2}) \\ &\quad + \varphi(\aleph_{v+1}, \aleph_\varsigma \varphi(\aleph_{v+2}, \aleph_\varsigma £_{bvcms}(\aleph_{v+2}, \aleph_\varsigma) \end{aligned}$$

$$
\begin{aligned}
\pounds_{bvcms}(\aleph_{\upsilon},\aleph_{\varsigma}) \precsim_{i_2} & \varphi(\aleph_{\upsilon},\aleph_{\upsilon+1})\pounds_{bvcms}(\aleph_{\upsilon},\aleph_{\upsilon+1}) \\
& + \varphi(\aleph_{\upsilon+1},\aleph_{\varsigma}\varphi(\aleph_{\upsilon+1},\aleph_{\upsilon+2})\pounds_{bvcms}(\aleph_{\upsilon+1},\aleph_{\upsilon+2}) \\
& + \varphi(\aleph_{\upsilon+1},\aleph_{\varsigma}\varphi(\aleph_{\upsilon+2},\aleph_{\varsigma}\varphi(\aleph_{\upsilon+2},\aleph_{\upsilon+3}))\pounds_{bvcms}(\aleph_{\upsilon+2},\aleph_{\varsigma}) \\
& + \varphi(\aleph_{\upsilon+1},\aleph_{\varsigma}\varphi(\aleph_{\upsilon+2},\aleph_{\varsigma}\varphi(\aleph_{\upsilon+3},\aleph_{\varsigma})\pounds_{bvcms}(\aleph_{\upsilon+2},\aleph_{\varsigma}) \\
\precsim_{i_2} \dots \precsim_{i_2} & \varphi(\aleph_{\upsilon},\aleph_{\upsilon+1})\pounds_{bvcms}(\aleph_{\upsilon},\aleph_{\upsilon+1}) \\
& + \sum_{\iota=\upsilon+1}^{\varsigma-2}\prod_{\mathrm{j}=\upsilon+1}^{\iota}\varphi(\aleph_{\mathrm{j}},\aleph_{\varsigma}\varphi(\aleph_{\iota},\aleph_{\iota+1})\pounds_{bvcms}(\aleph_{\iota},\aleph_{\iota+1}) \\
& + \prod_{\mathfrak{p}=\upsilon+1}^{\varsigma-1}\varphi(\aleph_{\mathfrak{p}},\aleph_{\varsigma}\pounds_{bvcms}(\aleph_{\varsigma-1},\aleph_{\varsigma}) \\
\precsim_{i_2} & \varphi(\aleph_{\upsilon},\aleph_{\upsilon+1})\mathfrak{a}^{\upsilon}\pounds_{bvcms}(\aleph_0,\aleph_1) \\
& + \sum_{\iota=\upsilon+1}^{\varsigma-2}\prod_{\mathrm{j}=\upsilon+1}^{\iota}\varphi(\aleph_{\mathrm{j}},\aleph_{\varsigma}\varphi(\aleph_{\iota},\aleph_{\iota+1})\mathfrak{a}^{\iota}\pounds_{bvcms}(\aleph_0,\aleph_1) \\
& + \prod_{\mathfrak{p}=\upsilon+1}^{\varsigma-1}\varphi(\aleph_{\mathfrak{p}},\aleph_{\varsigma})\mathfrak{a}^{\varsigma-1}\pounds_{bvcms}(\aleph_0,\aleph_1) \\
\precsim_{i_2} & \varphi(\aleph_{\upsilon},\aleph_{\upsilon+1})\mathfrak{a}^{\upsilon}\pounds_{bvcms}(\aleph_0,\aleph_1) \\
& + \sum_{\iota=\upsilon+1}^{\varsigma-2}\prod_{\mathrm{j}=\upsilon+1}^{\iota}\varphi(\aleph_{\mathrm{j}},\aleph_{\varsigma}\varphi(\aleph_{\iota},\aleph_{\iota+1})\mathfrak{a}^{\iota}\pounds_{bvcms}(\aleph_0,\aleph_1) \\
& + \prod_{\mathfrak{p}=\upsilon+1}^{\varsigma-1}\varphi(\aleph_{\mathfrak{p}},\aleph_{\varsigma})\mathfrak{a}^{\varsigma-1}\varphi(\aleph_{\varsigma-1},\aleph_{\varsigma})\pounds_{bvcms}(\aleph_0,\aleph_1) \\
= & \varphi(\aleph_{\upsilon},\aleph_{\upsilon+1})\mathfrak{a}^{\upsilon}\pounds_{bvcms}(\aleph_0,\aleph_1) \\
& + \sum_{\iota=\upsilon+1}^{\varsigma-1}\prod_{\mathrm{j}=\upsilon+1}^{\iota}\varphi(\aleph_{\mathrm{j}},\aleph_{\varsigma}\varphi(\aleph_{\iota},\aleph_{\iota+1})\mathfrak{a}^{\iota}\pounds_{bvcms}(\aleph_0,\aleph_1) \\
\precsim_{i_2} & \varphi(\aleph_{\upsilon},\aleph_{\upsilon+1})\mathfrak{a}^{\upsilon}\pounds_{bvcms}(\aleph_0,\aleph_1) \\
& + \sum_{\iota=\upsilon+1}^{\varsigma-1}\prod_{\mathrm{j}=\upsilon+1}^{\iota}\varphi(\aleph_{\mathrm{j}},\aleph_{\varsigma}\varphi(\aleph_{\iota},\aleph_{\iota+1})\mathfrak{a}^{\iota}\pounds_{bvcms}(\aleph_0,\aleph_1)
\end{aligned}
$$

Furthermore, using $\varphi(\aleph,\mathrm{i}) \geq 1$. Let

$$\mathcal{S}_{\flat} = \sum_{\iota=0}^{\flat}\prod_{\mathrm{j}=0}^{\iota}\varphi(\aleph_{\mathrm{j}},\aleph_{\varsigma}\varphi(\aleph_{\iota},\aleph_{\iota+1}))\mathfrak{a}^{\iota}.$$

Hence, we have

$$\pounds_{bvcms}(\aleph_{\upsilon},\aleph_{\varsigma}) \precsim_{i_2} \pounds_{bvcms}(\aleph_0,\aleph_1)[\mathfrak{a}^{\upsilon}\varphi(\aleph_{\upsilon},\aleph_{\upsilon+1}) + (\mathcal{S}_{\varsigma-1},\mathcal{S}_{\upsilon})]. \tag{4}$$

Applying the ratio test and (2), we obtain $\lim_{\varsigma,\upsilon\to\infty}\mathcal{S}_{\upsilon}$ exists and the sequence $\{\mathcal{S}_{\upsilon}\}$ is a real Cauchy sequence. Letting $\varsigma,\upsilon \to \infty$, we have

$$\lim_{\varsigma,\upsilon\to\infty}\pounds_{bvcms}(\aleph_{\upsilon},\aleph_{\varsigma}) = 0. \tag{5}$$

Then, $\{\aleph_{\upsilon}\}$ is a Cauchy sequence in a BVCMSs $(\zeta,\pounds_{bvcms})$; then $\{\aleph_{\upsilon}\}$ converges to $\aleph^* \in \zeta$. By the definition of continuity, we obtain

$$\aleph^* = \lim_{\upsilon\to\infty}\aleph_{\upsilon+1} = \lim_{\upsilon\to\infty}\eta\aleph_{\upsilon} = \eta(\lim_{\upsilon\to\infty}\aleph_{\upsilon}) = \eta\aleph^*.$$

Let $\aleph^*, \mathfrak{i}^* \in \text{fix}\,\eta$. Then,

$$\pounds_{bvcms}(\aleph^*, \mathfrak{i}^*) = \pounds_{bvcms}(\eta\aleph^*, \eta\mathfrak{i}^*) \precsim_{i_2} \varphi\pounds_{bvcms}(\aleph^*, \mathfrak{i}^*).$$

Therefore, $\pounds_{bvcms}(\aleph^*, \mathfrak{i}^*) = 0$; so $\aleph^* = \mathfrak{i}^*$. Hence, η has a UFP. □

Theorem 2. *Let $(\zeta, \pounds_{bvcms})$ be a complete BVCMS and $\eta \colon \zeta \to \zeta$ a map, such that*

$$\pounds_{bvcms}(\eta\aleph, \eta\mathfrak{i}) \precsim_{i_2} \varphi\pounds_{bvcms}(\aleph, \mathfrak{i}), \tag{6}$$

for all $\aleph, \mathfrak{i} \in \zeta$, where $0 < \mathfrak{a} < 1$. For $\aleph_0 \in \zeta$ we denote $\aleph_\upsilon = \eta^\upsilon\aleph_0$. Suppose that

$$\max_{\varsigma \geq 1} \lim_{\iota \to \infty} \frac{\varphi(\aleph_{\iota+1}, \aleph_{\iota+2})}{\varphi(\aleph_\iota, \aleph_{\iota+1})} \varphi(\aleph_{\iota+1}, \aleph_\varsigma) < \frac{1}{\mathfrak{a}}. \tag{7}$$

In addition, for each $\aleph \in \zeta$,

$$\lim_{\upsilon \to \infty} \varphi(\aleph_\upsilon, \aleph) \quad \text{and} \quad \lim_{\upsilon \to \infty} \varphi(\aleph, \aleph_\upsilon) \quad \text{exists and it is finite.} \tag{8}$$

Then, η has a UFP.

Proof. Using the proof of Theorem 1 and Lemma 2, we obtain a Cauchy sequence $\{\aleph_\upsilon\}$ in a complete BVCMS $(\zeta, \pounds_{bvcms})$. Then, the sequence $\{\aleph_\upsilon\}$ converges to $\aleph^* \in \zeta$. Therefore,

$$\pounds_{bvcms}(\aleph^*, \aleph_{\upsilon+1}) \precsim_{i_2} \varphi(\aleph^*, \aleph_\upsilon)\pounds_{bvcms}(\aleph^*, \aleph_\upsilon) + \varphi(\aleph_\upsilon, \aleph_{\upsilon+1})\pounds_{bvcms}(\aleph^*, \aleph_{\upsilon+1}).$$

Using (7), (8) and (18), we obtain

$$\lim_{\upsilon \to \infty} \pounds_{bvcms}(\aleph^*, \aleph_{\upsilon+1}) = 0. \tag{9}$$

Using the triangular inequality and (6),

$$\begin{aligned}\pounds_{bvcms}(\aleph^*, \eta\aleph^*) &\precsim_{i_2} \varphi(\aleph^*, \aleph_{\upsilon+1})\pounds_{bvcms}(\aleph^*, \aleph_{\upsilon+1}) + \varphi(\aleph_{\upsilon+1}, \eta\aleph^*)\pounds_{bvcms}(\aleph_{\upsilon+1}, \eta\aleph^*) \\ &\precsim_{i_2} \varphi(\aleph^*, \aleph_{\upsilon+1})\pounds_{bvcms}(\aleph^*, \aleph_{\upsilon+1}) + \mathfrak{a}\varphi(\aleph_{\upsilon+1}, \eta\aleph^*)\pounds_{bvcms}(\aleph_{\upsilon+1}, \eta\aleph^*)\end{aligned}$$

Taking the limit $\upsilon \to \infty$ from (8) and (19), we find that $\pounds_{bvcms}(\aleph^*, \eta\aleph^*) = 0$. By Lemma 1, the sequence $\{\aleph_\upsilon\}$ uniquely converges at $\aleph^* \in \zeta$. □

Example 4. *Let $\zeta = \{0, 1, 2\}$ and $\pounds_{bvcms} \colon \zeta \times \zeta \to \mathbb{C}_2$ be a symmetrical metric given by*

$$\pounds_{bvcms}(\aleph, \aleph) = 0, \quad \text{for each} \quad \aleph \in \zeta$$

and

$$\pounds_{bvcms}(0, 1) = 1 + i_2, \pounds_{bvcms}(1, 2) = 1 + i_2, \pounds_{bvcms}(0, 2) = 4 + 4i_2.$$

Define $\varphi \colon \zeta \times \zeta \to [1, \infty)$ by

$$\varphi(2, 2) = \frac{6}{5}, \varphi(0, 0) = 2, \varphi(1, 1) = \frac{4}{3},$$
$$\varphi(0, 2) = \frac{4}{3}, \varphi(0, 1) = \frac{3}{2}, \varphi(1, 2) = \frac{5}{4}.$$

Hence, it is a BVCMS.
Consider a map $\eta : \zeta \to \zeta$ is defined by $\eta(0) = 0, \eta(1) = 0, \eta(2) = 0$.
Letting $\mathfrak{a} = \frac{2}{5}$. Then,

Case 1. *If $\aleph = \mathfrak{i} = 0, \aleph = \mathfrak{i} = 1, \aleph = \mathfrak{i} = 2$, then the results is obvious.*

Case 2. *If* $\aleph = 0, \mathfrak{i} = 1$, *we obtain*

$$\begin{aligned}\pounds_{bvcms}(\eta\aleph, \eta\mathfrak{i}) &= \pounds_{bvcms}(\eta 0, \eta 1) = \pounds_{bvcms}(2, 2) = 0 \\ &\precsim_{i_2} \frac{2}{5}(1 + i_2) \\ &= \mathfrak{a}(\pounds_{bvcms}(0, 1)) = \mathfrak{a}(\pounds_{bvcms}(\aleph, \mathfrak{i})).\end{aligned}$$

Case 3. *If* $\aleph = 0, \mathfrak{i} = 2$, *we have*

$$\begin{aligned}\pounds_{bvcms}(\eta\aleph, \eta\mathfrak{i}) &= \pounds_{bvcms}(\eta 0, \eta 1) = \pounds_{bvcms}(2, 2) = 0 \\ &\precsim_{i_2} \frac{2}{5}(4 + 4i_2) = \mathfrak{a}(\pounds_{bvcms}(0, 2)) \\ &= \mathfrak{a}(\pounds_{bvcms}(\aleph, \mathfrak{i})).\end{aligned}$$

Case 4. *If* $\aleph = 1, \mathfrak{i} = 2$, *we have*

$$\begin{aligned}\pounds_{bvcms}(\eta\aleph, \eta\mathfrak{i}) &= \pounds_{bvcms}(\eta 1, \eta 2) = \pounds_{bvcms}(2, 2) = 0 \\ &\precsim_{i_2} \frac{2}{5}(1 + i_2) \\ &= \mathfrak{a}(\pounds_{bvcms}(1, 2)) = \mathfrak{a}(\pounds_{bvcms}(\aleph, \mathfrak{i})).\end{aligned}$$

Therefore, all axioms of Theorem 2 are fulfilled. Hence, η *has a UFP, which is* $\aleph^* = 0$.

Next, we show a Kannan-type contraction map.

Theorem 3. *Let* $(\zeta, \pounds_{bvcms})$ *be a complete BVCMS and* $\eta\colon \zeta \to \zeta$ *a continuous map, such that*

$$\pounds_{bvcms}(\eta\aleph, \eta\mathfrak{i}) \precsim_{i_2} \eta(\pounds_{bvcms}(\aleph, \eta\aleph) + (\pounds_{bvcms}(\mathfrak{i}, \eta\mathfrak{i})), \tag{10}$$

for all $\aleph, \mathfrak{i} \in \zeta$, *where* $0 \leq \eta < \frac{1}{2}$. *For* $\aleph_0 \in \zeta$ *we denote* $\aleph_v = \eta^v \aleph_0$. *Suppose that*

$$\max_{\varsigma \geq 1} \lim_{\iota \to \infty} \frac{\varphi(\aleph_{\iota+1}, \aleph_{\iota+2})}{\varphi(\aleph_\iota, \aleph_{\iota+1})} \varphi(\aleph_{\iota+1}, \aleph_\varsigma) < \frac{1}{\mathfrak{a}}, \quad \text{where} \quad \mathfrak{a} = \frac{\eta}{1 - \eta}. \tag{11}$$

Moreover, for each $\aleph \in \zeta$,

$$\lim_{v \to \infty} \varphi(\aleph_v, \aleph) \quad \text{and} \quad \lim_{v \to \infty} \varphi(\aleph, \aleph_v), \tag{12}$$

exists and is finite. Then, η *has a UFP.*

Proof. For $\aleph_0 \in \zeta$, consider a sequence $\{\aleph_v = \eta^v \aleph_0\}$. If $\exists\ \aleph_0 \in \mathtt{N}$ for which $\aleph_{v_0+1} = \aleph_{v_0}$, then $\eta\aleph_{v_0} = \aleph_{v_0}$. Thus, there is nothing to prove. Now we assume that $\aleph_{v+1} \neq \aleph_v$ for all $v \in \mathtt{N}$. By using (1) we obtain

$$\begin{aligned}\pounds_{bvcms}(\aleph_v, \aleph_{v+1}) &= \pounds_{bvcms}(\eta\aleph_{v-1}, \eta\aleph_v) \\ &\precsim_{i_2} \eta(\pounds_{bvcms}(\aleph_{v-1}, \eta\aleph_{v-1}) + \pounds_{bvcms}(\aleph_v, \eta\aleph_v)) \\ &= \eta(\pounds_{bvcms}(\aleph_{v-1}, \aleph_v) + \pounds_{bvcms}(\aleph_v, \aleph_{v+1})), \quad \text{which implies} \\ \pounds_{bvcms}(\aleph_v, \aleph_{v+1}) &\precsim_{i_2} \left(\frac{\eta}{1 - \eta}\right) \pounds_{bvcms}(\aleph_{v-1}, \aleph_v) \\ &= \mathfrak{a}\pounds_{bvcms}(\aleph_{v-1}, \aleph_v).\end{aligned}$$

In the same way

$$\begin{aligned}\pounds_{bvcms}(\aleph_{v-1},\aleph_v) &= \pounds_{bvcms}(\eta\aleph_{v-2},\eta\aleph_{v-1})\\ &\precsim_{i_2} \eta(\pounds_{bvcms}(\aleph_{v-2},\eta\aleph_{v-2}) + \pounds_{bvcms}(\aleph_{v-1},\eta\aleph_{v-1}))\\ &= \eta(\pounds_{bvcms}(\aleph_{v-2},\aleph_{v-1}) + \pounds_{bvcms}(\aleph_{v-1},\aleph_v)), \text{ which implies}\\ \pounds_{bvcms}(\aleph_{v-1},\aleph_v) &\precsim_{i_2} \left(\frac{\eta}{1-\eta}\right)\pounds_{bvcms}(\aleph_{v-2},\aleph_{v-1})\\ &= \mathfrak{a}\pounds_{bvcms}(\aleph_{v-2},\aleph_{v-1}).\end{aligned}$$

Continuing in the same way, we have

$$\begin{aligned}\pounds_{bvcms}(\aleph_v,\aleph_{v+1}) \precsim_{i_2} \mathfrak{a}\pounds_{bvcms}(\aleph_{v-1},\aleph_v) &\precsim_{i_2} \mathfrak{a}^2\pounds_{bvcms}(\aleph_{v-2},\aleph_{v-1})\\ &\precsim_{i_2} \dots \precsim_{i_2} \mathfrak{a}^v\pounds_{bvcms}(\aleph_0,\aleph_1).\end{aligned}$$

Thus, $\pounds_{bvcms}(\aleph_v,\aleph_{v+1}) \precsim_{i_2} \mathfrak{a}^v\pounds_{bvcms}(\aleph_0,\aleph_1)$ for all $v \geq 0$. For all $v < \varsigma$, where v and ς are natural numbers, we have

$$\begin{aligned}\pounds_{bvcms}(\aleph_v,\aleph_\varsigma) &\precsim_{i_2} \varphi(\aleph_v,\aleph_{v+1})\pounds_{bvcms}(\aleph_v,\aleph_{v+1}) + \varphi(\aleph_{v+1},\aleph_\varsigma)\pounds_{bvcms}(\aleph_{v+1},\aleph_\varsigma)\\ &\precsim_{i_2} \varphi(\aleph_v,\aleph_{v+1})\pounds_{bvcms}(\aleph_v,\aleph_{v+1})\\ &\quad + \varphi(\aleph_{v+1},\aleph_\varsigma\varphi(\aleph_{v+1},\aleph_{v+2})\pounds_{bvcms}(\aleph_{v+1},\aleph_{v+2})\\ &\quad + \varphi(\aleph_{v+1},\aleph_\varsigma)\varphi(\aleph_{v+2},\aleph_\varsigma)\pounds_{bvcms}(\aleph_{v+2},\aleph_\varsigma)\\ &\precsim_{i_2} \varphi(\aleph_v,\aleph_{v+1})\pounds_{bvcms}(\aleph_v,\aleph_{v+1})\\ &\quad + \varphi(\aleph_{v+1},\aleph_\varsigma\varphi(\aleph_{v+1},\aleph_{v+2})\pounds_{bvcms}(\aleph_{v+1},\aleph_{v+2})\\ &\quad + \varphi(\aleph_{v+1},\aleph_\varsigma\varphi(\aleph_{v+2},\aleph_\varsigma\varphi(\aleph_{v+2},\aleph_{v+3})\pounds_{bvcms}(\aleph_{v+2},\aleph_{v+3})\\ &\quad + \varphi(\aleph_{v+1},\aleph_\varsigma\varphi(\aleph_{v+2},\aleph_\varsigma\varphi(\aleph_{v+3},\aleph_\varsigma)\pounds_{bvcms}(\aleph_{v+3},\aleph_\varsigma)\\ &\quad \vdots\\ &\quad \precsim_{i_2} \varphi(\aleph_v,\aleph_{v+1})\pounds_{bvcms}(\aleph_v,\aleph_{v+1}) \sum_{\iota=v+1}^{\varsigma-2}\prod_{\mathrm{j}=v+1}^{\iota} \varphi(\aleph_{\mathrm{j}},\aleph_\varsigma\varphi(\aleph_\iota,\aleph_{\iota+1})\pounds_{bvcms}(\aleph_\iota,\aleph_{\iota+1})\\ &\quad + \prod_{\mathfrak{p}=v+1}^{\varsigma-1} \varphi(\aleph_{\mathfrak{p}},\aleph_\varsigma\pounds_{bvcms}(\aleph_{\varsigma-1},\aleph_\varsigma\\ &\precsim_{i_2} \varphi(\aleph_v,\aleph_{v+1})\mathfrak{a}^v\pounds_{bvcms}(\aleph_0,\aleph_1)\\ &\quad + \sum_{\iota=v+1}^{\varsigma-2}\prod_{\mathrm{j}=v+1}^{\iota} \varphi(\aleph_{\mathrm{j}},\aleph_\varsigma\varphi(\aleph_\iota,\aleph_{\iota+1})\mathfrak{a}^\iota\pounds_{bvcms}(\aleph_\iota,\aleph_{\iota+1})\\ &\quad + \prod_{\mathfrak{p}=v+1}^{\varsigma-1} \varphi(\aleph_{\mathfrak{p}},\aleph_\varsigma\mathfrak{a}^{\varsigma-1}\pounds_{bvcms}(\aleph_0,\aleph_1)\\ &\precsim_{i_2} \varphi(\aleph_v,\aleph_{v+1})\mathfrak{a}^v\pounds_{bvcms}(\aleph_0,\aleph_1)\\ &\quad + \sum_{\iota=v+1}^{\varsigma-2}\prod_{\mathrm{j}=v+1}^{\iota} \varphi(\aleph_{\mathrm{j}},\aleph_\varsigma\varphi(\aleph_\iota,\aleph_{\iota+1})\mathfrak{a}^\iota\pounds_{bvcms}(\aleph_\iota,\aleph_{\iota+1})\\ &\quad + \prod_{\mathfrak{p}=v+1}^{\varsigma-1} \varphi(\aleph_{\mathfrak{p}},\aleph_\varsigma\varphi(\aleph_{\varsigma-1},\aleph_\varsigma\pounds_{bvcms}(\aleph_0,\aleph_1)\end{aligned}$$

$$\begin{aligned}\pounds_{bvcms}(\aleph_v, \aleph_\varsigma) =& \varphi(\aleph_v, \aleph_{v+1})\mathfrak{a}^v \pounds_{bvcms}(\aleph_0, \aleph_1) \\ &+ \sum_{\iota=v+1}^{\varsigma-1} \prod_{\mathrm{j}=v+1}^{\iota} \varphi(\aleph_{\mathrm{j}}, \aleph_\varsigma \varphi(\aleph_\iota, \aleph_{\iota+1})\mathfrak{a}^\iota \pounds_{bvcms}(\aleph_\iota, \aleph_{\iota+1}) \\ \precsim_{i_2}& \varphi(\aleph_v, \aleph_{v+1})\mathfrak{a}^v \pounds_{bvcms}(\aleph_0, \aleph_1) \\ &+ \sum_{\iota=v+1}^{\varsigma-1} \prod_{\mathrm{j}=v+1}^{\iota} \aleph(\aleph_{\mathrm{j}}, \aleph_\varsigma \varphi(\aleph_\iota, \aleph_{\iota+1})\mathfrak{a}^\iota \pounds_{bvcms}(\aleph_\iota, \aleph_{\iota+1}).\end{aligned}$$

Furthermore, using $\varphi(\aleph, \mathrm{i}) \geq 1$. Let

$$\mathcal{S}_\flat = \sum_{\iota=0}^{\flat} \prod_{\mathrm{j}=0}^{\iota} \varphi(\aleph_{\mathrm{j}}, \aleph_\varsigma \varphi(\aleph_\iota, \aleph_{\iota+1})\mathfrak{a}^\iota.$$

Hence, we have

$$\pounds_{bvcms}(\aleph_v, \aleph_\varsigma) \precsim_{i_2} \pounds_{bvcms}(\aleph_0, \aleph_1)[\mathfrak{a}^v \varphi(\aleph_v, \aleph_{v+1}) + (\mathcal{S}_{\varsigma-1}, \mathcal{S}_v)]. \tag{13}$$

By applying the ratio test, we obtain $\lim_{\varsigma,v\to\infty} \mathcal{S}_v$ exists and so the sequence $\{\mathcal{S}_v\}$ is a Cauchy sequence. Letting $\varsigma, v \to \infty$, we have

$$\lim_{\varsigma,v\to\infty} \pounds_{bvcms}(\aleph_v, \aleph_\varsigma) = 0.$$

Then $\{\aleph_v\}$ is a Cauchy sequence in a complete BVCMS $(\zeta, \pounds_{bvcms})$. This means the sequence $\{\aleph_v\}$ converges to some $\aleph^* \in \zeta$. By the definition of continuity, we obtain

$$\aleph^* = \lim_{v\to\infty} \aleph_{v+1} = \lim_{v\to\infty} \eta\aleph_v = \eta(\lim_{v\to\infty} \aleph_v) = \eta\aleph^*.$$

Let $\aleph^*, \mathrm{i}^* \in \mathrm{fix}\,\eta$. Then,

$$\begin{aligned}\pounds_{bvcms}(\aleph^*, \mathrm{i}^*) &= \pounds_{bvcms}(\eta\aleph^*, \eta\mathrm{i}^*) \\ &\precsim_{i_2} \eta[\pounds_{bvcms}(\aleph^*, \eta\aleph^*) + \pounds_{bvcms}(\mathrm{i}^*, \eta\mathrm{i}^*)] \\ &\precsim_{i_2} \eta[\pounds_{bvcms}(\aleph^*, \aleph^*) + \pounds_{bvcms}(\mathrm{i}^*, \mathrm{i}^*)] = 0.\end{aligned}$$

Therefore, $\pounds_{bvcms}(\aleph^*, \mathrm{i}^*) = 0$, then $\aleph^* = \mathrm{i}^*$. Hence, η has a UFP. □

Theorem 4. *Let $(\zeta, \pounds_{bvcms})$ be a complete BVCMS and $\eta : \zeta \to \zeta$ a map, such that*

$$\pounds_{bvcms}(\eta\aleph, \eta\mathrm{i}) \precsim_{i_2} \eta(\pounds_{bvcms}(\aleph, \eta\aleph) + \pounds_{bvcms}(\mathrm{i}, \eta\mathrm{i})) \tag{14}$$

for all $\aleph, \mathrm{i} \in \zeta$ where $0 \leq \eta < \frac{1}{2}$. For $\aleph_0 \in \zeta$ we denote $\aleph_v = \eta^v \aleph_0$. Suppose that

$$\max_{\varsigma\geq 1} \lim_{\iota\to\infty} \frac{\varphi(\aleph_{\iota+1}, \aleph_{\iota+2})}{\varphi(\aleph_\iota, \aleph_{\iota+1})} \varphi(\aleph_{\iota+1}, \aleph_\varsigma) < \frac{1}{\mathfrak{a}}, \quad \textit{where} \quad \mathfrak{a} = \frac{\eta}{1-\eta}. \tag{15}$$

Moreover, for each $\aleph \in \zeta$,

$$\lim_{v\to\infty} \varphi(\aleph_v, \aleph) \quad \textit{and} \quad \lim_{v\to\infty} \varphi(\aleph, \aleph_v), \tag{16}$$

exists and is finite. Then η has a UFP.

Proof. By proving Theorem 3 and using Lemma 2, we show a Cauchy sequence $\{\aleph_v\}$ in a complete BVCMS $(\zeta, \pounds_{bvcms})$. Then the sequence $\{\aleph_v\}$ converges to a $\aleph^* \in \zeta$. Then,

$$\pounds_{bvcms}(\aleph^*, \aleph_{v+1}) \precsim_{i_2} \varphi(\aleph^*, \aleph_v)\pounds_{bvcms}(\aleph^*, \aleph_v) + \varphi(\aleph_v, \aleph_{v+1})\pounds_{bvcms}(\aleph_v, \aleph_{v+1})$$

Using (2), (3) and (18), we deduce

$$\lim_{\upsilon \to \infty} \aleph_{bvcms}(\aleph^*, \aleph_{\upsilon+1}) = 0.$$

Using the triangular inequality and (1), we obtain

$$\begin{aligned}\pounds_{bvcms}(\aleph^*, \eta\aleph^*) &\precsim_{i_2} \varphi(\aleph^*, \aleph_{\upsilon+1})\pounds_{bvcms}(\aleph^*, \aleph_{\upsilon+1}) \\ &\quad + \varphi(\aleph_{\upsilon+1}, \eta\aleph^*)\pounds_{bvcms}(\aleph_{\upsilon+1}, \eta\aleph^*) \\ &\precsim_{i_2} \varphi(\aleph^*, \aleph_{\upsilon+1})\pounds_{bvcms}(\aleph^*, \aleph_{\upsilon+1}) \\ &\quad + \varphi(\aleph_{\upsilon+1}, \eta\aleph^*)[\eta(\pounds_{bvcms}(\aleph_\upsilon, \aleph_{\upsilon+1}) + \pounds_{bvcms}(\aleph^*, \eta\aleph^*))].\end{aligned}$$

As $\upsilon \to \infty$ from (3) and (19), we conclude that $\pounds_{bvcms}(\aleph^*, \eta\aleph^*) = 0$. From Lemma 1, the sequence $\{\aleph_\upsilon\}$ uniquely converges at $\aleph^* \in \zeta$. □

Example 5. *Let $\zeta = \{0, 1, 2\}$ and $\pounds_{bvcms} \colon \zeta \times \zeta \to \mathbb{C}_2$ be a symmetrical metric as follows*

$$\pounds_{bvcms}(\aleph, \aleph) = 0 \quad \textit{for each} \quad \aleph \in \zeta$$

and

$$\pounds_{bvcms}(0, 1) = 1 + i_2, \pounds_{bvcms}(1, 2) = 1 + i_2, \pounds_{bvcms}(0, 2) = 4 + 4i_2.$$

Define $\varphi \colon \zeta \times \zeta \to [1, \infty)$ by

$$\begin{aligned}\varphi(2, 2) = \frac{9}{5}, \varphi(0, 0) = 5, \varphi(1, 1) = \frac{7}{3}, \\ \varphi(1, 2) = 2, \varphi(0, 1) = 3, \varphi(0, 2) = \frac{7}{3}.\end{aligned}$$

A self-map η on ζ can be defined by $\eta(0) = \eta(1) = \eta(2) = 2$.
Taking $\eta = \frac{2}{5}$; then,

Case 1. *If $\aleph = \mathfrak{i} = 0, \aleph = \mathfrak{i} = 1, \aleph = \mathfrak{i} = 2$, then the result is obvious.*

Case 2. *If $\aleph = 0, \mathfrak{i} = 1$, we obtain*
$\pounds_{bvcms}(\eta\aleph, \eta\mathfrak{i}) = \pounds_{bvcms}(\eta 0, \eta 1) = \pounds_{bvcms}(2, 2) = 0 \precsim_{i_2} \frac{10}{5}(1 + i_2)$
$= \frac{2}{5}(4 + 4i_2 + (1 + i_2)) = \eta(\pounds_{bvcms}(0, 2) + \pounds_{bvcms}(1, 2))$
$= \eta(\pounds_{bvcms}(\aleph, \eta\aleph) + \pounds_{bvcms}(\mathfrak{i}, \eta\mathfrak{i}))$.

Case 3. *If $\aleph = 0, \mathfrak{i} = 2$, we have*
$\pounds_{bvcms}(\eta\aleph, \eta\mathfrak{i}) = \pounds_{bvcms}(\eta 0, \eta 2) = \pounds_{bvcms}(2, 2) = 0 \precsim_{i_2} \frac{8}{5}(1 + i_2)$
$= \frac{2}{5}(4 + 4i_2 + 0) = \eta(\pounds_{bvcms}(0, 2) + \pounds_{bvcms}(2, 2))$
$= \eta(\pounds_{bvcms}(\aleph, \eta\aleph) + \pounds_{bvcms}(\mathfrak{i}, \eta\mathfrak{i}))$.

Case 4. *If $\aleph = 1, \mathfrak{i} = 2$, we have*
$\pounds_{bvcms}(\eta\aleph, \eta\mathfrak{i}) = \pounds_{bvcms}(\eta 1, \eta 2) = \pounds_{bvcms}(2, 2) = 0 \precsim_{i_2} \frac{2}{5}(1 + i_2)$
$= \frac{2}{5}((1 + i_2) + 0) = \eta(\pounds_{bvcms}(1, 2) + \pounds_{bvcms}(2, 2))$
$= \eta(\pounds_{bvcms}(\aleph, \eta\aleph) + \pounds_{bvcms}(\mathfrak{i}, \eta\mathfrak{i}))$.

Then, all hypothesis of Theorem 4 are fulfilled. Hence, $\mathcal{T}$ has a UFP, which is $\aleph^* = 2$.

Finally, we show that FPT in a Fisher-type contraction map.

Theorem 5. *Let $(\zeta, \pounds_{bvcms})$ be a complete BVCMS and $\eta \colon \zeta \to \zeta$ a continuous map, such that*

$$\pounds_{bvcms}(\eta\aleph, \eta\mathfrak{i}) \precsim_{i_2} \varpi\pounds_{bvcms}(\aleph, \mathfrak{i}) + f\frac{\pounds_{bvcms}(\aleph, \eta\aleph)\pounds_{bvcms}(\mathfrak{i}, \eta\mathfrak{i})}{1 + \pounds_{bvcms}(\aleph, \mathfrak{i})}, \tag{17}$$

for all $\aleph, \imath \in \zeta$, where $\varpi, f \in [0,1)$, such that $\nu = \frac{\varpi}{1-f} < 1$. For $\aleph_0 \in \zeta$ we denote $\aleph_v = \eta^v \aleph_0$. Suppose that

$$\max_{\varsigma \geq 1} \lim_{i_2 \to \infty} \frac{\varphi(\aleph_{i_2+1}, \aleph_{i_2+2})}{\varphi(\aleph_{i_2}, \aleph_{i_2+1})} \varphi(\aleph_{i_2+1}, \aleph_{\varsigma}) < \frac{1}{\nu}, \tag{18}$$

Moreover, suppose that for every $\varkappa \in \zeta$ we have

$$\lim_{v \to \infty} \varphi(\aleph_v, \aleph) \quad \text{and} \quad \lim_{v \to \infty} \varphi(\aleph, \aleph_v), \tag{19}$$

exist and are finite. Then η has a UFP.

Proof. For $\aleph_0 \in \zeta$. Let $\aleph_v = \eta^v \aleph_0$. If $\exists\ \aleph_0 \in \mathtt{N}$ for which $\aleph_{v_0+1} = \aleph_{v_0}$, then $\eta \aleph_{v_0} = \aleph_{v_0}$. Thus, there is nothing to prove. Now we assume that $\aleph_{v+1} \neq \aleph_v$ for all $v \in \mathtt{N}$. By using (1), we obtain

$$\begin{aligned}
\pounds_{bvcms}(\aleph_v, \aleph_{v+1}) &= \pounds_{bvcms}(\eta \aleph_{v-1}, \eta \aleph_v) \\
&\precsim_{i_2} \varpi \pounds_{bvcms}(\aleph_{v-1}, \aleph_v) + f \frac{\pounds_{bvcms}(\aleph_{v-1}, \eta \aleph_v) \pounds_{bvcms}(\aleph_v, \eta \aleph_v)}{1 + \pounds_{bvcms}(\aleph_{v-1}, \aleph_v)} \\
&= \varpi \pounds_{bvcms}(\aleph_{v-1}, \aleph_v) + f \frac{\pounds_{bvcms}(\aleph_{v-1}, \aleph_v) \pounds_{bvcms}(\varkappa_v, \aleph_v)}{1 + \pounds_{bvcms}(\aleph_{v-1}, \aleph_v)} \\
&\precsim_{i_2} \pounds_{bvcms}(\varkappa_{v-1}, \aleph_v) + f \pounds_{bvcms}(\aleph_v, \aleph_{v+1})
\end{aligned}$$

which implies

$$\begin{aligned}
\pounds_{bvcms}(\aleph_v, \aleph_{v+1}) &\precsim_{i_2} \left(\frac{\varpi}{1-f}\right) \pounds_{bvcms}(\aleph_{v-1}, \aleph_v) \\
&= \nu \pounds_{bvcms}(\aleph_{v-1}, \aleph_v)
\end{aligned}$$

In the same way

$$\begin{aligned}
\pounds_{bvcms}(\aleph_{v-1}, \aleph_v) &= \pounds_{bvcms}(\eta \aleph_{v-2}, \eta \aleph_{v-1}) \\
&\precsim_{i_2} \varpi \pounds_{bvcms}(\aleph_{v-2}, \aleph_{v-1}) + f \frac{\pounds_{bvcms}(\aleph_{v-2}, \eta \aleph_{v-2}) \pounds_{bvcms}(\aleph_{v-1}, \eta \aleph_{v-1})}{1 + \pounds_{bvcms}(\aleph_{v-2}, \aleph_{v-1})} \\
&= \varpi \pounds_{bvcms}(\aleph_{v-2}, \aleph_{v-1}) + f \frac{\pounds_{bvcms}(\aleph_{v-2}, \aleph_{v-1}) \pounds_{bvcms}(\aleph_{v-1}, \aleph_v)}{1 + \pounds_{bvcms}(\aleph_{v-2}, \aleph_{v-1})} \\
&\precsim_{i_2} \varpi \pounds_{bvcms}(\aleph_{v-2}, \aleph_{v-1}) + f \pounds_{bvcms}(\aleph_{v-1}, \aleph_v)
\end{aligned}$$

which implies

$$\begin{aligned}
\pounds_{bvcms}(\aleph_{v-1}, \aleph_v) &\precsim_{i_2} \left(\frac{\varpi}{1-f}\right) \pounds_{bvcms}(\aleph_{v-2}, \aleph_{v-1}) \\
&= \nu(\aleph_{v-2}, \aleph_{v-1})
\end{aligned}$$

Continuing in the same way, we have

$$\begin{aligned}
\pounds_{bvcms}(\aleph_v, \aleph_{v+1}) &\precsim_{i_2} \nu \pounds_{bvcms}(\aleph_{v-1}, \aleph_v) \\
&\precsim_{i_2} \nu^2 \pounds_{bvcms}(\aleph_{v-2}, \aleph_{v-1}) \\
&\vdots \\
&\precsim_{i_2} \nu^v \pounds_{bvcms}(\aleph_0, \aleph_1).
\end{aligned} \tag{20}$$

Thus, $\pounds_{bvcms}(\aleph_v, \aleph_{v+1}) \precsim_{i_2} \nu^v \pounds_{bvcms}(\aleph_0, \aleph_1)$ for all $v \geq 0$. For all $v < \varsigma$, where v and ς are natural numbers, giving

$$
\begin{aligned}
\pounds_{bvcms}(\aleph_v,\aleph_\varsigma) \precsim_{i_2} & \varphi(\aleph_v,\aleph_{v+1})\pounds_{bvcms}(\aleph_v,\aleph_{v+1}) + \varphi(\aleph_{v+1},\aleph_\varsigma \pounds_{bvcms}(\aleph_{v+1},\aleph_\varsigma \\
\precsim_{i_2} & \varphi(\aleph_v,\aleph_{v+1})\pounds_{bvcms}(\aleph_v,\aleph_{v+1}) \\
& + \varphi(\aleph_{v+1},\aleph_\varsigma\varphi(\aleph_{v+1},\aleph_{v+2})\pounds_{bvcms}(\aleph_{v+1},\aleph_{v+2}) \\
& + \varphi(\aleph_{v+1},\aleph_\varsigma\varphi(\aleph_{v+2},\aleph_\varsigma\pounds_{bvcms}(\aleph_{v+2},\aleph_\varsigma \\
\precsim_{i_2} & \varphi(\aleph_v,\aleph_{v+1})\pounds_{bvcms}(\aleph_v,\aleph_{v+1}) \\
& + \varphi(\aleph_{v+1},\aleph_\varsigma\varphi(\aleph_{v+1},\aleph_{v+2})\pounds_{bvcms}(\aleph_{v+1},\aleph_{v+2}) \\
& + \varphi(\aleph_{v+1},\aleph_\varsigma\varphi(\aleph_{v+2},\aleph_\varsigma\varphi(\aleph_{v+2},\aleph_{v+3})\pounds_{bvcms}(\aleph_{v+2},\aleph_{v+3}) \\
& + \varphi(\aleph_{v+1},\aleph_\varsigma\varphi(\aleph_{v+2},\aleph_\varsigma\varphi(\aleph_{v+3},\aleph_\varsigma\pounds_{bvcms}(\aleph_{v+3},\aleph_\varsigma \\
\precsim_{i_2} & \cdots \precsim_{i_2} \varphi(\aleph_v,\aleph_{v+1})\pounds_{bvcms}(\aleph_v,\aleph_{v+1}) \\
& + \sum_{\iota=v+1}^{\varsigma-2}(\prod_{\mathrm{j}=v+1}^{\iota}\varphi(\aleph_\mathrm{j},\aleph_\varsigma)\varphi(\aleph_\iota,\aleph_{\iota+1})\pounds_{bvcms}(\aleph_\iota,\aleph_{\iota+1}) \\
& + \prod_{\mathfrak{p}=v+1}^{\varsigma-1}\varphi(\aleph_\mathfrak{p},\aleph_\varsigma\pounds_{bvcms}(\aleph_{\varsigma-1},\aleph_\varsigma \\
\precsim_{i_2} & \varphi(\aleph_v,\aleph_{v+1})\mathfrak{a}^v\pounds_{bvcms}(\aleph_0,\aleph_1) \\
& + \sum_{\iota=v+1}^{\varsigma-2}(\prod_{\mathrm{j}=v+1}^{\iota}\varphi(\aleph_\mathrm{j},\aleph_\varsigma)\varphi(\aleph_\iota,\aleph_{\iota+1})\mathfrak{a}^\iota\pounds_{bvcms}(\aleph_0,\aleph_1) \\
& + \prod_{\mathfrak{p}=v+1}^{\varsigma-1}\varphi(\aleph_\mathfrak{p},\aleph_\varsigma\mathfrak{a}^{\varsigma-1}\pounds_{bvcms}(\aleph_0,\aleph_1) \\
\precsim_{i_2} & \varphi(\aleph_v,\aleph_{v+1})\mathfrak{a}^v\pounds_{bvcms}(\aleph_0,\aleph_1) \\
& + \sum_{\iota=v+1}^{\varsigma-2}(\prod_{\mathrm{j}=v+1}^{\iota}\varphi(\aleph_\mathrm{j},\aleph_\varsigma)\varphi(\aleph_\iota,\aleph_{\iota+1})\mathfrak{a}^\iota\pounds_{bvcms}(\aleph_0,\aleph_1) \\
& + (\prod_{\mathfrak{p}=v+1}^{\varsigma-1}\varphi(\aleph_\mathfrak{p},\aleph_\varsigma)\mathfrak{a}^{\varsigma-1}\varphi(\aleph_{\varsigma-1},\aleph_\varsigma\pounds_{bvcms}(\aleph_0,\aleph_1) \\
= & \varphi(\aleph_v,\aleph_{v+1})\mathfrak{a}^v\pounds_{bvcms}(\aleph_0,\aleph_1) \\
& + \sum_{\iota=v+1}^{\varsigma-1}(\prod_{\mathrm{j}=v+1}^{\iota}\varphi(\aleph_\mathrm{j},\aleph_\varsigma)\varphi(\aleph_\iota,\aleph_{\iota+1})\mathfrak{a}^\iota\pounds_{bvcms}(\aleph_0,\aleph_1) \\
\precsim_{i_2} & \varphi(\aleph_v,\aleph_{v+1})\mathfrak{a}^v\pounds_{bvcms}(\aleph_0,\aleph_1) \\
& + \sum_{\iota=v+1}^{\varsigma-1}(\prod_{\mathrm{j}=0}^{\iota}\varphi(\aleph_\mathrm{j},\aleph_\varsigma)\varphi(\aleph_\iota,\aleph_{\iota+1})\mathfrak{a}^\iota\pounds_{bvcms}(\aleph_0,\aleph_1).
\end{aligned}
$$

Furthermore, using $\varphi(\aleph,\mathrm{i}) \geq 1$. Let

$$\mathcal{S}_\flat = \sum_{\iota=0}^{\flat}(\prod_{\mathrm{j}=0}^{\iota}\varphi(\aleph_\mathrm{j},\aleph_\varsigma)\varphi(\aleph_\iota,\aleph_{\iota+1})\mathfrak{a}^\iota.$$

Hence, we have

$$\pounds_{bvcms}(\aleph_v,\aleph_\varsigma) \precsim_{i_2} \pounds_{bvcms}(\aleph_0,\aleph_1)[\nu^v\varphi(\aleph_v,\aleph_{v+1}) + (\mathcal{S}_{\varsigma-1},\mathcal{S}_v)]. \tag{21}$$

By using the ratio test, ensuring that $\lim_{\varsigma,v\to\infty}\mathcal{S}_v$ exists, the sequence $\{\mathcal{S}_v\}$ is a real Cauchy sequence. As $\varsigma, v \to \infty$, we conclude that

$$\lim_{\varsigma,v\to\infty}\pounds_{bvcms}(\aleph_v,\aleph_\varsigma) = 0,$$

Then, $\{\aleph_v\}$ is a Cauchy sequence in the complete BVCMS $(\zeta, £_{bvcms})$. Therefore, the sequence $\{\aleph_v\}$ converges to $\aleph^* \in \zeta$.

By the definition of continuity, we obtain

$$\aleph^* = \lim_{v\to\infty} \aleph_{v+1} = \lim_{v\to\infty} \eta\aleph_v = \eta(\lim_{v\to\infty} \aleph_v) = \eta\aleph^*.$$

Let $\aleph^*, \imath^* \in \text{fix}\,\eta$ as two fixed points of η. Then,

$$\begin{aligned} £_{bvcms}(\eta\aleph^*, \eta\imath^*) &\precsim_{i_2} \varpi £_{bvcms}(\aleph^*, \imath^*) + f\frac{£_{bvcms}(\aleph^*, \eta\imath^*)£_{bvcms}(\aleph^*, \eta\imath^*)}{1 + £_{bvcms}(\aleph^*, \imath^*)} \\ &\precsim_{i_2} \varpi £_{bvcms}(\aleph^*, \imath^*) + f\frac{£_{bvcms}(\aleph^*, \aleph^*)£_{bvcms}(\imath^*, \imath^*)}{1 + £_{bvcms}(\aleph^*, \imath^*)} \\ &\precsim_{i_2} \varpi £_{bvcms}(\aleph^*, \imath^*). \end{aligned}$$

Therefore, $£_{bvcms}(\aleph^*, \imath^*) = 0$; then $\aleph^* = \imath^*$. Hence, η has a UFP. □

If we drop the continuous condition, we obtain

Theorem 6. *Let $(\zeta, £_{bvcms})$ be a complete BVCMS and $\eta : \zeta \to \zeta$ a map, such that*

$$£_{bvcms}(\eta\aleph, \eta\imath) \precsim_{i_2} \varpi £_{bvcms}(\aleph, \imath) + f\frac{£_{bvcms}(\aleph, \eta\aleph)£_{bvcms(\imath,\eta\imath)}}{1 + £_{bvcms}(\aleph, \imath)}, \tag{22}$$

for all $\aleph, \imath \in \zeta$, where $\varpi, f \in [0,1)$, such that $\nu = \frac{\varpi}{1-f} < 1$. For $\aleph_0 \in \zeta$ we denote $\aleph_v = \eta^v\aleph_0$. Suppose that

$$\max_{\varsigma\geq 1} \lim_{i_2\to\infty} \frac{\varphi(\aleph_{i_2+1}, \aleph_{i_2+2})}{\varphi(\aleph_{i_2}, \aleph_{i_2+1})}\varphi(\aleph_{i_2+1}, \aleph_\varsigma < \frac{1}{\nu}, \tag{23}$$

In addition, assume that for every $\aleph \in \zeta$ we have

$$\lim_{v\to\infty} \varphi(\aleph_v, \aleph) \quad \text{and} \quad \lim_{v\to\infty} (\aleph, \aleph_v) \quad \text{exists.} \tag{24}$$

Therefore, it is finite. Then η has a UFP.

Proof. By proving Theorem 5 and using Lemma 2, we obtain a Cauchy sequence $\{\aleph_v\}$ which converges to $\aleph^* \in \zeta$. Then,

$$£_{bvcms}(\aleph^*, \aleph_{v+1}) \precsim_{i_2} \varphi(\aleph^*, \aleph_v)£_{bvcms}(\aleph^*, \aleph_v) + \varphi(\aleph_v, \aleph_{v+1})£_{bvcms}(\aleph_v, \aleph_{v+1}).$$

Using (2), (3) and (23), we deduce that

$$\lim_{v\to\infty} £_{bvcms}(\aleph^*, \aleph_{v+1}) = 0.$$

Using the triangular inequality and (1),

$$\begin{aligned} £_{bvcms}(\aleph^*, \eta\aleph^*) &\precsim_{i_2} \varphi(\aleph^*, \aleph_{v+1})£_{bvcms}(\aleph^*, \aleph_{v+1}) + \varphi(\aleph_{v+1}, \eta^*)£_{bvcms}(\aleph_{v+1}, \eta\aleph^*) \\ &\precsim_{i_2} \varphi(\aleph^*, \aleph_{v+1})£_{bvcms}(\aleph^*, \aleph_{v+1}) + \varphi(\aleph_{v+1}, \eta^*)[\varpi £_{bvcms}(\aleph_v, \aleph^*) \\ &\quad + f\frac{£_{bvcms}(\aleph_v, \eta\aleph_v)£_{bvcms}(\aleph^*, \eta\aleph^*)}{1 + £_{bvcms}(\aleph_v, \aleph^*)}] \\ &= \varphi(\aleph^*, \aleph_{v+1})£_{bvcms}(\aleph^*, \aleph_{v+1}) + \varphi(\aleph_{v+1}, \eta^*)[\varpi £_{bvcms}(\aleph_v, \aleph^*) \\ &\quad + f\frac{£_{bvcms}(\aleph_v, \eta\aleph_v)£_{bvcms}(\aleph^*, \eta\aleph^*)}{1 + £_{bvcms}(\aleph_v, \aleph^*)}]. \end{aligned}$$

As $\upsilon \to \infty$ in (3) and (24), we find that $\pounds_{bvcms}(\aleph^*, \eta\aleph^*) = 0$. From Lemma 1, the sequence $\{\aleph_\upsilon\}$ uniquely converge at $\aleph^* \in \zeta$. □

Example 6. *Let $\zeta = \{0, 1, 2\}$ and $\pounds_{bvcms} \colon \zeta \times \zeta \to \mathbb{C}$ be a symmetrical metric defined as*

$$\pounds_{bvcms}(\aleph, \aleph) = 0 \quad \text{for each} \quad \aleph \in \zeta$$

and

$$\pounds_{bvcms}(0,1) = 1 + i_2, \pounds_{bvcms}(1,2) = 1 + i_2, \pounds_{bvcms}(0,2) = 4 + 4i_2.$$

Defining $\varphi \colon \zeta \times \zeta \to [1, \infty)$ by

$$\varphi(2,2) = \frac{9}{5}, \varphi(0,0) = 5, \varphi(1,1) = \frac{7}{3},$$
$$\varphi(1,2) = 2, \varphi(0,1) = 3, \varphi(0,2) = \frac{7}{3}.$$

Clearly, $(\zeta, \pounds_{bvcms})$ is a BVCMS. A self-map η on ζ defined by $\eta(0) = \eta(1) = \eta(2) = 1$. If we assume that $\varpi = f = \frac{1}{5}$, we obtain

Case 1. *If $\aleph = \mathrm{i} = 0, \aleph = \mathrm{i} = 1, \aleph = \mathrm{i} = 2$ we have $\pounds_{bvcms}(\eta\aleph, \eta\mathrm{i}) = 0$.*

Case 2. *If $\aleph = 0, \mathrm{i} = 1$, we obtained $\pounds_{bvcms}(\eta\aleph, \eta\mathrm{i}) = 0 \precsim_{i_2} \varpi\pounds_{bvcms}(\aleph, \mathrm{i}) + f\frac{\pounds_{bvcms}(\aleph,\eta\aleph)\pounds_{bvcms}(\mathrm{i},\eta\mathrm{i})}{1+\pounds_{bvcms}(\aleph,\mathrm{i})}$*

Case 3. *If $\aleph = 0, \mathrm{i} = 2$, we have $\pounds_{bvcms}(\eta\aleph, \eta\mathrm{i}) = 0 \precsim_{i_2} \varpi\pounds_{bvcms}(\aleph, \mathrm{i}) + f\frac{\pounds_{bvcms}(\aleph,\eta\aleph)\pounds_{bvcms}(\mathrm{i},\eta\mathrm{i})}{1+\pounds_{bvcms}(\aleph,\mathrm{i})}$*

Case 4. *If $\aleph = 1, \mathrm{i} = 2$, we have $\pounds_{bvcms}(\eta\aleph, \eta\mathrm{i}) = 0 \precsim_{i_2} \varpi\pounds_{bvcms}(\aleph, \mathrm{i}) + f\frac{\pounds_{bvcms}(\aleph,\eta\aleph)\pounds_{bvcms}(\mathrm{i},\eta\mathrm{i})}{1+\pounds_{bvcms}(\aleph,\mathrm{i})}$*

Therefore, all axioms of Theorem 6 are fulfilled. Hence, η has a UFP, which is $\aleph^* = 1$.

Application

Now, we see some basic definitions from the fractional calculus.

Let $\varkappa \in \mathcal{C}[0,1]$ be a function, the Rieman–Liouville fractional derivatives of order $\delta > 0$ are defined as:

$$\frac{1}{\Gamma(n-\delta)} \frac{d^n}{d\flat^n} \int_0^\flat \frac{\varkappa(c)dc}{(\flat - c)^{\delta - n + 1}} = \mathcal{D}^\delta \varkappa(\flat),$$

presenting that the right-hand side is point-wise on $[0,1]$, where Γ is the Euler Γ function and $[\delta]$ is the integer part of δ.

Consider the following FDE

$$\begin{aligned} &{}^c\mathcal{D}^\xi \varkappa(\flat) + f(\flat, \varkappa(\flat)) = 0, \quad 1 \leq \flat \leq 0, \quad 2 \leq \xi > 1; \\ &\varkappa(0) = \varkappa(1) = 0, \end{aligned} \tag{25}$$

where ${}^c\mathcal{D}^\xi$ represents the order of ξ as the Caputo fractional derivatives and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ as a continuous map defined by

$${}^c\mathcal{D}^\xi = \frac{1}{\Gamma(n-\xi)} \int_0^\flat \frac{\varkappa^n(c)dc}{(\flat - c)^{\xi - n + 1}}.$$

The given FDE (25) is equivalent to

$$\varkappa(\flat) = \int_0^1 \Omega(\flat, c) f(\flat, \varkappa(c)) dc,$$

for all $\varkappa \in \zeta$ and $\flat \in [0,1]$, where

$$\Omega(\flat, c) = \begin{cases} \frac{[\flat(1-c)]^{\xi-1} - (\flat-c)^{\xi-1}}{\Gamma(\xi)}, & 0 \le c \le \flat \le 1, \\ \frac{[\flat(1-c)]^{\xi-1}}{\Gamma(\xi)}, & 0 \le \flat \le c \le 1. \end{cases}$$

Consider $\mathcal{C}([0,1],\mathbb{R}) = \zeta$ as the space of the continuous map described by $[0,1]$, and $\pounds_{bvcms}\colon \zeta \times \zeta \to \mathbb{C}_2$ a bicomplex-valued controlled metric, such that

$$\pounds_{bvcms}(\varkappa, \gamma) = \sup_{\flat \in [0,1]} |\varkappa(\flat) - \gamma(\flat)|^2 + i_2 \sup_{\flat \in [0,1]} |\varkappa(\flat) - \gamma(\flat)|^2,$$

for all $\varkappa, \gamma \in \zeta$. Let $\varphi_\flat \colon \zeta \times \zeta \to [1, \infty)$ be defined by

$$\varphi_\flat(\varkappa, \gamma) = 2,$$

for all $\varkappa, \gamma \in \zeta$. Then, $(\zeta, \pounds_{bvcms})$ is a complete BVCMS.

Theorem 7. *Consider the non-linear FDE (25). Suppose that the following assertions are satisfied:*

(i) *There exists* $\mathfrak{m} \in [0,1]$ *and* $\varkappa, \gamma \in \mathcal{C}([0,1],\mathbb{R})$, *such that*

$$|f(\flat, \varkappa) - f(\flat, \gamma)| \le \sqrt{\mathfrak{m}}|\varkappa(\flat) - \gamma(\flat)|;$$

(ii)

$$\sup_{\flat \in [0,1]} \int_0^1 \Omega(\flat, c)dc < 1.$$

Then, FDE (25) has a unique solution in ζ.

Proof. Consider the map $\eta \colon \zeta \to \zeta$ defined by

$$\eta\varkappa(\flat) = \int_0^1 \Omega(\flat, c) f(\flat, \varkappa(c))dc.$$

Now, for all $\varkappa, \gamma \in \zeta$, we deduce

$$\begin{aligned} |\eta\varkappa(\flat) - \eta\gamma(\flat)|^2(1+i_2) &= |\int_0^1 \Omega(\flat,c)f(\flat,\varkappa(c))dc - \int_0^1 \Omega(\flat,c)f(\flat,\gamma(c))dc|^2(1+i_2) \\ &\le \left(\int_0^1 \Omega(\flat,c)|f(\flat,\varkappa(c)) - f(\flat,\gamma(c))|dc\right)^2(1+i_2) \\ &\le \left(\int_0^1 \Omega(\flat,c)dc\right)^2 \int_0^1 \mathfrak{m}|\varkappa(\flat) - \gamma(\flat)|^2 dc(1+i_2). \end{aligned}$$

Taking the supreme, we obtain

$$\pounds_{bvcms}(\eta\varkappa, \eta\gamma) \le \mathfrak{m}\pounds_{bvcms}(\varkappa, \gamma).$$

Therefore, all conditions of Theorem 1 are fulfilled and the operator η has a UFP. □

4. Conclusions

In this paper we introduced the concept of BVCMS and FPTs for Banach-, Kannan- and Fisher-type contractions concepts. Furthermore, we presented examples that elaborated the usability of our results. Meanwhile, we provided an application for the existence of a solution to an FDE using one of our results. This concept can be applied for further investigations into studying BVCMSs for other structures in metric spaces.

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